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1. The Real Numbers

We will take the view that we know what the real numbers, denoted here by \( \mathbb{R} \), are and in this section we simply review some important properties.

1.1. Sets. The following notations will be used for the natural numbers, integers, positive integers and rational numbers, respectively.

\[
\mathbb{N} = \{0, 1, 2, \ldots\} \\
\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \\
\mathbb{N}^+ = \{1, 2, 3, \ldots\} \\
\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}^+ \right\}.
\]

**Example 1.1.** The square root of 2 is not rational; i.e., there is no rational number \( s > 0 \) such that \( s^2 = 2 \).

The cartesian product of two sets \( A \) and \( B \) is the set of ordered pairs \((a, b)\) with \( a \in A \) and \( b \in B \),

\[
A \times B = \{(a, b) : a \in A, b \in B\}.
\]

For instance \( \mathbb{R}^2 := \mathbb{R} \times \mathbb{R} \) is, geometrically, the cartesian plane.

1.2. Functions.

**Definition 1.2.** A function \( f \) consists of sets \( A \) and \( B \), called the domain and codomain of \( f \) respectively, and a rule that assigns to each \( a \in A \) a unique \( b = f(a) \in B \). We write, \( f : A \to B \).

The function \( f \) is one-one if \( f(x) = f(y) \) implies \( x = y \); and \( f \) is onto if \( \{f(x) : x \in A\} = B \).

In the case that \( B \) is a subset of \( \mathbb{R} \) we say that \( f \) is real-valued.

**Example 1.3.** Here are a couple examples of functions.

(a) Define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = x^2 \). Here both the domain and codomain of \( f \) is \( \mathbb{R} \).
(b) Define \( g : [0, \infty) \to \mathbb{R} \) by \( g(x) = x^2 \). Note, \( g \) is one-one, whereas \( f \) above is not.
(c) Define \( h : [0, 1] \to [0, 1] \) by \( h(x) = 1 \) if \( x \in \mathbb{Q} \cap [0, 1] \) and \( h(x) = 0 \) otherwise.

1.3. Field Axioms.

**Definition 1.4.** A field \( F \) is a triple, \((F, +, \cdot)\), where \( F \) is a set and

\[
+ : F \times F \to F \\
\cdot : F \times F \to F
\]

are functions, called addition and multiplication respectively and written \( x + y = +(x, y) \) and \( xy = \cdot(x, y) \), satisfying the following (long list) of axioms

(i) \( x + y = y + x \) for every \( x, y \in F \);
(ii) \( xy = yx \) for every \( x, y \);
(iii) \( (x + y) + z = x + (y + z) \) for every \( x, y, z \);
(iv) \( (xy)z = x(yz) \) for every \( x, y, z \);
(v) there is an element \( 0 \in F \) such that \( 0 + w = w \) for every \( x \in F \);
(vi) there is an element \(1 \in \mathbb{F}\), distinct from 0, such that \(1w = w\) for every \(w \in \mathbb{F}\);
(vii) for each \(x \in \mathbb{F}\) there is an element \(u \in \mathbb{F}\) such that \(x + u = 0\);
(viii) for each \(x \neq 0\), there is a \(y\) such that \(xy = 1\); and
(ix) \((x + y)z = xz + yz\) for every \(x, y, z\).

Note that only the property of item (ix) involves both operations. It is customary to write \(\mathbb{F}\) instead of \((\mathbb{F}, +, \cdot)\) to denote a field. Our primary example of a field is \(\mathbb{R}\) (with its usual operations).

**Proposition 1.5** (Cancellation). Suppose \(\mathbb{F}\) is a field. Given \(x, y, z \in \mathbb{F}\), if \(x + y = x + z\), then \(y = z\).

**Proof.** Let \(x, y, z \in \mathbb{F}\) such that \(x + y = x + z\) be given. By item (vii) there exists a \(u \in \mathbb{F}\) such that \(x + u = 0\). Thus,

\[
y = 0 + y = (u + x) + y = u + (x + y) = u + (x + z) = (u + x) + z = 0 + z = z,
\]

where we have used, in order, items (v), (i), (iii), the hypothesis \(x + y = x + z\), and items (iii), (v).

**Remark 1.6.** It follows that, given \(x \in \mathbb{F}\) there is exactly one \(u \in \mathbb{F}\) such that \(x + u = 0\). We call \(u\) the additive inverse of \(x\) and, for reasons that become clear after Proposition 1.7 below, denote it by \(-x\) and write \(y - x\) instead of \(y + (-x)\). Thus \(x + (-x) = x - x = -x + x = 0\). Similarly, if \(x \in \mathbb{F}\) and \(x \neq 0\), then there is exactly one \(y \in \mathbb{F}\) \((y \neq 0)\) such that \(xy = 1 = yx\). We call \(y\) the multiplicative inverse of \(x\). Both \(x^{-1}\) and \(\frac{1}{x}\) are used to denote the multiplicative inverse of an \(x \in \mathbb{F}\), \(x \neq 0\).

**Proposition 1.7.** Suppose \(\mathbb{F}\) is a field. If \(x \in \mathbb{F}\), then \(0x = 0\) and \(-x = (-1)x\).

**Proof.** Since \(0 + 0x = 0x = (0 + 0)x = 0x + 0x\), cancellation gives \(0 = 0x\).

Using \(0x = 0\) gives \(x + (-1)x = 1x + (-1)x = (1 + (-1))x = 0x = 0\). Thus \((-1)x\) is the additive inverse for \(x\); that is \((-1)x = -x\).

**Remark 1.8.** From here on we will use freely, without proof or further comment, the many routine properties of fields that follow from the axioms.

**Definition 1.9.** A subset \(\mathbb{G}\) of a field \(\mathbb{F}\) is a subfield if it satisfies the axioms,

(a) \(1 \in \mathbb{G}\);
(b) \(\mathbb{G}\) is closed under addition and scalar multiplication;
(c) if \(x \in \mathbb{G}\), then \(-x \in \mathbb{G}\)
(d) if \(x \in \mathbb{G}\) and \(x \neq 0\), then \(x^{-1} \in \mathbb{G}\).

Here the operations on \(\mathbb{G}\) are those inherited from \(\mathbb{F}\).

**Proposition 1.10.** If \(\mathbb{F}\) is a field and \(\mathbb{G} \subset \mathbb{F}\) is a subfield, then \(\mathbb{G}\) is a field.
For instance, by (a) and (c), \(-1 \in \mathbb{G}\). By (b), \(0 = 1 - 1 \in \mathbb{G}\).

**Example 1.11.** The rationals \(\mathbb{Q} \subset \mathbb{F}\) is easily seen to be a subfield of \(\mathbb{R}\) and hence a field. The complex numbers

\[
\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}
\]

with the usual operations is also a field. Note that \(\mathbb{R}\) can be viewed as a subfield of \(\mathbb{C}\).

**Example 1.12.** Show \(\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} \subset \mathbb{R}\) is a field.

**Example 1.13.** Let \(\mathbb{Z}_2 = (\{0, 1\}, +, \cdot)\) where

\[
x + y = x + y \text{ modulo } 2
\]

\[
xy = xy \text{ modulo } 2.
\]

Here the + on the left hand side is addition in \(\mathbb{Z}_2\), whereas + on the right hand side is addition in \(\mathbb{N}\).

The residue modulo 2 is the remainder after dividing by 2.

It is easy, but tedious, to verify that \(\mathbb{Z}_2\) is a field with neutral elements 0, 1.

**Example 1.14.** \(\mathbb{Z}\) (with the usual operations; that is those inherited from \(\mathbb{R}\)) is not a field. The smallest field containing \(\mathbb{Z}\) is \(\mathbb{Q}\).

1.4. **Ordered Fields.**

**Definition 1.15.** An ordered set \((S, <)\) consists of a (nonempty) set \(S\) and an order (relation \(^1\)) \(<\) on \(S\) satisfying

(i) (trichotomy) for each \(x, y \in S\), exactly one of the following hold,

\[
x < y, \quad y < x, \quad x = y;
\]

(ii) (transitivity) if \(x < y\) and \(y < z\), then \(x < z\).

Again, it is customary to write \(S\) for the ordered set instead of the more formal \((S, <)\).

**Example 1.16.** The usual order \(<\) on \(\mathbb{R}\) (and thus on any subset of \(\mathbb{R}\)) produces an example of an ordered set. In particular \(\mathbb{Q}\) is an order set and so is \(\mathbb{Q}(\sqrt{2})\) (with the order they inherits from \(\mathbb{R}\)).

**Definition 1.17.** An ordered field \(\mathbb{F} = (\mathbb{F}, +, \cdot, <)\) consists of a field \((\mathbb{F}, +, \cdot)\) that is also an ordered set \((\mathbb{F}, <)\) such that,

(i) if \(x, y, z \in \mathbb{F}\) and \(x < y\), then \(x + z < y + z\);

(ii) if \(x, y \in \mathbb{F}\) and \(x, y > 0\), then \(xy > 0\).

An element \(x \in \mathbb{F}\) is positive if \(x > 0\).

**Example 1.18.** Both \(\mathbb{R}\) and \(\mathbb{Q}\) with the usual ordering are ordered fields.

**Proposition 1.19.** Suppose \(\mathbb{F}\) is an ordered field and \(x \in \mathbb{F}\).

(i) If \(x < 0\), then \(-x > 0\);

(ii) if \(x > 0\), then \(-x < 0\);

(iii) if \(x \neq 0\), then \(x^2 > 0\);

\(^1\)A relation on a set \(S\) is a subset \(R\) of \(S \times S\) that satisfies certain axioms. In the setting here \(x < y\) indicates \((x, y) \in R\).
(iv) in particular, $1 > 0$ in any ordered field.

**Proof.** If $x < 0$, then, by item (i) of Definition 1.17, $0 = x - x < 0 - x$. Likewise, if $x > 0$, then $0 = x - x > 0 - x = -x$.

If $x > 0$, then, by (ii) of Definition 1.17, $x^2 = xx > 0$. If $x < 0$, then, by (ii) of Definition 1.17, $-x > 0$ and thus $x^2 = (-x)^2 > 0$. If $x \neq 0$, then either $x < 0$ or $x > 0$ by trichotomy. Thus, in either case $x^2 > 0$. Finally $1 = 1^2$ and hence $1 > 0$. □

**Remark 1.20.** We will not state (much less) prove all the usual facts about the order structure in an ordered field, but rather use them without comment.

**Example 1.21.** Prove that there is no order on $\mathbb{Z}_2$ that makes it an ordered field.

**Solution.** Arguing by contradiction, suppose $<$ is an order on $\mathbb{Z}_2$ that makes $\mathbb{Z}_2$ an ordered field. By Proposition 1.19 items (iv) and (ii) $1 > 0$ and hence $-1 < 0$. On the other hand, $-1 = 1$ since 1 is its own additive inverse in $\mathbb{Z}_2$. Thus $1 < 0$ and we have now reached a contradiction to the trichotomy property of an order (as both $1 > 0$ and $1 < 0$). □

1.5. **The least upper bound property.**

**Definition 1.22.** Let $S$ be a subset of an ordered field $\mathbb{F}$.

(i) The set $S$ is **bounded above** if there is a $b \in \mathbb{F}$ such that $b \geq s$ for all $s \in S$.

(ii) Any $b \in \mathbb{F}$ such that $b \geq s$ for all $s \in S$ is an **upper bound** for $S$.

**Example 1.23.** Identify the set of upper bounds for the following subsets of the ordered field $\mathbb{R}$.

(a) $[0, 1)$;  
(b) $[0, 1]$;  
(c) $\mathbb{Q}$;  
(d) $\emptyset$.

**Lemma 1.24.** Let $S$ be a subset of an ordered field $\mathbb{F}$ and suppose both $b$ and $b'$ are upper bounds for $S$. If $b$ and $b'$ both have the property that if $c \in \mathbb{F}$ is an upper bound for $S$, then $c \geq b$ and $c \geq b'$, then $b = b'$.

**Proof.** With $c = b'$ it follows that $b' \geq b$. Likewise with $c = b$ it follows that $b \geq b'$. By trichotomy $b = b'$. □

**Definition 1.25.** The **least upper bound** for a subset $S$ of an ordered field $\mathbb{F}$, if it exists, is a $b \in \mathbb{F}$ such that

(i) $b$ is an upper bound for $S$; and  
(ii) if $c \in \mathbb{F}$ is an upper bound for $S$, then $c \geq b$.

**Remark 1.26.** Lemma 1.24 justifies the use of the (as opposed to an) in describing the least upper bound.

The condition (ii) can be replaced with either of the following conditions

(ii)' if $c < b$, then there exists an $s \in S$ such that $c < s$; or  
(ii)" for each $\epsilon > 0$ there is an $s \in S$ such that $b - \epsilon < s$. 

DRAFT
The notions of bounded below, lower bound and greatest lower bound are defined analogously.

Least upper bound is often abbreviated lub. The term supremum, often abbreviated sup, is synonymous with lub. Likewise glb and inf for greatest lower bound and infimum.

**Example 1.27.** Here is a list of examples.

(i) The least upper bound of $S = [0, 1) \subset \mathbb{R}$ is 1.
(ii) The least upper bound of $V = [0, 1] \subset \mathbb{R}$ is also 1.
(iii) The set $\mathbb{Q} \subset \mathbb{R}$ has no upper bound and thus no least upper bound;
(iv) Every real number is an upper bound for the set $\emptyset \subset \mathbb{R}$. Thus $\emptyset$ has no least upper bound.

**Theorem 1.28.** Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

**Remark 1.29.** In fact, $\mathbb{R}$ is essentially the unique ordered field satisfying the conclusion of Theorem 1.28. This property, which thus distinguishes $\mathbb{R}$ from all other ordered fields, is a completeness property.

**Example 1.30.** Suppose $\mathbb{Q} \subset \mathbb{F} \subset \mathbb{R}$ and $\mathbb{F}$ is a (sub)field of $\mathbb{R}$. Let

$$S = \{ q \in \mathbb{Q} : q > 0, \ q^2 < 2 \} \subset \mathbb{Q} \subset \mathbb{F}.$$ Observe that $1 \in S$ and hence $S$ is nonempty. Next we show, if $0 < f \in \mathbb{Q} \subset \mathbb{F}$ and $f^2 \geq 2$, then $f$ is an upper bound for $S$. To this end, suppose $s \in \mathbb{F}$ and $s > f$, then $s^2 > f^2$ and hence $s \notin S$. Thus $f$ is an upper bound for $S$. In particular, by choosing $f = 2$, it follows that $2 \in \mathbb{F}$ is an upper bound for $S$ and hence $S$ is bounded above (in $\mathbb{F}$).

Now suppose $S$ has a least upper bound $\alpha \in \mathbb{F}$. (In the case that $\mathbb{F} = \mathbb{R}$ it does by Theorem 1.28.) We claim that $\alpha = \sqrt{2}$; that is $\alpha \geq 0$ and $\alpha^2 = 2$. To prove this claim first note that as $S \ni 1 \leq \alpha$, we have $\alpha \geq 1 > 0$. Further $\alpha \leq 2$ as 2 is an upper bound for $S$. Hence it suffices to show that if $1 \leq \alpha \leq 2$, then $\alpha^2 \neq 2$ and $\alpha^2 \neq 2$. We prove each of these two statements arguing by contradiction.

Suppose, by way of contradiction, suppose $2 \geq \alpha \geq 1$ and $\alpha^2 > 2$. Let

$$\beta = \alpha - \frac{\alpha^2 - 2}{4}$$

and note that $\beta \in \mathbb{F}$ and $\alpha > \beta > 0$, since $1 \leq \alpha \leq 2$. Moreover,

$$\beta^2 = \alpha^2 - \frac{\alpha^2 - 2}{2} + \frac{\alpha^4 - 4\alpha + 4}{16}$$

$$= \alpha^2 + 1 + \frac{\alpha^4}{16} + \frac{1}{4}$$

$$> \frac{1}{2} + 1 + \frac{1}{4} + \frac{1}{4} = 2.$$ Hence $\alpha > \beta$ and $\beta$ is an upper bound for $S$, contradicting the assumption that $\alpha$ is least among upper bounds for $S$.

Now suppose $2 > \alpha \geq 1$. In this case let

$$\beta = \alpha + \frac{2 - \alpha^2}{2} \geq \alpha.$$
Note that $\beta \in \mathbb{F}$ and $\beta > \alpha > 0$. Further,

$$
2 - \beta^2 = 2 - \left[ \frac{\alpha^2}{2} + 2 - \frac{\alpha^2}{4} + \frac{4 - 4\alpha^2 + \alpha^4}{16} \right] = 2 - \left[ \frac{1}{4}\alpha^2 + \frac{5}{4} + \frac{1}{16}\alpha^4 \right] = 2 - \left[ \frac{2}{4} + \frac{5}{4} + \frac{1}{4} \right] = 0.
$$

By Theorem 1.32 item (iii) (below), there is a an $s \in \mathbb{Q}$ such that $\beta > r > \alpha$. It follows that $s > 0$ and $s^2 < \beta^2 < 2$ and thus $s \in S$. Hence $\alpha$ is not an upper bound for $S$, a contradiction that completes the proof that if $S$ has a least upper bound $\alpha$, then $\alpha > 0$ and $\alpha^2 = 2$. In particular, $S$, viewed as a subset of the ordered field $\mathbb{Q}$, does not have a least upper bound. On the other hand, if $\mathbb{F} = \mathbb{R}$, then $S$ does have a least upper bound and this least upper bound is $\sqrt{2}$.

The argument used in Example 1.30 above can be generalized to prove the following proposition.

**Proposition 1.31.** If $y \in \mathbb{R}$, $y > 0$ and $n \in \mathbb{N}^+$, then there is a unique positive real number $s$ such that $s^n = y$.

**Theorem 1.32** (Archemedian properties of $\mathbb{R}$). Suppose $x, y \in \mathbb{R}$.

1. There is a natural number $n$ so that $n > x$;
2. If $y > 0$, then there is an $n \in \mathbb{N}^+$ such that $\frac{1}{n} < y$; and
3. If $x < y$, then there is a $q \in \mathbb{Q}$ such that $x < q < y$.

**Remark 1.33.** Item (iii) of Theorem is sometimes expressed as saying $\mathbb{Q}$ is dense in $\mathbb{R}$.

**Proof.** To prove (i), by arguing by contradiction, suppose no such natural number exists. In that case $x$ is an upper bound for $\mathbb{N}$. It follows that $\mathbb{N}$ has a lub $\alpha$. If $n \in \mathbb{N}$, then $n + 1 \leq \alpha$ and thus $n \leq \alpha - 1$ for all $n \in \mathbb{N}$. Thus, $\alpha - 1$ is an upper bound for $\mathbb{N}$, contradicting the least property of $\alpha$. Hence $\mathbb{N}$ is not bounded above and there is an $n > x$, which proves item (i).

Item (ii) follows by applying (i) to $x = \frac{1}{y}$.

To prove (iii), choose $n \in \mathbb{N}^+$ so that $1 < n(y - x)$. Choose $m \in \mathbb{Z}$ so that

$$m - 1 \leq nx < m.$$

Rearranging the inequalities gives,

$$nx < m < nx + 1 < ny.$$ 

Hence $x < \frac{m}{n} < y$. \[\square\]

**Example 1.34.** Suppose $0 < a < 1$. Show the set $A = \{a^n : n \in \mathbb{N}\}$ is bounded below and its infimum is 0.

**Solution.** Since $a \geq 0$ each $a^n \geq 0$. Thus $A$ is bounded below by 0. The set $A$ is not empty. It follows that $A$ has an infimum. Let $\alpha = \inf(A)$ and note $\alpha \geq 0$. Since $\alpha \leq a^n$ for $n = 0, 1, 2, \ldots$, $\alpha \leq a^{n+1}$ for $n \in \mathbb{N}$ and therefore $\frac{a}{n} \leq a^n$ for $n \in \mathbb{N}$. Thus, $\frac{a}{n}$ is a lower bound for $A$. It follows that $\frac{a}{\alpha} \leq \alpha$. Since $a < 1$ and $\alpha \geq 0$, $\alpha = 0$.
1.6. Accumulation points and the Balzano-Weierstrass Theorem.

**Definition 1.35.** Let $S$ be a given subset of $\mathbb{R}$. A point $a \in \mathbb{R}$ is an accumulation point (synonymously limit point) of $S$ if for each $\epsilon > 0$ there is an $s \in S$ such that $0 < |a - s| < \epsilon$ (equivalently, $s \neq a$ and $|a - s| < \epsilon$).

**Example 1.36.** The point 0 is an accumulation point of the set $S = \{\frac{1}{n} : n \in \mathbb{N}^+\} \subset \mathbb{R}$. The point 0 is also an accumulation point of the set $T = S \cup \{0\}$. Hence an accumulation point of a set may, or may not, be in the set.

Given a set $X$ and subsets $A, B \subset S$, the set difference $A \setminus B = \{x \in X : x \in A, x \notin B\}$.

**Lemma 1.37.** Suppose $S, T$ are subsets of $\mathbb{R}$ and $a \in \mathbb{R}$.

(i) If $a$ is an accumulation point of $S$, then $a$ is an accumulation point of $S \setminus \{a\}$.

(ii) If $S \subset T$ and $a$ is an accumulation point of $S$, then $a$ is an accumulation point of $T$.

(iii) The point $a$ is an accumulation point of $S$ if and only if for every $\epsilon > 0$ the set $(a-\epsilon, a+\epsilon) \cap S$ is infinite.

**Proof.** The proofs of items (i) and (ii) are routine and left to the gentle reader. To prove item (iii), first suppose that $\epsilon > 0$ the set $(a-\epsilon, a+\epsilon) \cap S$ is infinite. In particular, for each $\epsilon > 0$ there is a $t \neq a$ such that $t \in (a-\epsilon, a+\epsilon) \cap S$; that is $t \in S \setminus 0 < |a - t| < \epsilon$. It follows that if for every $\epsilon > 0$ the set $(a-\epsilon, a+\epsilon) \cap S$ is infinite, then $a$ is an accumulation point of $S$. To prove the converse, we prove the contrapositive. Namely, if $a \in \mathbb{R}$ and there is an $\delta > 0$ such that $T = (a-\delta, a+\delta) \cap S$ is finite, then $a$ is not an accumulation point of $S$. Indeed, in this case, $\{|a - t| : t \in T, t \neq a\}$ is a finite set of positive numbers. Hence it has a a positive minimum $\epsilon$. In particular, there is no $s \in S$ such that $s \neq a$ and $|s - a| < \epsilon$ and so $a$ is not a limit point of $S$. \hfill $\Box$

**Example 1.38.** (i) If $F \subset \mathbb{R}$ is finite, then $F$ has no accumulation points.

(ii) The set of accumulation points of $S = \{\frac{1}{n} : n \in \mathbb{N}^+\}$ is exactly $\{0\}$; i.e., if $r \neq 0$, then $r$ is not an accumulation point of $S$.

(iii) The set $\mathbb{Z}$ has no accumulation points.

(iv) The set of accumulation points of the set $(0, 1)$ is the set $[0, 1]$.

(v) The set of accumulation points of $Q$ is $\mathbb{R}$, a fact that is equivalent to the statement that between any two real numbers there is a rational, and often expressed by saying the rationals are dense in the real numbers. See Theorem 1.32.

**Theorem 1.39** (Balzano-Weierstrass). If $S$ is an infinite and bounded subset of $\mathbb{R}$, then $S$ has an accumulation point.

**Proof.** Since $S$ is bounded, there exists a $C > 0$ such that $S \subset [-C, C]$. Let $$ T = \{r \in \mathbb{R} : S \cap (-\infty, r] \text{ is finite}\}. $$ Note that $-C \in T$ since $S \cap (-\infty, -C] \subset \{-C\}$. Thus $T$ is nonempty. On the other hand if $r \geq C$, then $S \cap (-\infty, r] = S$ and thus, since $S$ is infinite, $r \notin T$. Hence $T \subset (-\infty, C]$; that is $C$ is an upper bound for $T$. Hence $T$ has a least upper bound $\alpha$.

Let $\beta < \alpha$ be given. By the least property of $\alpha$, there is a $\beta < \gamma < \alpha$ such that $\gamma \in T$. In particular, $S \cap (-\infty, \gamma]$ is a finite set and thus so is $S \cap (-\infty, \beta]$. On the other hand, if $\delta > \alpha$, then $\delta \notin T$ as $\alpha$ is an upper bound for $T$. In particular, $S \cap (-\infty, \delta]$ is an infinite set. It follows that $S \cap (\beta, \delta]$ is an infinite set. Summarizing: if $\beta < \alpha$, then $S \cap (\beta, \alpha]$ is an infinite
set. Given $\epsilon > 0$, let $\beta = \alpha - \epsilon$ and note $S \cap (\alpha - \epsilon, \alpha + \epsilon) = S \cap (\beta, \alpha]$ is infinite. Hence $\alpha$ is an accumulation point of $S$ and the proof is complete.

1.7. The Cauchy-Schwarz and triangle inequalities. Let $\mathbb{R}^d$ denote the set of matrices of size $d \times 1$. Thus an element of $a \in \mathbb{R}^d$ has the form

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix},$$

for real numbers $a_1, \ldots, a_d$.

Given $a, b \in \mathbb{R}^d$, define the inner product of $a$ and $b$ by

$$\langle a, b \rangle = \sum a_j b_j.$$ 

The inner product is also called the dot product and scalar product. The norm of $a \in \mathbb{R}^d$ is

$$\|a\| = \sqrt{\langle a, a \rangle}.$$ 

The interpretation of the dot product and norm should be familiar in the cases $d = 2$ and $d = 3$.

Proposition 1.40 (Cauchy-Schwarz inequality). Given $a, b \in \mathbb{R}^d$,

$$|\langle a, b \rangle| \leq \|a\| \|b\|.$$ 

Proof. Consider, for $t \in \mathbb{R}$,

$$0 \leq \|a + tb\|^2 = \|a\|^2 + 2t \langle a, b \rangle + t^2 \|b\|^2.$$ 

It follows that the discriminant of this polynomial is non-positive; i.e.,

$$|\langle a, b \rangle|^2 - \|a\|^2 \|b\|^2 \leq 0.$$

Proposition 1.41. If $a, b \in \mathbb{R}^d$, then

$$\|a + b\| \leq \|a\| + \|b\|.$$ 

Proof. From

$$\|a + b\|^2 = \|a\|^2 + 2 \langle a, b \rangle + \|b\|^2$$

and Proposition 1.40 it follows that

$$\|a + b\|^2 \leq \|a\|^2 + 2\|a\| \|b\| + \|b\|^2 = (\|a\| + \|b\|)^2.$$ 

\[\text{This subsection is not used elsewhere and hence optional.}\]
1.8. Problems.

Problem 1.1. Let $\mathbb{Z}_3 = (\{0, 1, 2\}, +, \cdot)$ where
\[
x + y = x + y \mod 3 \\
x \cdot y = x \cdot y \mod 3
\]
It is easy, but tedious, to verify that $\mathbb{Z}_3$ is a field.

Find the additive inverse for 1 and multiplicative inverse for 2.

Show there is no order $<$ on $\{0, 1, 2\}$ such that $(\mathbb{Z}_3, <)$ is an ordered field. (Suggestion: Arguing by contradiction, show the additive inverse for 1 would have to be both positive and negative.)

Problem 1.2. See the wikipedia page on the field of complex numbers $\mathbb{C}$. Suppose $z \in \mathbb{C}$ is not zero.

(a) Show if and $z = x + iy$ is the rectangular representation of $z$, then
\[
z^{-1} = \frac{\overline{z}}{|z|^2},
\]
where $|z|^2 = x^2 + y^2$.

(b) Show, if $z = r(\cos(\theta) + i \sin(\theta))$ is the polar representation of $z$, then
\[
z^{-1} = \frac{1}{r}(\cos(\theta) - i \sin(\theta)).
\]
Interpret geometrically.

Problem 1.3. Show there is no order $<$ on $\mathbb{C}$ such that $(\mathbb{C}, <)$ is an ordered field. (Suggestion: Consider $i^2$.)

Problem 1.4. The greatest lower bound (glb) or infimum (inf) is defined by simply reversing the inequalities in the definition of least upper bound.

Prove, using Theorem 1.28, that if $S \subset \mathbb{R}$ is nonempty and bounded below, then $S$ has a unique greatest lower bound.

Problem 1.5. Find, with proof, the greatest lower bound of the set
\[
\left\{ \frac{1}{n} : n \in \mathbb{N}^+ \right\}.
\]

Problem 1.6. Show, if $S \subset \mathbb{R}$ is nonempty and bounded (meaning bounded above and below), then $\inf(S) \leq \sup(S)$.

Problem 1.7. Suppose $S \subset T \subset \mathbb{R}$. Show, if $T$ is bounded above and $S$ is non-empty, then both $S$ and $T$ have least upper bounds and moreover,
\[
\sup(S) \leq \sup(T).
\]

Problem 1.8. Suppose $S \subset \mathbb{R}$ is non-empty and bounded above (and hence has a least upper bound). Given $a \in \mathbb{R}$, let
\[
T = a + S := \{a + s : s \in S\}.
\]
Prove that $T$ is non-empty and bounded above and moreover,
\[
\sup(T) = a + \sup(S).
\]
Problem 1.9. Show, if \( S \) and \( T \) are both nonempty and bounded above, then so is
\[
S + T = \{ s + t : s \in S, \ t \in T \}
\]
and moreover,
\[
\sup(S + T) = \sup(S) + \sup(T).
\]
[Suggestion. Given \( s \in S \), note that \( \sup(s + T) \leq \sup(S + T) \). On the other hand, by the previous problem, \( \sup(s + T) = s + \sup(T) \). Proceed.]

Problem 1.10. Given a positive real number \( y \) and positive integers \( m \) and \( n \), show
\[
(y^{\frac{1}{n}})^m = (y^m)^{\frac{1}{n}}.
\]
Thus, \( y^{\frac{m}{n}} \) is unambiguously defined.

Problem 1.11. Verify the claims in Example 1.38.

Problem 1.12. Give an example of a set with exactly two accumulation points.

Problem 1.13. Let \( S' \) denote the set of accumulation points of a subset \( S \) of \( \mathbb{R} \). Show, \( (S')' \subset S' \).

Use the set \( S \) from Item (ii) of Example 1.38 to show that inclusion can be proper.

Show, if \( \mathbb{Q} \subset S' \), then \( S' = \mathbb{R} \).

As a challenge question: What about \( S'' = ((S')')' \)?

Problem 1.14. Show, if \( a, b \) are non-negative real numbers, then
   (i) \( a < b \) if and only if \( a^2 < b^2 \); and
   (ii) \( ab \leq \frac{a^2 + b^2}{2} \).

Problem 1.15 (optional). Show, if \( a, b \in \mathbb{R}^d \), then \( \|a\| - \|b\| \leq \|a - b\| \).

2. Sequences

Definition 2.1. A sequence is a function \( a \) whose domain is \( \mathbb{N} \) or more generally a set of the form \( \{ n \in \mathbb{Z} : n \geq k \} \) for some \( k \in \mathbb{Z} \). It is commonly denoted as \( (a_n)_{n=k}^\infty \) where \( a_n = a(n) \) is the value of \( a \) at \( n \). In these notes, generally \( a \) is assumed to take real values so that each \( a_n \in \mathbb{R} \).

Example 2.2. Here are a few examples of sequences.
   (i) \( (a_n = \frac{1}{n})_{n=1}^\infty \);
   (ii) \( (a_n = (-1)^n) \);
   (iii) \( (a_n = n) \).

2.1. Limits.

Definition 2.3. Suppose \( A \in \mathbb{R} \) and \( (a_n) \) is a sequence of real numbers. The sequence \( (a_n) \) converges to \( A \) if for every \( \epsilon > 0 \) there is an \( N \) such that if \( n \geq N \), then \( |a_n - A| < \epsilon \). The notations \( (a_n) \to A \) and
\[
\lim_{n \to \infty} a_n = A
\]
are short hand for the statement \( (a_n) \) converges to \( A \).

The sequence \( (a_n) \) converges if there is an \( A \in \mathbb{R} \) such that \( (a_n) \) converges to \( A \). Otherwise, the sequence diverges.
Proposition 2.4. If \((a_n)\) converges to both \(A\) and \(B\), then \(A = B\).

Definition 2.5. Suppose \((a_n)\) is a sequence from \(\mathbb{R}\) and \(A \in \mathbb{R}\). If \((a_n)\) converges to \(A\), then \(A\) is the limit of the sequence.

Example 2.6. Show that \((\frac{1}{n})\) converges to 0.

Given \(\epsilon > 0\) choose, by the Archimedean property of \(\mathbb{R}\) (Theorem 1.32), an \(N \in \mathbb{N}^+\) such that \(\frac{1}{N} < \epsilon\). Now, if \(n \geq N\), then,

\[ \left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon. \]

Example 2.7. Show that the sequence \((a_n)\) defined by

\[ a_n = \frac{n}{n+2} \]

converges to 1.

Solution. Given \(\epsilon > 0\) choose, by the Archimedean property, \(N \in \mathbb{N}^+\) such that \(\frac{1}{N} < \frac{\epsilon}{2}\). Now, if \(n \geq N\), then

\[ \left| \frac{n}{n+2} - 1 \right| = \frac{2}{n+2} \leq \frac{2}{N} < \epsilon. \]

Example 2.8. Show that the sequence \((b_n)\) defined by

\[ b_n = \frac{n^2 + 2}{2n^3 - n - 2} \]

converges to 0.

Given \(\epsilon > 0\) choose, by the Archimedean property, \(N \in \mathbb{N}^+\) so that \(N \geq \max\{2, \frac{\epsilon}{2}\}\). Now, if \(n \geq N\), then

\[ \left| b_n - 0 \right| = \frac{n^2 + 2}{2n^3 - n - 2} \leq \frac{2n^2}{n^3} \leq \frac{2}{n} \leq \frac{2}{N} < \epsilon. \]

Example 2.9. Fix \(0 \leq a < 1\) and let \(a_n = a^n\) (for \(n \geq 0\)). To show that \((a^n)\) converges to 0, recall, from Example 1.34, that the greatest lower bound of the set \(A = \{a^n : n \in \mathbb{N}\}\) is 0. In particular, given \(\epsilon > 0\), there is an \(b \in A\) such that \(0 \leq b < \epsilon\). There is an \(N\) such that \(b = a^N\). If \(n \geq N\), then \(0 \leq a^n \leq a^N = b < \epsilon\). Hence, if \(n \geq N\), then \(|0 - a^n| < \epsilon\) and thus \((a^n)\) converges to 0.

Another proof that \((a^n)\) converges to 0 is given in Example 2.20.

Here is a list of simple properties of limits.

Proposition 2.10. Let \((a_n)^\infty_k\) be a sequence from \(\mathbb{R}\) and suppose \(A \in \mathbb{R}\).

(a) The sequence \((a_n)^\infty_k\) converges if and only if for each \(\ell > k\) the sequence \((a_n)^\infty_\ell\) converges;
(b) if there is an \(M \in \mathbb{N}\) and a \(c \in \mathbb{R}\) such that \(a_n = c\) for \(n \geq N\), then the sequence \((a_n)\) converges to \(c\);
(c) if there is an $N$ and an $\ell$ such that for $n \geq N$, $b_n = a_{n+\ell}$, then $(a_n)$ converges if and only if $(b_n)$ converges and in that case they converge to the same value;
(d) if $(a_n)$ converges to $A$ and $c \in \mathbb{R}$, then $(ca_n)$ converges to $cA$;
(e) The sequence $(a_n)$ converges to $A$ if and only if the sequence $(a_n - A)$ converges to 0.

2.2. Cauchy Sequences.

Definition 2.11. A sequence $(a_n)$ from $\mathbb{R}$ is Cauchy if for every $\epsilon > 0$ there is an $N$ so that if $m, n \geq N$, then $|a_n - a_m| < \epsilon$.

Proposition 2.12. If $(a_n)$ converges, then $(a_n)$ is Cauchy.

Proof. Let $A$ denote the limit of the sequence $(a_n)$. Let $\epsilon > 0$ be given. There is an $N$ so that if $n \geq N$, then $|a_n - A| < \frac{\epsilon}{2}$. Hence, if both $m, n \geq N$, then

$$|a_n - a_m| \leq |a_n - A| + |A - a_m| < 2\frac{\epsilon}{2} = \epsilon.$$ 

Example 2.13. The sequence $((-1)^n)$ diverges.

Using the contra-positive of Proposition 2.12 it suffices to show there exists an $\epsilon_0 > 0$ such that for every $N$ there exists $m, n \geq N$ such that $|a_n - a_m| \geq \epsilon_0$ (with $a_n = (-1)^n$).

Choose $\epsilon_0 = 1$. Given $N$, let $n = N$ and $m = N + 1$. Since $m, n$ have different parities, $|(-1)^n - (-1)^m| = 2 \geq \epsilon_0 = 1$. Note that this argument shows if $a \leq -1$, then $(a^n)$ does not converge.

Definition 2.14. A sequence $(a_n)$ is bounded above if the set $S = \{a_n : n \in \mathbb{N}\}$ is bounded above; i.e., there is an $M$ so that $a_n \leq M$ for all $n$. It is bounded if it is bounded above and below; i.e., if there is an $M$ so that $|a_n| \leq M$ for all $n$.

Proposition 2.15. If $(a_n)$ is Cauchy, then $(a_n)$ is bounded. In particular, convergent sequences are bounded.

The proof does not use anywhere near the full strength of the Cauchy condition.

Proof. With $\epsilon = 1$ there is an $N$ so that if $n, m \geq N$, then $|a_n - a_m| < 1$. Hence, $|a_n - a_N| < 1$ for all $n \geq N$ and thus $|a_n| \leq |a_N| + 1$ for $n \geq N$. There is a $C$ such that $|a_n| \leq C$ for all $n \leq N - 1$. Hence, $|a_n| \leq |a_N| + 1 + C$ for all $n$. □

Example 2.16. The sequence $(a_n = n)$ diverges, since the set $\mathbb{N}$ is not bounded above by Theorem 1.32.

Theorem 2.17. If $(a_n)$ is Cauchy, then $(a_n)$ converges.

Proof. Let $S$ denote the range of the sequence. Thus $S = \{a_n : n \in \mathbb{N}\}$. By Proposition 2.15, the set $S$ is bounded. If $S$ is finite, then there is an $A$ such that $a_n = A$ for infinitely many $n$. In particular, the set $I = \{m : a_m = A\}$ is infinite. To prove $(a_n)$ converges to $A$, let $\epsilon > 0$ be given. There is an $N$ such that for $m, n \geq N$, $|a_n - a_m| < \epsilon$. Thus, if $n \geq N$, then, choosing $m \in I$ and $m \geq N$, it follows that $|A - a_n| = |a_m - a_n| < \epsilon$.

Now suppose $S$ is infinite. Then, by Theorem 1.39, $S$ has an accumulation point $A$. To prove that $(a_n)$ converges to $A$, let $\epsilon > 0$ be given. There is an $N$ such that if $m, n \geq N$, then $|a_n - a_m| < \frac{\epsilon}{2}$. On the other hand, the set $(A - \epsilon, A + \epsilon) \cap S$ is infinite by Lemma 1.37 item
(iii), so there is an \( m \geq N \) such that \( a_m \in (A - \epsilon, A + \epsilon) \); i.e., \( |A - a_m| < \frac{1}{2}\epsilon \). Hence, if \( n \geq N \), then
\[
|A - a_n| \leq |A - a_m| + |a_m - a_n| < \epsilon.
\]

\[\blacksquare\]

### 2.3. Monotone Sequences.

**Definition 2.18.** The sequence \((a_n)\) is **increasing** if \(a_{n+1} \geq a_n\) for all \(n\) and it is **strictly increasing** if \(a_{n+1} > a_n\) for all \(n\). The notions of decreasing and strictly decreasing are defined analogously. A **monotone sequence** is one that is either increasing or decreasing.

Similarly, a sequence \((a_n)_k^\infty\) is **eventually monotone** if there is an \(M\) so that \((a_n)_M^\infty\) is monotone.

**Theorem 2.19.** If \((a_n)_n^\infty\) is eventually increasing and bounded above, then it converges.

**Proof.** There is an \(M\) so that \((a_n)_n^M\) is increasing. The set \(S = \{a_n : n \geq M\}\) is non-empty and, by hypothesis, bounded above. Hence \(S\) has a supremum \(A\). To show that \((a_n)_n^\infty\) converges to \(A\), let \(\epsilon > 0\) be given. There is \(s \in S\) such that \(A - \epsilon < s\). There is an \(N\) so that \(s = a_N\). Now, if \(n \geq N\), then
\[
0 \leq A - a_n \leq A - a_N = A - s < \epsilon.
\]

An application of Proposition 2.10 completes the proof. \[\blacksquare\]

**Example 2.20.** Fix \(0 < a < 1\) and consider the sequence \((a_n = a^n)\). Since \(a_{n+1} = aa_n\), the sequence is decreasing. It is also bounded below by \(0\). Hence \((a^n)\) converges to some \(L\). To see that \(L = 0\), note that \((a_{n+1}) = (aa_n)\) and \((a_{n+1})\) converges to \(L\); whereas, \((aa_n)\) converges to \(aL\). Thus, \(L = aL\) and since \(a \neq 1\), \(L = 0\).

Evidently, if \(a = 1\), then the sequence \((a_n = a^n = 1)\) converges to \(1\). If \(a > 1\), then, for \(n \geq 0\) we have \(|a^{n+1} - a^n| = a^n(a - 1) \geq a - 1 > 0\). Thus, in this case \((a^n)\) is not Cauchy and therefore doesn’t converge.

**Example 2.21.** Let \(a_1 = \sqrt{2}\) and define, recursively, \(a_{n+1} = \sqrt{2 + \sqrt{a_n}}\). The following induction argument shows that \((a_n)\) is increasing.

First, note that \(a_2 = \sqrt{2 + \sqrt{2}} \geq \sqrt{2} = a_1\). Now suppose that \(a_n \geq a_{n-1}\). In this case, \(\sqrt{a_n} \geq \sqrt{a_{n-1}}\) and hence,
\[
a_{n+1} = \sqrt{2 + \sqrt{a_n}} \geq \sqrt{2 + \sqrt{a_{n-1}}} = a_n,
\]
and the induction argument is complete.

An induction argument shows that \((a_n)\) is bounded above by \(2\). Hence, by Theorem 2.19, the sequence \((a_n)\) converges to some \(A\).

**Definition 2.22.** The sequence \((a_n)\) **diverges to \(\infty\)** if for each \(C > 0\) there is an \(N\) so that if \(n \geq N\), then \(a_n > C\).

**Example 2.23.** The sequence \((a_n = n)\) diverges to \(\infty\).

To see that the sequence \((b_n = \sqrt{n})\) diverges to \(\infty\), let \(C > 0\) be given. Choose, by Theorem 1.32, an \(N\) so that \(N > C^2\). If \(n \geq N\), then
\[
\sqrt{n} \geq \sqrt{N} > C.
\]
Example 2.24. Show the sequence \((a_n = \frac{n^2 - 1}{n + 2})\) diverges to \(\infty\).

Observe, for \(n \geq 2\), that \(n^2 - 1 \geq \frac{1}{2}n^2\) and at the same time \(n + 2 \leq 2n\). Thus, for \(n \geq 2\),
\[
\frac{n^2 - 1}{n + 2} \geq \frac{1}{4} n = \frac{1}{4n}.
\]

Given \(C > 0\) choose \(N\) such that \(N \geq \max\{2, \frac{1}{4C}\}\). With this choice of \(N\), if \(n \geq N\), then
\[
\frac{n^2 - 1}{n + 2} \geq \frac{1}{4n} \geq \frac{1}{4N} > C.
\]

Theorem 2.25. An eventually increasing sequence \((a_n)\) converges if and only if it is bounded above.

If \((a_n)\) is eventually increasing, but not bounded above (equivalently diverge), then \((a_n)\) diverges to \(\infty\).

Thus, if \((a_n)\) is eventually increasing, then either \((a_n)\) converges or diverges to \(\infty\) depending on whether it is bounded above or not.

Proof. That a bounded increasing sequence converges has already been established (Theorem 2.19). On the other hand, convergent sequences are bounded. Thus, assuming \((a_n)\) is increasing, convergence and boundedness are equivalent.

To prove the second statement, suppose \((a_n)\) is not bounded and let \(C > 0\) be given. Since \((a_n)\) is not bounded, there is an \(N\) so that \(a_N > C\). Now, if \(n \geq N\), then \(a_n \geq a_N > C\) and so \((a_n)\) diverges to \(\infty\). \(\square\)

Definition 2.26. A set \(A\) is at most countable if there is an onto mapping \(f : \mathbb{N} \to A\). Otherwise \(A\) is uncountable.

Theorem 2.27. The set \(\mathbb{R}\) is uncountable.

Proof. It suffices to show if \(f : \mathbb{N} \to \mathbb{R}\), then \(f\) is not onto. For notational ease, let \(x_j = f(j)\).

Choose \(b_0 > a_0\) such that \(x_0 \notin I_0 := [a_0, b_0]\). Next choose \(a_1 < b_1\) such that \(a_0 \leq a_1 < b_1 \leq b_0\) and \(x_1 \notin I_1 = [a_1, b_1]\). Continuing in this fashion, construct, by the principle of recursion, a sequence of intervals \(I_j = [a_j, b_j]\) such that

(i) \(I_0 \supset I_1 \supset I_2 \supset \cdots\);
(ii) \(b_j - a_j > 0\); and
(iii) \(x_j \notin I_k\) for \(j \leq k\).

Observe that the recursive construction of the sequences of endpoints \((a_j)\) and \((b_j)\) implies that \(a_0 \leq a_1 \leq a_2 < \cdots < b_2 \leq b_1 \leq b_0\); i.e., \((a_j)\) is increasing and is bounded above by each \(b_m\). By Theorem 2.19 \((a_j)\) converges to
\[
y = \sup\{a_j : j \in \mathbb{N}\}.
\]

In particular, \(a_m \leq y \leq b_m\) for each \(m\). Thus \(y \in I_m\) for all \(m\). On the other hand, for each \(k\),
\[
x_k \notin I_k
\]
and so \(y \neq x_k\). Hence \(y\) is not in the set \(\{x_k : k \in \mathbb{N}\}\) which is the range of \(f\). \(\square\)
2.4. Limit Theorems.

**Theorem 2.28.** Let \((a_n, b_n)\) be sequences from \(\mathbb{R}\) that converges to \(A\) and \(B\) respectively.

(a) The sequence \((a_n + b_n)\) converges to \(A + B\);
(b) The sequence \((a_nb_n)\) converges to \(AB\);
(c) If \(b_n \neq 0\) for all \(n\) and \(B \neq 0\), then \(\left(\frac{1}{b_n}\right)\) converges to \(\frac{1}{B}\); and
(d) if there is a \(K\) so that \(a_n \leq b_n\) for all \(n \geq K\), then \(A \leq B\).

**Proof.** Item (a) is left as an exercise.

To prove item (b) first observe that \((a_n)\) is a bounded sequence (since it converges) and hence there is a \(C > 0\) such that \(|a_n| \leq C\) for all \(n\). Now, given \(\epsilon > 0\) there is an \(N_a\) such that if \(n \geq N_a\), then
\[
|a_n - A| < \frac{\epsilon}{2(|B| + 1)}.
\]
Similarly, there is a \(N_b\) such that, for \(n \geq N_b\),
\[
|b_n - B| < \frac{\epsilon}{2C}.
\]
Let \(N = \max N_a, N_b\). If \(n \geq N\), then
\[
|a_nb_n - AB| \leq |a_n(b_n - B)| + |(a_n - A)B| \leq C|b_n - B| + |B||a_n - A| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}.
\]

To prove item (c), first note that, with \(\epsilon = \frac{|B|}{2}\), there is a \(K\) so that \(|b_n - B| \leq \frac{|B|}{2}\) for \(n \geq K\). Thus, \(|b_n| \geq \frac{|B|}{2}\) for \(n \geq K\). Now, given \(\epsilon > 0\) there is an \(M\) such that for \(n \geq N\),
\[
|b_n - B| < \frac{\epsilon |B|^2}{2}.
\]
Choose \(N = \max\{K, M\}\). If \(n \geq N\), then
\[
\left|\frac{1}{b_n} - \frac{1}{B}\right| = \frac{|b_n - B|}{|b_nB|} \leq \frac{2|b_n - B|}{|B|^2} < \epsilon.
\]

To prove item (iv), let \(\epsilon > 0\) be given. There exists \(N_a\) and \(N_b\) so that \(a_n > A - \epsilon\) and \(b_n < B - \epsilon\) for \(n \geq N_a\) and \(n \geq N_b\) respectively. Hence, for any \(m \geq \max\{N_a, N_b\}\)
\[
A - \epsilon < a_m \leq b_m < B + \epsilon.
\]
Hence \(A - B < 2\epsilon\). Since \(\epsilon > 0\) is arbitrary, it follows that \(A - B \leq 0\).

**Proposition 2.29.** Suppose \((a_n)\) is a sequence of non-negative numbers and \(q \in \mathbb{Q}^+\) (\(q\) is a positive rational number). If \((a_n)\) converges to \(A\), then \((a_n^q)\) converges to \(A^q\).

The proof exploits the identity, for \(m \in \mathbb{N}^+\),
\[
(y - x) \sum_{j=0}^{m-1} y^j x^{m-j-1} = y^m - x^m.
\]
Proof. Fix \( m \in \mathbb{N}^+ \). Since \((a_n)\) converges, there is a \( C \geq 1 \) such that \( C \geq A \) and \( C \geq a_n \) for all \( n \). In particular,
\[
\sum_{j=0}^{m-1} a_n^j A^{m-1-j} \leq mC^{\frac{m-1}{m}} \leq mC.
\]
Given \( \epsilon > 0 \), there is an \( N \) such that if \( n \geq N \), then
\[
|A - a_n| \leq \frac{\epsilon}{mC}.
\]
Hence, for such \( n \),
\[
|A^m - a_n^m| = |A - a_n| \sum_{j=0}^{m-1} a_n^j A^{m-1-j} \leq \frac{\epsilon}{mC}mC = \epsilon.
\]

Suppose \( A = 0 \); i.e., \((a_n)\) converges to 0. In this case, to see \((\sqrt{a_n})\) converges to 0, given \( \epsilon > 0 \), note that there is an \( N \) such that
\[
0 \leq a_n < \epsilon^n
\]
for \( n \geq N \). Thus, for \( n \geq N \),
\[
|\sqrt[1/n]{a_n}| \leq \epsilon.
\]
Now suppose \( A > 0 \). Replace \( y \) by \( A^{\frac{1}{n}} \) and \( x \) by \( \sqrt[n]{a_n} \) gives, using (1),
\[
|A^{\frac{1}{n}} - a_n^{\frac{1}{n}}| = \frac{|A - a_n|}{\sum_{j=0}^{m-1} A^{\frac{m-j-1}{m}} a_n^{\frac{m-j-1}{m}}} \leq |A - a_n| \frac{1}{A^{\frac{1}{m}}}.
\]
From here it is easy to show that \((\sqrt[n]{a_n})\) converges to \( A^{\frac{1}{n}} \). Finally, given the rational number \( q = \frac{m}{K} \), note that, from what has already been proved, \( b_n = a_n^m \) converges to \( B = A^m \). Thus, again by what has already been proved, \( b_n^{\frac{1}{m}} \) converges to \( B^{\frac{1}{m}} \) and the proof is complete. \( \square \)

**Example 2.30.** Recall that the sequence \((a_n)\) defined recursively in Example 2.21 converges to some \( A \). By Proposition 2.29, \((\sqrt[a_n]{a_n})\) converges to \( \sqrt{A} \). Hence, \((2 + \sqrt{a_n})\) converges to \((2 + \sqrt{A})\) by Theorem 2.28. Another application of Proposition 2.29 implies that
\[
\lim \sqrt{2 + \sqrt{a_n}} = \sqrt{2 + \sqrt{A}}.
\]
Now, the recursive definition of \( a_{n+1} \) and Proposition 2.10, imply
\[
A = \sqrt{2 + \sqrt{A}}.
\]
Hence \( 1 \leq A \leq 2 \) is a solution to
\[
A^4 - 4A^2 - A + 4 = 0.
\]

**Proposition 2.31** (Squeeze Theorem). Let \((a_n)\), \((b_n)\), and \((c_n)\) be given sequences from \( \mathbb{R} \).

If there is a \( K \) so that \( a_n \leq b_n \leq c_n \) for \( n \geq K \) and if there is an \( L \) so that both \((a_n)\) and \((c_n)\) converge to \( L \), then \((b_n)\) converges to \( L \).

If there is a \( K \) so that \( a_n \leq b_n \) for \( n \geq K \) and if \((a_n)\) diverges to \( \infty \), then so does \((b_n)\).
Proof. To prove the first statement of the proposition, let \( \epsilon > 0 \) be given. There exists \( N_a \) and \( N_c \) such that \( |a_n - L| < \epsilon \) and \( |c_n - L| < \epsilon \) for \( n \geq N_a \) and \( n \geq N_c \) respectively. Let \( N = \max\{N_a, N_c, K\} \). For \( n \geq N \),

\[-\epsilon < a_n - L \leq b_n - L \leq c_n - L < \epsilon.\]

Hence \( |b_n - L| < \epsilon \) for \( n \geq N \) and thus \((b_n)\) converges to \( L \).

The proof of the second statement is left as an exercise. \( \square \)

**Proposition 2.32.** Suppose \((a_n)\) is a sequence of positive numbers and that

\[ R = \lim_{n} \frac{a_{n+1}}{a_n} \]

exists. If \( R < 1 \), then the sequence \((a_n)\) converges to 0. If \( R > 1 \), then the sequence \((a_n)\) diverges to \( \infty \).

Proof. First suppose \( R < 1 \). Choose \( \rho \) such that \( R < \rho < 1 \). There exists an \( N \) so that

\[ 0 < \frac{a_{n+1}}{a_n} < \rho \]

for \( n \geq N \). In particular, \( a_{N+1} \leq \rho a_N \). Iterating this inequality gives \( a_{N+k} \leq \rho^k a_N \). Since, by Example 2.20, the sequence \((a_n \rho^n)_n\) converges to 0 (as does the sequence \((0)\)), Proposition 2.31 implies that \((a_{N+n})_n\) converges to 0. Hence \((a_n)\) itself converges to 0.

The case of \( R > 1 \) is left as an exercise. See Problem 2.9. \( \square \)

**Proposition 2.33.** Suppose \((a_n)\) and \((b_n)\) are sequences of positive numbers. If \((b_n)\) converges to \( B > 0 \) and \( \frac{a_n}{b_n} \) converges to \( L \), then \((a_n)\) converges to \( BL \).

Similarly, if \((b_n)\) diverges to \( \infty \) and if \( \frac{a_n}{b_n} \) converges to some \( L > 0 \), then \((a_n)\) diverges to \( \infty \).

There are other variations of this proposition.

Proof. Let \( c_n = \frac{a_n}{b_n} \). The sequences \((c_n)\) and \((b_n)\) converge to \( B \) and \( L \) respectively. Hence the sequence \((b_n c_n) = (a_n)\) converges to \( BL \).

The second part of the proposition is proved similarly. \( \square \)

**Example 2.34.** Revisiting Example 2.21, show \((a_n = \frac{n^2-1}{n+2})\) diverges to \( \infty \).

Let \( b_n = n \) and

\[ c_n := \frac{a_n}{b_n} = \frac{1 - \frac{1}{n^2}}{1 + \frac{2}{n}}.\]

Using various limits theorems, \((c_n)\) converges to 1. Since \((b_n)\) diverges to \( \infty \) by Example 2.23, it follows that \((a_n)\) does too by Proposition 2.33.

This section closes with a couple of concrete limits.

**Proposition 2.35.** Fix a positive number \( c \). Both sequence \((c^{\frac{1}{n}})\) and \((n^{\frac{1}{n}})\) converge to 1.
Proof. For a real number $x$, the Binomial Theorem gives,

\[(1 + x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j.\]

For $x > 0$ it follows that

\[(1 + x)^n \geq 1 + nx.\]

Thus, if $c > 1$ and $x = c^{\frac{1}{n}} - 1$, then

\[c - 1 \geq n(c^{\frac{1}{n}} - 1).\]

Dividing by $n$ and using the fact that $\frac{1}{n}$ converges to 0 and proves $(c^{\frac{1}{n}} - 1)$ converges to 0. Hence, by Proposition 2.10, $(c^{\frac{1}{n}})$ converges to 1.

If $0 < c < 1$, then $\frac{1}{c} > 0$ and from what has already been proved, $(\frac{1}{c^{\frac{1}{n}}})$ converges to 1. Hence, by Theorem 2.28, $(c^{\frac{1}{n}})$ converges to $\frac{1}{\frac{1}{c}} = 1$ too.

To prove the second part of the Proposition, note that the Binomial Theorem gives, for $x > 0$,

\[(1 + x)^n \geq \frac{n(n-1)}{2} x.\]

Thus, with $x = n^{\frac{1}{n}} - 1$,

\[n \geq \frac{n(n-1)}{2} x.\]

Hence,

\[\frac{2}{n-1} \geq n^{\frac{1}{n}} - 1 \geq 0,\]

from which it follows that $(n^{\frac{1}{n}})$ converges to 1. \hfill \Box

2.5. Super Cauchy sequences and the contraction principle.

Definition 2.36. A sequence $(a_n)$ is super Cauchy if there is a $C$ such that

\[\sum_{j=1}^{n} |a_{j+1} - a_j| \leq C\]

for all $n$.

Lemma 2.37. If $(a_n)$ is super Cauchy, then $(a_n)$ is Cauchy.

Proof. The sequence,

\[s_n = \sum_{j=1}^{n-1} |a_{j+1} - a_j|\]

is increasing and bounded above by $C$. Hence $(s_n)$ is convergent and therefore Cauchy. In particular, given $\epsilon > 0$ there is an $N$ so that if $n, m \geq N$, then $|s_m - s_n| < \epsilon$. Hence, for $m \geq n \geq N$,

\[|a_m - a_n| \leq \sum_{j=n}^{m-1} |a_{j+1} - a_j| = |s_{m+1} - s_n| < \epsilon.\]

\hfill \Box
Proposition 2.38 (Contraction Principle). Suppose \((a_n)\) is a sequence from \(\mathbb{R}\). If there is an \(N\) and an \(0 < r < 1\) such that

\[(2) \quad |a_{n+2} - a_{n+1}| \leq r|a_{n+1} - a_n|\]

for all \(n \geq N\), then \((a_n)\) is super Cauchy and hence converges.

Proof. It can be assumed that equation (2) holds for all \(n\). In that case, an induction argument shows,

\[|a_{j+1} - a_j| \leq r^j|a_1 - a_0|\]

Summing over \(j\) gives,

\[\sum_{j=1}^{n} |a_{j+1} - a_j| \leq |a_1 - a_0| \sum_{j=1}^{n} r^j.\]

On the other hand, by equation (1),

\[\sum_{j=1}^{n} r^j = r \frac{1 - r^n}{1 - r} \leq \frac{r}{1 - r}.\]

\[\square\]

Example 2.39. Define a sequence of real numbers recursively as follows. Let \(a_1 = 1\) and \(a_{n+1} = 1 + \frac{1}{1 + a_n}\). Show \(a_n \geq 1\) for each \(n\) and \((a_n)\) is not eventually monotonic (that is neither increasing or decreasing), but does converge.

2.6. Subsequences.

Definition 2.40. Suppose \((a_n)\) is a sequence from \(\mathbb{R}\). If \(n_1 < n_2 < \cdots\) is an increasing sequence of integers, then the sequence \((a_{n_j})_{j}\) is subsequence of \((a_n)\).

Example 2.41. Given \((a_n = (-1)^n)\) both \((b_j = a_{2j} = 1)\) and \((c_j = a_{2j+1} = -1)\) are subsequences of \((a_n)\).

Similarly, choosing \(n_j = j^2\), the sequence \((\frac{1}{2^j})\) is a subsequence of \((\frac{1}{j})\).

Definition 2.42. A point \(A\) is a subsequential limit of the sequence \((a_n)\) if there is a subsequence \((a_{n_j})\) of \((a_n)\) that converges to \(A\).

Example 2.43. The points 1 and \(-1\) are both subsequential limits of the sequence \((-1)^n)\).

Lemma 2.44. Let \((a_n)\) be a given sequence. If \(A\) is an accumulation point of \(S = \{a_n : n \in \mathbb{N}\}\), then \(A\) is a subsequential limit of the sequence \((a_n)\).

Proof. The set \((A - 1, A + 1) \cap S\) is infinite. Hence, there exists an \(n_1\) such that \(|a_{n_1} - A| < 1/2\). The set \(S \cap (A - \frac{1}{2}, A + \frac{1}{2})\) is infinite. Hence, there is an \(n_2 > n_1\) such that \(|a_{n_2} - A| < 1/4\). Continuing in this fashion (recursively), constructs \(n_1 < n_2 < n_3 < \cdots\) such that \(|a_{n_j} - A| < \frac{1}{2^j}\). Thus \((a_{n_j})\) is a subsequence of \((a_n)\) that converges to \(A\). \(\square\)

Theorem 2.45. A bounded sequence has a convergent subsequence.
Proof. Suppose \((a_n)\) is a bounded sequence. Thus, there is a \(C\) such that \(|a_n| \leq C\) for all \(n\). Let \(S = \{a_n : n\}\) denote the range of the sequence. Suppose \(S\) is infinite. In this case \(S\) has an accumulation point \(A\) by Theorem 1.39. By Lemma 2.44, a subsequence of \((a_n)\) converges to \(A\); i.e., \((a_n)\) has a convergent subsequence.

If \(S\) is finite, then there is an \(A\) such that \(A = a_n\) for infinitely many \(n\). It is an easy exercise, left to the reader, to show that there is a subsequence of \((a_n)\) that converges to \(A\). \(\square\)

**Proposition 2.46.** If \((a_n)\) converges to \(A\) and \((a_{n_j})\) is a subsequence of \((a_n)\), then \((a_{n_j})\) converges to \(A\).

**Example 2.47.** The sequence \(((−1)^n)\) diverges.

2.6.1. The limits superior and inferior. \(^3\)

**Definition 2.48.** Suppose \((a_n)\) is a bounded sequence. Let, for \(m \in \mathbb{N}\),

\[ b_m = \sup\{a_n : n \geq m\}. \]

The sequence \((b_m)\) is decreasing and bounded below (by any lower bound for \((a_n)\)). Hence \((b_m)\) converges to some \(L\) which is called the limit superior or \(\limsup\) of \((a_n)\) and is denoted by \(\limsup a_n\) or \(\lim a_n\).

The \(\liminf\), denoted \(\lim\) or \(\lim inf\) is defined analogously.

**Example 2.49.** Find the \(\limsup\) of the sequence \((a_n = \sin(n\frac{\pi}{2}))\).

First observe that the range of the sequence is the bounded set \(S = \{0, 1, −1\}\). Hence the sequence has both a \(\limsup\) and a \(\liminf\). Further Given an \(m\), the set \(\{a_n : n \geq m\} = S\). Hence, in the notation above, \(b_m = 1\) for all \(m\). It follows that \(\limsup a_n = \lim b_m = 1\). Similarly, \(\liminf a_n = −1\).

The proofs Propositions 2.50 and 2.51 below are left to the interested reader.

**Proposition 2.50.** A sequence \((a_n)\) converges if and only if it is bounded and \(\limsup a_n = \liminf a_n\).

The following proposition says that \(\limsup a_n\) is the largest subsequential limit of the sequence \((a_n)\) (and in particular asserts that a largest exists). This gives another rationale for the name \(\limsup\).

**Proposition 2.51.** Suppose \((a_n)\) is bounded. There is a subsequence \((a_{n_j})\) of \((a_n)\) that converges to \(\limsup a_n\). Moreover, if \((a_{n_k})\) is any convergent subsequence, then \(\lim k a_{n_k} \leq \limsup a_n\).

**Example 2.52.** Find the \(\limsup\) of the sequence \(a_n = (−1)^n(1 + \frac{1}{n})\).

Let \(c_n = 1 + \frac{1}{n}\) and observe that \(c_n\) converges to 1. Suppose \(A\) is a subsequential limit of \((a_n)\). Hence, there is a subsequence \((a_{n_j})\) of \((a_n)\) that converges to \(A\). In this case \(a_{n_j} \leq |a_{n_j}| = (1 + \frac{1}{n_j}) = c_{n_j}\). It follows that \(A = \lim_j a_{n_j} \leq \lim_j c_{n_j} = 1\). On the other hand, the subsequence \(a_{2n} = 1 + \frac{1}{2n}\) of \((a_n)\) converges to 1. Thus, 1 is a subsequential limit of \((a_n)\). Hence \(1 = \limsup a_n\).

\(^3\)This section is optional.
2.7. Problems.

Problem 2.1. Let \( a_n = \frac{n-2}{2n+3} \). Show, directly from the definition of limit, that \((a_n)\) converges to \(\frac{1}{2}\).

Problem 2.2. Let \( a_n = \frac{2n^2-n+1}{n^2+n+3} \). Show \((a_n)\) converges.

Problem 2.3. Let \( b_n = \frac{n+3}{n^2+n+3} \). Show \((b_n)\) converges to 0.

Problem 2.4. For \( n \geq 2 \), let \( b_n = \frac{n+3}{n^2-n-1} \). Prove, directly from the definition of limit, that \((b_n)\) converges.

Problem 2.5. In Problem 2.4, rewrite
\[
b_n = \frac{\frac{1}{n} - \frac{3}{n^2}}{1 - \frac{1}{n} - \frac{1}{n^2}}
\]
and use both known limits and limit theorems to show \((b_n)\) converges to 0. Carry out a similar program with Problem 2.1

Problem 2.6. Let \( a_0 = 1 \) and define, recursively, \( a_{n+1} = \sqrt{2 + a_n} \). Prove, by induction, that \( a_n \leq 2 \) for all \( n \) and that \((a_n)\) is increasing. Conclude that \((a_n)\) converges. Identify the limit.

Problem 2.7. Fix \( r > 1 \). Let \( a_1 = 1 \) and define recursively,
\[
a_{n+1} = \frac{1}{r}(a_n + r + 1),
\]
Show, that \((a_n)\) is increasing. Show by induction that \((a_n)\) is bounded above by \( \frac{r+1}{r-1} \). Does the sequence converge? If so, identify the limit.

Problem 2.8. Fix \( a > 1 \). Show that the sequence \((a^n)\) diverges to \(\infty\). (Suggestion: use the fact that \((\left(\frac{1}{a}\right)^n)\) converges to 0.)

Problem 2.9. Complete the proof of Proposition 2.32.

Problem 2.10. Let \( a_n = \sin(\frac{n\pi}{4}) \). Show \((a_n)\) is not Cauchy. Conclude that \((a_n)\) doesn’t converge.

Problem 2.11. Show that \( a \) is an accumulation point of a set \( D \) if and only if there is a sequence \((a_n)\) from \( D \setminus \{a\} \) that converges to \( a \). Perhaps this result explains the reason that limit point is a synonym for accumulation point.

Problem 2.12. Let \( F_0 = 0 \) and \( F_1 = 1 \) and define, recursively,
\[
F_{n+1} = F_n + F_{n-1}
\]
(the Fibonacci sequence). Let \( a_n = \frac{F_{n+1}}{F_n} \). Is the sequence \((a_n)\) monotone? Show \( a_{n+1}a_n \geq 2 \). Show,
\[
|a_{n+1} - a_n| = \frac{|a_{n-1} - a_n|}{a_na_{n-1}}.
\]
Conclude that \((a_n)\) converges. Identify the limit.
Problem 2.13. Let $a_0 = 1$ and define, recursively,

$$a_{n+1} = \frac{a_n + \frac{2}{a_n}}{2}.$$ 

Is the sequence $(a_n)$ monotone?

Show, by induction, that

(i) $a_n^2 \geq 1$ for all $n$;

(ii) $a_n^2 \leq 3$; and

(iii) $\frac{5}{2} \geq a_{n+1}a_n \geq \frac{3}{2}$.

Conclude that

$$\left| \frac{1}{2} - \frac{1}{a_{n+1}a_n} \right| \leq \frac{1}{2}.$$ 

Show

$$|a_{n+1} - a_n| \leq \frac{1}{2}|a_n - a_{n-1}|.$$ 

Explain why $(a_n)$ converges and find, if possible, its limit.

Problem 2.14. Let $(a_n)$ be a sequence of real numbers. If there is an $A$ such that every subsequence of $(a_n)$ has a further subsequence that converges to $A$, then $(a_n)$ converges to $A$.

Problem 2.15. Define a real sequence recursively as follows. Let $a_1 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$.

Show that $(a_n)$ converges to the golden ratio $\frac{\sqrt{5}+1}{2}$.

Likewise define a real sequence recursively by letting $b_1 = \frac{1}{2}$ and $b_{n+1} = \sqrt{1 - b_n}$. (In particular, it needs to be shown that $1 - b_n \geq 0$ for all positive integers $n$.)

Problem 2.16. Given $0 < b_1 < a_1$, define sequences $(a_n)$ and $(b_n)$ recursively by

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_nb_n}.$$ 

Show,

(1) $a_n^2 - b_n^2 > 0$ and hence $a_n > b_n$;

(2) $(a_n)$ is decreasing and $(b_n)$ is increasing;

(3) explain why both sequences converge and then deduce that they have a common limit $L$. (Suggestion: estimate $a_n - b_n$ in terms of $a_1 - b_1$.)

Problem 2.17. Define, for positive integers $n$,

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = \sum_{j=1}^{n} \frac{1}{j}.$$ 

Show,

$$a_{2^n} \geq 1 + \frac{n}{2}.$$ 

Deduce $(a_n)$ diverges (to $\infty$).
3. Limits

3.1. Definitions and examples.

**Definition 3.1.** Suppose \( D \subset \mathbb{R} \), the real number \( a \) is an accumulation point of \( D \), and \( f : D \to \mathbb{R} \). We say that \( f \) has a limit at \( a \) if there exists a real number \( L \) such that for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( s \in D \) and \( 0 < |s - a| < \delta \), then \( |f(s) - L| < \epsilon \). In this case we write,

\[
L = \lim_{x \to a} f(x)
\]

and call \( L \) the limit of \( f \) at \( a \).

It is an exercise to show \( L \), if it exists, is unique and hence can be called the limit.

**Example 3.2.** Let \( D = \mathbb{R} \) and \( f(x) = x^2 \). Show \( \lim_{x \to 1} f(x) = 1 \).

Given \( \epsilon > 0 \) choose \( \delta = \min\{1, \frac{\epsilon}{3}\} \). If \( |x - 1| < \delta \), then

\[
|f(x) - 1| = |x + 1||x - 1| \leq 3|x - 1| < \epsilon.
\]

Notice that, in Definition 3.1, \( f \) may, or may not, being defined at 1 and if it is defined at 1, the limit doesn’t depend upon the value of \( f \) at 1. The following example illustrates this point.

**Example 3.3.** Let \( D = (\infty, 1) \cup (1, \infty) = \mathbb{R} \setminus \{1\} \) and define \( g : D \to \mathbb{R} \) by \( g(x) = x^2 \). Show, \( \lim_{x \to 1} g(x) = 1 \).

Let \( D = \mathbb{R} \) and define \( h : D \to \mathbb{R} \) by \( h(x) = x^2 \) for \( x \neq 1 \) and \( h(1) = 0 \). Show, \( \lim_{x \to 1} h(x) = 1 \).

**Definition 3.4.** Suppose \( f : D \to \mathbb{R} \) and \( E \subset D \). The function \( f|_E : E \to \mathbb{R} \) defined by \( f|_E(x) = f(x) \) (for \( x \in E \)) is the restriction of \( f \) to \( E \).

**Proposition 3.5.** Suppose \( f : D \to \mathbb{R} \) and \( E \subset D \) and \( a \) is a limit point of \( E \). If \( f \) has a limit \( L \) at \( a \), then so does \( f|_E \).

**Example 3.6.** Define \( f : (0, \infty) \to \mathbb{R} \) by \( f(x) = \frac{x}{|x|} \). Thus, \( f \) is the restriction of \( g \) from Example 3.3 to the set \( E = (0, \infty) \subset D \). Show, \( \lim_{x \to 0} f(x) = 1 \).

**Example 3.7.** Let \( D = \mathbb{R} \setminus \{0\} \) and define \( g : D \to \mathbb{R} \) by \( g(x) = \frac{x}{|x|} \). Show, \( \lim_{x \to \infty} g(x) \) doesn't exist.

Let \( L \) be given. First, suppose \( L < 0 \). Choose \( \epsilon_0 = 1 \). Given \( \delta > 0 \), choose \( x = \frac{\delta}{2} \). Then \( x \in D \) and \( 0 < |x - 0| < \delta \), but \( |f(x) - 0| = 1 \geq \epsilon_0 \). Thus, \( L \neq \lim_{x \to 0} g(x) \).

A similar argument shows if \( L \geq 0 \), then \( \lim_{x \to 0} g(x) \neq L \). Hence \( g \) does not have a limit at 0.

**Example 3.8.** Define \( F : (0, \infty) \setminus \{1\} \to \mathbb{R} \) by \( F(x) = \frac{1 - x}{1 - \sqrt{x}} \). Show \( F \) has a limit at 1.

First, observe that

\[
\left| \frac{1 - x}{1 - \sqrt{x}} - 2 \right| = \left| 1 + \sqrt{x} - 2 \right| = \left| 1 - \sqrt{x} \right| = \left| \frac{1 - x}{1 + \sqrt{x}} \right| \leq |1 - x|.
\]
Thus, given $\epsilon > 0$, if we choose $\delta = \epsilon$, then if $x \in D$ and $|x - 1| < \delta$, then $|f(x) - 2| < \epsilon$. Hence $f$ has a limit at 1 and this limit is 2.

Examples 3.6, 3.7 and 3.8 illustrate that the definition of limit does not require $a$ to be in the domain of $f$ (it just needs to be a limit point of the domain), nor does it depend upon the value of $f$ at $a$ if $a$ is in the domain of $f$. They also demonstrate that the limit depends crucially on the domain.

Example 3.9. Define $f : [0, 1] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p, q > 0 \text{ and } \gcd(p, q) = 1. \end{cases}$$

Show, for each $a \in (0, 1)$, that $\lim_{x \to a} f(x) = 0$.

Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}^+$ so that $\frac{1}{N} < \epsilon$. Consider the set

$$S_N = \left\{ \frac{m}{n} : m, n \in \mathbb{N}^+, \ n \leq N, \ m \leq n \right\}.$$  

Since $S_N$ is finite,

$$\delta = \min\{|s - a| : s \in S, \ s \neq a\} > 0.$$  

In particular, $(a - \delta, a + \delta) \cap S_N \subset \{a\}$. Hence, if $0 < |a - x| < \delta$, then either $x \notin \mathbb{Q}$ in which case $|f(x) - 0| = |0 - 0| < \epsilon$; or $x \in \mathbb{Q}$ and $x = \frac{p}{q}$ where $q > N$, in which case $|f(x) - 0| = \left|\frac{1}{q}\right| < \frac{1}{N} < \epsilon$.

Proposition 3.10. Suppose $f : D \rightarrow \mathbb{R}$ and $a$ is an accumulation point of $D$. If $f$ has a limit as $x$ approaches $a$, then for every $\epsilon > 0$ there is an $\eta > 0$ such that if $x, y \in D$ and both $|x - a|, |y - a| < \eta$, then $|f(x) - f(y)| < \epsilon$.

The Proposition is an analog of the fact that convergent sequences are Cauchy (Proposition 2.12). Its proof of the proposition is left as an exercise.

Example 3.11. Define $f : (0, \infty) \rightarrow \mathbb{R}$ by

$$f(x) = \sin\left(\frac{1}{x}\right).$$

Show $f$ does not have a limit at 0.

Observe that 0 is in fact an accumulation point of $D = (0, \infty)$, the domain of $f$. With $\epsilon_0 = 1$, given $\eta > 0$ there exists $n \in \mathbb{N}^+$ such that $\frac{1}{2n\pi} < \eta$. With $x = \frac{1}{2n\pi}$ and $y = \frac{1}{(2n + \frac{1}{2})\pi}$, we have $0 < x, y < \eta$, but $|f(x) - f(y)| = 1 \geq \epsilon_0$. Hence, by the (contrapositive of the) proposition, $f$ does not have a limit at 0.

3.2. The sequential formulation of limit of a function.

Proposition 3.12. Suppose $f : D \rightarrow \mathbb{R}$ and $a$ is an accumulation point of $D$.

If $\lim_{x \to a} f(x)$ exists and equals $L$ and if $(a_n)$ is a sequence from $D \setminus \{a\}$ that converges to $a$, then $(f(a_n))$ converges to $L$.

Conversely, if there is an $L$ such that for any sequence $(a_n)$ from $D \setminus \{a\}$ that converges to $L$ the limit $\lim_{n \to \infty} f(a_n) = L$, then $\lim_{x \to a} f(x) = L$.  

Suppose \( \lim_{x \to a} f(x) = L \) and that \((a_n)\) is a sequence from \(D \setminus \{a\}\) that converges to \(a\). To prove \((f(a_n))\) converges to \(L\), let \(\epsilon > 0\) be given. There is a \(\delta > 0\) such that if \(0 < |x - a| < \delta\) and \(x \in D\), then \(|f(x) - L| < \epsilon\). There is an \(N\) so that if \(n \geq N\), then \(0 < |a_n - a| < \delta\). Hence, if \(n \geq N\), then \(|f(a_n) - L| < \epsilon\).

To prove the second statement, suppose \(\lim_{x \to a} f(x) \neq L\). Thus, there is an \(\epsilon_0 > 0\) such that for every \(\delta > 0\) there is a point \(x \in D\) such that \(0 < |x - a| < \delta\), but \(|f(x) - L| \geq \epsilon_0\).

Thus, with \(\delta_n = \frac{1}{n}\), there exists \(a_n \in D\) such that \(0 < |a_n - a| < \delta_n\) and \(|f(a_n) - L| \geq \epsilon_0\). It follows that \((a_n)\) converges to \(a\), but \((f(a_n))\) does not converge to \(L\). □

**Corollary 3.13.** Suppose \(D \subset \mathbb{R}\), \(f : D \to \mathbb{R}\) and \(a\) is an accumulation point of \(D\). If there exists a sequence \((a_n)\) from \(D \setminus \{a\}\) such that \((a_n)\) converges to \(a\), but \((f(a_n))\) diverges, then \(f\) does not have a limit as \(x\) tends to \(a\).

Similarly, if there exists sequences \((a_n)\) and \((b_n)\) from \(D \setminus \{a\}\) such that both converge to \(a\), but \((f(a_n))\) and \((f(b_n))\) don’t converge to the same value (which includes the case that one or both diverges), then \(f\) does not have a limit as \(x\) approaches \(a\).

**Example 3.14.** Let \(D = \mathbb{R} \setminus \{0\}\) and define \(f : D \to \mathbb{R}\) by \(f(x) = \sin(\frac{1}{x})\). Show that \(f\) does not have a limit at 0.

Choose \(a_n = \frac{1}{(n+\frac{1}{2})\pi}\) for \(n \in \mathbb{N}\). Note that \((a_n)\) converges to 0, but \((f(a_n)) = (-1)^n\) diverges.

**Example 3.15.** Let \(f : [0,1] \to \mathbb{R}\) denote the indicator function (synonymously characteristic function) of \(\mathbb{Q} \cap [0,1]\) defined by

\[
f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}
\]

Show, for each \(a \in (0,1)\) that \(\lim_{x \to a} f(x)\) does not exists.

Fix \(a\). Since \(a\) is a limit point of \(\mathbb{Q}\), there is a sequence \((a_n)\) from \(\mathbb{Q} \cap (0,1)\) which converges to \(a\) and such that for each \(n\), \(a_n \neq a\). Since \(a_n \in \mathbb{Q}\), we have \((f(a_n)) = (1)\) converges to 1. There is also a sequence \((b_n)\) from \([0,1] \setminus \mathbb{Q}\) such that \((b_n)\) converges \(a\) and for each \(n\), \(b_n \neq a\). We have \((f(b_n)) = (0)\) converges to 0. Thus, \(f\) does not have a limit at \(a\).

**Proposition 3.16.** If \(D \subset (0,\infty)\), \(a\) is a limit point of \(D\) and \(f : D \to \mathbb{R}\) is defined by \(f(x) = x^a\), then

\[
\lim_{x \to a} f(x) = a^a
\]

**Proof.** Suppose \((a_n)\) is a sequence from \(D\) and \((a_n)\) converges to \(a\). By Proposition 2.29, \((a_n^a = f(a_n))\) converges to \(a^a\). An application of Proposition 3.12 completes the proof. □

3.3. Infinite limits and limits at infinity.

**Definition 3.17.** Suppose \(f : D \to \mathbb{R}\) and \(a\) is an accumulation point of \(D\). The **limit of \(f\) as \(x\) approaches \(a\) is \(\infty\)** if for every \(C > 0\) there is a \(\delta\) such that if \(x \in D\) and \(0 < |x - a| < \delta\), then \(f(x) > C\), denoted

\[
\lim_{x \to a} f(x) = \infty.
\]

Given \(D \subset \mathbb{R}\), we say \(\infty\) is a limit point of \(D\) if for every \(C > 0\) there is an \(x \in D\) such that \(x > C\).
Suppose \( f : D \to \mathbb{R} \) and \( \infty \) is a limit point of \( D \) and \( A \in \mathbb{R} \). The limit of \( f \) as \( x \) approaches \( \infty \) is \( A \) if for every \( \epsilon > 0 \) there is a \( C > 0 \) such that if \( x \in D \) and \( x > C \), then \( |f(x) - A| < \epsilon \), denoted

\[
\lim_{x \to \infty} f(x) = A.
\]

The expression,

\[
\lim_{x \to \infty} f(x) = \infty
\]

is defined similarly.

**Example 3.18.** Define \( f : (0, \infty) \to \mathbb{R} \) by \( f(x) = x^2 - 2 \). Show \( \lim_{x \to \infty} f(x) = 0 \).

Implicitly, \( D = \mathbb{R} \setminus \{0\} \) and \( f(x) = x^2 - 2 \). Given \( \epsilon > 0 \) choose \( C = \epsilon^{-\frac{1}{2}} \). Now, if \( x > C \), then \( 0 < x^2 - 2 < C^{-2} = \epsilon \).

**Example 3.19.** Let \( D = (0, \infty) \) and define \( g : D \to \mathbb{R} \) by \( g(x) = x - 1 \). Show \( \lim_{x \to 0^+} g(x) = \infty \).

Note, this is sometimes expressed as

\[
\lim_{x \to 0^+} \frac{1}{x} = \infty
\]

Given \( C > 0 \), choose \( \epsilon = \frac{1}{C} > 0 \). If \( x \in D \) and \( |x - 0| < \epsilon \), then \( 0 < x < \frac{1}{C} \) and hence \( 0 < g(x) = x - 1 > C \).

**Example 3.20.** Assuming knowledge of the log function, show,

\[
\lim_{x \to \infty} \log(x) = \infty.
\]

Recall, \( \log(2^k) = k \log(2) \) for \( k \in \mathbb{N} \) and \( \log(2) > 0 \). In particular, the sequence \( (\log(2^k)) \) diverges to \( \infty \). Moreover, if \( y > x > 0 \), then \( \log(y) > \log(x) \). Given \( K > 0 \) choose \( k \) such that \( k \log(2) > K \). Choose \( C = 2^k \). If \( x > C \), then \( \log(x) > \log(C) = k \log(2) > K \).

**Proposition 3.21.** Suppose \( D \subset (0, \infty) \) and let \( E = \left\{ \frac{1}{x} : x \in D \right\} \subset (0, \infty) \). Then \( \infty \) is a limit point of \( D \) if and only if 0 is a limit point of \( E \).

Suppose \( f : D \to \mathbb{R} \) and \( g : E \to \mathbb{R} \) is defined by \( g(x) = f\left(\frac{1}{x}\right) \). Then \( f \) has a limit at 0 if and only if \( g \) has a limit at 0 and in this case

\[
\lim_{x \to \infty} f(x) = \lim_{x \to 0} g(x).
\]

**Remark 3.22.** The conclusion is that either both limits exist and are equal or they both fail to exist. Proposition 3.21 is a variant of Proposition 3.35.

**Example 3.23.** Redo Example 3.18. In that example \( D = (0, \infty) \) and \( f : D \to \mathbb{R} \) is defined by \( f(x) = x^{-2} \). Let \( E = D \) and define \( g : E \to \mathbb{R} \) by \( g(x) = f\left(\frac{1}{x}\right) = x^2 \). To see that \( \lim_{x \to \infty} f(x) = 0 \) it suffices to show

\[
\lim_{x \to 0} g(x) = 0.
\]

To prove this statement, let \( \epsilon > 0 \) be given. Choose \( \delta = \min\{1, \epsilon\} \). Now, if \( x \in D \) and \( |x - 0| < \delta \), then

\[
|g(x) - 0| = x^2 < x < \delta \leq \epsilon,
\]
where we have used \(0 < x < 1\) in the first inequality.

Observe that a similar argument shows \(\lim_{x \to \infty} \frac{1}{x} = 0\) too.

There are sequential formulations of limits at infinity and infinite limits. Rather than state all the variations, we offer the following proposition as a sample result.

**Proposition 3.24.** Suppose \(f : D \rightarrow \mathbb{R}\) and \(a\) is an accumulation point of \(D\). If \(\lim_{x \to a} f(x) = \infty\) and if \((a_n)\) is a sequence from \(D \setminus \{a\}\) that converges to \(a\), then \(\lim_{n \to \infty} f(a_n) = \infty\).

**Example 3.25.** Show \(\lim_{x \to 0} \frac{1}{x}\) does not exists and is not either \(\pm \infty\). Implicitly, here \(D = \mathbb{R} \setminus \{0\}\) and \(f(x) = \frac{1}{x}\).

Consider the sequence \((a_n = \frac{1}{n})_{n=1}^{\infty}\). The sequence \((b_n = f(a_n)) = (n)\) diverges to \(\infty\). Hence, no real number or \(-\infty\) can be the limit. On the other hand, with \((c_n = -a_n)\) the sequence \((f(c_n)) = (-n)\) diverges to \(-\infty\) and thus \(\infty\) can not be the limit. Hence the limit fails to exist even in the sense of \(\pm \infty\).

This subsection closes with two simple examples.

**Example 3.26.** Suppose \(k\) is a real number and \(f : D \rightarrow \mathbb{R}\) is the constant function \(f(x) = k\). If \(a\) is an accumulation point of \(D\), then \(\lim_{x \to a} f(x) = k\).

**Example 3.27.** If \(f : D \rightarrow \mathbb{R}\) is the identity function, \(f(x) = x\), and if \(a\) is an accumulation point of \(D\), then \(\lim_{x \to a} f(x) = a\).

### 3.4. Limit Theorems

The following theorem says, like in the case of limits of sequences, limits of functions are compatible with the algebraic operations on \(\mathbb{R}\).

**Proposition 3.28.** Suppose \(f, g, h : D \rightarrow \mathbb{R}\) and \(a\) is an accumulation point of \(D\). If the limits \(\lim_{x \to a} f(x)\), \(\lim_{x \to a} g(x)\) and \(\lim_{x \to a} h(x)\) exist and equal \(A, B, C\) respectively, then

(i) \(\lim_{x \to a} (f + g)(x) = A + B\);  
(ii) \(\lim_{x \to a} fg(x) = AB\); and  
(iii) \(\lim_{x \to a} \frac{1}{h(x)} = \frac{1}{C}\) if \(C \neq 0\) and \(h\) is never 0, then \(\lim_{x \to a} \frac{1}{h(x)} = \frac{1}{C}\).

**Proof.** Suppose \((a_n)\) is a sequence from \(D \setminus \{a\}\) that converges to \(a\). By Proposition 3.12, \(\lim_{n \to \infty} f(a_n) = A\) and similarly, \(\lim_{n \to \infty} g(a_n) = B\). It follows, from Theorem 2.28, that both \(\lim_{n \to a} f g(a_n) = AB\) and \(\lim_{n \to \infty} (f + g)(a_n) = A + B\). Hence another application of Proposition 3.12 proves both items (i) and (ii).

The proof of item (iii) is similar. The details are left as an exercise. \(\square\)

**Remark 3.29.** Analogous results hold for limits at infinity and infinite limits, with, in the latter case, the obvious caveats.

**Example 3.30.** Find

\[
\lim_{x \to \infty} \frac{2x - 1}{x + 3},
\]

if it exists.

We interpret the limit as \(\lim_{x \to \infty} f(x)\) where \(D = \mathbb{R} \setminus \{-3\}\) and \(f : D \to \mathbb{R}\) is defined by

\[
f(x) = \frac{2x - 1}{x + 3}.
\]
Rewrite $f$ as

$$f(x) = \frac{2 - \frac{1}{x}}{1 + \frac{3}{x}}.$$ 

As an exercise, show that $\lim_{x \to \infty} \frac{1}{x} = 0$. Using this fact, $\lim_{x \to \infty} \frac{3}{x} = 0$ and hence,

$$\lim_{x \to \infty} f(x) = \frac{2 - \lim_{x \to \infty} \frac{1}{x}}{1 + \lim_{x \to \infty} \frac{3}{x}} = \frac{2 - 0}{1 + 0} = 2.$$ 

**Remark 3.31.** From Examples 3.26 and 3.27 and Proposition 3.28, it follows that if $p$ is a polynomial, then, for any $a \in \mathbb{R}$ that

$$\lim_{x \to a} p(x) = p(a).$$

For instance, choosing $f = g$ to be the function in Example 3.27 and part (ii) of the proposition,

$$\lim_{x \to a} x^2 = a^2.$$ 

### 3.4.1. Order and limits.

**Proposition 3.32.** Suppose $f, g : D \to \mathbb{R}$ and $a$ is a limit point of $D$. If there is an $\eta > 0$ such that if $x \in D$ and $0 < |x - a| < \eta$, then $f(x) \leq g(x)$ and if both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

**Proof.** For notational ease, let $A = \lim_{x \to a} f(x)$ and $B = \lim_{x \to a} g(x)$. Since $a$ is a limit point of $D$, there exists a sequence $(a_n)$ from $D$ such that $0 < |a_n - a| < \eta$ for all $n$ and $(a_n)$ converges to $a$. It follows that $f(a_n) \leq g(a_n)$ and the sequences $(f(a_n))$ and $(g(a_n))$ converge to $A$ and $B$ respectively. Thus, by Theorem 2.28, $A \leq B$. \hfill \Box

**Proposition 3.33.** Suppose $f, g, h : D \to \mathbb{R}$ and $a$ is an accumulation point of $D$. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} h(x)$ exist and equal $L$ and if there is an $\eta > 0$ such that $f(x) \leq g(x)$ for $x \in D$ and $0 < |x - a| < \eta$, then $\lim_{x \to a} g(x) = L$.

**Proof.** Let $\epsilon > 0$ be given. There exists a $0 < \delta < \eta$ such that if $x \in D$ and $0 < |x - a| < \delta$, then $L - f(x) < \epsilon$ and $h(x) - L < \epsilon$. Hence, for such $x$,

$$-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon$$

and the conclusion follows. \hfill \Box

**Example 3.34.** Show,

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} = 1.$$ 

Here choose $D = (0, \infty)$ and define $g : (0, \infty) \to \mathbb{R}$ by

$$g(x) = \frac{x}{\sqrt{x^2 + 1}}.$$ 

Define $f, h : (0, \infty) \to \mathbb{R}$ by $h(x) = 1$ and

$$f(x) = 1 - \frac{1}{x}.$$
Verify that the inequalities \( f(x) \leq g(x) \leq h(x) \) hold for \( x > 1 \). Since both \( f \) and \( h \) approach 1 as \( x \) tends to \( \infty \),

\[
\lim_{x \to \infty} g(x) = 1
\]

too.

3.4.2. Compositions.

**Proposition 3.35.** Suppose \( D, E \subset \mathbb{R} \) and \( a \) and \( b \) are limit points of \( D \) and \( E \) respectively and \( g : D \to E \) and \( f : E \to \mathbb{R} \). If

(i) \( \lim_{x \to a} g(x) \) exists and is equal \( b \);

(ii) \( \lim_{y \to b} f(y) \) exists and is say \( L \);

(iii) either \( b \not\in E \) or \( f(b) = L \),

then \( f \circ g : D \to \mathbb{R} \) has limit \( L \) at \( a \).

**Proof.** Let \( \epsilon > 0 \) be given. There is a \( \delta > 0 \) such that if \( 0 < |y - b| < \delta \), and \( y \in E \), then \( |f(y) - L| < \epsilon \). With this \( \delta > 0 \) there is an \( \eta > 0 \) such that if \( 0 < |x - a| < \eta \) and \( x \in D \), then \( |g(x) - b| < \delta \). Hence, if \( 0 < |x - a| < \eta \) and \( x \in D \), then \( g(x) \in E \) and either \( 0 < |g(x) - b| < \delta \) or \( g(x) = b \) and in either case \( |f(g(x)) - L| < \epsilon \). \( \square \)

**Example 3.36.** Find

\[
\lim_{x \to 0} \frac{\sin(4x)}{\sin(5x)}.
\]

We will assume the fact that, with \( D = \mathbb{R} \setminus \{0\} \) and \( f : D \to \mathbb{R} \) defined by \( f(x) = \frac{\sin(x)}{x} \), that

\[
\lim_{x \to 0} f(x) = 1.
\]

Let \( g(x) = 4x \) defined on \( \mathbb{R} \setminus \{0\} \). Thus \( g \) maps into the domain \( D \) of \( f \) and the hypotheses of the Proposition 3.35 are satisfied. Hence,

\[
\lim_{x \to 0} f(g(x)) = \lim_{t \to 0} f(t) = 1.
\]

Thus,

\[
\lim_{x \to 0} \frac{\sin(4x)}{4x} = 1.
\]

Similarly,

\[
\lim_{x \to 0} \frac{\sin(5x)}{5x} = 1.
\]

Finally,

\[
\frac{\sin(4x)}{\sin(5x)} = 4 \frac{\sin(4x)}{4x} \frac{4x}{5x} \frac{5x}{\sin(5x)}.
\]

Using rules of limits (mostly notably the limit of a quotient is the quotient of the limits provided both limits exist and the limit in the denominator is not zero), we find,

\[
\lim_{x \to 0} \frac{\sin(4x)}{\sin(5x)} = \frac{4}{5}.
\]
**Proposition 3.37.** Suppose \( f : D \to [0, \infty) \), \( a \) is an accumulation point of \( D \) and \( q \in \mathbb{Q}^+ \). If \( \lim_{x \to a} f(x) \) exists and equals \( L \), then,

\[
\lim_{x \to a} f(x)^q = L^q.
\]

**Example 3.38.** Find

\[
\lim_{x \to 4} \frac{x}{\sqrt{x^2 + 1}}.
\]

**Example 3.39.** Find,

\[
\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}}.
\]

With \( D = (0, \infty) \) and \( f : D \to \mathbb{R} \) defined by

\[
f(x) = \frac{x}{\sqrt{x^2 + 1}},
\]

we will show \( \lim_{x \to \infty} f(x) = 1 \). First, rewrite \( f \) as

\[
f(x) = \frac{1}{\sqrt{1 + \frac{1}{x^2}}}.
\]

Now, by Example 3.23, \( \lim_{x \to \infty} \frac{1}{x^2} = 0 \) and thus \( \lim_{x \to \infty} 1 + \frac{1}{x^2} = 1 \). It follows that

\[
\lim_{x \to \infty} \sqrt{1 + \frac{1}{x^2}} = 1
\]

and thus,

\[
\lim_{x \to \infty} \frac{1}{\sqrt{1 + \frac{1}{x^2}}} = \lim_{x \to \infty} \sqrt{1 + \frac{1}{x^2}} = \frac{1}{1} = 1.
\]

**3.5. One sided limits.**

**Definition 3.40.** Suppose \( D \subset \mathbb{R} \). Given a point \( a \in \mathbb{R} \), let

\[
D_{a+} = \{ x \in D : x > a \}.
\]

Given \( L \in \mathbb{R} \), if \( a \) is a limit point of \( D_{a+} \) we say \( f \) has limit \( L \) at \( a \) from the right (or above) if

\[
\lim_{x \to a} f|_{D_{a+}} = L.
\]

In this case we write,

\[
L = \lim_{x \to a^+} f(x).
\]

The notion of the limit of \( f \) at \( a \) from the left is defined similarly. These limits, to the extent they exist, are one-sided limits of \( f \) at \( a \).

**Example 3.41.** Define \( h : \mathbb{R} \to \mathbb{R} \) by

\[
h(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
1 & \text{if } x > 0.
\end{cases}
\]

Show both limits \( \lim_{x \to 0^+} h(x) \) and \( \lim_{x \to 0^-} h(x) \) exist.
**Proposition 3.42.** Suppose \( f : D \to \mathbb{R} \) and \( a \in \mathbb{R} \). If \( a \) is a limit point of both \( D_{a^+} \) and \( D_{a^-} \), then \( \lim_{x \to a} f(x) \) exists if and only if both one sided limits at \( a \) exists and are equal. In this case,

\[
\lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x).
\]

**Remark 3.43.** The proof is left as an easy exercise.

A similar result holds for infinite limits.

**Example 3.44.** The function \( h \) in Example 3.41 does not have a limit at 0.

**Example 3.45.** Define \( g : \mathbb{R} \to \mathbb{R} \) by

\[
g(x) = \begin{cases} 
  x^3 & \text{if } x \leq 0 \\
  x^2 & \text{if } x > 0.
\end{cases}
\]

Show \( g \) has limit 0 at 0.

Let \( p \) denote the function \( p : \mathbb{R} \to \mathbb{R} \) defined by \( p(x) = x^2 \). Since \( p \) is a polynomial,

\[
\lim_{x \to 0} p(x) = p(0) = 0.
\]

Hence, \( \lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} p(x) = 0 \). Similarly, \( \lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} x^3 = 0 \). Thus \( \lim_{x \to 0} g(x) = 0 \).

### 3.6. Monotone functions.

**Definition 3.46.** A function \( f : D \to \mathbb{R} \) is increasing if \( x, y \in D \) and \( x < y \) implies \( f(x) \leq f(y) \).

**Proposition 3.47.** If \( f : D \to \mathbb{R} \) is increasing and \( a \in D \) is an accumulation point of \( D_{a^-} \), then \( \lim_{x \to a^-} f(x) \) exists.

Informally, Proposition 3.47 says if \( f \) is monotone, then \( f \) has one-sided limits.

**Proof.** Note that if \( x < a \) and \( x \in D \), then \( f(x) \leq f(a) \). Hence the set

\[
S = \{ f(x) : x \in D, \ x < a \}
\]

is bounded above by \( f(a) \). Since \( a \) is an accumulation point of \( D_{a^-} \) the set \( D_{a^-} \), and hence \( S \), is nonempty. Therefore \( S \) has a least upper bound \( L \). To see that \( \lim_{x \to a^-} f(x) = L \), let \( \epsilon > 0 \) be given. By the least property of \( L \), there exists a \( z \in D \) with \( z < a \) and \( f(z) > L - \epsilon \). Let \( \delta = a - z \). If \( 0 < a - x < \delta \) and \( x \in D \), then \( z < x < a \) so that \( f(z) \leq f(x) \leq f(a) \). Thus \( L - \epsilon < f(z) \leq f(x) \leq L \) so that

\[
|f(x) - L| < \epsilon.
\]

**Remark 3.48.** By a similar argument, if \( a \) is also a limit point of \( D_{a^+} \), then \( \lim_{x \to a^+} f(x) \) exists and moreover,

\[
\lim_{x \to a^-} f(x) \leq f(a) \leq \lim_{x \to a^+} f(x).
\]

### 3.7. Problems.

**Problem 3.1.** In each case show, directly from the definition of limit, that, for \( f : \mathbb{R} \to \mathbb{R} \),

(a) if \( f(x) = x^2 \), then \( \lim_{x \to 2} f(x) = 4 \);
(b) if \( f(x) = x^3 \), then \( \lim_{x \to 1} f(x) = 1 \);
(c) if \( f(x) = \frac{x^2 - 4}{x - 2} \), then \( \lim_{x \to 2} f(x) = 4 \).
Problem 3.2. Define \( f : \mathbb{R} \setminus \{4\} \rightarrow \mathbb{R} \) by \( f(x) = \frac{x-4}{\sqrt{x^2-2}} \). Show 
\[
\lim_{x \to 4} f(x) = 4.
\]

Problem 3.3. Define \( f : \mathbb{R} \rightarrow \mathbb{R} \) by
\[
f(x) = \begin{cases} 
x & \text{if } x \in \mathbb{Q} \\
0 & \text{if } x \notin \mathbb{Q}.
\end{cases}
\]

For which \( a \in \mathbb{R} \) does \( \lim_{x \to a} f(x) \) exist? Prove your answer.

Problem 3.4. Suppose \( f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) is bounded (that is, there is a \( C > 0 \) such that \( |f(x)| \leq C \) for all \( x \in \mathbb{R} \setminus \{0\} \)). Show
\[
\lim_{x \to 0} xf(x) = 0.
\]

Problem 3.5. Show (assuming the usual properties of the \( \sin \) function),
\begin{enumerate}
\item \( \lim_{x \to 0} x \sin(x) = 0; \)
\item \( \lim_{x \to \infty} \sin(x) = 0 \) (here take the domain as \((0, \infty)\));
\item \( \lim_{x \to \infty} \sin(x) \) does not exist.
\end{enumerate}

Problem 3.6. In each establish the limit directly from the definitions.
\begin{enumerate}
\item \( \lim_{x \to \infty} f(x) = 2 \), where \( f : \mathbb{R} \rightarrow \mathbb{R} \) is defined by \( f(x) = \frac{2x^2 - 3x + 1}{x^4 + 2} \).
\item \( \lim_{x \to \infty} f(x) = \infty \), where \( f : \mathbb{R} \rightarrow \mathbb{R} \) is defined by \( f(x) = \frac{2x^3 - 3x + 1}{x^2 + 2} \).
\end{enumerate}

Problem 3.7. Suppose \( D \subset \mathbb{R} \), for each \( K > 0 \) there is a point \( s \in D \) such that \( s > K \). Show, if there is an \( R \) such that if \( R < x < y \) and \( x, y \in D \), then \( f(x) \leq f(y) \), then \( \lim_{x \to \infty} f(x) \) exists or \( \lim_{x \to \infty} f(x) = \infty \). Informally, the problem says, if \( f \) is \textit{eventually increasing}, then \( \lim_{x \to \infty} f(x) \) exists as an \textit{extended real number}.

Problem 3.8. Fill in the following outline that if \( f : [0, 1] \rightarrow \mathbb{R} \) is monotone, then \( f \) has a limit at all but a most countably many points; precisely, there is a set \( J \subset [0,1] \) that is a countable union of finite sets such that if \( y \in [0,1] \setminus J \), then \( f \) has a limit at \( y \).
\begin{enumerate}
\item Given \( a \in (0, 1) \), \( f \) does not have a limit at \( a \) if and only if
\[
f(a-) := \lim_{x \to a-} f(x) < \lim_{x \to a+} f(x) =: f(a+).
\]
Let \( J = \{ a \in (0,1) : f(a+) - f(a-) > 0 \} \cup \{0,1\} \) and, for positive integers \( n \), let \( J_n = \{ a \in (0,1) : f(a+) - f(a-) > \frac{1}{n} \} \). Thus
\[
J = \bigcup_{n=1}^{\infty} J_n \cup \{0,1\}
\]
and if \( y \in [0,1] \setminus J \), then \( f \) has a limit at \( y \). Thus it suffices to show each \( J_n \) is finite. To this end, let \( T = f(1) - f(0) \).
\item Show \( |J_n| \leq nT \). Here \( |J_n| \) is the cardinality (number of points in) of \( J_n \). (Suggestion: list the set \( J_n \) as \( 0 < a_1 < a_2 < \cdots < a_N \) and argue that
\[
\sum_{j=1}^{N} |f(a_{j+}) - f(a_{j-})| \geq f(1) - f(0).
\]

Problem 3.9. Given an example of (nonempty) sets $D, E \subset \mathbb{R}$ with accumulation points $a$ and $b$ respectively and functions $g : D \to E$ and $f : E \to \mathbb{R}$ such that $g$ has limit $b$ at $a$ and $f$ has a limit at $b$, but $f \circ g : D \to \mathbb{R}$ does not have a limit at $a$. Explain why Proposition 3.35 does not apply to the example given.

Problem 3.10. Provide a careful interpretation of, and compute, the limits

1. $\lim_{x \to 1} \frac{\sqrt{x^2 + 7}}{\sqrt{x^2 + 1}}$;
2. $\lim_{x \to \infty} \frac{\sqrt{x + 7}}{\sqrt{x^2 + 1}}$;
3. $\lim_{x \to 0} \frac{\sin(5x)}{\sin(3x)}$.

Problem 3.11. A geometric argument shows, for $|x| < \frac{\pi}{2}$, that $|\sin(x)| \leq |x|$. Use this inequality to prove $\lim_{x \to 0} \sin(x) = 0$. Use the identity $\cos(x)^2 = 1 - \sin(x)^2$ to prove $\lim_{x \to 0} \cos(x) = 1$. Now use the identities

$$\cos(x + h) = \cos(x) \cos(h) - \sin(x) \sin(h)$$
$$\sin(x + h) = \cos(x) \sin(h) + \sin(x) \cos(h)$$

to prove, for each $a \in \mathbb{R}$, that

$$\lim_{x \to a} \cos(x) = \cos(a)$$
$$\lim_{x \to a} \sin(x) = \sin(a).$$

Problem 3.12. What is wrong with the statement: if $f$ is bounded in some neighborhood of $x = a$, then $\lim_{x \to a} f(x)$ is bounded. Can you fix it and prove the intended result?

Problem 3.13. Given a subset $S$ of a set $X$, the indicator function of $S$, denoted $1_S$ is the function $1_S : X \to \mathbb{R}$ defined by

$$1_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

Now suppose $S \subset \mathbb{R}$ and $a \in \mathbb{R}$. Prove $1_S$ has a limit at $a$ if and only if $a$ is not a limit point of both $S$ and $\tilde{S}$. Here $\tilde{S} = \mathbb{R} \setminus S$. In general, if $S \subset X$, then $\tilde{S} = X \setminus S$ is the complement of $S$ (in $X$).

4. Continuity

Definition 4.1. Given $D \subset \mathbb{R}$, a point $a \in D$, the function $f : D \to \mathbb{R}$ is continuous at $a$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $x \in D$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

The function $f$ is continuous if it is continuous at each point $a \in D$.

Remark 4.2. If $a$ is an accumulation point of $D$, then $f$ is continuous at $a$ if and only if

$$\lim_{x \to a} f(x) = f(a).$$

If $a$ is not an accumulation point of $D$, then $f$ is continuous at $a$. In particular, and as an example, every function $f : \mathbb{Z} \to \mathbb{R}$ is continuous.

Example 4.3. Show that the function $h$ from Example 1.3 is nowhere continuous.
Example 4.4. The function from Example 3.9 is continuous at the irrational points in \((0,1)\) and discontinuous at the rational points in \((0,1)\).

Many of our facts about limits can be interpreted in terms of continuity.

Example 4.5. Fix \(k \in \mathbb{R}\) and define \(h : \mathbb{R} \to \mathbb{R}\) by \(h(x) = k\). Show that \(h\) is continuous.

Fix a point \(a \in \mathbb{R}\). By example 3.26, \(\lim_{x \to a} h(x) = k\). Thus, \(\lim_{x \to a} h(x) = h(a)\) and \(h\) is continuous at \(a\). Since \(a\) was arbitrary, \(h\) is continuous.

Example 4.6. Show that \(f : \mathbb{R} \to \mathbb{R}\) defined by \(f(x) = x\) (the identity function) is continuous.

Fix a point \(a \in \mathbb{R}\). By Example 3.27, \(\lim_{x \to a} f(x) = a\). Thus, \(\lim_{x \to a} f(x) = f(a)\) and \(f\) is continuous at \(a\). Hence \(f\) is continuous.

Definition 4.7. Given polynomials \(p\) and \(q\), let \(D = \{x \in \mathbb{R} : q(x) \neq 0\}\). The function \(r : D \to \mathbb{R}\) defined by \(r(x) = \frac{p(x)}{q(x)}\) is a rational function.

Example 4.8. Polynomials and rational functions are continuous (on their domains).

Example 4.9. Given \(q \in \mathbb{Q}^+\), then function \(s : [0, \infty) \to [0, \infty)\) defined by \(s(x) = x^q\) is continuous.

Remark 4.10. We will accept, without proof, that the functions \(e^x\), \(\cos(x)\), \(\sin(x)\), \(\log(x)\) and other standard functions from calculus are continuous on their domains. See for instance Problem 3.11. A suitable definition for the log function must wait until after the Riemann integral.

Continuity behaves well when restricting the domain of a function. The proof follows readily from the definitions and prior facts about limits.

Proposition 4.11. Suppose \(E \subset D \subset \mathbb{R}\), \(f : D \to \mathbb{R}\) and \(a \in E\). If \(f\) is continuous at \(a\), then so is \(f|_E\). If \(f\) is continuous, then \(g = f|_E\) is continuous.

Conversely, if there is a \(\delta > 0\) such that, with \(G = D \cap (a - \delta, a + \delta)\), the function \(f|_G\) is continuous at \(a\), then \(f\) is continuous at \(a\).

That continuity behaves well with respect to the algebraic operations on \(\mathbb{R}\) again follows from the corresponding facts about limits.

Proposition 4.12. Suppose \(D \subset \mathbb{R}\), \(f, g : D \to \mathbb{R}\) and \(a \in D\). If both \(f\) and \(g\) are continuous at \(a\), then so are

(i) \(f + g\);
(ii) \(fg\); and
(iii) \(\frac{1}{g}\), assuming that \(g\) is never 0.

Moreover, if \(f\) takes non-negative values and \(q\) is a positive rational number, then \(h : D \to \mathbb{R}\) defined by \(h(x) = f(x)^q\) is continuous at \(a\).

Proof. To prove item (iii), first observe if \(a\) is not an accumulation point of \(D\), then \(fg\) is continuous at \(a\). Now suppose \(a\) is an accumulation point of \(D\). By assumption both,

\[
\lim_{x \to a} f(x) = f(a), \quad \lim_{x \to a} g(x) = g(a).
\]

By Theorem 2.28,

\[
\lim_{x \to a} fg(x) = f(a)g(a) = fg(a).
\]
Hence \(fg\) is continuous at \(a\).

The proofs of the other items are similar. \(\square\)

4.1. Compositions of continuous functions.

Proposition 4.13. Suppose \(f : D \rightarrow E\) and \(g : E \rightarrow \mathbb{R}\).

If \(f\) is continuous at \(a \in D\) and \(g\) is continuous at \(b = f(a) \in E\), then \(g \circ f : D \rightarrow \mathbb{R}\) is continuous at \(a\).

If both \(f\) and \(g\) are continuous, then \(g \circ f\) is continuous.

Proof. Let \(\epsilon > 0\) be given. There is a \(\eta > 0\) such that if \(y \in E\) and \(|y - b| < \eta\), then \(|g(y) - g(b)| < \epsilon\). There is a \(\delta > 0\) such that if \(x \in D\) and \(|a - x| < \delta\), then \(|f(x) - f(a)| < \eta\). Hence if \(x \in D\) and \(|a - x| < \delta\), then \(|g(f(x)) - g(f(a))| < \epsilon\).

The second part follows immediately from the first. \(\square\)

Example 4.14. The function \(f : \mathbb{R} \rightarrow \mathbb{R}\) defined by \(f(x) = \sqrt{x^2 + 1}\) is continuous.

Continuity at a point also has a sequential characterization.

Proposition 4.15. Suppose \(f : D \rightarrow \mathbb{R}\) and \(a \in D\). If \(f\) is continuous at \(a\) and if \((a_n)\) is a sequence from \(D\) that converges to \(a\), then \((f(a_n))\) converges to \(f(a)\).

Proof. First suppose \(f\) is continuous at \(a\) and \((a_n)\) is a sequence from \(D\) that converges to \(a\). In this case, given \(\epsilon > 0\) there is a \(\delta > 0\) such that if \(x \in D\) and \(|a - x| < \delta\), then \(|f(x) - f(a)| < \epsilon\). There is an \(N \in \mathbb{N}^+\) such that if \(n \geq N\), then \(|a - a_n| < \delta\). Hence, if \(n \geq N\), then \(|f(a_n) - f(a)| < \epsilon\) and it follows that \((f(a_n))\) converges to \(f(a)\).

To prove the converse, suppose \(f\) is not continuous at \(a\). In this case there exists an \(\eta > 0\) such that for every \(\delta > 0\) there is an \(x\) such that \(x \in D\) and \(|x - a| < \delta\), but \(|f(x) - f(a)| \geq \eta\). Choosing, for \(n \in \mathbb{N}^+\), \(\delta = \frac{1}{n}\) there is an \(x_n \in D\) such that \(|x_n - a| < \frac{1}{n}\), but \(|f(x_n) - f(a)| \geq \eta\). By construction \((x_n)\) is a sequence from \(D\) that converges to \(a\), but \((f(x_n))\) does not converge to \(f(a)\). \(\square\)

4.2. The extreme and intermediate value theorems.

Lemma 4.16. If \(f : [a, b] \rightarrow \mathbb{R}\) is continuous, then \(f\) is bounded; i.e., the set \(f([a, b]) = \{f(x) : a \leq x \leq b\}\) is a bounded both above and below.

Proof. We will argue by contradiction to show that \(f\) is bounded above. Accordingly, suppose \(f\) is not bounded above. In this case, for each \(n \in \mathbb{N}\), there is an \(x_n \in [a, b]\) such that \(f(x_n) \geq n\). There is a subsequence \((x_{n_k})\) that converges to some \(y\) by Theorem 2.45. Since \(a \leq x_{n_k} \leq b\), it follows that \(y \in [a, b]\). By Proposition 4.15, \((f(x_{n_k}))\) converges to \(f(y)\), a contradicting the fact that \((f(x_{n_k}))\) is unbounded. \(\square\)

Theorem 4.17 (Extreme Value Theorem (EVT)). If \(f : [a, b] \rightarrow \mathbb{R}\) is continuous, then there exists \(a \leq y \leq b\) such that \(f(y) \geq f(x)\) for all \(a \leq x \leq b\).

Remark 4.18. Under the hypotheses of the theorem, the conclusion says that the range of \(f\), namely the set \(S = \{f(x) : a \leq x \leq b\}\) has a largest value. Thus the set \(S\) has a maximum and of course this maximum is the lub of \(S\). Informally, the theorem is stated as: a continuous function on a closed bounded interval attains its maximum.
Of course \( f \) also attains its minimum.

The maximum and minimum of \( f \) are the \textit{extrema} or \textit{extreme values} of \( f \).

\textbf{Proof.} The set \( S \) is bounded by Lemma 4.16 and it thus has a least upper bound, say \( M \). For each \( n \in \mathbb{N}^+ \) there is an \( y_n \in S \) such that \( M - \frac{1}{n} < y_n \leq M \), by the least property of \( M \).

For each \( n \) there is an \( x_n \) in \([a, b]\) so that \( f(x_n) = y_n \). By Proposition 2.45, there is a \( z \) and a subsequence \( (x_{n_k}) \) of \( (x_n) \) that converges to \( z \). Since \( f \) is continuous, \( (f(x_{n_k})) \) converges to \( f(z) \). On the other hand, from the construction \( (y_n = f(x_n)) \) converges to \( M \). Hence \( f(z) = M \) and the proof is complete. \( \square \)

\textbf{Theorem 4.19} (Intermediate Value Theorem (IVT)). Suppose \( f : [a, b] \to \mathbb{R} \) is continuous and \( f(a) < f(b) \). If \( f(a) < k < f(b) \), then there exists \( a < c < b \) such that \( f(c) = k \).

\textbf{Proof.} Let

\[ E = \{ x \in [a, b] : f(x) \leq k \}. \]

Note that \( a \in E \) and \( E \) is bounded above by \( b \). Thus \( E \) has a least upper bound \( a < c \leq b \).

For each \( n \in \mathbb{N}^+ \) there is a point \( x_n \in E \) such that \( x_n > c - \frac{1}{n} \). In particular, \( (x_n) \) converges to \( c \) and \( f(x_n) \leq k \). By continuity of \( f \), the sequence \( (f(x_n)) \) converges to \( f(c) \) and moreover, \( f(c) \leq k \). In particular, \( c < b \). On the other hand, if \( t > c \), then \( f(t) > k \) as otherwise \( t \in E \).

Choosing any sequence \( (t_n) \) such that \( t_n > c \) and \( (t_n) \) converges to \( c \), it follows that \( (f(t_n)) \) converges to \( f(c) \) (by continuity of \( f \) again) and moreover \( f(c) \geq k \).

\textbf{Corollary 4.20} (Brower’s fixed point theorem). If \( f : [a, b] \to [a, b] \) is continuous, then there exists a point \( p \in [a, b] \) such that \( f(p) = p \).

\textbf{Proof.} Define \( g : [a, b] \to [a, b] \) by \( g(x) = x - f(x) \). Observe that \( g(a) = a - f(a) \leq 0 \) and \( g(b) = b - f(b) \geq 0 \). Hence, by Theorem 4.19, there is a point \( p \) such that \( g(p) = 0 \). Hence \( p - f(p) = 0 \) and the corollary is proved. \( \square \)

\textbf{Corollary 4.21.} If \( f : [a, b] \to [a, b] \) is continuous, then there exists \( c \leq d \) such that \( f([a, b]) = [c, d] \).

\textbf{Proof.} Let \( c \) and \( d \) denote the minimum and maximum values of \( f \) that exist by Theorem 4.17. In particular, \( f(C) \subset [c, d] \). Moreover, there exists points \( u \) and \( v \) in \([a, b]\) such that \( f(u) = c \) and \( f(v) = d \). Assuming \( u < v \), given any \( c \leq k \leq d \), there is a point \( u \leq z \leq v \) such that \( f(z) = k \). Hence \( f(C) \supset [c, d] \).

\( \square \)

4.3. Open and closed sets - elementary topology of the real line.

\textbf{Definition 4.22.} The \textit{complement} of a subset \( C \) of \( \mathbb{R} \) is the set

\[ \bar{C} = \{ x \in \mathbb{R} : x \notin C \}. \]

If \( B \) is also a subset of \( C \), then

\[ B \setminus C = B \cap \bar{C}. \]

In particular,

\[ \bar{C} = \mathbb{R} \setminus C. \]
**Definition 4.23.** Given \( \delta > 0 \), the **\( \delta \)-neighborhood** of a point \( a \in \mathbb{R} \) is the set
\[
N_{\delta}(x) = (a - \delta, a + \delta).
\]

A subset \( O \) of \( \mathbb{R} \) is **open** if for each \( p \in O \) there exists a \( \delta > 0 \) such that \( N_{\delta}(x) \subset O \).

A subset \( C \) of \( \mathbb{R} \) is **closed** if \( \bar{C} \) is open.

**Example 4.24.** Given \( r > 0 \) and \( a \in \mathbb{R} \) the set \( N_r(a) \) is open.

To prove that \( N_r(a) \) is open, let \( b \in N_r(a) \) be given. Choose \( \delta = r - |a - b| > 0 \). Now, if \( x \in N_{\delta}(b) \), then \( |x - a| \leq |x - b| + |b - a| < \delta + |a - b| = r \). Thus, \( N_{\delta}(b) \subset N_r(a) \).

**Example 4.25.** The set \( [0, 1) \subset \mathbb{R} \) is not open.

Observe that \( 0 \in [0, 1) \), but for each \( \delta > 0 \) we have \( N_{\delta}(0) \not\subset [0, 1) \).

**Proposition 4.26.** Suppose \( I \) is a set. If for each \( i \in I \) the set \( O_i \subset \mathbb{R} \) is open, then
\[
O = \bigcup_{i \in I} O_i
\]
is open; i.e., arbitrary unions of open sets are open.

**Proof.** Let \( x \in O \) be given. There is a \( j \in I \) such that \( x \in O_j \). Since \( O_j \) is open and \( x \in O_j \), there is a \( \delta > 0 \) such that \( N_{\delta}(x) \subset O_j \). Since \( O_j \subset O \), it follows that \( N_{\delta}(x) \subset O \) and hence \( O \) is open. \( \square \)

**Example 4.27.** The set \( V = \bigcup_{i=0}^{\infty} (i, i+1) \) is open.

For any \( a \) the sets \((-\infty, a)\) and \((a, \infty)\) are open.

**Example 4.28.** To see that the set \( C = [0, 1] \) is closed, simply observe that \( \bar{C} = (-\infty, 0) \cup (1, \infty) \) is the union of open sets and is hence open.

**Proposition 4.29.** Suppose \( n \in \mathbb{N}^+ \). If \( O_1, \ldots, O_n \) are open sets, then so is
\[
O = \bigcap_{j=1}^{n} O_j.
\]

**Example 4.30.** Let \( U_n = (0, 1 + \frac{1}{n}) \). Then each \( U_n \) is open, but
\[
(0, 1] = \bigcap_{n=1}^{\infty} U_n
\]
is not.

Suppose \( f : X \to Y \) is a function and \( T \subset Y \). The **inverse image** of \( T \) under \( f \), is
\[
f^{-1}(T) = \{ x \in X : f(x) \in Y \}.
\]

**Theorem 4.31.** Suppose \( D \subset \mathbb{R} \) is open and \( f : D \to \mathbb{R} \). The function \( f \) is continuous if and only if \( f^{-1}(U) \) is open for every open set \( U \subset \mathbb{R} \).

**Proof.** Suppose \( f \) is continuous and \( U \subset \mathbb{R} \) is open. To prove that \( V = f^{-1}(U) \) is open, let \( x \in V \) be given. Thus \( y = f(x) \in U \). Since \( U \) is open, there is an \( \epsilon > 0 \) such that if \( |z - y| < \epsilon \), then \( z \in U \). Since \( f \) is continuous (at \( x \)), there is a \( \delta > 0 \) such that if \( |x - s| < \delta \), then \( |f(x) - f(s)| < \epsilon \). Hence, \( f(s) \in U \) and therefore \( s \in V \). Thus \( (x - \delta, x + \delta) \subset V \) and \( V \) is open.

The converse is left as an exercise. See Problem 4.3. \( \square \)
4.4. Closed sets.

**Proposition 4.32.** A subset $C$ of $\mathbb{R}$ is closed if and only if $C$ contains all its accumulation points.

**Proof.** First suppose $a$ is an accumulation point of $C$, but $a \notin C$. Because $a$ is an accumulation point, given $\delta > 0$ the set

$$N_\delta(a) \cap C \neq \emptyset.$$ 

Hence, for every $\delta > 0$, $N_\delta(a) \nsubseteq \tilde{C}$. Thus $\tilde{C}$ is not open and thus $C$ is not closed.

Conversely, suppose $C$ is not closed in which case $\tilde{C}$ is not open. Hence, there exists a point $a \in \tilde{C}$ such that for every $\delta > 0$,

$$N_\delta(a) \nsubseteq \tilde{C}.$$ 

Thus, for every $\delta > 0$,

$$N_\delta(a) \cap C \neq \emptyset.$$ 

Since $a \notin C$, it follows that $a$ is an accumulation point of $C$. \[\square\]

**Proposition 4.33.** If $C$ is nonempty, bounded above, and closed, then $C$ contains its least upper bound.

**Proof.** The hypotheses imply that $C$ has a least upper bound, say $b$. Given $\delta > 0$, there is a $c \in C$ such that $b - \delta < c \leq b$. Thus, $b$ is either in $C$ or $b$ an accumulation point of $C$. By Proposition 4.32, $b \in C$. \[\square\]

**Proposition 4.34.** If $C$ is closed and $(a_n)$ is a sequence from $C$ that converges (in $\mathbb{R}$), then $\lim a_n \in C$.

**Proof.** Let $A = \lim a_n$. If $A = a_n$ for some $n$, then $A \in C$. Otherwise, by Problem 2.11, $A$ is a limit point of $C$. Since $C$ is closed, Proposition 4.32 implies that $A \in C$. \[\square\]

**Proposition 4.35.** Suppose $C$ is closed and bounded. If $(a_n)$ is a sequence from $C$, then $(a_n)$ has a subsequence that converges to some point of $C$.

**Proof.** Since $C$ is bounded, so is $(a_n)$. Hence, by Theorem 2.45, there is a subsequence $(a_{n_k})$ of $(a_n)$ that converges to some $a$. By Proposition 4.34, $a \in C$. \[\square\]

4.5. Continuous functions on closed bounded sets.

**Proposition 4.36.** Suppose $C$ is nonempty, closed and bounded. If $f : C \to \mathbb{R}$ is continuous, then $f(C)$ is closed and bounded too.

**Proof.** Suppose $f(C)$ is not bounded above. In this case, for each $n$ there exists $y_n \in C$ such $y_n \geq n$. For each $n$ there is an $x_n \in C$ such that $f(x_n) = y_n$. There is a subsequence $(x_{n_k})$ of $(x_n)$ converging to some $z \in C$ by Proposition 4.35. By continuity of $f$, the sequence $(f(x_{n_k}) = y_{n_k})$ is convergent and thus bounded, a contradiction which shows $f(C)$ is in fact bounded above. A similar argument shows $f(C)$ is bounded below.

By Proposition 4.32, to see that $f(C)$ is closed it suffices to show that it contains all its accumulation points. Accordingly, suppose $p$ is an accumulation point of $f(C)$. In particular, there is a sequence $(y_n)$ from $f(C)$ that converges to $p$. For each $n$ there is an $x_n \in C$ such that $y_n = f(x_n)$. By Proposition 4.35, there is a subsequence $(x_{n_k})$ that converges to some
z \in C. By continuity of \( f \), the sequence \( (y_{nk} = f(x_{nk})) \) converges to \( f(z) \). But \( (y_{nk}) \) converges to \( p \). Hence \( f(z) = p \) and thus \( p \in f(C) \).

4.6. Inverse functions.

**Definition 4.38.** Given a set \( A \), the function \( \text{id}_A : A \to A \) defined by \( \text{id}_A(a) = a \) is called the identity function.

**Proposition 4.39.** Given a function \( f : A \to B \), there exists a function \( g : B \to A \) such that \( f \circ g = \text{id}_B \) and \( g \circ f = \text{id}_A \) if and only if \( f \) is one-one and onto. Moreover, in this case, \( g \) is unique.

**Proof.** First suppose that \( f \) is one-one and onto. Define \( g : B \to A \) as follows. Given \( b \in B \) there is a unique \( a \in A \) such that \( b = f(a) \) (because \( f \) is both one-one and onto). Let \( g(b) = a \). Then \( g(f(a)) = g(b) = a \) and \( f(g(b)) = f(a) = b \).

Conversely, suppose there is a \( g \) such that both \( f \circ g = \text{id}_B \) and \( g \circ f = \text{id}_A \). To prove that \( f \) is one-one, suppose \( f(x) = f(y) \). Then \( x = g(f(x)) = g(f(y)) = y \). To prove that \( f \) is onto, let \( b \in B \) be given and observe that \( f(g(b)) = b \).

Finally, to see that \( g \) is unique suppose also that \( f \circ h = \text{id}_B \). It follows that \( g \circ f \circ h = h \) and also \( g \circ f \circ h = g \circ \text{id}_B = g \).

**Definition 4.40.** The function \( g \) in Proposition 4.39 (assuming it exists) is called the inverse of \( f \) and is denoted \( f^{-1} \).

**Remark 4.41.** Of course if \( f \) has an inverse \( g \), then by Proposition 4.39, \( g \) has an inverse and \( g^{-1} = f \).

**Example 4.42.** Define \( f : [0, \infty) \to [0, \infty) \) by \( f(x) = x^2 \). From Proposition 1.14 it follows that \( f \) is one-one. On the other hand, for each \( b > 0 \) the Intermediate Value Theorem (Theorem 4.19) implies that \( f([0, b]) = [0, b^2] \). Consequently, \( f \) is onto and thus has an inverse. Of course, the inverse of \( f \) is the square root function.

The functions \( \log \) and \( \exp \) are inverses of each other.

The function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = \sin(x) \) does not have an inverse. However, the function \( F : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1] \) defined by \( F(x) = \sin(x) \) is one-one and, using the Intermediate Value Theorem (and continuity of \( \sin \)) it is onto. Thus \( f \) has an inverse we call the arcsin. Thus \( \arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}] \) and for \( x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \), \( \arcsin(F(x)) = x \) and \( F(\arcsin(y)) = y \) for \( -1 \leq y \leq 1 \).

**Theorem 4.43.** If \( C \) is nonempty, closed and bounded and if \( f : C \to f(C) \) is one-one, then \( f^{-1} : f(C) \to C \) is continuous.

**Proof.** For notational ease, let \( g = f^{-1} \). Fix \( w \in f(C) \) and, arguing by contradiction, suppose \( g \) is not continuous at \( w \). In this case, there exists an \( \eta > 0 \) such that for each \( n \in \mathbb{N}^+ \) there exists \( y_n \in f(C) \) such that \( |y_n - w| < \frac{1}{n} \), but

\[
|g(y_n) - g(w)| \geq \eta.
\]
Let \( x_n = g(y_n) \). Since \((x_n)\) is a sequence from the closed and bounded set \( C \), it has a subsequence \((x_{n_k})\) that converges to some \( z \in C \) by Proposition 4.35. By continuity of \( f \), the sequence \((f(x_{n_k}) = y_{n_k})\) converges to \( f(z) \). But \((y_{n_k})\) converges to \( w \) by construction. Thus \( w = f(z) \) so that \( z = g(w) \). Hence \((g(y_{n_k}) = x_{n_k})\) converges to \( z = g(w) \), contradicting (3). \( \square \)

4.7. Uniform continuity.

Definition 4.44. Given \( D \subset \mathbb{R} \), a function \( f : D \to \mathbb{R} \) is uniformly continuous if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( x, y \in D \) and \( |x - y| < \delta \), then \( |f(x) - f(y)| < \epsilon \).

Example 4.45. The function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x \) is uniformly continuous.

Example 4.46. The function \( g : \mathbb{R} \to \mathbb{R} \) defined by \( g(x) = x^2 \) is not uniformly continuous.

To show \( g \) is not uniformly continuous, choose \( \epsilon_0 = 1 \). Given \( \delta > 0 \) choose \( x = \frac{1}{\delta} \) and \( y = x + \frac{\delta}{2} \). Then \( |x - y| < \delta \), but
\[
|f(y) - f(x)| = 1 \geq \epsilon_0.
\]

Example 4.47. Define \( h : [1, \infty) \to \mathbb{R} \) by \( h(x) = \sqrt{x} \). To see that \( h \) is uniformly continuous, let \( \epsilon > 0 \) be given. Choose \( \delta = 2\epsilon \). If \( 1 \leq x, y \) and \( |x - y| < \delta \), then
\[
|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}|
\]
\[
= \sqrt{x} + \sqrt{y} \cdot |x - y|
\]
\[
\leq \frac{2}{\delta} |x - y|
\]
\[
< \frac{\delta}{2} = \epsilon.
\]

Theorem 4.48. If \( C \) is nonempty, closed and bounded and if \( f : C \to \mathbb{R} \) is continuous, then \( f \) is uniformly continuous.

Proof. Arguing by contradiction, suppose \( f \) is not uniformly continuous. In this case there is an \( \epsilon_0 > 0 \) such that for every \( n \in \mathbb{N}^+ \) there exists points \( x_n, y_n \in C \) such that
\[
|x_n - y_n| < \frac{1}{n},
\]
but
\[
|f(x_n) - f(y_n)| \geq \epsilon_0.
\]
By Proposition 4.35 there is a point \( z \in C \) and a subsequence \((x_{n_k})\) of \((x_n)\) that converges to \( z \). As an exercise for the reader, use (4) to show that \((y_{n_k})\) converges to \( z \) also.

By continuity of \( f \), the sequences \((f(x_{n_k}))\) and \((f(y_{n_k}))\) both converge to \( f(z) \) which contradicts (5). \( \square \)

Example 4.49. Given \( a < b \), the function \( f : [a, b] \to \mathbb{R} \) defined by \( f(x) = x^2 \) is uniformly continuous.

Similarly, for any \( b > 0 \), the function \( g : [0, b] \to \mathbb{R} \) defined by \( g(x) = \sqrt{x} \) is continuous.

Proposition 4.50. If \( f : D \to \mathbb{R} \) is uniformly continuous and if \((x_n)\) is a Cauchy sequence from \( D \), then \((f(x_n))\) is a Cauchy sequence.

The proof is left as an exercise for the gentle reader.
4.8. Lipschitz continuity. 4

Definition 4.51. A function \( f : D \to \mathbb{R} \) is Lipschitz continuous if there exists a \( C \) such that

\[ |f(x) - f(y)| \leq C|x - y| \]

for all \( x, y \in D \).

If \( C \) can be chosen such that \( 0 \leq C < 1 \), then \( f \) is a contraction or contraction mapping.

Remark 4.52. If \( f \) is Lipschitz continuous, then \( f \) is uniformly continuous.

Example 4.53. The function \( f : [0, 1] \to \mathbb{R} \) defined by \( f(x) = \sqrt{x} \) is uniformly continuous, but not Lipschitz continuous.

Because \( f \) is a continuous function on a closed and bounded set it is uniformly continuous. To see that \( f \) is not Lipschitz continuous, let \( C > 0 \) be given. With \( 0 < x < C^{-2} \), we have \( C < \frac{1}{\sqrt{x}} \) and hence

\[ C|0 - x| = Cx < \sqrt{x} = |f(0) - f(x)|. \]


Problem 4.1. Give a proof of Proposition 4.13 based on Proposition 3.35.

Problem 4.2. Prove finite sets are closed.

Problem 4.3. Prove Proposition 4.31.

Problem 4.4. Show, if \( f : [a, b] \to \mathbb{R} \) is an increasing function and the range of \( f \) is an interval, then \( f \) is continuous.

Problem 4.5. Prove the converse of Proposition 4.34.

Problem 4.6. Define \( f : (0, 1) \to \mathbb{R} \) by \( f(x) = \frac{1}{x} \). Show \( f \) is not uniformly continuous.

Problem 4.7. Suppose \( f : [0, \infty) \to \mathbb{R} \) is continuous. Show, if there is a \( b > 0 \) such that \( f|_{[0, b]} \) is uniformly continuous, then \( f \) is uniformly continuous.

Show that \( f : [0, \infty) \to \mathbb{R} \) defined by \( f(x) = \sqrt{x} \) is uniformly continuous. (Compare with Example 4.47.)

Problem 4.8. Show, if \( f : [a, b] \to \mathbb{R} \) is continuous and \( \lim_{x \to b} f(x) \) exists, then \( f \) is uniformly continuous and bounded. (Suggestion: Let \( L \) denote the limit and define \( g : [a, b] \to \mathbb{R} \) by \( g(x) = f(x) \) if \( a \leq x < b \) and \( g(b) = L \) and prove \( g \) is continuous.)

Problem 4.9. Show \( f \) in Problem 4.6 is not uniformly continuous.

Problem 4.10. Show if \( f : [a, b] \to \mathbb{R} \) is uniformly continuous, then \( \lim_{x \to b} f(x) \) exists. Conclude that \( f \) is bounded.

5. Differentiation

5.1. Definitions and examples.

\[ \text{This section is optional.} \]
**Definition 5.1.** Suppose \( f : D \to \mathbb{R} \) and \( a \in D \) is an accumulation point of \( D \). Define \( g : D \setminus \{a\} \to \mathbb{R} \) by
\[
g(x) = \frac{f(x) - f(a)}{x - a}.
\]
If \( \lim_{x \to a} g(x) \) exists, then \( f \) is **differentiable at \( a \)** and the limit is the **derivative of \( f \) at \( a \)**, denoted \( f'(a) \).

**Example 5.2.** Define \( f : \mathbb{R} \to \mathbb{R} \) by
\[
f(x) = \begin{cases} 
x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\
0 & x = 0.
\end{cases}
\]
To see that \( f \) is differentiable at 0, consider \( g \), defined for all real numbers except 0 by
\[
g(x) = \frac{f(x) - f(0)}{x - 0} = x \sin\left(\frac{1}{x}\right).
\]
A routine argument shows \( \lim_{x \to 0} g(x) = 0 \). Thus \( f \) is differentiable at 0 and \( f'(0) = 0 \).

**Example 5.3.** Show \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = |x| \) is not differentiable at 0.

**Proposition 5.4.** Suppose \( f : D \to \mathbb{R} \) and \( a \) is an accumulation point of \( D \). If \( f \) is differentiable at \( a \), then \( f \) is continuous at \( a \).

**Remark 5.5.** Example 5.3 shows the converse of the proposition is false.

**Proof.** Note that
\[
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)
\]
exists by hypothesis. Thus,
\[
0 = f'(a) \lim_{x \to a} (x - a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a} (x - a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.
\]
Hence,
\[
f(a) = \lim_{x \to a} [f(x) - f(a)] + \lim_{x \to a} f(a) = \lim_{x \to a} f(x),
\]
so that the limit on the right hand side exists and is equal \( f(a) \). Hence \( f \) is continuous at \( a \). \( \square \)

**Definition 5.6.** Suppose \( f : D \to \mathbb{R} \) and every point of \( D \) is an accumulation point of \( D \). If \( f \) is differentiable at each \( a \in D \), then \( f \) is differentiable.

**Remark 5.7.** Often, when discussing differentiation, \( D \) is an open interval.

If \( f \) is differentiable, then we obtain a function \( f' : D \to \mathbb{R} \).

**Example 5.8.** Show, if \( c \) is a constant, then \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = c \) is differentiable and \( f'(x) = 0 \).

Show \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x \) is differentiable and \( f'(x) = 1 \).

Show \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x^2 \) is differentiable.
5.2. Properties of the derivative.

**Theorem 5.9.** Suppose \( f, g : D \to \mathbb{R} \) and \( a \in D \) is an accumulation point of \( D \). If both \( f \) and \( g \) are differentiable at \( a \), then

(i) \( f + g \) is differentiable at \( a \) and \( (f + g)'(a) = f'(a) + g'(a) \);

(ii) \( fg \) is differentiable at \( a \) and \( (fg)'(a) = f'(a)g(a) + f(a)g'(a) \); and

(iii) if \( g \) is never 0 and \( g'(a) \neq 0 \), then \( h = \frac{1}{g} \) is differentiable at \( a \) and \( h'(a) = -\frac{g'(a)}{g^2(a)} \).

**Proof.** Here is the proof of item (2). Using properties of limits,

\[
(f'(a)g(a) + f(a)g'(a)) = g(a) \lim_{x \to a} \frac{f(x) - f(a)}{x - a} + f(a) \lim_{x \to a} \frac{g(x) - g(a)}{x - a}.
\]

\[
= g(a) \lim_{x \to a} \frac{f(x) - f(a)}{x - a} + f(a) \lim_{x \to a} \frac{g(x) - g(a)}{x - a}.
\]

\[
= \lim_{x \to a} \frac{g(x)(f(x) - f(a))}{x - a} + f(a) \lim_{x \to a} \frac{g(x) - g(a)}{x - a}.
\]

\[
= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a}.
\]

The proofs of the other items are similar and omitted. \( \square \)

**Remark 5.10.** It follows from the Theorem and Example 5.8 that a rational function is differentiable (on its domain).

**Theorem 5.11 (Chain Rule).** Suppose \( f : D \to E \), \( g : E \to \mathbb{R} \) and \( a \in D \) and \( b = f(a) \in E \) are accumulation points of \( D \) and \( E \) respectively. If \( f \) is differentiable at \( a \) and \( g \) is differentiable at \( b \), then \( h = g \circ f \) is differentiable at \( a \) and \( h'(a) = g'(f(a))f'(a) \).

**Proof.** The assumption that \( g \) is differentiable at \( a \) is equivalent to continuity of

\[
F(y) = \begin{cases} 
\frac{g(y) - g(b)}{y - b} & y \neq b \\
g'(b) & y = b,
\end{cases}
\]

at \( b \). Thus, Proposition 4.13 gives

\[
\lim_{x \to a} F(f(x)) = F(b) = g'(b).
\]

Note that

\[
F(f(x)) \frac{f(x) - f(a)}{x - a} = \frac{h(x) - h(a)}{x - a}
\]

(for \( x \neq a \) of course). Thus, routine properties of limits gives,

\[
\lim_{x \to a} \frac{g(f(x)) - g(b)}{x - a} = \lim_{x \to a} F(f(x)) \frac{f(x) - f(a)}{x - a}
\]

\[
= \lim_{x \to a} F(f(x)) \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

\[
= g'(b)f'(a).
\]

\( \square \)
**Proposition 5.12** (Inverse Function Theorem). Suppose \( f: [a, b] \to \mathbb{R} \) is continuous, strictly increasing, and differentiable at \( a < c < b \). If \( f'(c) \neq 0 \), then \( f^{-1} : [f(a), f(b)] \to [a, b] \) is differentiable at \( f(c) \) and
\[
(f^{-1})'(f(c)) = \frac{1}{f'(c)}.
\]

**Proof.** For notational ease, let \( g = f^{-1} \) and \( d = f(c) \). The function
\[
F(x) = \begin{cases} 
\frac{x-c}{f(x)-f(c)} & x \neq c \\
\frac{1}{f'(c)} & x = c
\end{cases}
\]
is defined and continuous, including at \( c \). Since also \( g(y) \) is continuous and the composition of continuous functions is continuous, it follows that
\[
\lim_{y \to d} F(g(y)) = F(g(d)) = F(c).
\]

Noting that \( F(g(y)) = \frac{g(y)-d}{y-d} \) completes the proof. \( \square \)

**Problem 5.1.** Define \( f: \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \to [-1, 1] \) by \( f(x) = \sin(x) \). Assuming we know that \( f \) is differentiable (and its derivative is \( \cos(x) \)), the hypotheses of the Inverse Function Theorem are satisfied for \( f \) and any point \( c \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \). Let \( g = f^{-1} \) and find \( g'(f(c)) \).

### 5.3. The Mean Value Theorem

**Definition 5.13.** A function \( f: D \to \mathbb{R} \) has a local (relative) minimum at \( c \in D \) if there is a \( \delta > 0 \) such that if \( y \in D \) and \( |c-y| < \delta \), then \( f(c) \leq f(y) \).

**Lemma 5.14.** Suppose \( c \in D \subseteq \mathbb{R} \) and there is an \( \eta > 0 \) such that \( (c-\eta, c+\eta) \subseteq D \). If \( f: D \to \mathbb{R} \) has a local minimum at \( c \) and if \( f \) is differentiable at \( c \), then \( f'(c) = 0 \).

**Lemma 5.15** (Rolle’s Theorem). Suppose \( f: [a, b] \to \mathbb{R} \) is continuous. If \( f(a) = f(b) \) and if \( f \) is differentiable on the open interval \( (a, b) \), then there is a point \( a < c < b \) such that \( f'(c) = 0 \).

**Proof.** Without loss of generality, it can be assumed that \( f \) is not constant. Since \( f \) is continuous on the closed bounded interval \( [a, b] \), it attains its extrema. Since \( f(a) = f(b) \) and \( f \) is not constant, \( f \) attains either its maximum or minimum at some point \( a < c < b \). From Lemma 5.14 it follows that \( f'(c) = 0 \). \( \square \)

**Theorem 5.16** (Cauchy Mean Value Theorem). If \( f, g: [a, b] \to \mathbb{R} \) are continuous and differentiable at each point in \( (a, b) \), then there is a \( c \) with \( a < c < b \) so that \( (f(b) - f(a))g'(c) = f'(c)(g(b) - g(a)) \).

**Proof.** Let \( F(x) = (f(x) - f(a))(g(b) - g(a)) - (f(b) - f(a))(g(x) - g(a)) \). Then \( F(a) = F(b) = 0 \) and \( F \) satisfies the hypotheses of Rolle’s Theorem. Hence there is a \( a < c < b \) such that \( F'(c) = 0 \); i.e., \( f'(c)(g(b) - g(a)) = f'(x)(g(b) - g(a)) \). \( \square \)

Choosing \( g(x) = x \) in the Cauchy Mean Value Theorem captures the usual Mean Value Theorem.

**Corollary 5.17** (Mean Value Theorem). If \( f: [a, b] \to \mathbb{R} \) is continuous and differentiable at each point in \( (a, b) \), then there is a \( c \) with \( a < c < b \) so that \( f(b) - f(a) = f'(c)(b-a) \).
Corollary 5.18. Suppose \( f : (u, v) \to \mathbb{R} \) is differentiable.

The function \( f \) is increasing if and only if \( f' \geq 0 \) (meaning \( f'(x) \geq 0 \) for all \( x \in (a, b) \)).

The function \( f \) is constant if and only if \( f' = 0 \).

Example 5.19. Define \( f : \mathbb{R} \to \mathbb{R} \) by
\[
 f(x) = \begin{cases} 
 x + 2x^2\sin\left(\frac{1}{x}\right) & x \neq 0, \\
 0 & x = 0.
\end{cases}
\]
Then \( f'(0) = \frac{1}{2} > 0 \), but there is no interval properly containing 0 on which \( f \) is increasing. Indeed, for \( n \in \mathbb{N}^+ \), let
\[
x_n = \frac{1}{2n\pi}
\]
and note that \( f'(x_n) = -1 < 0 \) which implies there is no interval properly containing 0 on which \( f' \geq 0 \).

Problem 5.2. Suppose \( f : [a, b] \to \mathbb{R} \) is differentiable. Show, if \( f' > 0 \) (so \( f'(x) > 0 \) for all \( x \in [a, b] \)), then \( f \) is strictly increasing. (In particular, the hypotheses of the Inverse Function Theorem are satisfied).

Show the result remains true if \( f'(x) > 0 \) for all except possibly one \( x \in [a, b] \).

5.4. Further topics.

Theorem 5.20 (Taylor’s Theorem). Let \( I = (u, v) \subset \mathbb{R} \) be an open interval, \( n \in \mathbb{N} \), and suppose \( f : I \to \mathbb{R} \) is \((n+1)\) times differentiable. If \( a < a < b < v \), then there is a \( c \) such that \( a < c < b \) and
\[
f(b) = \sum_{j=0}^{n} \frac{f^{(j)}(a)(b-a)^j}{j!} + \frac{f^{(n+1)}(c)(b-a)^{n+1}}{(n+1)!}.
\]

Proof. Define \( R_n : I \to \mathbb{R} \) by
\[
 R_n(x) = f(b) - \sum_{j=0}^{n} \frac{f^{(j)}(x)(b-x)^j}{j!}.
\]
There is a \( K \) so that \( R_n(a) = K\frac{(b-a)^{n+1}}{(n+1)!} \) and the goal is to prove there is a \( a < c < b \) such that \( K = f^{(n+1)}(c) \).

Let
\[
 \varphi(x) = R_n(x) - K\frac{(b-x)^{n+1}}{(n+1)!}.
\]
Note that \( \varphi : [a, b] \to \mathbb{R} \) is continuous and differentiable on \( (a, b) \). Moreover, \( \varphi(a) = 0 = \varphi(b) \). Thus, by the MVT, there is a \( a < c < b \) such that \( \varphi'(c) = 0 \). Since,
\[
 \varphi'(x) = -f^{(n+1)}(x)\frac{(b-x)^n}{n!} + K\frac{(b-x)^n}{n!},
\]
it follows that
\[
 0 = (-f^{(n+1)}(c) + K)\frac{(b-c)^n}{n!}.
\]
The conclusion of the theorem follows.

Proposition 5.21 (A version of L’hopital’s rule). Let \( I = (a, b) \) and \( f, g : I \to \mathbb{R} \) and suppose
(i) both \( f \) and \( g \) are differentiable;
(ii) \[
\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x); \quad \text{and;}
\]
(iii) both \( g \) and \( g' \) are never 0.

If
\[
\lim_{x \to a^+} \frac{f'(x)}{g'(x)}
\]
exists, then
\[
\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.
\]

Proof. The functions \( f \) and \( g \) extend to be continuous on \([a, b)\) by defining \( f(a) = g(a) = 0 \).

Let
\[
L = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.
\]
Given \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( a < y < a + \delta \), then
\[
|L - \frac{f'(y)}{g'(y)}| < \epsilon.
\]

From the Cauchy mean value theorem and hypothesis (iii), given \( a < x < a + \delta \) there is a \( a < c < x \) such that
\[
\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}.
\]
Thus, if \( a < x < a + \delta \), then,
\[
|L - \frac{f(x)}{g(x)}| = |L - \frac{f'(c)}{g'(c)}| < \epsilon.
\]
\( \Box \)
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