Linear Matrix Inequalities vs Convex Sets

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Outline

SDP and LMIs

A **linear pencil** is a matrix valued function L of the form

$$L(x) := L_0 + L_1 x_1 + \cdots + L_g x_g,$$

where $L_0, L_1, L_2, \cdots, L_g$ are symmetric matrices and $x := \{x_1, \cdots, x_g\}$ are m real parameters.

A **Linear Matrix Inequality (LMI)** is one of the form:

$$L(x) \succeq 0.$$

The set of solutions

$$\mathcal{G} := \{(x_1, x_2, \cdots, x_g) : L_0 + L_1 x_1 + \cdots + L_g x_g \quad \text{is PosSD}\}$$

is a convex set. Solutions can be found numerically for problems of modest size. This is called

Semidefinite Programming SDP

Notation: a monic LMI is one with $L_0 = I$.

SDP is Everywhere

Main development in convex programming in the last 20 years.

- **Many problems from many areas can be converted to SDPs.**
- **Max -Cut, etc, etc is approximable by SDP**
- **Linear systems theory, classic problems convert to Linear Matrix Inequalities LMIs. Compromises opens many uses of SDP**
- **Computational Real Algebraic Geometry RAG (RAG is critical in this talk)**
- **Lyapunov functions for Nonlin Sys.** **Statistics**
- **Types of Chemical Problems** **Many others**

See Nemirovski's Plenary ICM address 2006
"Semidefinite Programming" gets about 70,000 hits on google
"Symmetric Matrix" gets about 300,000 hits on google

In linear engineering systems a special class of SDP is fundamental: SDP with matrix (not scalar) variables.

OLD FASHION ORDINARY LMI's

COMMUTATIVE VARIABLES
NOT DIM FREE

sliLMIrepVVPart

Which Sets have LMI REPRESENTATIONS?

QUESTION (Vague):
ARE CONVEX PROBLEMS ALL TREATABLE WITH LMI's?

DEFINITION:

A set $\mathcal{C} \subset \mathbb{R}^g$ has an **Linear Matrix Inequality (LMI) Representation** provided that there are symmetric matrices $L_0, L_1, L_2, \cdots, L_g$ for which the **Linear Pencil**, $L(x) := L_0 + L_1 x_1 + \cdots + L_g x_g$, has positivity set,

$$\mathcal{G} := \{(x_1, x_2, \cdots, x_g) : L_0 + L_1 x_1 + \cdots + L_g x_g \quad \text{is PosSD}\}$$

equals the set $\mathcal{C}$; that is, $\mathcal{C} = \mathcal{G}$.

Which convex sets have an LMI representation?
Parrilo and Sturmfels preprint 2000 (open for $g = 2$).

EXAMPLE

$$\mathcal{C} := \{(x_1, x_2) : 1 + 2x_1 + 3x_2 - (3x_1 + 5x_2)(3x_1 + 2x_2) \geq 0\}$$

has the LMI Rep

$$\mathcal{C} = \{x : L(x) \succeq 0\} \qquad \text{here } x := (x_1, x_2)$$

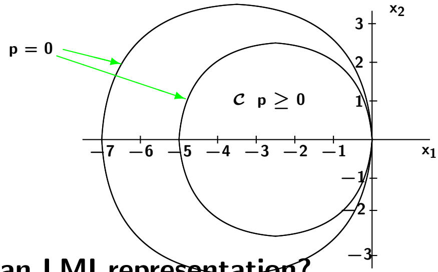
with

$$L(x) = \begin{pmatrix} 1 + 2x_1 + 3x_2 & 3x_1 + 5x_2 \\ 3x_1 + 2x_2 & 1 \end{pmatrix}$$

Pf: The determinant of $L(x)$ is pos iff $L(x)$ is PosSD.

QUESTION 1

Does this set $\mathcal{C}$ which is the inner component of



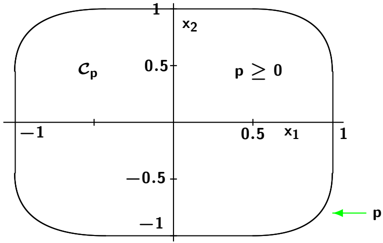
have an LMI representation?

$$p(x_1, x_2) = (x_1{}^2 + x_2{}^2)(x_1{}^2 + x_2{}^2 + 12x_1 - 1) + 36x_1{}^2 \geq 0$$

$$\mathcal{C} := \text{inner component of}\{x \in \mathbb{R}^2 : p(x) \geq 0\}$$

QUESTION 2

Does this set have an LMI representation?



$$p(x_1, x_2) = 1 - x_1{}^4 - x_2{}^4 \geq 0$$

$\mathcal{C}_p := \{x \in R^2 : p(x) \geq 0\}$ **has degree 4.**

Rigid Convexity

DEFINE: A convex set $\mathcal{C}$ in $\mathbb{R}^g$ with minimal degree (denote degree by d) defining polynomial p to be rigidly convex provided

> **for every point x^0 in $\mathcal{C}$ and every line ℓ through x^0 (except for maybe a finite number of lines), the line ℓ intersects the the zero set $\{x \in \mathbb{R}^g : p(x) = 0\}$ of p in exactly d points [3].**

The "line test"

[3] In this counting one ignores lines which go thru $\mathbf{x^0}$ and hit the boundary of $\mathcal{C}$ at ∞.

DEF: A **REAL ZERO polynomial** in g variables satisfies for each $x \in \mathbb{R}^g$,

(RZ) $\quad p(\mu x) = 0$ implies μ is real

LEMMA: If $\mathcal{C}$ contains 0 in its interior, and p is the minimal degree defining polynomial for $\mathcal{C}$, then p satisfies the Real Zeroes Condition if and only if $\mathcal{C}$ is rigidly convex.

If $\mathcal{C}$ does not contain 0 shift it.

IN $\mathbf{R}^2$ RIGID CONVEXITY RULES

THM (Vinnikov + H). IF $\mathcal{C}$ is a closed convex set in $\mathbf{R}^g$ with an LMI representation, THEN $\mathcal{C}$ is rigidly convex.

When $g = 2$, the converse is true, namely, a rigidly convex degree d set has a LMI representation with symmetric matrices $L_j \in \mathbf{R}^{d \times d}$.

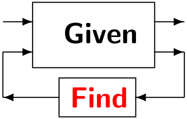
The Proof of necessity is trivial. The Proof of sufficiency ($g = 2$) is not at all elementary. Uses algebraic geometry methods of Vinnikov. These have been refined and extended by Vinnikov and Joe Ball.

NC (free) LMI

LMI in Matrix Variables

OUTLINE

Systems problems $\longrightarrow$ Matrix Ineq



Many such problems Eg. H^∞ control

Freq. domain	**Nev-Pick**		**1975-85**
State space	**tricks**	**Riccatis**	**1985-95**
Matrix ineqs	**tricks**	**LMIs**	**1990 -**

Solve numerical LMIs, up to 50×50 (unstructured) matrices, via semidef prog., bundle methods, etc.

$$\left.\begin{array}{lcl} F_1(X) := AX + XA^T + BB^T & \succeq & 0 \\ F_2(X, Y) & \succeq & 0 \end{array}\right\} \qquad \text{PosSD}$$

MATRIX INEQUALITIES

Polynomial or Rational function of matrices are PosSDef.
Example: Get Riccati expressions like

$$AX + XA^T - XBB^TX + CC^T \succ 0$$

OR Linear Matrix Inequalities (LMI) like

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \succ 0$$

which is equivalent to the Riccati inequality.

ncDimlessPartB

H^∞ Control

ALGEBRA PROBLEM:

Given the polynomials:

$$H_{xx} = E_{11} A + A^T E_{11} + C_1^T C_1 + {E_{12}}^T b C_2 + C_2^T b^T {E_{12}}^T + E_{11} B_1 b^T {E_{12}}^T + E_{11} B_1 B_1^T E_{11} + E_{12} b b^T {E_{12}}^T + E_{12} b B_1^T E_{11}$$

$$H_{xz} = E_{21} A + \frac{a^T (E_{21} + {E_{12}}^T)}{2} + c^T C_1 + E_{22} b C_2 + c^T B_2^T {E_{11}}^T + \frac{E_{21} B_1 b^T (E_{21} + {E_{12}}^T)}{2} + E_{21} B_1 B_1^T {E_{11}}^T + \frac{E_{22} b b^T (E_{21} + {E_{12}}^T)}{2} + E_{22} b B_1^T {E_{11}}^T$$

$$H_{zx} = A^T {E_{21}}^T + C_1^T c + \frac{(E_{12} + {E_{21}}^T) a}{2} + E_{11} B_2 c + C_2^T b^T {E_{22}}^T + E_{11} B_1 b^T {E_{22}}^T + E_{11} B_1 B_1^T {E_{21}}^T + \frac{(E_{12} + {E_{21}}^T) b b^T {E_{22}}^T}{2} + \frac{(E_{12} + {E_{21}}^T) b B_1^T {E_{21}}^T}{2}$$

$$H_{zz} = E_{22} a + a^T {E_{22}}^T + c^T c + E_{21} B_2 c + c^T B_2^T {E_{21}}^T + E_{21} B_1 b^T {E_{22}}^T + E_{21} B_1 B_1^T {E_{21}}^T + E_{22} b b^T {E_{22}}^T + E_{22} b B_1^T {E_{21}}^T$$

(PROB) A, B_1, B_2, C_1, C_2 **are knowns.**

Solve the inequality $\begin{pmatrix} H_{xx} & H_{xz} \\ H_{zx} & H_{zz} \end{pmatrix} \preceq 0$ **for unknowns**

a, b, c **and for** E_{11}, E_{12}, E_{21} **and** E_{22}

Engineering problems defined entirely by signal flow diagrams and L^2 performance specs are equivalent to Polynomial Matrix Inequalities

How and why is a long story but the correspondence [4] between linear systems and noncommutative algebra is on the next slides:

[4] Ask after the talk if you would like more detail.

Notation

$\mathbf{x} = (x_1, \cdots, x_g)$ **algebraic noncommuting variables**

p(x) and nc polynomial
Eg. $p(x) = x_1 x_2 + x_2 x_1$

$X = (X_1, \cdots X_g)$ **a tuple of matrices.**

Substitute a matrix for each variable
Eg. $p(X) = X_1 X_2 + X_2 X_1$

Linear Systems and Algebra Synopsis

A Signal Flow Diagram with L^2 based performance, eg H^∞ gives precisely a nc polynomial

$$p(a, x) := \begin{pmatrix} p_{11}(a, x) & \cdots & p_{1k}(a, x) \\ \vdots & \ddots & \vdots \\ p_{k1}(a, x) & \cdots & p_{kk}(a, x) \end{pmatrix}$$

The linear systems problem becomes exactly:

> **Given matrices A.**
> **Find matrices X so that $P(A, X)$ is PosSemiDef.**

WHY? Turn the crank using quadratic storage functions.

BAD Typically p is a mess, until a hundred people work on it and maybe convert it to convex Matrix Inequalities.

QUESTIONS (Vague) :

WHICH SUCH PROBLEMS "ARE" LMI PROBLEMS.

Clearly, such a problem must be convex and "semialgebraic".

Which convex nc problems are NC LMIS?

WHICH PROBLEMS ARE TREATABLE WITH LMI's?

This requires some kind of change of variables theory.

ARE LMIs INTERESTING MATHEMATICAL OBJECTS?

Free Real Algebraic Geometry: Polynomials positive on NC Convex sets

These are topics of this talk

We shall suppress the two classes of variables a, x.

Outline

Semidefinite Programming and LMIs

Linear Systems give nc poly inequalities

LMIs and Convex Sets

NC Real Algebraic Geometry: Positivstellensatz

Change of Variables to achieve Free Convexity

- nc maps
- Proper nc maps
- Pencil balls and maps
- NC Proper maps are bianalytic

Summary

SETS

HOW DOES NC CONVEXITY compare to classical CONVEXITY?

Q? Can we treat many more problems with convex techniques than LMI techniques?

Scott McCullough + H

sliConvSetHBB10.tex

NC SEMIALGEBRAIC SET

EXAMPLE: p is a symmetric NC polynomial in g variables
$p(0_n) = I_n$

$\mathcal{D}_p^n :=$ component of 0 of

$$\{X \in (\mathbb{SR}^{n\times n})^g : \ p(X) \succ 0\}$$

$\mathcal{D}_p :=$ Positivity Domain $= \cup_n \mathcal{D}_p^n$.
p is a defining polynomial for the domain $\mathcal{D}_p$.

Example: $x = (x^1, x^2)$ and $x = (X^1, X^2)$

$$p = 1 - x_1^4 - x_2^4$$

$$\mathcal{D}_p^2 = \{X \in (\mathbb{SR}^{2\times 2})^2 : \ I - X_1^4 - X_2^4 \succ 0\}$$

NC BASIC SEMIALGEBRAIC SET

p is a symmetric $\delta \times \delta-$ matrix of NC polynomials
$p(0_n)$ is Pos Def. Eg.

$$p = \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix}$$

$\mathcal{D}_p^n :=$ component of 0 of

$$\{X \in \mathbb{SR}^{n\times n} : \ p(X) \succ 0\}$$

$\mathcal{D}_p :=$ Positivity Domain $= \cup_n \mathcal{D}_p^n$.

A NC basic open semialgebraic set is one of the form $\mathcal{D}_p$,
p is called a defining polynomial for $\mathcal{D}_p$.

NC LIN MATRIX INEQS vs NC CONVEXITY

NC LINEAR MATRIX INEQUALITIES LMIs

GIVEN a linear pencil $L(x) := L_0 + L_1 x_1 + \cdots + L_g x_g$ symmetric matrices

FIND matrices $X := \{X_1, X_2, \cdots, X_g\}$ making $L(X)$ Pos SemiDef. $L(X) := L_0 + L_1 \otimes X_1 + \cdots + L_g \otimes X_g$

QUESTION:

Which Convex nc semialgebraic sets have an NC LMI representation?

What is the NC line test?

LMI REPS for NC CONVEX SETS

THM: McCullough-H

SUPPOSE p is an NC symmetric polynomial, $p(0) \succ 0$

IF $\mathcal{D}_p^n := \{X \in \mathbb{SR}^{n \times n} : \; p(X) \succ 0\}$ is "convex" for all n,

IF $\mathcal{D}_p$ is bounded.

THEN there is a monic linear pencil $L(x)$ making

$$\mathcal{D}_p = \mathcal{D}_L.$$

THM says: Every bounded nc basic open semialgebraic set has an NC LMI representation.

Proof: Long; its for another time.

Uses mostly recent methods NC Real Algebraic Geometry.

Separating one point with an LMI is pretty easy.

THE HARD PROBLEM: find a finite set of separating L's. □

LMIs GIVE NC SEPARATING HYPERPLANES

THM: (Winkler + Effros) $\sim$
GIVEN A bounded closed "NC convex set" $\mathcal{C}$.
Consider the $n \times n$ matrix level of this set $\mathcal{C}_n \subset R^{n \times n}$. If $M = (M_1, \cdots, M_g)$ is in $\partial\mathcal{C}$, then there is a linear pencil $L(x)$ with

$$L(M) \text{ not invertible and } L(X) \succ 0$$

for all X inside $\mathcal{C}$. Here X can be of any dimension.
Pf: Complete Positivity - Straight forward Arvesonism, 1970ish.

sliConvSetHBB10.tex

NC convexity yields a Perfect Positvstellensatz

OUTLINE Joint work with Klep and McCullough

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Linear Systems give nc poly inequalities

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NC Real Algebraic Geometry: Positivstellensatz

Change of Variables to achieve Free Convexity

- nc maps
- Proper nc maps
- Pencil balls and maps
- NC Proper maps are bianalytic

Summary

NC convex Positivstellensatz

It is perfect

THM **SUPPOSE** **L is a monic linear pencil;** $\mathcal{D}_L$ **has interior.**

$L(X) \succeq 0$ implies $p(X) \succeq 0$ **IFF**

$$p(x) = s(x)^* s(x) + \sum_{j}^{\text{finite}} f_j(x)^* L(x) f_j(x),$$

where s, f_j are vectors of noncommutative polynomials each of degree no greater than $\frac{\deg(p)}{2}]_+$.

Compare to commutative PosSS

Commutativity has blemishes

This result contrasts sharply with the commutative setting:

THM (COMMUTATIVE)
IF $L(X) \succeq 0$ **implies** $p(X) \succ 0$, **THEN**

$$p(x) = s(x)^* s(x) + \sum_{j}^{\text{finite}} f_j(x)^* L(x) f_j(x),$$

where s, f_j **are vectors of polynomials each of (high) degree with bound depending on** $\frac{1}{\min_{\{x : L(x) \succeq 0\}} p(x)}$.

Thus we call the nc convex PosSS a " Perfect PosSS"

Duality: PosSS $\longleftarrow\!\!\longrightarrow$ Moment Problems

Matrices of Moments have Hankel structure

The main ingredient of the proof is an analysis of rank preserving extensions of truncated noncommutative Hankel matrices.

Voilà

nc Hankel Matrices

$$
\underbrace{\begin{bmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & x_1^3 & x_1^2x_1 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4
\end{bmatrix}}
$$

$$= \quad \text{first column} \times (\text{first column})^T$$

Hankel: Replace each monomial x^α by a number $y_\alpha = \ell(x^\alpha)$, we get:

$$
\text{Hank} := \begin{bmatrix}
1 & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\
y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\
y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\
y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\
y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\
y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04}
\end{bmatrix}
$$

True confession- the Hankels above are commutative.

nc Hankels need lots of laptop battery on the plane

The nc world is flat

THM **Every positive definite noncommutative Hankel matrix has a "1-step flat" extension.**

meaning

Given y_{ij} up to a certain degree, find the y_{ij} making

$$\text{Hank} := \begin{bmatrix} 1 & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix}$$

Hank is PosSemiDef,

and

rank Hankel = rank Hank.

Proof of free convex PosSS

Patch flat extension for nc Hankel theorem together with the usual Hahn Banach separating linear functional plus GNS (Putinar) proof.

Long story.

Changing variables to achieve NC convexity

Semidefinite Programming and LMIs

Linear Systems give nc poly inequalities

LMIs and Convex Sets

NC Real Algebraic Geometry: Positivstellensatz

Change of Variables to achieve Free Convexity

- nc maps
- Proper nc maps
- Pencil balls and maps
- NC Proper maps are bianalytic

Summary

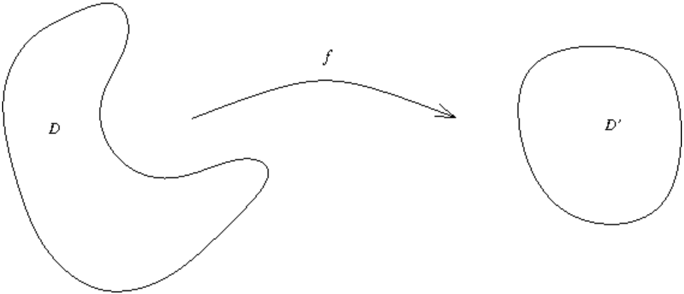
Change of variables

Main issues

- **Which sets can be transformed into nc convex sets?**

Equivalently

- **Which sets can be transformed into nc LMI domains?**



Based on joint work with Igor Klep and Scott McCullough.

NC (free) analytic maps

Change of variables

(1) **Analytic nc polynomials have no $\mathbf{x}^*$.**

NC (free) analytic maps

Change of variables

(1) **Analytic nc polynomials have no x^*.**

(2) **$f(x_1) = x_1^*$ is not an nc analytic map.**

NC (free) analytic maps

Change of variables

(1) **Analytic nc polynomials have no x^*.**

(2) **$f(x_1) = x_1^*$ is not an nc analytic map.**

(3) **Convergent power series**

$$f = \sum_{w \in \langle x \rangle} c_w w, \quad c_w \in \mathbb{C}$$

are nc analytic maps. Here w are words in x_j.

Convergence works manageably, but we do not discuss convergence in this talk. Think polynomials! See Dan Voiculescu Dima KV and Victor V, or see our HKMc paper for a light version

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_{\tilde{g}} \end{pmatrix}$$

Change of variables

Proper nc maps

$$\mathbf{M}(\mathbb{C})^{g} := \cup_n \, (\mathbb{C}^{n\times n})^{g}$$

Let $\mathcal{U} \subset \mathbf{M}(\mathbb{C})^{g}$ and $\mathcal{V} \subset \mathbf{M}(\mathbb{C})^{\tilde{g}}$ be given nc basic semialgebraic domains.

- **An nc map $f : \mathcal{U} \to \mathcal{V}$ is proper if each $f[n] : \mathcal{U}(n) \to \mathcal{V}(n)$ is proper. That is, if $K \subset \mathcal{V}(n)$ is compact, then $f^{-1}(K)$ is compact.**

Change of variables

Proper nc maps

$$M(\mathbb{C})^g := \cup_n \ (\mathbb{C}^{n\times n})^g$$

Let $\mathcal{U} \subset M(\mathbb{C})^g$ and $\mathcal{V} \subset M(\mathbb{C})^{\tilde{g}}$ be given nc basic semialgebraic domains.

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- **In other words: for all n, if (z_j) is a sequence in $\mathcal{U}(n)$ and $z_j \to \partial\mathcal{U}(n)$, then $f(z_j) \to \partial\mathcal{V}(n)$.**

Change of variables

Proper nc maps

$$\mathbf{M}(\mathbb{C})^{\mathbf{g}} := \cup_{\mathbf{n}} \; (\mathbb{C}^{\mathbf{n}\times\mathbf{n}})^{\mathbf{g}}$$

Let $\mathcal{U} \subset \mathrm{M}(\mathbb{C})^{\mathrm{g}}$ and $\mathcal{V} \subset \mathrm{M}(\mathbb{C})^{\tilde{\mathrm{g}}}$ be given nc basic semialgebraic domains.

- **An nc map $f : \mathcal{U} \to \mathcal{V}$ is proper if each $f[n] : \mathcal{U}(n) \to \mathcal{V}(n)$ is proper. That is, if $K \subset \mathcal{V}(n)$ is compact, then $f^{-1}(K)$ is compact.**
- **In other words: for all n, if (z_j) is a sequence in $\mathcal{U}(n)$ and $z_j \to \partial\mathcal{U}(n)$, then $f(z_j) \to \partial\mathcal{V}(n)$.**
- **In the case $g = h$ and both f and f^{-1} are (proper) nc maps, we say f is a bianalytic nc map.**

Change of variables

Pencil balls

- **Let** $\mathbb{B}^g := \{X \in M(\mathbb{C})^g : \ \|X\| < 1\}$ **be the operator ball.**
- **To a linear pencil**

$$\Lambda(x) := L_1 x_1 + \cdot + L_g x_g,$$

L_j not necc symmetric, we associate its pencil ball

$$\mathcal{B}_\Lambda := \{X \mid \|\Lambda(X)\| < 1\} = \mathcal{D}_{\begin{pmatrix} I & \Lambda \\ \Lambda^* & I \end{pmatrix}}.$$

Change of variables

Pencil balls

- Let $\mathbb{B}^{g} := \{X \in M(\mathbb{C})^{g} : \ \|X\| < 1\}$ be the **operator ball**.
- To a linear pencil

$$\Lambda(x) := L_1 x_1 + \cdot + L_g x_g,$$

L_j not necc symmetric, we associate its **pencil ball**

$$\mathcal{B}_\Lambda := \{X \mid \|\Lambda(X)\| < 1\} = \mathcal{D}_{\begin{pmatrix} I & \Lambda \\ \Lambda^* & I \end{pmatrix}}.$$

THM

SUPPOSE Λ is a **minimal dimensional defining pencil** for $\mathcal{B}_\Lambda$. **IF** $f : \mathcal{B}_\Lambda \to \mathbb{B}^{\tilde{g}}$ is a proper analytic nc map (with matrix coefficients) with $f(0) = 0$, **THEN** there is a contraction-valued analytic $J : \mathcal{B}_\Lambda \to \mathbb{B}$ such that

$$f(x) = U \begin{pmatrix} \Lambda(x) & 0 \\ 0 & J(x) \end{pmatrix} V^*$$

for some unitaries U, V.

Change of variables

Pencil balls. An important special case

- **Let**

$$\Lambda = \sum_{i,j} E_{ij}\, x_{ij},$$

where E_{ij} are the matrix units.

Change of variables

Pencil balls. An important special case

- **Let**

$$\Lambda = \sum_{i,j} E_{ij}\, x_{ij},$$

  where E_{ij} are the matrix units.
- **Then**

$$\mathcal{B}_\Lambda = \{X = (X_{ij})_{i,j} \mid \|X\| < 1\}.$$

- **The nc automorphism group of $\mathcal{B}_\Lambda$ is transitive and contains maps of the form**

$$\mathcal{F}_v(u) := v - (I - vv^*)^{1/2} u (I - v^*u)^{-1} (I - v^*v)^{1/2},$$

  where v is a scalar matrix of norm < 1.

Generalizes Muhly -Solel fin dim path algebras result.

Proper nc maps tend to be bianalytic

Theorem

Let $\mathcal{U}$ be an nc domain in $\mathbf{g}$ variables,
let $\mathcal{V}$ be an nc domain in $\tilde{\mathbf{g}}$ variables,
and suppose $\mathbf{f} : \mathcal{U} \to \mathcal{V}$ is an nc analytic map.

(1) *If $\mathbf{f}$ is proper, then it is one-to-one, and $\mathbf{f}^{-1} : \mathbf{f}(\mathcal{U}) \to \mathcal{U}$ is an nc map.*

(2) *If $\mathbf{g} = \tilde{\mathbf{g}}$ and $\mathbf{f} : \mathcal{U} \to \mathcal{V}$ is proper and continuous, then $\mathbf{f}$ is bianalytic.*

Change of variables $\mathbf{g} = \tilde{\mathbf{g}}$ is rigid

Circular domains:

- **A subset S of a complex vector space is circular if $\exp(it)s \in S$ whenever $s \in S$ and $t \in \mathbb{R}$.**
- **An nc domain $\mathcal{U}$ is circular if each $\mathcal{U}(n)$ is circular.**

Change of variables $\mathbf{g} = \tilde{\mathbf{g}}$ is rigid

Proper nc self-maps tend to be linear

- **A subset S of a complex vector space is circular if $\exp(it)s \in S$ whenever $s \in S$ and $t \in \mathbb{R}$.**
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Theorem

Let $\mathcal{U}, \mathcal{V}$ be nc domains in $\mathbf{g}$ variables containing $\mathbf{0}$, and suppose $\mathbf{f} : \mathcal{U} \to \mathcal{V}$ is a proper analytic nc map with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.
If $\mathcal{U}$ and $\mathcal{V}$ are circular, then $\mathbf{f}$ is linear.

Generalizes:
Gelu Popescu map result for Reinhardt Domains
Muhly- Solel path algebras.

Summary

Highlights to remember:

(1) **A free convex bounded basic open semialgebraic set is the set of solutions to some LMI.**

(2) **Much of classical real algebraic geometry goes thru to free cases. For free convex situations there is a "perfect" PosSS.**

(3) **A proper free analytic map is one-to-one.(Partly in progress.) A proper free analytic map between domains of the same dimension is bianalytic. If the domains are circular and $f(0) = 0$, it is linear.**

(4) **A proper free analytic map f on pencil balls with $f(0) = 0$ is the direct sum of a linear pencil ball isometry plus junk.**

(5)

(6)