On a problem of Halmos: Unitary equivalence of a matrix to its transpose

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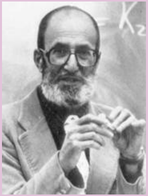
Abstract

Halmos asked whether every square complex matrix is unitarily equivalent to its transpose (UET). Ad hoc examples indicate that the answer is no. In this talk, we give a complete characterization of matrices which are UET. Surprisingly, the naïve conjecture that a matrix is UET if and only if it is unitarily equivalent to a complex symmetric (i.e., self-transpose) matrix is true in dimensions $n \leq 7$ but false for $n \geq 8$. In particular, unexpected building blocks begin to appear in dimensions 6 and 8. This is joint work with James E. Tener (UC Berkeley).

Partially supported by NSF Grant DMS-1001614 (SRG) and a NSF Graduate Research Fellowship (JET).

Problem (Halmos, LAPB - Problem 159)



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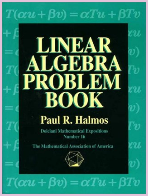

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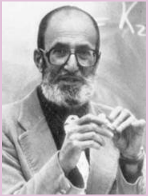

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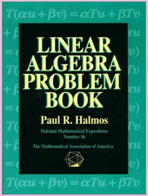

Solution (Halmos)

Ad-hoc methods show that

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is not UET.

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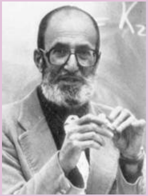

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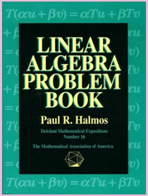

Solution (Halmos)

Ad-hoc methods show that

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

is not UET. Can we <u>characterize</u> those matrices which are UET?

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- It has an obvious "answer," which is totally wrong.
- Funny things happen once you hit dimensions 6 and 8.

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If T is unitarily equivalent to a complex symmetric (i.e., self-transpose) matrix, then T is *UECSM*.

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Halmos' matrix

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$$UET \quad \Longleftrightarrow \quad UECSM \qquad (\Leftarrow \text{trivial})$$

UECSMs are Tricky

Exactly one of the following is UECSM:

$$\begin{pmatrix} 0 & 7 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 0 & 7 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 0 & 7 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 0 & 7 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 0 & 7 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 6 \end{pmatrix}$$

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Which one is UECSM?

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Which one is UECSM?

The fourth matrix is unitarily equivalent to

$$\begin{pmatrix} 2+\sqrt{\frac{57}{2}} & 0 & -\frac{1}{2}i\sqrt{37-73\sqrt{\frac{2}{57}}} \\ 0 & 2-\sqrt{\frac{57}{2}} & -\frac{1}{2}i\sqrt{37+73\sqrt{\frac{2}{57}}} \\ -\frac{1}{2}i\sqrt{37-73\sqrt{\frac{2}{57}}} & -\frac{1}{2}i\sqrt{37+73\sqrt{\frac{2}{57}}} & 3 \end{pmatrix}$$

A Gallery of UECSMs

$$\begin{pmatrix} 3 & 6 & 6 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{pmatrix} \cong \begin{pmatrix} \frac{1}{2}\left(7-\sqrt{74}\right) & 3i\sqrt{\frac{2}{109}\left(74-\sqrt{74}\right)} & \frac{1}{2}i\sqrt{\frac{1}{109}\left(2629+72\sqrt{74}\right)} \\ 3i\sqrt{\frac{2}{109}\left(74-\sqrt{74}\right)} & 3+\frac{36\sqrt{74}}{109} & \frac{3}{109}\sqrt{5476+218\sqrt{74}} \\ \frac{1}{2}i\sqrt{\frac{1}{109}\left(2629+72\sqrt{74}\right)} & \frac{3}{109}\sqrt{5476+218\sqrt{74}} & \frac{1}{218}\left(763+37\sqrt{74}\right) \end{pmatrix}$$

$$\begin{pmatrix} 3 & 5 & 4 \\ 0 & 8 & 4 \\ 0 & 0 & 3 \end{pmatrix} \cong \begin{pmatrix} \frac{1}{2}\left(11-\sqrt{82}\right) & 2i\sqrt{\frac{2}{73}\left(82-5\sqrt{82}\right)} & \frac{1}{2}i\sqrt{\frac{1}{73}\left(1537+160\sqrt{82}\right)} \\ 2i\sqrt{\frac{2}{73}\left(82-5\sqrt{82}\right)} & 3+\frac{16\sqrt{82}}{73} & \frac{2}{73}\sqrt{6724+730\sqrt{82}} \\ \frac{1}{2}i\sqrt{\frac{1}{73}\left(1537+160\sqrt{82}\right)} & \frac{2}{73}\sqrt{6724+730\sqrt{82}} & \frac{1}{146}\left(803+41\sqrt{82}\right) \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2 & 2 \\ 7 & 0 & 0 \\ 7 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} \frac{1}{2}\left(5-\sqrt{187}\right) & -5i\sqrt{\frac{561+5\sqrt{187}}{1658}} & -i\sqrt{\frac{3350}{829}-\frac{125\sqrt{187}}{1658}} \\ -5i\sqrt{\frac{561+5\sqrt{187}}{1658}} & \frac{1}{829}\left(1870+293\sqrt{187}\right) & \frac{9}{829}\sqrt{\frac{1}{2}\left(173723+7075\sqrt{187}\right)} \\ -i\sqrt{\frac{3350}{829}-\frac{125\sqrt{187}}{1658}} & \frac{9}{829}\sqrt{\frac{1}{2}\left(173723+7075\sqrt{187}\right)} & \frac{81}{-5+3\sqrt{187}} \end{pmatrix}$$

$$\begin{pmatrix} 9 & 8 & 9 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix} \cong \begin{pmatrix} 8-\frac{\sqrt{149}}{2} & \frac{9}{2}i\sqrt{\frac{16837+64\sqrt{149}}{13093}} & i\sqrt{\frac{133672}{13093}-\frac{1296\sqrt{149}}{13093}} \\ \frac{9}{2}i\sqrt{\frac{16837+64\sqrt{149}}{13093}} & \frac{207440+9477\sqrt{149}}{26186} & \frac{18\sqrt{3978002+82324\sqrt{149}}}{13093} \\ i\sqrt{\frac{133672}{13093}-\frac{1296\sqrt{149}}{13093}} & \frac{18\sqrt{3978002+82324\sqrt{149}}}{13093} & \frac{92675+1808\sqrt{149}}{13093} \end{pmatrix}$$

Another Contender?

Skew-Hamiltonian Matrices

Definition

If $B^t = -B$ and $D^t = -D$, then

$$T = \begin{pmatrix} A & B \\ D & A^t \end{pmatrix}$$

is a *skew-Hamiltonian* matrix (SHM).

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UESHM $\Rightarrow$ UET

If T is skew-Hamiltonian, then $T = JT^tJ^*$ where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Reducibility of Skew-Hamiltonian Matrices

Theorem (Waterhouse, 2004)

If T is skew-Hamiltonian, then T is *similar* to $A \oplus A^t$ for some A.

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Pf. Sketch, 6×6 case.

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- Exploiting symmetries and using various reductions, one obtains an equivalent 30×21 real linear system.
- Mathematica's symbolic `MatrixRank` command shows that the rank is ≤ 20, whence a nonscalar Q exists. □

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Numerical Evidence

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Proposition

For $d \geq 4$, the matrix $T = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}$, where

$$A = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & d \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & 1 & \ddots & 1 & 1 \\ -1 & -1 & 0 & \ddots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -1 & -1 & -1 & \ddots & 0 & 1 \\ -1 & -1 & -1 & \cdots & -1 & 0 \end{pmatrix},$$

is skew-Hamiltonian and irreducible.

Putting It All Together

An Instructive Example

A Potential Pitfall

$$T = \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ 0 & 0 & 2 & & & \\ 0 & 0 & 0 & & & \\ \hline & & & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & 0 & 2 & 0 \end{array}\right)$$

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Remarks

- If $T = A \oplus A^t$, then T is UECSM, UESHM, and UET.

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Remarks

- If $T = A \oplus A^t$, then T is UECSM, UESHM, and UET.
- Neither class UECSM nor UESHM is closed under restriction to direct summands.
- This issue is important only in dimensions ≥ 6

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Moreover, the unitary orbits of these classes are pairwise disjoint.

Corollary

If $T \in M_n(\mathbb{C})$ is UET and $n \leq 7$, then T is UECSM.

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Moreover, the unitary orbits of these classes are pairwise disjoint.

Corollary

If $T \in M_n(\mathbb{C})$ is UET and $n \leq 7$, then T is UECSM. If $n \leq 5$, then T is unitarily equivalent to a direct sum of irreducible CSMs.

A Trace Criterion

Theorem (Pearcy, 1962 + Sibirskiĭ, 1968)

For 3×3 *matrices,* $A \cong B$ *if and only if the traces of*

$$X, \quad X^2, \quad X^3, \quad X^*X, \quad X^*X^2, \quad X^{*2}X^2, \quad X^*X^2X^{*2}X$$

agree for $X = A$ *and* $X = B$.

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Proposition (SRG, JET)

If T is 3×3, then T is UECSM if and only if

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Proof.

Check if $T \cong T^t$. Only the final word is not satisfied trivially. □

Bibliography

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