MF C*-algebras, MF Traces and Lower Bounds for Topological Free Entropy Dimension

Joint with: Qihui Li, Weihua Li, Junhao Shen

1 Let's take a moment

$(\mathcal{M}, \tau)$ von Neumann algebra with a faithful trace τ

$\mathcal{M} = W^* (x_1, \ldots, x_n), \vec{x} = (x_1, \ldots, x_n)$

$m = m(t_1, \ldots, t_n)$ $*$-monomial in n free variables

$\tau(m(x_1, \ldots, x_n)) = m^{th}$ moment of $\vec{x}$.

Suppose $(\mathcal{N}, \rho)$ and $\mathcal{N} = W^* (y_1, \ldots, y_n)$.

THM $\tau(m(\vec{x})) = \rho(m(\vec{y}))$ for all $*$-monomials m $\Leftrightarrow$ $\exists$ normal $*$-isomorphism $\pi : \mathcal{M} \to \mathcal{N}$ such that $\rho \circ \pi = \tau$.

If $N \in \mathbb{N}$ and $\varepsilon > 0$, we say that $(\vec{x}, \tau)$ and $(\vec{y}, \rho)$ are $(\mathbf{N}, \varepsilon)$**-close** if

$$|\tau(m(\vec{x})) - \rho(m(\vec{y}))| < \varepsilon$$

for all $*$-monomials m with $\mathbf{deg}\, m \leq N$.

Voiculescu's Free Entropy Dimension

Given $\mathcal{M} = W^*(x_1, \ldots, x_n)$ and faithful trace τ.

Let $R > \max\{\|x_1\|, \ldots, \|x_n\|\} = \|\vec{x}\|$

$\mathcal{M}_k(\mathbb{C}) = k \times k$ complex matrices, $\tau_k = \frac{1}{k} Tr$

For $\varepsilon > 0$, $N \in \mathbb{N}$ and $k \in \mathbb{N}$ define

$\Gamma_R(\vec{x}, N, \varepsilon, k) \subseteq \left\{\vec{A} \in \mathcal{M}_k(\mathbb{C})^n : \left\|\vec{A}\right\| < R\right\}$ by:

$\vec{A} \in \Gamma_R(\vec{x}, N, \varepsilon, k) \Leftrightarrow (\vec{x}, \tau), \left(\vec{A}, \tau_k\right)$ are (N, ε)-close

We want to "measure" the sets $\Gamma_R(\vec{x}, N, \varepsilon, k)$.

COVERING NUMBERS

(X,d) is a totally bounded metric space,

$\omega > 0$

$\nu_d(X,\omega) =$ minimal number of ω-balls needed to cover X.

Ex: $(1/\omega)^m \leq \nu_{\|\|}(ball\mathbb{R}^m,\omega) \leq (3/\omega)^m$

$\mathcal{A}$ C*-algebra $\dim_{\mathbb{C}} \mathcal{A} = d < \infty \Longrightarrow$

$$(1/\omega)^{d^2} \leq \nu_{\|\|}(\mathcal{U}(\mathcal{A}),\omega) \leq (9\pi e/\omega)^{d^2}$$

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BOX DIMENSION

The **Box Dimension** (**Minkowski Dimension**) is defined as

$$\dim_{\text{box}}(X) = \limsup_{\omega \to 0^+} \frac{\log \nu_d(X, \omega)}{-\log \omega}.$$

If $X \subseteq \mathbb{R}^n$, X bounded with positive Lebesgue measure, then

$$\dim_{\text{box}}(X) = n$$

Voiculescu's free entropy dimension

Define for $\vec{x} = (x_1, \ldots, x_n)$

$$\delta_0(x_1, \ldots, x_n) =$$

$$\frac{\log\left(v(\Gamma_R(\vec{x}; N, \varepsilon, , k), \omega\right)}{-\log \omega}$$

Voiculescu's free entropy dimension

Define for $\vec{x} = (x_1, \ldots, x_n)$

$$\delta_0(x_1, \ldots, x_n) =$$

$$\frac{\log\left(v(\Gamma_R(\vec{x}; N, \varepsilon, , k), \omega\right)}{-k^2 \log \omega}$$

Voiculescu's free entropy dimension

Define for $\vec{x} = (x_1, \ldots, x_n)$

$$\delta_0(x_1, \ldots, x_n) =$$

$$\limsup_{k \to \infty} \frac{\log \left(v(\Gamma_R(\vec{x}; N, \varepsilon, , k), \omega \right)}{-k^2 \log \omega}$$

Voiculescu's free entropy dimension

Define for $\vec{x} = (x_1, \ldots, x_n)$

$$\delta_0(x_1, \ldots, x_n) =$$

$$\inf_{N\in\mathbb{N},\varepsilon>0} \limsup_{k\to\infty} \frac{\log\left(v(\Gamma_R(\vec{x}; N, \varepsilon, , k), \omega\right)}{-k^2 \log \omega}$$

Voiculescu's free entropy dimension

Define for $\vec{x} = (x_1, \ldots, x_n)$

$$\delta_0(x_1, \ldots, x_n) =$$

$$\limsup_{\omega \to 0^+} \inf_{N \in \mathbb{N}, \varepsilon > 0} \limsup_{k \to \infty} \frac{\log\left(v(\Gamma_R(\vec{x}; N, \varepsilon, , k), \omega\right)}{-k^2 \log \omega}$$

This is independent of $R > \|\vec{x}\|$.

Szarek: δ_0 unchanged if we use the covering numbers with respect to $\| \|_2$.

APPLICATIONS

Many results of the form: If $\mathcal{M} = W^*(x_1, \ldots, x_n)$ has some particular property, then $\delta_0(\vec{x}) \leq 1$.

conclusion: If $\delta_0(\vec{x}) > 1$ (e.g., $\mathcal{L}_{\mathbb{F}_n}$, $n \geq 2$), then $\mathcal{M}$ does not have the property.

Voiculescu: Cartan subalgebra

Ge: Prime Factors, Simple Masas

Ge, Shen: Deep results absorbing all previous results

Shen, H: New invariant $\mathfrak{K}_2$ involving covering by unitary orbits of ω-balls that gives elementary proofs of extended Ge-Shen results.

Many exciting recent results: Jung, Dykema, Shlyakhtenko, etc.

Thm (H 1998: Shanghai Theorem) Suppose $\mathcal{M} = W^*(x_1, \ldots, x_n)$ is **hyperfinite** with a faithful normal trace τ and suppose $\omega > 0$ and $R > \max_{1 \le j \le n} \left\| x_j \right\|$. Then there are $\varepsilon > 0$ and $N \in \mathbb{N}$ such that, for every finite factor $(\mathbb{N}, \rho)$ and every $\vec{A}, \vec{B} \in \mathcal{N}^n$ with $\left\| \vec{A} \right\| \le R$, if $\left(\vec{A}, \rho \right)$ and $\left(\vec{B}, \rho \right)$ are both (N, ε)-close to $(\vec{x}, \tau)$, then there is a unitary $U \in \mathcal{N}$ such that

$$\sum_{j=1}^{n} \left\| B_j - U A_j U^* \right\|_2 < \omega.$$

Note: R is unnecessary.

COR. $W^*(x_1, \ldots, x_n)$ hyperfinite $\Longrightarrow$

$\delta_0(x_1, \ldots, x_n) \le 1.$

Note: If you find a nice A then any B is almost unitarily equivalent to A

The proof is based on **tracial ultraproducts**.

Sakai An ultraproduct of finite factors is a factor von Neumann algebra

Weihua Li & H: A tracial ultraproduct of C*-algebras is a von Neumann algebra.

Sample application:

For every $\varepsilon > 0$ and every $n \in \mathbb{N}$, there is a $\delta > 0$ such that, for every tracial C*-algebra $(\mathcal{A}, \tau)$ and every $T \in \mathcal{M}_n(\mathcal{A})$, if $\|T\| \leq 1$ and $\|T^*T - TT^*\|_2 < \delta$, then there are normal contractions $A_1, \ldots, A_n \in \mathcal{A}\, and$ a unitary $U \in \mathcal{M}_n(\mathcal{A})$ such that

$$\left\| T - U \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{pmatrix} U^* \right\|_2 < \varepsilon.$$

C*-algebra Version

$\mathcal{A} = \mathcal{C}^*(x_1, \ldots, x_n)$, $\mathcal{B} = \mathcal{C}^*(y_1, \ldots, y_n)$

$\{p_1(t_1, \ldots, t_n), p_2(t_1, \ldots, t_n), p_2(t_1, \ldots, t_n), \ldots\}$= all $*$-polynomials over $\mathbb{Q} + i\mathbb{Q}$.

THM There is a unital $*$-isomorphism $\pi : \mathcal{A} \to \mathcal{B}$ sending each x_j to y_j iff

$$\|p_m(\vec{x})\| = \|p_m(\vec{y})\|$$

for all $m \geq 1$.

$\vec{x}, \vec{y}$ are **topologically** (N, ε)**-close** $\Leftrightarrow$ top-(N, ε)-close $\Leftrightarrow$

$$\left| \left\| p_j(\vec{x}) \right\| - \left\| p_j(\vec{y}) \right\| \right| < \varepsilon$$

for $1 \leq j \leq N$.

Voiculescu's Topological Free Entropy Dimension

Given $\mathcal{A} = C^*(x_1, \ldots, x_n)$.

For $\varepsilon > 0$, $N \in \mathbb{N}$ and $k \in \mathbb{N}$ define

$\Gamma^{\text{top}}(\vec{x}, N, \varepsilon, k) \subseteq \mathcal{M}_k(\mathbb{C})^n$ by:

$$\vec{A} \in \Gamma^{\text{top}}(\vec{x}, N, \varepsilon, k) \Leftrightarrow \vec{x}, \vec{A} \text{ are top-}(N, \varepsilon)\text{-close}$$

$$\delta_{\text{top}}(x_1, \ldots, x_n) =$$

$$\limsup_{\omega \to 0^+} \inf_{N \in \mathbb{N}, \varepsilon > 0} \limsup_{k \to \infty} \frac{\log\left(v(\Gamma_R(\vec{x}; N, \varepsilon, , k), \omega\right)}{-k^2 \log \omega}$$

Szarek: δ_{top} unchanged if we use the covering numbers with respect to $\|\|_2$.

Junhao Shen & H: $\mathcal{A} = C^*(\vec{x})$ commutative or finite-dimensional $\Longrightarrow$

$$\delta_{\text{top}}(x_1, \ldots, x_n) = 1 - \frac{1}{\dim \mathcal{A}}$$

Shen & H: Formula for direct sums

Shen, Qihui Li & H: δ_{top} additive for full free products

Shen, Qihui Li & H:

$\mathcal{A}$ nuclear $\Longrightarrow \delta_{\text{top}}(x_1, \ldots, x_n) \leq 1$.

Domain of δ_0

$\delta_0(x_1, \ldots, x_n; \tau)$ defined $\Leftrightarrow \forall\ \varepsilon > 0, N \in \mathbb{N}\ \exists k \in \mathbb{N}$ such that $\Gamma(\vec{x}, N, \varepsilon, k) \neq \varnothing$

$\Longleftrightarrow$

$W^*(x_1, \ldots, x_n)$ embeddable in a tracial ultrapower of the hyperfinite II_1 factor (Connes' embedding problem)

$\Longleftrightarrow$

$W^*(x_1, \ldots, x_n)$ embeddable in a tracial ultraproduct $\prod_{k \in \mathbb{N}}^{\alpha} (\mathcal{M}_k(\mathbb{C}), \tau_k)$

Domain of $\boldsymbol{\delta}_{\text{top}}$

$\delta_{\text{top}}(x_1, \ldots, x_n)$ defined $\Leftrightarrow \forall \varepsilon > 0, N \in \mathbb{N}\ \exists k \in \mathbb{N}$ such that $\Gamma(\vec{x}, N, \varepsilon, k) \neq \varnothing$.

$\Leftrightarrow$

$C^*(\vec{x})$ is an **MF algebra** (Blackadar-Kirchberg), i.e.,

$C^*(\vec{x})$ can be embedded in

$$\prod_{1 \leq s < \infty} \mathcal{M}_{k_s}(\mathbb{C}) \Big/ \sum_{1 \leq s < \infty} \mathcal{M}_{k_s}(\mathbb{C})$$

for some increasing sequence $\{k_s\}$ of positive integers

$\Longleftrightarrow$

$C^*(\vec{x})$ can be embedded in a C*-ultraproduct $\prod_{k \in \mathbb{N}}^{\alpha} \mathcal{M}_k(\mathbb{C})$.

$C^*(x_1, \ldots, x_n)$ is an MF-algebra if there are sequences $\{k_s\}$ of positive integers and $\vec{A}_s = (A_{1s}, \ldots, A_{ns}) \in \mathcal{M}_{k_s}(\mathbb{C})^n$ such that, for every $*$-polynomial p,

$$\lim_{s \to \infty} \|p(A_{1s}, \ldots, A_{ns})\| = \|p(x_1, \ldots, x_n)\|.$$

We call this **convergence in topological distribution**, and write

$$\vec{A}_s \xrightarrow{\text{top}} \vec{x}.$$

In the tracial von Neumann algebra case, we write

$$\left(\vec{A}_s, \tau_{k_s}\right) \xrightarrow{\text{dist}} (\vec{x}, \tau)$$

if $\lim_{s \to \infty} \tau_{k_s}\left(m\left(\vec{A}_s\right)\right) = \tau(m(\vec{x}))$ for every $*$-monomial m.

2 MF-traces

Definition 1 *Suppose $\mathcal{A} = \mathcal{C}^* (x_1, \ldots, x_n)$ is an MF C*algebra. A tracial state τ on $\mathcal{A}$ is an MF-*trace *if there are sequences $\{k_s\}$ in $\mathbb{N}$ and $\vec{A}_s$ in $\mathcal{M}_{k_s} (\mathbb{C})^n$ such that, for every $*$-polynomial p,*

1. $\lim_{k \to \infty} \| p (A_{1k}, \ldots, A_{nk}) \| = \| p (x_1, \ldots, x_n) \|$, *and*

2. $\lim_{k \to \infty} \tau_{m_k} (p (A_{1k}, \ldots, A_{nk})) = \tau (p (x_1, \ldots, x_n))$.

$\mathcal{T} (\mathcal{A}) =$ the set of all tracial states on $\mathcal{A}$

$\mathcal{T}_{MF} (\mathcal{A}) =$ the set of all MF-traces on $\mathcal{A}$.

Notation: τ a tracial state, $\tau (a) = (\pi_\tau (a) e, e) =$ GNS representation for τ, $\hat{\tau} : \pi_\tau (\mathcal{A})'' \to \mathbb{C}$, $\hat{\tau} (T) = (Te, e)$ faithful normal trace

τ is **finite-dimensional** if $\dim \pi_\tau (\mathcal{A}) < \infty$.

Proposition 2 *If $\mathcal{A} = \mathcal{C}^*(x_1, \ldots, x_n)$ is MF, then*

1. *$\mathcal{T}_{MF}(\mathcal{A})$ is a nonempty weak*-compact convex set.*

2. *Every finite-dimensional tracial state on $\mathcal{A}$ is in $\mathcal{T}_{MF}(\mathcal{A})$,*

3. *If π is a unital $*$-homomorphism on $\mathcal{A}$ and $\pi(\mathcal{A})$ is an MF-algebra, then*
$$\{\varphi \circ \pi : \varphi \in \mathcal{T}_{MF}(\pi(\mathcal{A}))\} \subseteq \mathcal{T}_{MF}(\mathcal{A}).$$

4. *$\tau \in \mathcal{T}_{MF}(\mathcal{A}) \iff$ there is a free ultrafilter on $\mathbb{N}$, and embedding $\pi : \mathcal{A} \to \prod_{k\in\mathbb{N}}^{\alpha} \mathcal{M}_k(\mathbb{C})$ such that for every $a \in \mathcal{A}$,*
$$\tau(a) = \lim_{k\to\alpha} \tau_k(A_k),$$
where $\pi(a) = \{A_k\}_\alpha$.

3 MF-nuclear C*-algebras

$\mathcal{A} = \mathcal{C}^*(x_1, \ldots, x_n)$ is **nuclear** if, for every representation $\pi : \mathcal{A} \to B(H)$ we have $\pi(\mathcal{A})''$ is hyperfinite.

$\mathcal{A}$ is **MF-nuclear** if $\mathcal{A}$ is MF and, for every $\tau \in \mathcal{T}_{MF}(\mathcal{A})$, we have $\pi_\tau(\mathcal{A})''$ is hyperfinite.

The class of MF-nuclear C*-algebras is **closed under**: **direct sums**, **nice tensor products**, **direct limits**, and **MF-representations**.

THM If$\mathcal{A} = \mathcal{C}^*(x_1, \ldots, x_n)$ is MF and

$$\{x_1, \ldots, x_n\} = \bigcup_{j=1}^{m} E_j,$$

and $C^*\left(E_j\right)$ is MF-nuclear for $1 \leq j \leq m$, then

$$\delta_{\text{top}}(x_1, \ldots, x_n) \leq m.$$

Lower Bounds

If $A = (A_1, \ldots, A_n) \in \mathcal{M}_k^n(\mathbb{C})$, we define the *unitary orbit* of $\vec{A}$ by

$$\mathcal{U}\left(\vec{A}\right) = \{(UA_1U, UA_2U^*, \ldots, UA_nU^*) : U \in \mathcal{U}_k\}.$$

THM (Dostal, H.) If $W^*(x_1, \ldots, x_n)$ is hyperfinite and if there are sequences $\{k_s\}$ in $\mathbb{N}$ and $\vec{A}_s$ in $\mathcal{M}_{k_s}(\mathbb{C})^n$such that $\vec{A}_s \vec{x}$ and $\left\{\left\|\vec{A}_s\right\|\right\}$ is bounded, then

$$\delta_0(\vec{x}) = \lim_{\omega \to 0^+} \sup \lim_{s \to \infty} \sup \frac{\log\left(v(\mathcal{U}\left(\vec{A}_s\right), \omega\right)}{-k_s^2 \log \omega}$$

. **THM** (Li,Li,Shen,H.) If $\mathcal{A} = C^*(\vec{x})$ is MF-nuclear, then, for every $\tau \in \mathcal{T}_{MF}$

$$\delta_{\text{top}}(x_1, \ldots, x_n) \geq \delta_0(x_1 \ldots, x_n; \tau).$$

Theorem 3 *Suppose* $\mathcal{A} = C^*(x_1, \ldots, x_n)$ *is a unital* MF*-algebra and* $\tau \in \mathcal{T}_{MF}(\mathcal{A})$*. Suppose* $b = b^* \in \pi_\tau(\mathcal{A})''$*. Then*

$$\delta_{top}(x_1, \ldots, x_n) \geq \delta_0(b, \tau).$$

Voiculescu: $x = x^* \Longrightarrow$

$$\delta_0(x; \tau) = 1 - \sum_{\lambda \in \sigma_p(x)} \tau\left(\chi_{\{\lambda\}}(x)\right)^2.$$

so $\delta_0(x; \tau) = 1 \Leftrightarrow x$ has no eigenvalues.

In a finite von Neumann algebra every selfadjoint element has an eigenvalue $\Leftrightarrow$

it has a normal finite-dimensional representation $\Longleftrightarrow$

it has no minimal projection $\Longleftrightarrow$

it is **diffuse.**

We call a unital C*-algebra **C*-diffuse** if it has no finite-dimensional representations.

Theorem 4 *Suppose $\mathcal{A} = \mathcal{C}^*(x_1, \ldots, x_n)$ is an MF-algebra and either*

1. *$\mathcal{A}$ C*-diffuse, or*

2. *$\mathcal{A}$ has infinitely many non-unitarily equivalent finite-dimensional representations.*

Then $\delta_{top}(x_1, \ldots, x_n) \geq 1$.

Corollary 5 *If $\mathcal{A}$ is a unital residually finite-dimensional C*-algebra, then, for any generating set $\{x_1, \ldots, x_n\}$ of $\mathcal{A}$, we have*

$$\delta_{top}(x_1, \ldots, x_n) \geq 1 - \frac{1}{\dim \mathcal{A}}.$$

If, in addition, $\mathcal{A}$ is MF-nuclear, then equality holds.

Corollary 6 *Suppose $\mathcal{A}$ is a unital finitely generated MF C*-algebra and G is a finitely generated infinite abelian group and $\alpha : G \to Aut(\mathcal{A})$ is a group homomorphism. If $\mathcal{A} \rtimes_\alpha G$ is MF, then, for every set $\{x_1, \ldots, x_n\}$ of generators for $\mathcal{A} \rtimes_\alpha G$, we have*

$$\delta_{top}(x_1, \ldots, x_n) \geq 1.$$

If, in addition, $\mathcal{A}$ is MF-nuclear, then

$$\delta_{top}(x_1, \ldots, x_n) = 1.$$

Corollary 7 *If $\mathcal{A}$ is a simple, MF-nuclear C*algebra, then, for any generating set $\{x_1, \ldots, x_n\}$ of $\mathcal{A}$, we have*

$$\delta_{top}(x_1, \ldots, x_n) = 1 - \frac{1}{\dim \mathcal{A}}.$$

The MF-Ideal

$$I_{MF}(\mathcal{A}) = \{a \in \mathcal{A} : \tau(a^*a) = 0 \forall \tau \in \mathcal{T}_{MF}(\mathcal{A})\}$$

Theorem 8 *Suppose* $\mathcal{A} = \mathcal{C}^*(x_1, \ldots, x_n)$ *is an* MF*-algebra and* $\dim \mathcal{A}/\mathcal{J}_{MF}(\mathcal{A}) = d < \infty$*. Then*

$$\delta_{top}(x_1, \ldots, x_n) = 1 - \frac{1}{d}.$$

Theorem 9 *If* $\mathcal{A} = C^*(x_1, \ldots, x_n)$ *is an MF-nuclear and* $\mathcal{A}/\mathcal{J}_{MF}(\mathcal{A})$ *is RFD algebra, then*

$$\delta_{top}(x_1, \ldots, x_n) = 1 - \frac{1}{\dim \mathcal{A}/\mathcal{I}_{MF}}.$$

Example 10 *Suppose $\mathcal{B}$ is a unital separable MF-C*-algebra that is not nuclear, $e.g.$, $\mathcal{B} = C_r^*(\mathbb{F}_2)$,.and let $\mathcal{D} = \mathcal{B} \otimes \mathcal{K}\left(\ell^2\right)$. Then $\mathcal{D}$ is singly generated, and every tracial state vanishes on $\mathcal{D}$. Let $\mathcal{D}^+$ be the C*-algebra obtained by adjoining the identity to $\mathcal{D}$ and suppose $\mathcal{N}$ is a finitely generated nuclear MF C*-algebra. Then $\mathcal{A} = \mathcal{N} \otimes \mathcal{D}^+$ is finitely generated and MF, but not nuclear. However, $1 \otimes \mathcal{D}^+ \subseteq \mathcal{I}_{MF}(\mathcal{A})$, so*

$$\mathcal{I}_{MF}(\mathcal{A}) = \mathcal{I}_{MF}(\mathcal{N}) \otimes \mathcal{D}.$$

Thus $\mathcal{A}/\mathcal{I}_{MF}(\mathcal{A})$ is isomorphic to $\mathcal{N}/\mathcal{I}_{MF}(\mathcal{N})$, which is nuclear. Hence, for every set $\{x_1, \ldots, x_n\}$ of generators of $\mathcal{A}$, we have

$$\delta_{top}(x_1, \ldots, x_n) \leq 1.$$

We now consider the class $\mathcal{S}$ of separable MF C*-algebras for which every trace is an MF-trace, i.e., $\mathcal{TS}(\mathcal{A}) = \mathcal{T}_M(\mathcal{A})$.

Lemma 11 *The following are true.*

1. *If $\mathcal{A}, \mathcal{B} \in \mathcal{S}$, then $\mathcal{A} \oplus \mathcal{B} \in \mathcal{S}$.*

2. *If $\mathcal{A} \in \mathcal{S}$, then $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}) \in \mathcal{S}$ for every $n \geq 1$.*

3. *$\mathcal{S}$ is closed under direct (inductive) limits.*

4. *Every separable commutative C*-algebra is in $\mathcal{S}$.*

5. *Every AH C*-algebra is in $\mathcal{S}$.*

6. *$\mathcal{A} \in \mathcal{S} \Leftrightarrow$ every* **factor** *tracial state on $\mathcal{A}$ is in $\mathcal{T}_{MF}(\mathcal{A})$*

7. *If* $\mathcal{A}, \mathcal{B} \in \mathcal{S}$, *then* $\mathcal{A} \otimes_{\mathsf{min}} \mathcal{B} \in \mathcal{S}$.

8. *Connes' embedding counterexample* $\Longrightarrow$ *There is MF* $\mathcal{A} \notin \mathcal{S}$

Theorem 12 *Suppose* $\mathcal{A} = C^*(x_1, \ldots, x_n) \in \mathcal{S}$ *. Then either*

1. *There is a* $\tau \in \mathcal{T}_{MF}(\mathcal{A})$ *and an* $A \in \pi_\tau(\mathcal{A})''$ *such that* $\delta_0(A) = 1$*, or*

2. $\mathcal{A}/\mathcal{I}_{MF}(\mathcal{A})$ *is RFD.*

Therefore, either $\delta_{top}(x_1, \ldots, x_n) \geq 1$ *or* $\dim \mathcal{A}/\mathcal{I}_{MF}(\mathcal{A}) < \infty$*.*

Corollary 13 *If* $\mathcal{A} = C^*(x_1, \ldots, x_n)$ *is MF-nuclear and* $\mathcal{A} \in \mathcal{S}$*, then*

$$\delta_{top}(x_1, \ldots, x_n) = 1 - \frac{1}{\dim \mathcal{A}}$$

We now want to focus on the class $\mathcal{W}$ of all separable MF C*-algebras $\mathcal{A}$ such that $\mathcal{I}_{MF}(\mathcal{A}) = \{0\}$. The main reason is the following.

Proposition 14 *Suppose $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is MF-nuclear and $\mathcal{A} \in \mathcal{S} \cap \mathcal{W}$. Then*

$$\delta_{top}(x_1, \ldots, x_n) = 1 - \frac{1}{\dim \mathcal{A}}.$$

Theorem 15 *The following are true.*

1. *If $\{\mathcal{A}_i : i \in I\} \subseteq \mathcal{W}$, and $\mathcal{A}$ is a separable unital subalgebra of the C*-direct product $\prod_{i \in I} \mathcal{A}_i$, then $\mathcal{A} \in \mathcal{W}$.*

2. *If $\mathcal{A}$,$\mathcal{B} \in \mathcal{W}$, then $\mathcal{A} \otimes_{\min} \mathcal{B} \in \mathcal{W}$.*

3. *Every separable unital simple MF C*-algebra is in $\mathcal{W}$.*

4. *Every separable unital RFD C*-algebra is in $\mathcal{W}$.*

5. *$\mathcal{W}$ is not closed under direct limits.*