Some algebraic properties of Toeplitz operators on the Segal-Bargmann space

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Introduction

- Let $\mathcal{H}$ denote the Segal-Bargmann space on $\mathbb{C}^n$, which consists of all entire functions h for which

$$\|h\| = \Big(\int_{\mathbb{C}^n} |h|^2 d\mu \Big)^{1/2} < \infty,$$

  here $d\mu(z) = \pi^{-n} \exp(-|z|^2) dV(z)$ is the Gaussian measure, dV being the usual Lebesgue measure on $\mathbb{C}^n$.
- It is well known that $\mathcal{H}$ is a closed subspace of $L^2(\mathbb{C}^n, d\mu)$, hence $\mathcal{H}$ is a Hilbert space itself. It is also referred to by the name "Fock space".

Introduction

- Monomials are orthogonal in $\mathcal{H}$. In fact, $\mathcal{H}$ has the standard orthonormal basis

$$\mathcal{B} = \left\{ e_\alpha(z) = \frac{z^\alpha}{\sqrt{\alpha!}} : \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \right\}.$$

  Here for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, we write $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $\alpha! = \alpha_1! \cdots \alpha_n!$.

- Let P denote the orthogonal projection from $L^2(\mathbb{C}^n, d\mu)$ onto $\mathcal{H}$. For a measurable function f, the Toeplitz operator T_f is defined by

$$T_f h = PM_f h = P(f \cdot h),$$

  for all $h \in \mathcal{H}$ for which $f \cdot h \in L^2(\mathbb{C}^n, d\mu)$.

Introduction

- If f is bounded, then since M_f is bounded, $T_f = PM_f$ is bounded and $\|T_f\| \leq \|f\|_\infty$.
- For a general f, the operator T_f may not even have a dense domain. For example, if $f(z) = e^{|z|^2}$, then the domain of T_f contains only the zero function.
- We need to deal with unbounded functions, and hence unbounded operators because on $\mathbb{C}^n$, even nicest functions (polynomials) are unbounded.

Introduction

- We will restrict our attention to functions that have at most polynomial growth at infinity: there exist constants $C, M > 0$ so that
$$|f(z)| \leq C(1+|z|^2)^M \text{ for all } z \in \mathbb{C}^n.$$
For such an f, the domain of T_f contains all polynomials, which is dense in $\mathcal{H}$.
- In fact, for $f_1, \ldots, f_k$ belonging to the above class of functions, the product $T_{f_1} \cdots T_{f_k}$ is densely defined with domain containing the polynomials.

Radial symbols

- We call a function f on $\mathbb{C}^n$ radial if $f(z) = f(|z|)$ for all $z \in \mathbb{C}^n$.
- If f is radial, then T_f is diagonal with respect to the standard orthonormal basis $\mathcal{B} = \{e_\alpha : \alpha \in \mathbb{N}_0^n\}$.
- $T_f e_\alpha = \lambda(f, \alpha) e_\alpha$ for all α. In fact, $\lambda(f, \alpha)$ depends only on f and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The problem

- Let f_1 and f_2 be two radial functions that have at most polynomial growth at infinity. We are interested in the operator equation $T_{f_1} T_g = T_g T_{f_2}$.
- Solving this problem will give important applications to the commuting problem (when $f_1 = f_2$) and the zero product problem (when either $f_1 = 0$ or $f_2 = 0$).
- Our work here was motived by previous works on similar problems on the Bergman space of the unit ball.
- To solve the equation $T_{f_1} T_g = T_g T_{f_2}$, we need to investigate the relation between the eigenvalues of T_{f_1} and T_{f_2}.

Main result

Theorem (Bauer-L.)

Let f_1 and f_2 be two radial functions that have at most polynomial growth at infinity; at least one is non-constant.
Then exactly one of the following two cases occurs.

1. *For any g having at most polynomial growth, $T_{f_1} T_g = T_g T_{f_2}$ if and only if $g = 0$ a.e. in $\mathbb{C}^n$.*
2. *There is an integer d such that for any g having at most polynomial growth,*

$$T_{f_1} T_g = T_g T_{f_2} \text{ if and only if } g(\gamma z_1, \ldots, \gamma z_n) = \gamma^d g(z)$$

for a.e. $\gamma \in \mathbb{T}, z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

Example 1

- Consider $f_1(z) = |z|^2$ and $f_2(z) = |z|^2 + b$, where b is a fixed complex number.
- It turns out that if b is not an integer, then the first case occurs: if $T_{f_1} T_g = T_g T_{f_2}$, then $g = 0$ a.e.
- If b is an integer, then the second occurs with $d = b$: if $T_{f_1} T_g = T_g T_{f_2}$, then $g(\gamma z_1, \ldots, \gamma z_n) = \gamma^b g(z)$ for a.e. $\gamma \in \mathbb{T}$ and $z \in \mathbb{C}^n$.
- This example shows that the derivation map $T_g \mapsto [T_g, T_{|z|^2}]$ has only integer eigenvalues.

Example 2

- Consider $f_1(z) = |z|^4$ and $f_2(z) = |z|^4 + b$, where b is a fixed complex number.
- It turns out that if $b \neq 0$, then the first case occurs: if $T_{f_1} T_g = T_g T_{f_2}$, then $g = 0$ a.e.
- If $b = 0$, then the second case occurs with $d = 0$: if $T_{f_1} T_g = T_g T_{f_2}$, then $g(\gamma z_1, \ldots, \gamma z_n) = g(z)$ for a.e. $\gamma \in \mathbb{T}$ and $z \in \mathbb{C}^n$.

Consequences of the main result

1. $f_1 = f_2 = f$, non-constant radial: the second case must occur and the integer d is 0.

$$T_f T_g = T_g T_f \text{ iff } g(\gamma z_1, \ldots, \gamma z_n) = g(z) \quad a.e. \quad \gamma \in \mathbb{T}, z \in \mathbb{C}^n.$$

   This result is the same as the result on the Bergman space of the unit ball obtained by Čučković and Rao for $n = 1$, and later by L. for all n.

2. $f_1 = 0$ and $f_2 = f$ non-constant radial: the first case occurs and we have

$$T_g T_f = 0 \text{ if and only if } g = 0 \quad \text{a.e.}$$

3. Similarly (or by taking adjoint), if f is non-constant radial, then

$$T_f T_g = 0 \text{ if and only if } g = 0 \quad \text{a.e.}$$

Consequences of the main result

- We have thus obtained an affirmative answer to a special case (when one symbol is radial) of the *zero product problem*: if $T_f T_g = 0$, does it follow that $f = 0$ or $g = 0$?
- This result is also the same as the result on the Bergman space of the unit ball obtained earlier by Ahern and Čučković for $n = 1$, and later by L. for all n.

Zero products of three Toeplitz operators

- On the Bergman space of the unit ball, the following is true: if $f_1, \ldots, f_k$ are bounded functions all of which, except possibly one, are radial and $T_{f_1} \cdots T_{f_k} = 0$, then one of these functions must be zero.
- It turns out that the situation is completely different for the Segal-Bargmann space: there are zero products of three non-zero Toeplitz operators.
- We consider one dimension case $n = 1$. For $j = 0, 1, 2$, take

$$f_j(z) = |z|^{2j} e^{-|z|^2} \sin(2\sqrt{3}|z|^2), \quad z \in \mathbb{C}.$$

Note that the functions f_j go to zero at exponential rate as $|z| \to \infty$.

Zero products of three Toeplitz operators

- Each T_{f_j} is diagonal with respect to $\mathcal{B} = \{e_m : m \in \mathbb{N}_0\}$.
- The m^{th} eigenvalue of T_{f_j} is a non-zero multiple of $\sin\left(\frac{(j+m+1)\pi}{3}\right)$.
- It follows that the product $T_{f_0} T_{f_1} T_{f_2}$ is diagonal and the m^{th} eigenvalue is a multiple of the product

$$\sin\left(\frac{(m+1)\pi}{3}\right)\sin\left(\frac{(1+m+1)\pi}{3}\right)\sin\left(\frac{(2+m+1)\pi}{3}\right).$$

But the last thing is zero for all integers m, so $T_{f_1} T_{f_2} T_{f_3} = 0$.

Some questions

1. The commuting problem: on the Bergman space of the unit disk, Axler and Čučković showed that if both f and g are bounded harmonic functions and $T_f T_g = T_g T_f$, then there are three possibilities: both f, g are analytic; or both $\bar{f}, \bar{g}$ are analytic; or one is a linear combination of the other with the constant function 1. We do not know what happens in the Segal-Bargmann space on $\mathbb{C}$.
2. The zero-product problem: on the Bergman space of the unit disk, Ahern and Čučković showed that if f and g are bounded harmonic functions and $T_f T_g = 0$, then one of the functions must be zero. We do not know if this holds in the Segal-Bargmann space on $\mathbb{C}$.