Schwarzian Norms and Two-Point Distortion

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Schwarzian Norms and Two-Point Distortion

f analytic, $f'(z) \neq 0$. Schwarzian derivative

$$\mathcal{S}f = (f''/f')' - \tfrac{1}{2}(f''/f')^2.$$

$\mathcal{S}(T \circ f) = \mathcal{S}f$ for any Möbius trsf T.

Schwarzian norm

$$\|\mathcal{S}f\| = \sup_{z\in\mathbb{D}} (1-|z|^2)^2 |\mathcal{S}f(z)|\,.$$

Möbius invariance: $\|\mathcal{S}(f\circ\varphi)\| = \|\mathcal{S}f\|\,,$
φ any Möbius automorphism of unit disk $\mathbb{D}$.

Nehari: $\|\mathcal{S}f\| \le 2 \implies f$ univalent in $\mathbb{D}$.

B. Schwarz: $\|\mathcal{S}f\| \le 2(1+\delta^2),\ \delta > 0$
$\implies f$ uniformly locally univalent in $\mathbb{D}$:
$f(\alpha) = f(\beta),\ \alpha \neq \beta \implies d(\alpha,\beta) \ge \pi/\delta$,
where $d(\alpha,\beta)$ is hyperbolic distance,

$$d(\alpha,\beta) = \tfrac{1}{2}\log\frac{1+\rho(\alpha,\beta)}{1-\rho(\alpha,\beta)}, \quad \rho(\alpha,\beta) = \left|\frac{\alpha-\beta}{1-\overline{\alpha}\beta}\right|.$$

Main Results

For f analytic, $f'(z) \neq 0$ in $\mathbb{D}$, $\alpha, \beta \in \mathbb{D}$, let

$$\Delta_f(\alpha,\beta) = \frac{|f(\alpha) - f(\beta)|}{\sqrt{(1-|\alpha|^2)|f'(\alpha)|}\,\sqrt{(1-|\beta|^2)|f'(\beta)|}}.$$

Chuaqui & Pommerenke (1999):

$$\|\mathcal{S}f\| \leq 2 \iff \Delta_f(\alpha,\beta) \geq d(\alpha,\beta) \quad \forall \alpha,\beta \in \mathbb{D}.$$

Theorem. For each $\delta > 0$, $\|\mathcal{S}f\| \leq 2(1+\delta^2)$

$$\iff \Delta_f(\alpha,\beta) \geq (1/\delta)\sin(\delta\, d(\alpha,\beta)) \quad \forall \alpha,\beta \in \mathbb{D} \text{ with } d(\alpha,\beta) \leq \tfrac{\pi}{\delta}$$

$$\iff \Delta_f(\alpha,\beta) \leq \tfrac{1}{\sqrt{2+\delta^2}} \sinh\!\big(\sqrt{2+\delta^2}\, d(\alpha,\beta)\big) \quad \forall \alpha,\beta \in \mathbb{D}.$$

Key to Proof

Comparison Lemma. Let $Q(x) > 0$ and continuous for $x \in [0, 1)$. Let $v(x)$ and $w(x)$ be the solutions of

$$v''(x) + Q(x)v(x) = 0\,, \quad v(0) = 0\,,\ v'(0) = 1\,;$$
$$w''(x) - Q(x)w(x) = 0\,, \quad w(0) = 0\,,\ w'(0) = 1\,,$$

respectively. Suppose $v(x) > 0$ in $(0, \xi)$, where $0 < \xi \leq 1$. Let $p(z)$ be analytic and satisfy $|p(z)| \leq Q(|z|)$ for all $z \in \mathbb{D}$. Then sol'n of

$$u''(z) + p(z)u(z) = 0\,, \qquad u(0) = 0\,,\ u'(0) = 1$$

satisfies the inequalities

$$v(|z|) \leq |u(z)|\,, \quad |z| < \xi\,; \quad |u(z)| \leq w(|z|)\,, \quad z \in \mathbb{D}.$$

Sketch of Proof of Theorem

Take $Q(x) = \frac{1+\delta^2}{(1-x^2)^2}$ in comparison lemma. Then

$$v(x) = \frac{1}{\delta}\sqrt{1-x^2}\,\sin\left(\frac{\delta}{2}\log\frac{1+x}{1-x}\right),$$
$$w(x) = \frac{\sqrt{1-x^2}}{\sqrt{2+\delta^2}}\,\sinh\left(\frac{\sqrt{2+\delta^2}}{2}\log\frac{1+x}{1-x}\right).$$

First positive zero of $v(x)$ is $\xi = \tanh(\pi/\delta)$.

Can show that

$$\Delta_{T\circ f}(\alpha,\beta) = \Delta_f(\alpha,\beta)$$

for every Möbius transformation T, and

$$\Delta_{f\circ\varphi}(\alpha,\beta) = \Delta_f(\varphi(\alpha),\varphi(\beta)),$$

for every Möbius automorphism φ of $\mathbb{D}$.

Sketch of Proof (cont'd)

Let $\|\mathcal{S}f\| \le 2(1+\delta^2)$. Suppose WLOG that $f(0)=0$ and $f'(0)=1$. Let

$$g(z) = -1/f(z), \quad u(z) = [g'(z)]^{-1/2}.$$

Then $u(0)=0$, $u'(0)=1$, and

$$u'' + [\tfrac{1}{2}\,\mathcal{S}f]\,u = 0, \quad \text{since } \ \mathcal{S}g = \mathcal{S}f.$$

But $|\frac{1}{2}\,\mathcal{S}f(z)| \le Q(|z|)$, so comparison lemma gives

$$\Delta_f(0,z) \ge \frac{1}{\delta}\,\sin\left(\delta\, d(0,z)\right), \quad d(0,z) \le \frac{\pi}{\delta}\,;$$
$$\Delta_f(0,z) \le \frac{1}{\sqrt{2+\delta^2}}\,\sinh\bigl(\sqrt{2+\delta^2}\,d(0,z)\bigr)\,.$$

Deduce general inequality for $\alpha,\beta \in \mathbb{D}$ by Möbius invariance.

Proof of Converse

Either 2-pt dist cond $\implies \|\mathcal{S}f\| \le 2(1+\delta^2)$.
Suppose, for instance, that

$$\Delta_f(\alpha,\beta) \le \frac{1}{\sqrt{2+\delta^2}}\,\sinh\bigl(\sqrt{2+\delta^2}\,d(\alpha,\beta)\bigr)$$

for all $\alpha,\beta \in \mathbb{D}$. Take $\alpha = 0$, assume WLOG that $f(0) = 0$ and $f'(0) = 1$. Then write

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots .$$

No loss of info to take $\alpha = 0$. Then hyp is

$$\frac{|f(z)|^2}{|f'(z)|} \le \frac{1-|z|^2}{2+\delta^2}\,\sinh^2\bigl(\sqrt{2+\delta^2}\,d(0,z)\bigr)\,, \ \ z \in \mathbb{D}.$$

Enough to show $|\mathcal{S}f(0)| \le 2(1+\delta^2)$.
But $\mathcal{S}f(0) = 6(a_3 - a_2^2)$, so need to show

$$|a_3 - a_2^2| \le \tfrac{1}{3}(1+\delta^2)\,.$$

Proof of Converse (cont'd)

Calculations give

$$\frac{f(z)^2}{f'(z)} = z^2[1 + (a_2^2 - a_3)z^2 + \dots] \qquad \text{and}$$

$$\frac{1-|z|^2}{2+\delta^2}\,\sinh^2\!\big(\sqrt{2+\delta^2}\,d(0,z)\big)$$

$$= r^2[1 + \tfrac{1}{3}(1+\delta^2)r^2 + \dots]\,, \quad r = |z|\,.$$

Therefore, hypothesis implies

$$\operatorname{Re}\big\{(a_2^2 - a_3)z^2 + O(r^3)\big\} \le \tfrac{1}{3}(1+\delta^2)r^2 + O(r^3),$$

which gives as $r \to 0$

$$\operatorname{Re}\big\{(a_2^2 - a_3)e^{2i\theta}\big\} \le \tfrac{1}{3}(1+\delta^2),$$

where $z = re^{i\theta}$. This for all θ implies

$$|a_3 - a_2^2| \le \tfrac{1}{3}(1+\delta^2),$$

as desired.