A Brief Survey of Closed-Range Composition Operators on Bergman and Bloch Spaces

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This is a more detailed version of the talk I presented at the Gainesville SEAM. I have learnt a lot from all my co-authors. When it comes to interesting examples, the lion's share belongs to J. Akeroyd. I will always be grateful to Wayne Smith for his kind and consistent interest.

1 Notation

The spaces of analytic functions of interest to us are the classical Bloch, Bergman and Hardy spaces.

$\mathbb{D}$ is the open unit disk whose area measure is assumed to be 1.
$\mathcal{B}$ is the classical Bloch space and it is a Banach space under the norm,
$||f|| = \sup\{(1-|z|^2)|f'(z)|, z \in \mathbb{D}\} + |f(0)|$.
The little Bloch space $\mathcal{B}_0$ consisting of the closure of polynomials in the Bloch-norm can be described as $\{f \in \mathcal{B}, (1-|z|^2)|f'(z)| \to 0 \text{ as } |z| \to 1\}$.
It is easy to see that $\mathcal{B}$ is contained in the Bergman space $\mathcal{A}^2 = \left\{f, \int_D |f(z)|^2 dA(z) < \infty\right\}$.
The pseudo-hyperbolic metric ρ on $\mathbb{D}$ is defined by $\rho(z,w) = |\sigma_z(w)|$, where σ_z is the automorphism of $\mathbb{D}$, which interchanges z with 0.
$D(w,r)$ is the pseudo-hyperbolic disk $\{z, \rho(z,w) < r\}$, and its area $|D(w,r)| \sim (1-|w|)^2$.
If φ is an analytic self-map of $\mathbb{D}$ and $a \in \mathbb{D}$, then $h_a = \sigma_{\varphi(a)} \circ \varphi \circ \sigma_a$, is an analytic self-map of $\mathbb{D}$, $h_a(0) = 0$ and $h'_a(0) = \dfrac{(1-|a|^2)\varphi'(a)}{1-|\varphi(a)|^2}$.
We write $\tau_\varphi(z) = \dfrac{(1-|z|^2)\varphi'(z)}{1-|\varphi(z)|^2}$ and observe that by the Schwarz-Pick lemma, $|\tau_\varphi(z)| \leq 1$.
$|\tau_\varphi(z)| = 1 \ \forall \ z$ if φ is an automorphism.
It has been shown that τ_φ is Lipschitz, with respect to the pseudo-hyperbolic metric on the domain and Euclidean one on the range.

2 Regarding the Bloch space

Theorem 2.0.1. *The following are equivalent:*

(a) C_φ has closed-range on $\mathcal{B}$.

(b) Given $w \in \mathbb{D}, \varphi^{-1}(D(w,s))$ contains $D(z,s)$, and $|\varphi((D(z,s))| \sim |D(w,s)|$ for some $s \in (0,1)$.

(c) $\exists t \in (0,s)$ so that, given $w \in \mathbb{D}, D(z,t) \subseteq \varphi^{-1}(D(w,r))$, $|\varphi(D(z,t)| \sim |D(w,r)|$ and φ is univalent on $D(z,t)$.

The above result is natural in view of the fact that the Bloch norm is often defined in terms of Schlicht disks.

Application: If φ is a univalent self-map of $\mathbb{D}$, and C_φ has closed-range on $\mathcal{B}_0$, then C_φ is Fredholm on $\mathcal{B}_0$.

1. A univalent self-map of $\mathbb{D}$ belongs to the Dirichlet space $\mathcal{D}$, where $\mathcal{D} = \left\{ f, \int |f'(z)|^2 dA(z) < \infty \right\}$.

2. $\mathcal{D} \subset \mathcal{B}_0$, is well-known.

3. $\mathcal{B}_1 = \left\{ \sum a_n \sigma_{w_n}, \sum |a_n| < \infty, w_n \in \overline{\mathbb{D}} \right\}$ is called the Besov space.
   If $< f, g >= \int f'(z)\overline{g'(z)} dA(z)$, $\mathcal{B}_1$ is the dual space of $\mathcal{B}_0$.
   Now if φ is a univalent self-map of $\mathbb{D}$, then C_φ maps $\mathcal{B}_0 1$ into itself and hence ${C_\varphi}^*$ maps $\mathcal{B}_1$.
   Moreover,${C_\varphi}^*(\sigma_w) = \tau_\varphi(w)\sigma_{\varphi(w)}$ if $w \in \mathbb{D}$ and 0 if $w \in \overline{\mathbb{D}}$.
   In particular, ${C_\varphi}^*(\sigma_w) \neq 0$ if $w \in \mathbb{D}$. It is now easy to see that $\ker {C_\varphi}^*$ is one-dimensional if φ is univalent.

4. The Riemann map of $\mathbb{D}$ onto $\mathbb{D} \setminus (0,1)$ gives a closed-range composition operator on $\mathcal{B}_0$.

   Thus there are non-automorphic univalent self-maps of $\mathbb{D}$ which induce Fredholm composition operators on $\mathcal{B}_0$. By P. Bourdon's result, this does not happen when $\mathcal{B}_0$ is replaced by a Hilbert space of analytic functions.

3 Regarding the Bergman space

Suppose $K = \overline{K} \subseteq \partial\mathbb{D}$ and $M \in (0,\infty)$ and assume that $\forall \zeta \in K, |\varphi'(\zeta)| \leq M$.
Let $W = \bigcup S(\zeta, \pi/2), \zeta \in K$ where $\Gamma(\zeta, \pi/2)$ is a Stolz angle at ζ and $\varphi(\zeta) = \omega \in \partial\mathbb{D}$ and $W_s = W \cap \{z, |z| > s\}$.
The following result seems strong, but is a consequence of the following facts.

1. By the Julia-Caratheòdory theorem, the rate at which $(1 - |\varphi(z)|) \div (1 - |z|)$ converges to $|\varphi'(\zeta)|$ is dependent on $|\varphi'(\zeta)|$, rather than the location of ζ.

2. By a result of Pommerenke, if $|\varphi'(\zeta)| = M < \infty$ and $\varphi(\zeta) = \eta$ where $|\zeta| = |\eta| = 1$, then $\exists \theta, r, s$, depending on M, satisfying the following:
   If $|w| > r$ and $w \in \Gamma(\eta, \theta)$ then $\exists z \in \Gamma(\zeta, \frac{\pi}{2})$ satisfying $\varphi(z) = w$.

Theorem 3.0.2. *If φ is a holomorphic self-map of $\mathbb{D}$, the following are equivalent:*

1. *C_φ is closed-range on $\mathcal{A}^2$.*
2. *$\exists K = \overline{K} \subseteq \partial\mathbb{D}$ such that φ has uniformly bounded angular derivatives at every point of K, φ extends continuously to K and $\varphi(K) = \partial\mathbb{D}$.*
3. *$\exists\, r \in (0,1)$ and $M \in (0,\infty)$ such that, $\mathbb{D} \setminus r\mathbb{D}$ is contained in $\varphi(W_r) \subseteq \varphi(\mathbb{D})$ and $\forall\, z \in W_r$, $M(1-|\varphi(z)|) \geq (1-|z|)$ and $|\tau_\varphi(z)| \geq 1/2$.*

Corollary 3.0.3. *No univalent self-map of $\mathbb{D}$ induces a closed-range composition operator on $\mathcal{A}^2(\Leftrightarrow$ on $\mathcal{H}^2)$ unless it is automorphic.*
It is a fact that in the univalent case, extension of φ to K is one-to-one. This makes K homeomorphic to $\partial\mathbb{D}$ and hence $K = \partial\mathbb{D}$.

Corollary 3.0.4. *C_φ is closed-range on $\mathcal{A}^2$, then C_φ is closed-range on $\mathcal{B}$.*

3.1 Examples

Example 1: J. Shapiro describes a Blaschke product B whose zeros are given by $a_n = r_n e^{i\omega_n}$ where $r_n = 1 - \dfrac{1}{n^2}$ and ω_n is the mid-point of an arc of length $\dfrac{1}{n}$. By dropping the zeros in a well-defined manner, one ensures that the resulting product is thin and has no angular derivatives. So, C_B is compact on $\mathcal{A}^2$, which is as far as one can get from having a closed range.
It is known that if B is a thin Blaschke product, (i.e. if $|\tau_(z_n)| \to 1$ whenever $|z_n| \to 1$) then $\mathbb{D} \subseteq B(\Omega_{1/2})$.

Example 2: There exists h, a conformal mapping of $\mathbb{D}$ onto an infinite ribbon G , which spirals to $\partial\mathbb{D}$ in such a way that C_h has closed range on $\mathcal{B}$, but $h(\partial\mathbb{D})$ does not intersect $\partial\mathbb{D}$.

On the positive side, we have the following:
Example 3: If B is a Blaschke product whose spectrum skips an arc of the unit circle, then $\exists n$ so that $z^n B(z)$ induces a closed-range composition operator on $\mathcal{A}^2$.

Example 4: There exists a Blaschke product B^*, for which C_{B^*} does not have a closed range on $\mathcal{A}^2$, but $zB^*(z)$ does.

Corollary 3.1.1. *If C_φ is bounded below on $\mathcal{A}^2$, then C_φ is bounded below on $\mathcal{H}^2$.*
Let $\nu(A) = m(\varphi^{-1}(A))$,a measure defined on $\partial\mathbb{D}$. Then $\nu \preceq m$.
By a well-known result, C_φ is bounded below on $\mathcal{H}^2$ if and only if $m \preceq \nu$.
This holds since $\varphi(K) = \partial\mathbb{D}$ and φ has uniformly bounded angular derivatives at every point of K.

If N_φ is the Nevalinna counting function and $\varphi(0) = 0$, then $N_\varphi(w) \leq c(1-|w|^2)$.

By Zorboska's theorem, if C_φ is bounded below on $\mathcal{H}^2$, and $G_c = \{w \in \mathbb{D}, N_\varphi(w) \geq c(1-|w|^2)\}$,

then G_c satisfies the reverse Carleson condition for some c.
i.e. $|D(w,r) \cap G_c| \geq \delta |D(w,r)|$ for some r and $\forall w \in \mathbb{D}$.

Thus, we only need to consider “inner-like” functions, when dealing with $\mathcal{A}^2$.
As shown in the examples above, not all inner functions give rise to closed-range composition operators on $\mathcal{A}^2$.