Fourier Integral Operators with singularities

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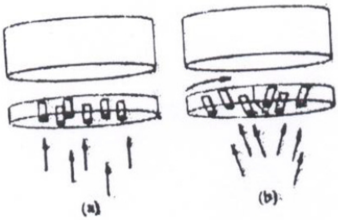
Motivation: Slant-Hole Single Photon Emission Computed Tomography (SPECT)

- Detect metabolic processes or body structure
- Backprojection is used
- Understand the added singularities
- Decrease their strength

Standard SPECT versus Slant Hole SPECT



- (a) detector moves around the body
- lines $\perp$ to axis of rotation
- (b) detector moves about its center
- lines at angle $\phi \in (0, \frac{\pi}{2})$
- local algorithm by Quinto, Bakhos and Chung

Geometry of SPECT

- S a simple curve $\theta : I \to S^2$
- $\theta^\perp = \{x \in R^3 | \; x \cdot \theta = 0\}$
- $Y_S = \{(y, \theta) | \; \theta \in S, y \;\; \in \theta^\perp\}$
- $L(y, \theta) = \{y + t\theta | \; t \in R\}$
- $P_m f(y, \theta) = \int_{x \in L(y,\theta)} f(x) m(y, \theta, x) dx$
- $P_m^* g(x) = \int_{a \in I} g(x - (x \cdot \theta)\theta, \theta) m da$
- Study $P_m^* P_m$

Assumptions

- $(1)\ \theta'' \cdot \theta \neq 0$
- $(2)\ \beta' = \theta \times \theta'' \neq 0, \quad \beta = \theta \times \theta'$
- $(3)\ \beta$ is a simple regular curve
- $(1) \Rightarrow P_m =$ FIO
- $(2) + (3) \Rightarrow P_m =$ FIO with singularities
- $\phi = \frac{\pi}{2} \Rightarrow \beta = e_3 \Rightarrow (2) + (3)$ fail

Fourier Integral Operators

- $Ff(x) = \int e^{i\phi(x,y,\theta)} a(x,y,\theta) f(y) d\theta dy$
- $F : \mathcal{E}'(Y) \to \mathcal{D}'(X)$
- ϕ is a nondegenerate **phase function**
- a is a **symbol** S^M: $|\partial^\alpha_{x,y} \partial^\beta_\theta a| < c(1 + |\theta|)^{M-|\beta|}$
- C is a **canonical relation** in $T^*X \setminus 0 \times T^*Y \setminus 0$
- $C = \{(x, d_x\phi; y, -d_y\phi); d_\theta\phi = 0\}$
- $I^m(X, Y, C)$, $m = M - \frac{N}{2} + \frac{n_X + n_Y}{4}$
- Adjoint F^*: $Ff(y) = \int e^{-i\phi(x,y,\theta)} \bar{a}(x,y,\theta) f(x) d\theta dx$
- If $F \in I^m(X, Y, C)$ then $F^* \in I^m(Y, X, C^t)$.

Composition of FIOs

- If $F_1 \in I^{m_1}(X, Y, C_1)$ and $F_2 \in I^{m_2}(Y, Z, C_2)$ then $F_1 \circ F_2 \longrightarrow C_1 \circ C_2$?

- **Duistermaat- Guillemin: clean intersection condition**

- $C_1 \times C_2$ intersects $T^*X \times \Delta_{T^*Y} \times T^*Z$ cleanly with excess e, then $F_1 \circ F_2 \in I^{m_1+m_2+\frac{e}{2}}(X, Z; C_1 \circ C_2)$

- $e = 0 \Rightarrow$ **Hormander: transverse intersection condition**

Nondegenerate versus degenerate geometry

- Geometry of $$C \in T^*X \setminus 0 \times T^*Y \setminus 0$$
$$\begin{array}{ccc} & \pi_L \swarrow \quad \searrow \pi_R & \\ T^*X \setminus 0 & & T^*Y \setminus 0 \end{array}$$

- π_L, π_R are local diffeomeorphisms:
- $F_1 \circ F_2 \longrightarrow$ transverse intersection condition
- degenerate : have singularities; Examples: folds; blowdowns
- Bolker condition (π_L is an injective immersion and π_R is an submersion) then F^*F is covered by the clean intersection condition and $F^*F = \Psi$DO

Singularities

- **Whitney Folds**

  $f : N \to M$ has a fold singularity along $\Sigma = \{x : det\ df = 0\}$ if Σ is smooth and if Ker $df \not\subseteq T\Sigma$.

- Ex: $f(x_1, x_2) = (x_1, x_2^2)$
- $\Sigma = \{x_2 = 0\}$; Ker $df = \frac{\partial}{\partial x_2}$
- **Blowdown**

  $f : N \to M$ has a blowdown singularity along $\Sigma = \{x : det\ df = 0\}$ if Σ is smooth, if Ker $df \subset T\Sigma$.

- Ex: $f(x_1, x_2) = (x_1, x_1 x_2)$
- $\Sigma = \{x_1 = 0\}$; Ker $df = \frac{\partial}{\partial x_2}$

Canonical relation of P_m

- $C = \{(r, s, a, \eta_r, \eta_s, x \cdot (\eta_r \alpha'(a) + \eta_s \beta'(a))$;
  $x, \eta_r \alpha(a) + \eta_s \beta(a) r) | \, (\eta_r, \eta_s) \neq 0, \; x \cdot \alpha(a) = r, \; x \cdot \beta(a) = s\}$

- π_R and π_L drop rank by 1 at $\Sigma = \{\eta_r = 0\}$

- Ker $d\pi_R = \partial_a - c\eta_s \partial_{\eta_r} \Rightarrow \pi_R$ is a fold
- Ker $d\pi_L = \theta \cdot (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) \Rightarrow \pi_L$ is a blowdown

- $C =$ fibered folding canonical relation
- A. Greenleaf $\&$ G. Uhlmann
- $F \in I^m(C) \Rightarrow F^* F \in I^{2m,0}(\Delta, \Lambda_{\pi_R(\Sigma)})$
- $\Lambda_\Gamma = \{(x, \xi; y, \eta) | (x, \xi) \in \Gamma = \{p = 0\}, \; (y, \eta) =$
  $\exp{(tH_p)}(x, \xi), \quad t \in R\}$

$I^{p,l}(C_0, C_1)$

- C_0, C_1 intersect cleanly.
- u has an oscillatory representation of certain type with a product type symbol
- $WF(u) \subset C_0 \cup C_1$
- $u \in I^{p,l}(C_0, C_1)$ then $u \in I^{p+l}(C_0 \setminus C_1)$ and $u \in I^{p}(C_1 \setminus C_0)$
- Ex: $\tilde{C}_0 = \Delta, \tilde{C}_1 = N^*\{x' - y' = 0\}$
- $u(x,y) = \int e^{i(x'-y')\cdot\xi' + (x''-y'')\cdot\xi''} a(x,\xi) d\xi' d\xi''$ where
- $|\partial_x^\alpha \partial_{\xi'}^\beta \partial_{\xi''}^\gamma a| < c(1+|\xi|)^{m-|\beta|}(1+|\xi''|)^{m'-|\gamma|}$

$$|\xi''| \leq |\xi'|$$

Backprojection

- $P_m \in I^{-\frac{1}{2}}(C)$
- $P_m^* P_m \in I^{-1,0}(\Delta, \Lambda_{\pi_R(\Sigma)})$
- $P_m^* P_m \in I^{-1}(\Delta \setminus \Lambda_{\pi_R(\Sigma)})$
- $P_m^* P_m \in I^{-1}(\Lambda_{\pi_R(\Sigma)} \setminus \Delta)$

De-emphasis of added singularities

- $Ł = P^*_{\frac{1}{m}} D P_m \in I^{1,0}(\Delta, \Lambda_{\pi_R(\Sigma)})$

- **Theorem** (RF, T. Quinto) If D has $\sigma(D)$ vanishing on $\pi_L(\Sigma)$ then $Ł \in I^{0,1}(\Delta, \Lambda_{\pi_R(\Sigma)})$

- $D = \partial^2_{y_1} - (\partial_{y_2} - c(y_3) y_1 \partial_{y_3})^2$

- $Ł \in I^1(\Delta \setminus \Lambda_{\pi_R(\Sigma)})$

- $Ł \in I^0(\Lambda_{\pi_R(\Sigma)} \setminus \Delta)$

- $\phi = \frac{\pi}{2}$; π_L and π_R have blowdown singularities

- Marhuenda $Ł \in I^{1,0}(\Delta, \Lambda_{\pi_R(\Sigma)})$