Some Rarita-Schwinger Type Operators

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Introduction

In representation theory for $O(n)$, one can consider the space of homogeneous harmonic polynomials. If one refines to the space of homogeneous monogenic polynomials when refining the representation theory to the covering group of $O(n)$, then the Rarita-Schwinger operators arise in this context. The Rarita-Schwinger operators are generalizations of the Dirac operator. They are also known as Stein-Weiss operators. We denote a Rarita-Schwinger operator by R_k, where $k = 0, 1, \cdots, m, \cdots$. When $k = 0$ it is the Dirac operator.

Preliminaries

A Clifford algebra, Cl_n, can be generated from $\mathbb{R}^n$ by considering the relationship

$$\underline{x}^2 = -\|\underline{x}\|^2$$

for each $\underline{x} \in \mathbb{R}^n$. We have $\mathbb{R}^n \subseteq Cl_n$. If $e_1, \ldots, e_n$ is an orthonormal basis for $\mathbb{R}^n$, then $\underline{x}^2 = -\|\underline{x}\|^2$ tells us that

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

where δ_{ij} is the Kroneker delta function.
An arbitrary element of the basis of the Clifford algebra can be written as $e_A = e_{j_1} \cdots e_{j_r}$, where $A = \{j_1, \cdots, j_r\} \subset \{1, 2, \cdots, n\}$ and $1 \leq j_1 < j_2 < \cdots < j_r \leq n$. Hence for any element $a \in Cl_n$, we have $a = \sum_A a_A e_A$, where $a_A \in \mathbb{R}$.

For $a \in Cl_n$, we will need the following anti-involutions:
Reversion:

$$\tilde{a} = \sum_A (-1)^{|A|(|A|-1)/2} a_A e_A,$$

where $|A|$ is the cardinality of A. In particular, $\widetilde{e_{j_1} \cdots e_{j_r}} = e_{j_r} \cdots e_{j_1}$. Also $\widetilde{ab} = \tilde{b}\tilde{a}$ for $a, b \in Cl_n$.
Clifford conjugation:

$$\bar{a} = \sum_A (-1)^{|A|(|A|+1)/2} a_A e_A.$$

Further, we have $\overline{e_{j_1} \cdots e_{j_r}} = (-1)^r e_{j_r} \cdots e_{j_1}$ and $\overline{ab} = \bar{b}\bar{a}$ for $a, b \in Cl_n$.

Pin(n) and Spin(n)

$$Pin(n) := \{a \in Cl_n : a = y_1 \dots y_p : y_1, \dots, y_p \in S^{n-1}, p \in \mathbb{N}\}$$

is a group under multiplication in Cl_n. As we can choose $y_1, \dots, y_p$ arbitrarily in S^{n-1} the group homomorphism

$$\theta : Pin(n) \longrightarrow O(n) : a \longmapsto O_a$$

with $a = y_1 \dots y_p$ and $O_a(x) = ax\tilde{a}$ is surjective. Further $-ax(-\tilde{a}) = ax\tilde{a}$, so $1, -1 \in ker(\theta)$. In fact $ker(\theta) = \{\pm 1\}$.
Let $Spin(n) := \{a \in Pin(n) : a = y_1 \dots y_p$ and p is even$\}$.
Then $Spin(n)$ is a subgroup of $Pin(n)$ and

$$\theta : Spin(n) \longrightarrow SO(n)$$

is surjective with kernel $\{1, -1\}$.

Dirac Operator

The Dirac Operator in $\mathbb{R}^n$ is defined to be

$$D := \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}.$$

Note $D^2 = -\Delta_n$, where Δ_n is the Laplacian in $\mathbb{R}^n$. The solution of the equation $Df = 0$ is called a left monogenic function.

Almansi-Fischer decomposition

Let $\mathcal{P}_k$ denote the space of Cl_n- valued monogenic polynomials, homogeneous of degree k and $\mathcal{H}_k$ be the space of Cl_n valued harmonic polynomials homogeneous of degree k. If $h_k \in \mathcal{H}_k$ then $Dh_k \in \mathcal{P}_{k-1}$. But $Dup_{k-1}(u) = (-n-2k+2)p_{k-1}(u)$, so the Almansi-Fischer decomposition is

$$\mathcal{H}_k = \mathcal{P}_k \bigoplus u\mathcal{P}_{k-1}.$$

That is, for any $h_k \in \mathcal{H}_k$, we obtain that $h_k(u) = p_k(u) + up_{k-1}(u)$. Note that if $Df(u) = 0$ then $\bar{f}(u)\bar{D} = -\bar{f}(u)D = 0$. So we can talk of right $k-$ monogenic polynomials and we have a right Almansi-Fisher decomposition, $\mathcal{H}_k = \bar{\mathcal{P}}_k \bigoplus \bar{\mathcal{P}}_{k-1}\bar{u}$.

Rarita-Schwiger operators R_k

Suppose U is a domain in $\mathbb{R}^n$. Consider a function

$$f : U \times \mathbb{R}^n \longrightarrow Cl_n$$

such that for each $x \in U, f(x, u)$ is a left monogenic polynomial homogeneous of degree k in u.

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Consider $D_x f(x, u) = f_{1,k}(x, u) + uf_{2,k-1}(x, u)$, where $f_{1,k}(x, u)$ and $f_{2,k-1}(x, u)$ are left monogenic polynomials homogeneous of degree k and $k-1$ in u respectively.

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Let P_k be the left projection map

$$P_k : \mathcal{H}_k(= \mathcal{P}_k \bigoplus u\mathcal{P}_{k-1}) \to \mathcal{P}_k,$$

then $R_k f(x,u)$ is defined to be $P_k D_x f(x,u)$.
The left Rarita-Schwinger equation is defined to be

$$R_k f(x,u) = 0.$$

We also have a right projection $P_{k,r} : \mathcal{H}_k \to \bar{\mathcal{P}}_k$, and a right Rarita-Schwinger equation $f(x,u)D_x P_{k,r} = f(x,u)R_k = 0$. Further we obtain that

$$P_k = (\frac{uD_u}{n+2k-2} + 1) \text{ and } R_k = (\frac{uD_u}{n+2k-2} + 1)D_x.$$

Are there any non-trivial solutions to the Rarita-Schwinger equation?

First for any k-monogenic polynomial $p_k(u)$ we have trivially $R_k p_k(u) = 0$. Now consider the fundamental solution $G(u) = \frac{1}{\omega_n} \frac{-u}{\|u\|^n}$ to the Dirac operator D, where ω_n is the surface area of the unit sphere, S^{n-1}.
Consider the Taylor series expansion of $G(v-u)$ and restrict to the kth order terms in $u_1, \ldots, u_n$ ($u = u_1 e_1 + \ldots + u_n e_n$). These terms have as vector valued coefficients

$$\frac{\partial^k}{\partial {v_1}^{k_1} \ldots \partial {v_n}^{k_n}} G(v) \quad (k_1 + \ldots + k_n = k).$$

As $DG(v) = \sum_{i=1}^{n} e_j \dfrac{\partial G(v)}{\partial v_j} = 0$, we can replace $\dfrac{\partial}{\partial v_1}$ by $-\sum_{j=2}^{n} e_1^{-1} e_j \dfrac{\partial}{\partial v_j}$. Doing this each time $\dfrac{\partial}{\partial v_1}$ occurs and collecting like terms we obtain a finite series of polynomials homogeneous of degree k in u

$$\sum_{\sigma} P_{\sigma}(u) V_{\sigma}(v)$$

where the summation is taken over all permutations of monogenic polynomials $(u_2 - u_1 e_1^{-1} e_2), \cdots, (u_n - u_1 e_1^{-1} e_n)$,

$P_{\sigma}(u) = \dfrac{1}{k!} \Sigma (u_{i_1} - u_1 e_1^{-1} e_{i_1}) \ldots (u_{i_k} - u_1 e_1^{-1} e_{i_k})$ and

$V_{\sigma}(v) = \dfrac{\partial^k G(v)}{\partial v_2^{j_2} \ldots \partial v_n^{j_n}}, j_2 + \cdots + j_n = k.$

Further $\int_{S^{n-1}} V_\sigma(u) u P_\mu(u) dS(u) = \delta_{\sigma,\mu}$ where $\delta_{\sigma,\mu}$ is the Kroneker delta and μ is a set of $n-1$ non-negative integers summing to k. Consequently, the expression

$$Z_k(u,v) := \sum_\sigma P_\sigma(u) V_\sigma(v) v$$

is the reproducing kernel of $\mathcal{P}_k$ with respect to integration over S^{n-1}. See [BDS]. Further as $Z_k(u,v)$ does not depend on x,

$$R_k Z_k = 0.$$

Are there any solutions to $R_k f(x, u) = 0$ that depends on x?

To answer this we need to look at the links to conformal transformations. Ahlfors [A] and Vahlen [V] show that given conformal transformation on $\mathbb{R}^n \bigcup \{\infty\}$ it can be expressed as $y = \phi(x) = (ax + b)(cx + d)^{-1}$ where $a, b, c, d \in Cl_n$ and satisfy the following conditions:

1. a, b, c, d are all products of vectors in $\mathbb{R}^n$.
2. $a\tilde{b}, c\tilde{d}, \tilde{b}c, \tilde{d}a \in \mathbb{R}^n$.
3. $a\tilde{d} - b\tilde{c} = \pm 1$.

When $c = 0$,
$\phi(x) = (ax + b)(cx + d)^{-1} = axd^{-1} + bd^{-1} = \pm ax\tilde{a} + bd^{-1}$.
Now assume $c \neq 0$, then
$\phi(x) = (ax + b)(cx + d)^{-1} = ac^{-1} \pm (cx\tilde{c} + d\tilde{c})^{-1}$, this is called an Iwasawa decomposition.

The following can be established.
If $Df(y) = 0$ and $y = ax\tilde{a}, a \in Pin(n)$, then $D\tilde{a}f(ax\tilde{a}) = 0$.
If $Df(y) = 0$ and $y = x^{-1}$ then $DG(x)f(x^{-1}) = 0$.
Using these and the Iwasawa decomposition we get $Df(y) = 0$ implies $DJ(\phi, x)f(\phi(x)) = 0$, where $J(\phi, x) = \frac{\widetilde{cx+d}}{\|cx+d\|^n}$.

Conformal invariance of P_k

1. Orthogonal transformation: Let $x = ay\tilde{a}$, and $u = aw\tilde{a}$, where $a \in Pin(n)$.

Lemma 1 $P_{k,w}\tilde{a}f(ay\tilde{a}, aw\tilde{a}) = \tilde{a}P_{k,u}f(x,u)$, where $P_{k,w}$ and $P_{k,u}$ are the projections with respect to w and u respectively.

2. Inversion: Let $x = y^{-1}, u = \frac{ywy}{\|y\|^2}$.

Lemma 2 $P_{k,w}\frac{y}{\|y\|^n}f(y^{-1}, \frac{ywy}{\|y\|^2}) = \frac{y}{\|y\|^n}P_{k,u}f(x,u)$, where $P_{k,w}$ and $P_{k,u}$ are the projections with respect to w and u respectively.

3. Translation: $x = y + a, a \in \mathbb{R}^n$. In order to keep the homogeneity of $f(x,u)$ in u, u does not change under translation.

Lemma 3 $P_k f(x,u) = P_k f(y+a, u)$.

4. Dilation: $x = \lambda y$, where $\lambda \in \mathbb{R}^+$.

Lemma 4 $P_k f(x,u) = P_k f(\lambda y, u)$.

Using the Iwasawa decomposition, we get the following result:
Theorem 3

$$P_{k,w}J(\phi,x)f(\phi(x),\frac{\widetilde{(cx+d)}w(cx+d)}{\|cx+d\|^2}) = J(\phi,x)P_{k,u}f(\phi(x),u),$$

where $u = \dfrac{\widetilde{(cx+d)}w(cx+d)}{\|cx+d\|^2}$, where $P_{k,w}$ and $P_{k,u}$ are the projections with respect to w and u respectively.

Conformal invariance of the equation $R_k f = 0$

(i) Inversion: Let $x = y^{-1}, (= \frac{-y}{\|y\|^2})$.

Theorem 4 If $R_k f(x, u) = 0$, then $R_k G(y) f(y^{-1}, \frac{ywy}{\|y\|^2}) = 0$.

(ii) Orthogonal transformation: $O \in O(n), a \in Pin(n)$

Theorem 5 If $x = ay\tilde{a}, u = aw\tilde{a}$ and $R_k f(x, u) = 0$ then $R_k \tilde{a} f(ay\tilde{a}, aw\tilde{a}) = 0$.

(iii) Dilation: $x = \lambda y, \lambda \in \mathbb{R}^+$.

If $R_k f(x, u) = 0$ then $R_k f(\lambda y, u) = 0$.

(iv) Translation: $x = y + \underline{a}, \quad \underline{a} \in \mathbb{R}^n$

If $R_k f(x, u) = 0$ then $R_k f(y + \underline{a}, u) = 0$.

Now using the Iwasawa decomposition of $(ax + b)(cx + d)^{-1}$, we obtain that

$$R_k f(x, u) = 0 \text{ implies } R_k J(\phi, x) f(\phi(x), \frac{\widetilde{(cx + d)} w (cx + d)}{\|cx + d\|^2}) = 0,$$

where $u = \frac{\widetilde{(cx + d)} w (cx + d)}{\|cx + d\|^2}$.

The Fundamental Solution of R_k

Applying inversion to $Z_k(u, v)$ from the right we obtain

$$E_k(y, u, v) := c_k Z_k(u, \frac{yvy}{\|y\|^2}) \frac{y}{\|y\|^n}$$

is a non-trivial solution to $f(y, v)R_k = 0$ on $\mathbb{R}^n \setminus \{0\}$, where $c_k = \dfrac{n - 2 + 2k}{n - 2}$.

Similarly, applying inversion to $Z_k(u, v)$ from the left we can obtain that

$$c_k \frac{y}{\|y\|^n} Z_k(\frac{yuy}{\|y\|^2}, v)$$

is a non-trivial solution to $R_k f(y, u) = 0$ on $\mathbb{R}^n \setminus \{0\}$. In fact, this function is $E_k(y, u, v)$, and $E_k(y, u, v)$ is the fundamental solution of R_k.

Basic Integral Formulas

Definition 1 For any Cl_n-valued polynomials $P(u), Q(u)$, the inner product $(P(u), Q(u))_u$ with respect to u is given by

$$(P(u), Q(u))_u = \int_{S^{n-1}} P(u)Q(u)dS(u).$$

For any $p_k \in \mathcal{P}_k$, one obtains

$$p_k(u) = (Z_k(u,v), p_k(v))_v = \int_{S^{n-1}} Z_k(u,v)p_k(v)dS(v).$$

See [BDS].

Basic Integral Formulas

Theorem 1 [BSSV] (Stokes' Theorem for R_k) Let Ω and Ω' be domains in $\mathbb{R}^n$ and suppose the closure of Ω lies in Ω'. Further suppose the closure of Ω is compact and $\partial\Omega$ is piecewise smooth. Then for $f, g \in C^1(\Omega', \mathcal{P}_k)$, we have

$$\int_\Omega [(g(x,u)R_k, f(x,u))_u + (g(x,u), R_k f(x,u))_u] dx^n$$

$$= \int_{\partial\Omega} (g(x,u), P_k d\sigma_x f(x,u))_u = \int_{\partial\Omega} (g(x,u) d\sigma_x P_{k,r}, f(x,u))_u ,$$

where $dx^n = dx_1 \wedge \cdots \wedge dx_n, d\sigma_x = \sum_{j=1}^{n} (-1)^{j-1} e_j d\hat{x}_j$, and $d\hat{x}_j = dx_1 \wedge \cdots dx_{j-1} \wedge dx_{j+1} \cdots \wedge dx_n$.

Basic Integral Formulas

Theorem 2 [BSSV](Borel-Pompeiu Theorem) Let Ω' and Ω be as in Theorem 1 and $y \in \Omega$. Then for $f \in C^1(\Omega', \mathcal{P}_k)$

$$f(y,u) = \int_{\partial\Omega} (E_k(x-y,u,v), P_k d\sigma_x f(x,v))_v$$

$$- \int_{\Omega} (E_k(x-y,u,v), R_k f(x,v))_v dx^n.$$

Here we will use the representation
$$E_k(x-y,u,v) = c_k Z_k(u, \frac{(x-y)v(x-y)}{\|x-y\|^2}) \frac{x-y}{\|x-y\|^n}.$$

Basic Integral Formulas

Theorem 3 (Cauchy Integral Formula) If $R_k f(x,v) = 0$, then for $y \in \Omega$,

$$f(y,v) = \int_{\partial\Omega} \left(E_k(x-y,u,v), P_k d\sigma_x f(x,v)\right)_v$$
$$= \int_{\partial\Omega} \left(E_k(x-y,u,v) d\sigma_x P_{k,r}, f(x,v)\right)_v .$$

Theorem 4 $\displaystyle\iint_{\mathbb{R}^n} -(E_k(x-y,u,v), R_k\psi(x,v))_v dx^n = \psi(y,u)$ for each $\psi \in C_0^\infty(\mathbb{R}^n)$.

Basic Integral Formulas

Definition 2 For a domain $\Omega \subset \mathbb{R}^n$ and a function $f : \Omega \times \mathbb{R}^n \longrightarrow Cl_n$, the Cauchy, or T_k-transform, of f is formally defined to be

$$(T_k f)(y, v) = -\int_{\Omega} \left(E_k(x - y, u, v), f(x, u)\right)_u dx^n, \qquad y \in \Omega.$$

Theorem 5 $R_k T_k \psi = \psi$ for $\psi \in C_0^{\infty}(\mathbb{R}^n)$. i.e

$$R_k \iint_{\mathbb{R}^n} \left(E_k(x - y, u, v), \psi(x, u)\right)_u dx^n = \psi(y, v).$$

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