A New View of Presentation Theory for C*-algebras

William Benjamin Grilliette

University of Nebraska - Lincoln

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Proposition (Failure of Freeness, [3])

Let $S \neq \emptyset$ and $\mathscr{C}$ any subcategory of normed vector spaces with contractive maps. If $\mathrm{Ob}(\mathscr{C})$ contains $V \ncong \mathbb{O}$, then S has no associated free object in $\mathscr{C}$.

The classical notions can be recovered by replacing **Set**, previously formulated by Gerbracht in [1].

Definition

A *crutched set* is a pair (S, f), where S is a set and f a function from S to $[0, \infty)$. The function f is called the *crutch function*.

Definition

Given two crutched sets (S, f) and (T, g), a function $\phi : S \to T$ is *constrictive* if $g(\phi(s)) \leq f(s)$ for all $s \in S$.

Given a crutched set (S, f),

1. Form the set $S_f := S \setminus f^{-1}(0)$.
2. Construct the free unital *-algebra $A_{S,f}$ over $\mathbb{C}$ on S_f.
3. Construct a C*-norm on $A_{S,f}$ from f.

Lemma (Universal C*-norm, [3])

For each $a \in A_{S,f}$, define

$$\mathscr{S}_a := \left\{ \|\pi(a)\|_{\mathcal{B}} : \begin{array}{l} \mathcal{B} \text{ a unital } C^*\text{-algebra}, \\ \pi : A_{S,f} \to \mathcal{B} \text{ a unital } {}^*\text{-homomorphism}, \\ \|\pi(s)\|_{\mathcal{B}} \leq f(s) \forall s \in S_f \end{array} \right\}.$$

and $\rho_{S,f} : A_{S,f} \to [0, \infty)$ by $\rho_{S,f}(a) := \sup \mathscr{S}_a$. Then, $\rho_{S,f}$ is a sub-multiplicative norm on $A_{S,f}$ satisfying the C-property.*

4. Complete $A_{S,f}$ into a unital C*-algebra $\mathcal{A}_{S,f}$.

Theorem (Universal Property, [3])

Let (S, f) be a crutched set, $\mathcal{B}$ a unital C-algebra, and $\phi : (S, f) \to \mathcal{B}$ a constriction. Then, there is a unique unital *-homomorphism $\hat{\phi} : \mathcal{A}_{S,f} \to \mathcal{B}$ such that $\hat{\phi}(s) = \phi(s)$ for all $s \in S$.*

Definition

A *C*-relation* on (S, f) is an element of $\mathcal{A}_{S,f}$.

Definition

For a crutched set (S, f) and C*-relations $R \subseteq \mathcal{A}_{S,f}$ on (S, f), let J_R be the two-sided, norm-closed ideal generated by R in $\mathcal{A}_{S,f}$. Then, the *unital C*-algebra presented on (S, f) subject to R* is

$$\langle S, f | R \rangle_{\mathbf{1C^*}} := \mathcal{A}_{S,f} / J_R.$$

Note: There are *more* “relations” than just *-polynomials.

For group theory, Tietze ([5], 1908) described canonical means of converting between presentations of the same group.
These same transformations exist for this presentation theory for $\mathbf{1C}^*$.

1. Adding/Removing C*-relations.
   (e.g. $\left\langle (x,\lambda) \,\middle|\, x = x^2 \right\rangle_{\mathbf{1C}^*} \leftrightarrow \left\langle (x,\lambda) \,\middle|\, x = x^2, x = x^5 \right\rangle_{\mathbf{1C}^*}$)
2. Adding/Removing generators.
   (e.g. $\left\langle (x,\lambda) \,\middle|\, x = x^2 \right\rangle_{\mathbf{1C}^*} \leftrightarrow \left\langle (x,\lambda), (y,\lambda^2) \,\middle|\, x = x^2, y = x^*x \right\rangle_{\mathbf{1C}^*}$)

One of these transformations is *elementary* if only one generator or C*-relation is altered.

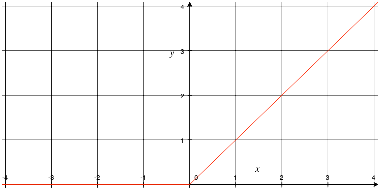
Theorem (Tietze Theorem for $\mathbf{1C}^*$, [2])

Given unital C-algebras $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \cong_{\mathbf{1C}^*} \mathcal{B}$ iff there is a sequence of four Tietze transformations changing the presentation of $\mathcal{A}$ into the presentation for $\mathcal{B}$.*

Corollary (Elementary Version, [2])

Given finitely presented unital C-algebras $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \cong_{\mathbf{1C}^*} \mathcal{B}$ iff there is a finite sequence of elementary Tietze transformations changing the presentation of $\mathcal{A}$ into the presentation for $\mathcal{B}$.*

Define $p : \mathbb{R} \to \mathbb{R}$ by $p(\mu) := \begin{cases} \mu, & \mu \geq 0, \\ 0, & \mu < 0. \end{cases}$



Proposition (C*-relation for Positivity, [2])

For a C-algebra $\mathcal{A}$ and $x \in \mathcal{A}$, $x \geq 0$ if and only if $x = p(\Re(x))$.*

Corollary (C*-relations for Invertibility, [4])

Let $\mathcal{A}$ be a unital C-algebra, $x \in \mathcal{A}$ and $\mu \in (0, \infty)$. Then, there is $y \in \mathcal{A}$ satisfying $\|y\|_{\mathcal{A}} \leq \mu$ and $xy = yx = \mathbb{1}$ if and only if $\mathbb{1} \leq \mu^2 x^* x$ and $\mathbb{1} \leq \mu^2 x x^*$. In this case, $y \in C^*(\mathbb{1}, x)$.*

Consider the C*-algebra of an invertible element.

$$\mathcal{I}_{\lambda,\mu} := \left\langle (x,\lambda) \,\middle|\, \mu^2 x^* x \geq \mathbb{1}, \mu^2 x x^* \geq \mathbb{1} \right\rangle_{\mathbf{1C^*}} .$$

If $\lambda\mu < 1$,

$$\|\mathbb{1}\|_{\mathcal{I}_{\lambda,\mu}} \leq \mu^2 \|x^* x\|_{\mathcal{I}_{\lambda,\mu}} = \mu^2 \|x\|^2_{\mathcal{I}_{\lambda,\mu}} \leq (\lambda\mu)^2 < 1.$$

Thus, $\mathbb{1} = 0$ so $\mathcal{I}_{\lambda,\mu} \cong_{\mathbf{1C^*}} \mathbb{0}$.

For $\lambda\mu \geq 1$,

$$\begin{aligned}
\mathcal{I}_{\lambda,\mu} = \; & \left\langle (x,\lambda) \,\middle|\, \mu^2 x^* x \geq \mathbb{1}, \mu^2 x x^* \geq \mathbb{1} \right\rangle_{\mathbf{1C^*}} \\
\cong_{\mathbf{1C^*}} & \left\langle \begin{matrix} (x,\lambda), \\ (q,\lambda),(u,\lambda\mu) \end{matrix} \,\middle|\, \begin{matrix} \mu^2 x^* x \geq \mathbb{1}, \mu^2 x x^* \geq \mathbb{1}, q = (x^* x)^{\frac{1}{2}}, \\ u = \mu x \left(p \left(\mu (x^* x)^{\frac{1}{2}} - \mathbb{1} \right) + \mathbb{1} \right)^{-1} \end{matrix} \right\rangle_{\mathbf{1C^*}} \\
\cong_{\mathbf{1C^*}} & \left\langle \begin{matrix} (x,\lambda), \\ (q,\lambda),(u,\lambda\mu) \end{matrix} \,\middle|\, \begin{matrix} \mu^2 x^* x \geq \mathbb{1}, \mu^2 x x^* \geq \mathbb{1}, q = (x^* x)^{\frac{1}{2}}, \\ u = \mu x \left(p \left(\mu (x^* x)^{\frac{1}{2}} - \mathbb{1} \right) + \mathbb{1} \right)^{-1}, \\ \mathbb{1} \leq \mu q, u^* u = u u^* = \mathbb{1}, x = uq \end{matrix} \right\rangle_{\mathbf{1C^*}}
\end{aligned}$$

$$\mathcal{I}_{\lambda,\mu} \cong_{\mathbf{1C}^*} \left\langle \begin{array}{c} (x,\lambda), \\ (q,\lambda),(u,\lambda\mu) \end{array} \middle| \; \mathbb{1} \leq \mu q, u^*u = uu^* = \mathbb{1}, x = uq \right\rangle_{\mathbf{1C}^*}$$

$$\cong_{\mathbf{1C}^*} \langle (q,\lambda),(u,\lambda\mu) \,|\, \mathbb{1} \leq \mu q, u^*u = uu^* = \mathbb{1} \rangle_{\mathbf{1C}^*}$$

$$\cong_{\mathbf{1C}^*} \langle (q,\lambda) \,|\, \mathbb{1} \leq \mu q \rangle_{\mathbf{1C}^*} *_{\mathbb{C}} \langle (u,\lambda\mu) \,|\, u^*u = uu^* = \mathbb{1} \rangle_{\mathbf{1C}^*}$$

$$\cong_{\mathbf{1C}^*} C\left[\frac{1}{\mu},\lambda\right] *_{\mathbb{C}} C(\mathbb{T})$$

In summary,

$$\left\langle (x,\lambda) \,\middle|\, \mu^2 x^*x \geq \mathbb{1}, \mu^2 xx^* \geq \mathbb{1} \right\rangle_{\mathbf{1C}^*} \cong_{\mathbf{1C}^*} \begin{cases} \mathbb{O}, & \lambda\mu < 1, \\ C(\mathbb{T}), & \lambda\mu = 1, \\ C[0,1] *_{\mathbb{C}} C(\mathbb{T}), & \lambda\mu > 1. \end{cases}$$

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