Algebraic Elements and Invariant Subspaces.

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27th SEAM

1. Algebraic Elements.

Definition. Let H be a Hilbert space.

A contraction $T \in L(H)$ is said to be ***completely nonunitary***

if there is no (non-zero) reducing subspace M for T such that the

restriction $T|M$ of T to the space M is a unitary operator.

Theorem. (Functional Calculus) Let $T \in L(H)$ be a completely nonunitary contraction. Then there is a unique algebra representation

$$\Psi_{\mathbf{T}} : \mathbf{H}^{\infty} \rightarrow \mathbf{L(H)}$$

such that : (i) $\Psi_T(1) = I_H$, where $I_H \in L(H)$ is the identity operator.

(ii) $\Psi_T(I_{\mathbf{D}}) = T$, where $\mathbf{D}$ is the open unit disc and $I_{\mathbf{D}}(z) = z \ \forall z \in \mathbf{D}$.

(iii) Ψ_T is continuous when H^∞ and $L(H)$ are given the $weak^*-$topology.

(iv) Ψ_T is contractive, that is, $\|\Psi_T(f)\| \leq \|f\|$ for all $f \in H^\infty$.

- From now on we will denote $\Psi_T(f)$ by $\mathbf{f(T)}$ for all $f \in H^\infty$.

Definition

(a) A completely nonunitary contraction $T \in L(H)$ is said to be of ***class*** $\mathbf{C_0}$ if $\ker \Psi_T \neq \{0\}$.

(b) If a completely nonunitary contraction $T \in L(H)$ is of class $C_{\mathbf{0}}$, then $\ker \Psi_T = m_T H^\infty$, and m_T is called the ***minimal function*** of T.

(c) Let $T \in L(H)$ be a completely non-unitary contraction. An element h of H is said to be ***algebraic with respect to*** T provided that $\theta(T)h = 0$ for some $\theta \in H^\infty \setminus \{0\}$.

If $h \neq 0$, then h is said to be a ***non-trivial algebraic element*** with respect to T.

(d) If h is not algebraic with respect to T, then h is said to be ***transcendental with respect to*** T.

(e) If T is a completely non-unitary contraction without a non-trivial algebraic element; that is, every non-zero element in H is transcendental with respect to T, then T is said to be a ***transcendental operator***.

2. Algebraic Elements and Invariant Subspaces

- H denotes a separable Hilbert space whose dim is infinite.

- If T is a contraction, then

(Case 1) T is a completely non-unitary contraction with a non-trivial algebraic element, or

(Case 2) T is a transcendental operator, or

(Case 3) T is not completely non-unitary. (It is clear for (Case 3).)

Lemma 1. If $T : H \to H$ is a transcendental operator, then, for any $\theta \in H^\infty \setminus \{0\}$, $\theta(T)$ is one-to-one.

Proposition 1. If $T : H \to H$ is a transcendental operator, then, for any non-zero element h in H, $M = \{T^n h : n = 0, 1, 2, \cdots\}$ is linearly independent.

Corollary 1. Under the same assumption as Proposition 1, for a given function $\theta \in H^\infty \setminus \{0\}$, $M' = \{\theta(T)^n h : n = 0, 1, 2, \cdots\}$ is linearly independent.

Proposition 2. Let $T : H \to H$ be a multiplicity-free operator of class C_0 and ϕ be an inner divisor of the minimal function m_T of T.

If ϕ is not a trivial inner divisor, then $\ker \phi(T)$ is a non-trivial invariant subspace for T.

Theorem 1. Let $T \in L(H)$ be a completely non-unitary contraction.

If T has a non-trivial algebraic element h, then T has a non-trivial invariant subspace.

Corollary 2. Every C_0-operator has a non-trivial invariant subspace.