Composition Operators on $S^2(\mathbb{D})$

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Composition Operators

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- Let $\mathcal{H}$ be a Hilbert space of analytic functions on the unit disc $\mathbb{D}$ in $\mathbb{C}$.

- For $\varphi : \mathbb{D} \to \mathbb{D}$, analytic, we define the composition operator on $\mathcal{H}$ by

$$C_\varphi(f) = f \circ \varphi$$

for $f \in \mathcal{H}$.

Weighted Hardy spaces

Weighted Hardy spaces

$$H^2(\beta) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : ||f||^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta(n)^2 < \infty \right\}$$

where $\{\beta(n)\}$ is a sequence with $\beta(0) = 1$, $\beta(n) > 0$ for all n, and $\liminf \beta(n)^{1/n} \geq 1$.

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$$A^2(\mathbb{D}) \quad : \quad ||f||_{A^2}^2 = |a_0|^2 + \sum_{n=1}^{\infty} \frac{|a_n|^2}{n}$$

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$$S^2(\mathbb{D}) \quad : \quad ||f||^2_{H^2(\beta)} = |a_0|^2 + \sum_{n=1}^{\infty} n^2|a_n|^2$$

$S^2(\mathbb{D})$

The space, $S^2(\mathbb{D})$, consists of analytic functions in the disc whose power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

satisfies

$$||f||_{S^2}^2 = |a_0|^2 + \sum_{n=1}^{\infty} n^2 |a_n|^2 < \infty.$$

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We can also define the norm on $S^2(\mathbb{D})$ in terms of an integral:

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$$||f||_{S^2}^2 = |f(0)|^2 + ||f'||_{H^2}^2$$

$$S^2(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D} : f' \in H^2(\mathbb{D})\}$$

Interesting Observations about $S^2(\mathbb{D})$

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Interesting Observations about $S^2(\mathbb{D})$

- $S^2(\mathbb{D})$ is a weighted Hardy space with $\beta^2(n) = n^2$
- Self-maps of the disc need not give rise to bounded composition operators on $S^2(\mathbb{D})$
- No nice closed form exists for the reproducing kernel functions, K_z, on $S^2(\mathbb{D})$

$$f(z) = \langle f, K_z \rangle \text{ for } f \in \mathcal{H}$$

Previous Results on $S^2(\mathbb{D})$

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- M. Contreras and A. Hernández-Díaz (2004) showed $W_{\psi,\varphi}$ is bounded (respectively, compact) on $S^2(\mathbb{D})$ if and only if $W_{\psi\varphi',\varphi}$ is bounded (respectively, compact) on $H^2(\mathbb{D})$.

Definition

We define the adjoint of C_φ, denoted C_φ^*, as the unique operator on $\mathcal{H}$ satisfying

$$\langle C_\varphi f, g\rangle = \langle f, C_\varphi^* g\rangle$$

for all $f, g \in \mathcal{H}$.

Previous Adjoint Results

Let $\varphi(z) = \dfrac{az+b}{cz+d}$ be a linear fractional self-map of $\mathbb{D}$.

- On $H^2(\mathbb{D})$ (Cowen, 1988) and $A^2(\mathbb{D})$ (Hurst, 1997):

$$C_\varphi^* = M_g C_\sigma M_h^*$$

with $g(z) = (-\overline{b}z + \overline{d})^{-p}$ and $h(z) = (cz+d)^p$, $p = 1$ or 2,

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- Proof uses that $K_w(z) = \dfrac{1}{(1-\overline{w}z)^p}$, $p = 1$ or 2

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- On $\mathcal{D}(\mathbb{D})$ (Gallardo-Gutiérrez and Montes-Rodríguez, 2003):

$$\begin{aligned} C_\varphi^* &= C_\sigma \pmod{\mathcal{K}} \\ &= C_\sigma + K, \end{aligned}$$

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$S^2(\mathbb{D})$ Results

Theorem

If $\varphi(z) = \frac{az+b}{cz+d}$, *then on* $S^2(\mathbb{D})$,

$$C_\varphi^* = M_G^* C_\psi M_{1/H} M_h^* C_\eta \pmod{\mathcal{K}}$$

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where

$$\psi(z) = \frac{\varphi(0) - z}{1 - \overline{\varphi(0)}z},$$
$$G(z) = -(\overline{b}/\overline{d})z + 1 = H(z),$$
$$h(z) = -(\gamma/\alpha)z + 1,$$
$$\eta(z) = (\sigma \circ \psi)(z),$$

with $\gamma = (c\overline{d} - \overline{b}a)d$ and $\alpha = (bc - ad)\overline{d}$.

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Let $\varphi(z) = \frac{az+b}{cz+d}$ be a self-map of the disc. We can write

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- Step 1. Let $\phi(z) = \frac{\alpha z}{\gamma z+\delta}$ be a self-map of the disc. Then

$$C_\phi^* = M_h^* C_\eta + K_1$$

where $h(z) = (-\gamma/\alpha)z + 1$ and $\eta(z) = (\overline{\alpha}/\overline{\delta})z - \overline{\gamma}/\overline{\delta}$.

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- Step 2. Let $\psi = \frac{\lambda(z+u)}{1+\overline{u}z}$ be an automorphism of the disc. Then

$$C_\psi^* = M_G^* C_{\psi^{-1}} M_{1/H} + K_2$$

where $G(z) = -\overline{\lambda u}z + 1$ and $H(z) = \overline{u}z + 1$.

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- Step 3. Let $\varphi = \psi \circ \phi$ where

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A Word on the Proof

An automorphism of the disc $\varphi(z) = \frac{\lambda(z+u)}{1+\overline{u}z}$ can be written as

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Then

$$\begin{aligned} C_\varphi^* &= C_\phi^* C_\eta^* \\ &= (M_G^* C_\mu + K)(C_\psi M_{1/H} + K) \end{aligned}$$

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An automorphism of the disc $\varphi(z) = \frac{\lambda(z+u)}{1+\overline{u}z}$ can be written as

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Finally, showed that $C_\varphi^* - M_G^* C_{\varphi-1} M_{1/H}$ is compact.

$S^2(\mathbb{D})$ Results

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\begin{aligned}
C_\varphi^* &= M_G^* C_\psi M_{1/H} M_h^* C_\eta \pmod{\mathcal{K}} \\
&= k_1 C_\psi M_z^* C_\eta + k_2 M_z^* C_\psi M_z^* C_\eta + k_3 M_z C_\psi M_z^* C_\eta \\
&\quad + k_4 C_\sigma + k_5 M_z^* C_\sigma + k_6 M_z C_\sigma \pmod{\mathcal{K}}
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where $\psi(z) = \frac{\varphi(0)-z}{1-\overline{\varphi(0)}z}$, $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}$ *and* $\eta = \sigma \circ \psi$*.*

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Theorem

If $\varphi(z) = \frac{az+b}{cz+d}$ is nonautomorphic with $\varphi(\zeta) = \omega$ for some $\zeta, \omega \in \partial\mathbb{D}$, then on $S^2(\mathbb{D})$,

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where $\sigma(z) = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}$.

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where $\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}$.

If $||\varphi||_\infty < 1$, then C_φ is compact on $S^2(\mathbb{D})$.

Definition

We use our formula for the adjoint, C_φ^*, to study the commutator

$$[C_\varphi^*, C_\psi] := C_\varphi^* C_\psi - C_\psi C_\varphi^*$$

for analytic self-maps of the disc, φ and ψ.

In $H^2(\mathbb{D})$, if φ is a non-automorphic, linear fractional self-map of the disc with $\varphi(\zeta) = \omega$ for some $\zeta, \omega \in \partial\mathbb{D}$, then

$$C_\varphi^* = kC_\sigma + K$$

where σ is the Krein adjoint of φ.

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Recall, on $S^2(\mathbb{D})$,

$$C_\varphi^* = k_1 M_z^* C_\sigma + k_2 M_z^* C_\sigma M_z^* + k_3 C_\sigma + k_4 M_z C_\sigma + K$$

Commutator Results on $S^2(\mathbb{D})$

Theorem

Let φ and ψ be automorphisms of the disc such that neither is the identity. Then on $S^2(\mathbb{D})$, the commutator $[C_\varphi^*, C_\psi]$ is compact if and only if $\varphi(z) = \lambda z$, $\psi(z) = \gamma z$, $|\lambda| = |\gamma| = 1$.

Commutator Results on $S^2(\mathbb{D})$

- We say that the commutator

$$[C_\varphi^*, C_\psi] = C_\varphi^* C_\psi - C_\psi C_\varphi^*$$

is non-trivially compact if it is compact, but neither of the operators $C_\varphi^* C_\psi$ or $C_\psi C_\varphi^*$ is individually compact.

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- If $||\varphi||_\infty < 1$, then C_φ is compact on $S^2(\mathbb{D})$ and the commutator is trivially compact.

Commutator Results on $S^2(\mathbb{D})$

Lemma

Let φ and ψ be non-automorphic, linear fractional self-maps of the disc. If the commutator $[C_\varphi^*, C_\psi]$ is non-trivially compact on $S^2(\mathbb{D})$ then $\varphi(\zeta) = \psi(\zeta) = \zeta$ for some $\zeta \in \partial\mathbb{D}$.

Commutator Results on $S^2(\mathbb{D})$

Theorem

Let φ and ψ be non-automorphic, linear fractional self-maps of the disc. Then $[C_\varphi^*, C_\psi]$ is non-trivially compact on $S^2(\mathbb{D})$ if and only if one of the following hold:

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1. *If φ and ψ are parabolic with the same boundary fixed point, or*
2. *If φ and ψ are hyperbolic with fixed points (ζ, a) and $(\zeta, 1/\overline{a})$, respectively, for $\zeta \in \partial\mathbb{D}$ and a in the complement of the unit circle.*

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$$\begin{aligned}[C_\sigma, C_\psi] &= C_\sigma C_\psi - C_\psi C_\sigma \\ &= C_{\psi\circ\sigma} - C_{\sigma\circ\psi}\end{aligned}$$

Therefore, ψ and σ commute and so they must have the same fixed point sets.

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Future Research

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$$(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$$

for $f \in \mathcal{H}$.