Algebraic Elements and Invariant Subspaces.

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1. Algebraic Elements.

**Definition.** Let $H$ be a Hilbert space.

A contraction $T \in L(H)$ is said to be *completely nonunitary* if there is no (non-zero) reducing subspace $M$ for $T$ such that the restriction $T|M$ of $T$ to the space $M$ is a unitary operator.
Theorem. (Functional Calculus) Let $T \in L(H)$ be a completely nonunitary contraction. Then there is a unique algebra representation

$$
\Psi_T : H^\infty \to L(H)
$$

such that:

(i) $\Psi_T(1) = I_H$, where $I_H \in L(H)$ is the identity operator.

(ii) $\Psi_T(I_D) = T$, where $D$ is the open unit disc and $I_D(z) = z \forall z \in D$.

(iii) $\Psi_T$ is continuous when $H^\infty$ and $L(H)$ are given the weak$^*$—topology.

(iv) $\Psi_T$ is contractive, that is, $\|\Psi_T(f)\| \leq \|f\|$ for all $f \in H^\infty$.

• From now on we will denote $\Psi_T(f)$ by $f(T)$ for all $f \in H^\infty$. 

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Definition

(a) A completely nonunitary contraction \( T \in L(H) \) is said to be of \textbf{class} \( C_0 \) if \( \ker \Psi_T \neq \{0\} \).

(b) If a completely nonunitary contraction \( T \in L(H) \) is of class \( C_0 \), then
\[
\ker \Psi_T = m_T H^\infty, \quad \text{and} \quad m_T \text{ is called the } \textbf{minimal function} \text{ of } T.
\]

(c) Let \( T \in L(H) \) be a completely non-unitary contraction. An element \( h \) of \( H \) is said to be \textit{algebraic with respect to} \( T \) provided that
\[
\theta(T)h = 0 \text{ for some } \theta \in H^\infty \setminus \{0\}.
\]
If \( h \neq 0 \), then \( h \) is said to be a **non-trivial algebraic element** with respect to \( T \).

(d) If \( h \) is not algebraic with respect to \( T \), then \( h \) is said to be **transcendental with respect to \( T \)**.

(e) If \( T \) is a completely non-unitary contraction without a non-trivial algebraic element; that is, every non-zero element in \( H \) is transcendental with respect to \( T \), then \( T \) is said to be a **transcendental operator**.
2. Algebraic Elements and Invariant Subspaces

• $H$ denotes a separable Hilbert space whose dim is infinite.

• If $T$ is a contraction, then

(Case 1) $T$ is a completely non-unitary contraction with a non-trivial algebraic element, or
(Case 2) $T$ is a transcendental operator, or

(Case 3) $T$ is not completely non-unitary. (It is clear for (Case 3).)

**Lemma 1.** If $T : H \to H$ is a transcendental operator, then, for any $\theta \in H^\infty \setminus \{0\}$, $\theta(T)$ is one-to-one.

**Proposition 1.** If $T : H \to H$ is a transcendental operator, then, for any non-zero element $h$ in $H$, $M = \{T^n h : n = 0, 1, 2, \ldots\}$ is linearly independent.
**Corollary 1.** Under the same assumption as Proposition 1, for a given function $\theta \in H^\infty \setminus \{0\}$, $M' = \{\theta(T)^n h : n = 0, 1, 2, \cdots\}$ is linearly independent.

**Proposition 2.** Let $T : H \to H$ be a multiplicity-free operator of class $C_0$ and $\phi$ be an inner divisor of the minimal function $m_T$ of $T$.

If $\phi$ is not a trivial inner divisor, then $\ker \phi(T)$ is a non-trivial invariant subspace for $T$. 
**Theorem 1.** Let $T \in L(H)$ be a completely non-unitary contraction.

If $T$ has a non-trivial algebraic element $h$, then $T$ has a non-trivial invariant subspace.

**Corollary 2.** Every $C_0$-operator has a non-trivial invariant subspace.