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1. The Real Numbers

We will take the view that we know what the real numbers, denoted here by \( \mathbb{R} \), are and in this section we simply review some important properties.

1.1. Sets. The following notations will be used for the natural numbers, integers, positive integers and rational numbers, respectively.

\[
\mathbb{N} = \{0, 1, 2, \ldots \} \\
\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \} \\
\mathbb{N}^+ = \{1, 2, 3, \ldots \} \\
\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, \ n \in \mathbb{N}^+ \right\}.
\]

Example 1.1. The square root of 2 is not rational; i.e., there is no rational number \( s > 0 \) such that \( s^2 = 2 \). □

The cartesian product of two sets \( A \) and \( B \) is the set of ordered pairs \((a, b)\) with \( a \in A \) and \( b \in B \),

\[
A \times B = \{(a, b) : a \in A, \ b \in B \}.
\]

For instance \( \mathbb{R}^2 := \mathbb{R} \times \mathbb{R} \) is, geometrically, the cartesian plane.

1.2. Functions.

Definition 1.2. A function \( f \) consists of sets \( A \) and \( B \), called the domain and codomain of \( f \) respectively, and a rule that assigns to each \( a \in A \) a unique \( b = f(a) \in B \). We write, \( f : A \to B \).

The function \( f \) is one-one if \( f(x) = f(y) \) implies \( x = y \); and \( f \) is onto if \( \{f(x) : x \in A\} = B \).

In the case that \( B \) is a subset of \( \mathbb{R} \) we say that \( f \) is real-valued. □

Given \( C \subset A \), let

\[
f(C) = \{f(x) : x \in C\}.
\]

The set \( f(A) \) is the range of \( f \), sometimes denoted \( \text{range}(f) \).

For the most part, \( A \) will be a subset of \( \mathbb{R} \) and often \( B = \mathbb{R} \).

Example 1.3. Here are a couple examples of functions.

(a) Define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = x^2 \). Here both the domain and codomain of \( f \) is \( \mathbb{R} \).

(b) Define \( g : [0, \infty) \to \mathbb{R} \) by \( g(x) = x^2 \). Note, \( g \) is one-one, whereas \( f \) above is not.

(c) Define \( h : [0, 1] \to [0, 1] \) by \( h(x) = 1 \) if \( x \in \mathbb{Q} \cap [0, 1] \) and \( h(x) = 0 \) otherwise. □

1.3. Field Axioms.

Definition 1.4. A field \( F \) is a triple, \((F, +, \cdot)\), where \( F \) is a set and

\[
+, \cdot : F \times F \to F
\]

are functions, called addition and multiplication respectively and written \( x + y = +(x, y) \) and \( xy = \cdot(x, y) \), satisfying the following (long list) of axioms
(i) \( x + y = y + x \) for every \( x, y \in F \);
(ii) \( xy = yx \), for every \( x, y \);
(iii) \( (x + y) + z = x + (y + z) \) for every \( x, y, z \);
(iv) \( (xy)z = x(yz) \) for every \( x, y, z \);
(v) there is an element \( 0 \in F \) such that \( 0 + w = w \) for every \( x \in F \);
(vi) there is an element \( 1 \in F \), distinct from 0, such that \( 1w = w \) for every \( w \in F \);
(vii) for each \( x \in F \) there is an element \( u \in F \) such that \( x + u = 0 \);
(viii) for each \( x \neq 0 \), there is a \( y \) such that \( xy = 1 \); and
(ix) \( (x + y)z = xz + yz \) for every \( x, y, z \).

Note that only the property of item (ix) involves both operations. It is customary to write \( F \) instead of \((F, +, \cdot)\) to denote a field. Our primary example of a field is \( \mathbb{R} \) (with its usual operations).

**Proposition 1.5 (Cancellation).** Suppose \( F \) is a field. Given \( x, y, z \in F \), if \( x + y = x + z \), then \( y = z \).

**Proof.** Let \( x, y, z \in F \) such that \( x + y = x + z \) be given. By item (vii) there exists a \( u \in F \) such that \( x + u = 0 \). Thus,

\[
y = 0 + y = (u + x) + y = u + (x + y) = u + (x + z) = (u + x) + z = 0 + z = z,
\]

where we have used, in order, items (v), (i), (iii), the hypothesis \( x + y = x + z \), and items (iii), (v).

**Remark 1.6.** It follows that, given \( x \in F \) there is exactly one \( u \in F \) such that \( x + u = 0 \). We call \( u \) the additive inverse of \( x \) and, for reasons that become clear after Proposition 1.7 below, denote it by \( -x \) and write \( y - x \) instead of \( y + (-x) \). Thus \( x + (-x) = x - x = -x + x = 0 \).

Similarly, if \( x \in F \) and \( x \neq 0 \), then there is exactly one \( y \in F \) (\( y \neq 0 \)) such that \( xy = 1 = yx \).
We call \( y \) the multiplicative inverse of \( x \). Both \( x^{-1} \) and \( \frac{1}{x} \) are used to denote the multiplicative inverse of an \( x \in F \), \( x \neq 0 \).

**Proposition 1.7.** Suppose \( F \) is a field. If \( x \in F \), then \( 0x = 0 \) and \( -x = (-1)x \).

**Proof.** Since \( 0 + 0x = 0x = (0 + 0)x = 0x + 0x \), cancellation gives \( 0 = 0x \).

Using \( 0x = 0 \) gives \( x + (-1)x = 1x + (-1)x = (1 + (-1))x = 0x = 0 \). Thus \((-1)x\) is the additive inverse for \( x \); that is \((-1)x = -x \).

**Remark 1.8.** From here on we will use freely, without proof or further comment, the many routine properties of fields that follow from the axioms.

**Definition 1.9.** A subset \( G \) of a field \( F \) is a **subfield** if it satisfies the axioms,

(a) \( 1 \in G \);
(b) \(G\) is closed under addition and multiplication;
(c) if \(x \in G\), then \(-x \in G\); 
(d) if \(x \in G\) and \(x \neq 0\), then \(x^{-1} \in G\).

Here the operations on \(G\) are those inherited from \(F\). 

\[\square\]

**Proposition 1.10.** If \(F\) is a field and \(G \subset F\) is a subfield, then \(G\) is a field.

For instance, by (a) and (c), \(-1 \in G\). By (b), \(0 = 1 - 1 \in G\).

**Example 1.11.** The rationals \(\mathbb{Q} \subset \mathbb{F}\) is easily seen to be a subfield of \(\mathbb{R}\) and hence a field. The complex numbers 
\[
\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}
\]
with the usual operations is also a field. Note that \(\mathbb{R}\) can be viewed as a subfield of \(\mathbb{C}\).

\[\square\]

**Example 1.12.** Show \(\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} \subset \mathbb{R}\) is a field.

**Example 1.13.** Let \(\mathbb{Z}_2 = (\{0, 1\}, +, \cdot)\) where 
\[
x + y = x + y \mod 2 \\
x y = x y \mod 2.
\]
Here the + on the left hand side is addition in \(\mathbb{Z}_2\), whereas + on the right hand side is addition in \(\mathbb{N}\).

The residue modulo 2 is the remainder after dividing by 2.

It is easy, but tedious, to verify that \(\mathbb{Z}_2\) is a field with neutral elements 0, 1. Since \(1 + 1 = 0\), the additive inverse of 1 is 1. In particular, since we use the notation \(-x\) to denote the additive inverse of an element \(x\) in a field, \(-1 = 1\) (read the additive inverse of 1 is 1).

\[\square\]

**Example 1.14.** \(\mathbb{Z}\) (with the usual operations; that is those inherited from \(\mathbb{R}\)) is not a field. The smallest field containing \(\mathbb{Z}\) is \(\mathbb{Q}\).

\[\square\]

1.4. Ordered Fields.

**Definition 1.15.** An ordered set \((S, <)\) consists of a (nonempty) set \(S\) and an order (relation \(^1\)) \(<\) on \(S\) satisfying

(i) (trichotomy) for each \(x, y \in S\), exactly one of the following hold,
\[
x < y, \quad y < x, \quad x = y;
\]

(ii) (transitivity) if \(x < y\) and \(y < z\), then \(x < z\).

\[\square\]

Again, it is customary to write \(S\) for the ordered set instead of the more formal \((S, <)\).

**Example 1.16.** The usual order \(<\) on \(\mathbb{R}\) (and thus on any subset of \(\mathbb{R}\)) produces an example of an ordered set. In particular \(\mathbb{Q}\) is an order set and so is \(\mathbb{Q}(\sqrt{2})\) (with the order they inherits from \(\mathbb{R}\)).

\[\square\]

**Definition 1.17.** An ordered field \(\mathbb{F} = (F, +, \cdot, <)\) consists of a field \((F, +, \cdot)\) that is also an ordered set \((\mathbb{F}, <)\) such that,

\[\text{A relation on a set } S \text{ is a subset } R \text{ of } S \times S \text{ that satisfies certain axioms. In the setting here } x < y \text{ indicates } (x, y) \in R.\]
(i) if $x, y, z \in \mathbb{F}$ and $x < y$, then $x + z < y + z$;
(ii) if $x, y \in \mathbb{F}$ and $x, y > 0$, then $xy > 0$.

An element $x \in \mathbb{F}$ is **positive** if $x > 0$.

**Example 1.18.** Both $\mathbb{R}$ and $\mathbb{Q}$ with the usual ordering are ordered fields.

**Proposition 1.19.** Suppose $\mathbb{F}$ is an ordered field and $x \in \mathbb{F}$.

(i) If $x < 0$, then $-x > 0$;
(ii) if $x > 0$, then $-x < 0$;
(iii) if $x \neq 0$, then $x^2 > 0$;
(iv) in particular, $1 > 0$ in any ordered field.

**Proof.** If $x < 0$, then, by item (i) of Definition 1.17, $0 = x - x < 0 - x$. Likewise, if $x > 0$, then $0 = x - x > 0 - x = -x$.

If $x > 0$, then, by (ii) of Definition 1.17, $x^2 = xx > 0$. If $x < 0$, then, by (ii) of Definition 1.17, $-x > 0$ and thus $x^2 = (-x)^2 > 0$. If $x \neq 0$, then either $x < 0$ or $x > 0$ by trichotomy. Thus, in either case $x^2 > 0$. Finally $1 = 1^2$ and hence $1 > 0$.

**Remark 1.20.** We will not state (much less) prove all the usual facts about the order structure in an ordered field, but rather use them without comment.

**Example 1.21.** Recall Example 1.13. Prove that there is no order on $\mathbb{Z}_2$ that makes it an ordered field.

**Solution.** Arguing by contradiction, suppose $<$ is an order on $\mathbb{Z}_2$ that makes $\mathbb{Z}_2$ an ordered field. Since $1$ is the multiplicative identity in $\mathbb{Z}_2$, by Proposition 1.19 items (iv) and (ii) $1 > 0$ and hence $-1 < 0$. On the other hand, $-1 = 1$ since $1$ is its own additive inverse in $\mathbb{Z}_2$. Thus $1 < 0$ and we have now reached a contradiction to the trichotomy property of an order (as both $1 > 0$ and $1 < 0$).

1.5. **The least upper bound property.**

**Definition 1.22.** Let $S$ be a subset of an ordered field $\mathbb{F}$.

(i) The set $S$ is **bounded above** if there is a $b \in \mathbb{F}$ such that $b \geq s$ for all $s \in S$.
(ii) Any $b \in \mathbb{F}$ such that $b \geq s$ for all $s \in S$ is an **upper bound** for $S$.

**Example 1.23.** Identify the set of upper bounds for the following subsets of the ordered field $\mathbb{R}$.

(a) $[0, 1)$;
(b) $[0, 1]$;
(c) $\mathbb{Q}$;
(d) $\emptyset$.

**Lemma 1.24.** Let $S$ be a subset of an ordered field $\mathbb{F}$ and suppose both $b$ and $b'$ are upper bounds for $S$. If $b$ and $b'$ both have the property that if $c \in \mathbb{F}$ is an upper bound for $S$, then $c \geq b$ and $c \geq b'$, then $b = b'$.

**Proof.** With $c = b'$ it follows that $b' \geq b$. Likewise with $c = b$ it follows that $b \geq b'$. By trichotomy $b = b'$. 

DRAFT
Definition 1.25. The least upper bound for a subset \( S \) of an ordered field \( F \), if it exists, is a \( b \in F \) such that

(i) \( b \) is an upper bound for \( S \); and
(ii) if \( c \in F \) is an upper bound for \( S \), then \( c \geq b \).

\( \square \)

Remark 1.26. Lemma 1.24 justifies the use of the (as opposed to an) in describing the least upper bound.

The condition (ii) can be replaced with either of the following conditions

(a) if \( d < b \), then there exists an \( s \in S \) such that \( d < s \); or
(b) for each \( \epsilon > 0 \) there is an \( s \in S \) such that \( b - \epsilon < s \).

The notions of bounded below, lower bound and greatest lower bound are defined analogously. Least upper bound is often abbreviated lub. The term supremum, often abbreviated sup, is synonymous with lub. Likewise glb and inf for greatest lower bound and infimum. \( \square \)

Example 1.27. Here is a list of examples.

(i) The least upper bound of \( S = [0, 1) \subset \mathbb{R} \) is 1.
(ii) The least upper bound of \( V = [0, 1] \subset \mathbb{R} \) is also 1.
(iii) As a consequence of Theorem 1.32 below, the set \( \mathbb{N} \subset \mathbb{R} \) has no upper bound and thus no least upper bound;
(iv) Every real number is an upper bound for the set \( \emptyset \subset \mathbb{R} \). Thus \( \emptyset \) has no least upper bound. \( \square \)

Theorem 1.28. Every non-empty subset of \( \mathbb{R} \) that is bounded above has a least upper bound.

Remark 1.29. In fact, \( \mathbb{R} \) is essentially the unique ordered field satisfying the conclusion of Theorem 1.28. This property, which thus distinguishes \( \mathbb{R} \) from all other ordered fields, is a completeness property.

As can be deduced from Theorem 1.28, it is also true that a nonempty subset \( S \subset \mathbb{R} \) that is bounded below has a greatest lower bound. See Problem 1.4. \( \square \)

Example 1.30. Let \( F = \mathbb{Q} \) or \( \mathbb{R} \).

\[ S = \{ q \in \mathbb{Q} : q > 0, \quad q^2 < 2 \} \subset \mathbb{Q} \subset \mathbb{F}. \]

Observe that \( 1 \in S \) and hence \( S \) is nonempty. Next we show, if \( 0 < f \in \mathbb{F} \) and \( f^2 \geq 2 \), then \( f \) is an upper bound for \( S \). To this end, suppose \( s \in \mathbb{F} \) and \( s > f \), then \( s^2 > f^2 \) and hence \( s \notin S \). Thus \( f \) is an upper bound for \( S \). In particular, by choosing \( f = 2 \), it follows that 2 is an upper bound for \( S \) and hence \( S \) is bounded above (in \( \mathbb{F} \)).

Now suppose \( S \) has a least upper bound \( \alpha \in \mathbb{F} \). (In the case that \( \mathbb{F} = \mathbb{R} \) it does by Theorem 1.28.) We claim that \( \alpha = \sqrt{2} \); that is \( \alpha \geq 0 \) and \( \alpha^2 = 2 \). To prove this claim first note that \( \alpha \geq 1 > 0 \).

Suppose \( \mathbb{F} \ni \delta > 0 \) satisfies \( \alpha > \delta \). It follows that

\[ 0 < \alpha - \delta < \alpha < \alpha + \delta \]

and therefore,

\[ (\alpha - \delta)^2 < \alpha^2 < (\alpha + \delta)^2. \]
By the least property of \( \alpha \), there is an \( r \in S \) such that \( \alpha - \delta < r \). In particular, \((\alpha - \delta)^2 < r^2 < 2\). On the other hand \( \alpha + \delta \notin S \) since \( \alpha \) is an upper bound for \( S \). Hence
\[
(\alpha - \delta)^2 < 2 \leq (\alpha + \delta)^2.
\]

It now follows that both
\[
(\alpha - \delta)^2 - (\alpha + \delta)^2 < \alpha^2 - 2 < (\alpha + \delta)^2 - (\alpha - \delta)^2.
\]

Equivalently,
\[
|\alpha^2 - 2| < (\alpha + \delta)^2 - (\alpha - \delta)^2 = 4\alpha\delta.
\]

Arguing by contradiction, suppose \( \alpha^2 - 2 = f \neq 0 \). By Theorem 1.32(ii), there is an \( n \in \mathbb{N}^+ \) such that
\[
0 < \delta = \frac{|f|}{4\alpha n} < \alpha.
\]

But then, \( |f| < \frac{|f|}{n} \), a contradiction. Hence \( \alpha^2 = 2 \).

Since there is no \( \alpha \in \mathbb{Q} \) such that \( \alpha > 0 \) and \( \alpha^2 = 2 \), it follows that \( S \) does not have a lub in \( \mathbb{Q} \). Thus \( \mathbb{Q} \) does not have the lub property (every nonempty set that is bounded above has a lub). On the other hand, since \( \mathbb{R} \) has the lub property, there is an \( \alpha \in \mathbb{R} \) such that \( \alpha > 0 \) and \( \alpha^2 = 2 \).

The argument used in Example 1.30 above can be generalized to prove the following proposition.

**Proposition 1.31.** If \( y \in \mathbb{R}, \ y > 0 \) and \( n \in \mathbb{N}^+ \), then there is a unique positive real number \( s \) such that \( s^n = y \).

**Theorem 1.32 (Archemedian properties of \( \mathbb{R} \)).** Suppose \( x, y \in \mathbb{R} \).

(i) There is a natural number \( n \) so that \( n > x \) (equivalently \( \mathbb{N} \) is not bounded above);

(ii) If \( y > 0 \), then there is an \( n \in \mathbb{N}^+ \) such that \( \frac{1}{n} < y \); and

(iii) If \( x < y \), then there is a \( q \in \mathbb{Q} \) such that \( x < q < y \).

**Remark 1.33.** Item (iii) of Theorem 1.32 is sometimes expressed as saying \( \mathbb{Q} \) is dense in \( \mathbb{R} \).

**Proof.** To prove item (i), arguing by contradiction, suppose no such natural number exists. In that case \( x \) is an upper bound for \( \mathbb{N} \). Since also \( 0 \in \mathbb{N} \), it follows that \( \mathbb{N} \) has a lub \( \alpha \) by Theorem 1.28. If \( n \in \mathbb{N} \), then \( n + 1 \leq \alpha \) and thus \( n \leq \alpha - 1 \) for all \( n \in \mathbb{N} \). Thus, \( \alpha - 1 \) is an upper bound for \( \mathbb{N} \), contradicting the least property of \( \alpha \). Hence \( \mathbb{N} \) is not bounded above and there is an \( n > x \), proving item (i).

Item (ii) follows by applying (i) to \( x = \frac{1}{y} \).

To prove item (iii), choose \( n \in \mathbb{N}^+ \) so that \( 1 < n(y - x) \). Choose \( m \in \mathbb{Z} \) so that
\[
m - 1 \leq nx < m.
\]

(Here the well ordering property of \( \mathbb{N} \) is used in the following form. If \( J \subset \mathbb{Z} \) is nonempty and bounded below, then \( J \) has a smallest element.) Rearranging the inequalities gives,
\[
nx < m < nx + 1 < ny.
\]

Hence \( x < \frac{m}{n} < y \).

□
Example 1.34. Suppose $0 < a < 1$. Show the set $A = \{a^n : n \in \mathbb{N}\}$ is bounded below and its infimum is 0.

Solution. Since $a \geq 0$ each $a^n \geq 0$. Thus $A$ is bounded below by 0. The set $A$ is not empty. It follows that $A$ has an infimum (see Problem 1.4). Let $\alpha = \inf(A)$ and note $\alpha \geq 0$, since 0 is a lower bound and $\alpha$ is the glb. Since $\alpha \leq a^n$ for $n = 0, 1, 2, \ldots$, $\alpha \leq a^{n+1}$ for $n \in \mathbb{N}$ and therefore $\frac{\alpha}{a} \leq a^n$ for $n \in \mathbb{N}$. Thus, $\frac{\alpha}{a}$ is a lower bound for $A$. It follows that $\frac{\alpha}{a} \leq \alpha$ as $\alpha$ is the glb. Since $a < 1$ and $\alpha \geq 0$, $\alpha = 0$. \qed

1.6. Accumulation points and the Bolzano-Weierstrass Theorem.

Definition 1.35. Let $S$ be a given subset of $\mathbb{R}$. A point $a \in \mathbb{R}$ is an accumulation point (synonymously limit point) of $S$ if for each $\epsilon > 0$ there is an $s \in S$ such that $0 < |a - s| < \epsilon$ (equivalently, $s \neq a$ and $|a - s| < \epsilon$).

Example 1.36. Show 0 is an accumulation point of both the sets $S = \{\frac{1}{n} : n \in \mathbb{N}^+\} \subset \mathbb{R}$ and $T = S \cup \{0\}$. (Hence an accumulation point of a set may, or may not, be in the set.)

Solution. To show that 0 is an accumulation point of $S$, let $\epsilon > 0$ be given. By Theorem 1.32(ii), there is a positive integer $n$ such that $\frac{1}{n} < \epsilon$. Hence $0 \neq \frac{1}{n} \in S$ and $|0 - \frac{1}{n}| < \epsilon$. The proof that 0 is an accumulation point of $T$ is essentially the same. It is also a consequence of Lemma 1.37(ii) below.

Given a set $X$ and subsets $A, B \subset S$, let $A \setminus B = \{x \in X : x \in A, x \notin B\}$. It is the the set difference.

Lemma 1.37. Suppose $S, T$ are subsets of $\mathbb{R}$ and $a \in \mathbb{R}$.

(i) If $a$ is an accumulation point of $S$, then $a$ is an accumulation point of $S \setminus \{a\}$.
(ii) If $S \subset T$ and $a$ is an accumulation point of $S$, then $a$ is an accumulation point of $T$.
(iii) The point $a$ is an accumulation point of $S$ if and only if for every $\epsilon > 0$ the set $(a - \epsilon, a + \epsilon) \cap S$ is infinite.

Proof. The proofs of items (i) and (ii) are routine and left to the gentle reader. To prove item (iii), first suppose that $\epsilon > 0$ the set $(a - \epsilon, a + \epsilon) \cap S$ is infinite. Thus there is a $t \neq a$ such that $t \in (a - \epsilon, a + \epsilon) \cap S$; that is $t \in S$ and $0 < |a - t| < \epsilon$. It follows that if for every $\epsilon > 0$ the set $(a - \epsilon, a + \epsilon) \cap S$ is infinite, then $a$ is an accumulation point of $S$. To prove the converse, we prove the contrapositive. Namely, if $a \in \mathbb{R}$ and there is an $\delta > 0$ such that $T = (a - \delta, a + \delta) \cap S$ is finite, then $a$ is not an accumulation point of $S$. Indeed, in this case, $\{|a - t| < \delta : t \in T, t \neq a\}$ is a finite set of positive numbers. Hence it has a smallest element $\epsilon$. In particular, there is no $s \in S$ such that $s \neq a$ and $|s - a| < \epsilon$ and so $a$ is not a limit point of $S$. \qed

Example 1.38. (i) If $F \subset \mathbb{R}$ is finite, then $F$ has no accumulation points.
(ii) The set of accumulation points of $S = \{\frac{1}{n} : n \in \mathbb{N}^+\}$ is exactly $\{0\}$.
(iii) The set $\mathbb{N}$ has no accumulation points.

Solution. Given $x \in \mathbb{R}$, let $J_x = \{n \in \mathbb{N} : x < n\}$.

By Theorem 1.32(i), $J_x \neq \emptyset$. By the well ordering property of $\mathbb{N}$, $J_x$ has a smallest element, say $m$. In particular, $m - 1 \leq x < m$. Choosing $1 > \epsilon = m - x > 0$, it follows that

$$(x - \epsilon, x + \epsilon) \cap \mathbb{N} \subset (m - 2, m) \cap \mathbb{N} = \{m - 1\}.$$
Hence \( x \) is not an accumulation point of \( N \) by Lemma 1.37(iii). □

(iv) The set of accumulation points of the set \((0, 1)\) is the set \([0, 1]\).

(v) The set of accumulation points of \( \mathbb{Q} \) is \( \mathbb{R} \), a fact that is equivalent to the statement that between any two real numbers there is a rational, and often expressed by saying the rationals are dense in the real numbers. See Theorem 1.32.

A subset \( S \) of \( \mathbb{R} \) is bounded if it is bounded above and below. Equivalently, \( S \) is bounded if there is a \( C \in \mathbb{R} \) such that \( S \subset [-C, C] \).

**Theorem 1.39** (Bolzano-Weierstrass). **If** \( S \) **is an infinite and bounded subset of** \( \mathbb{R} \), **then** \( S \) **has an accumulation point.**

**Proof.** Since \( S \) is bounded, there exists a \( C > 0 \) such that \( S \subset [-C, C] \). Let
\[
T = \{ r \in \mathbb{R} : S \cap (-\infty, r] \text{ is finite} \}.
\]
Note that \(-C \in T\) since \( S \cap (-\infty, -C] \subset \{-C\} \). Thus \( T \) is nonempty. On the other hand if \( r \geq C \), then \( S \cap (-\infty, r] = S \) and thus, since \( S \) is infinite, \( r \notin T \). Hence \( T \subset (-\infty, C) \). In particular, \( C \) is an upper bound for \( T \). Hence \( T \) has a least upper bound \( \alpha \).

Let \( \beta < \alpha \) be given. By the least property of \( \alpha \), there is a \( \beta < \gamma < \alpha \) such that \( \gamma \in T \). (See Remark 1.26(a).) In particular, \( S \cap (-\infty, \gamma] \) is a finite set and thus so is \( S \cap (-\infty, \beta] \). On the other hand, if \( \delta > \alpha \), then \( \delta \notin T \) as \( \alpha \) is an upper bound for \( T \). In particular, \( S \cap (-\infty, \delta] \) is an infinite set. It follows that \( S \cap [\beta, \delta] \) is an infinite set and hence so is \( S \cap (-\infty, \beta] \). Summarizing: if \( \beta < \alpha \), then \( S \cap (\beta, \delta) \) is an infinite set. Given \( \epsilon > 0 \), let \( \beta = \alpha - \epsilon \) and \( \delta = \alpha + \epsilon \) and note \( S \cap (\alpha - \epsilon, \alpha + \epsilon) = S \cap (\beta, \delta) \) is infinite. Hence \( \alpha \) is an accumulation point of \( S \) by Lemma 1.37(iii) and the proof is complete. □

1.7. **The Cauchy-Schwarz and triangle inequalities.** \(^2\) Let \( \mathbb{R}^d \) denote the set of matrices of size \( d \times 1 \). Thus an element of \( a \in \mathbb{R}^d \) has the form
\[
a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix},
\]
for real numbers \( a_1, \ldots, a_d \).

Given \( a, b \in \mathbb{R}^d \), define the inner product of \( a \) and \( b \) by
\[
\langle a, b \rangle = \sum a_j b_j.
\]
The inner product is also called the dot product and scalar product. The norm of \( a \in \mathbb{R}^d \) is
\[
\|a\| = \sqrt{\langle a, a \rangle}.
\]
The interpretation of the dot product and norm should be familiar in the cases \( d = 2 \) and \( d = 3 \).

**Proposition 1.40** (Cauchy-Schwarz inequality). **Given** \( a, b \in \mathbb{R}^d \),
\[
|\langle a, b \rangle| \leq \|a\| \|b\|.
\]

\(^2\)This subsection is not used elsewhere and hence optional.
Proof. Consider, for \( t \in \mathbb{R} \),
\[
0 \leq \|a + tb\|^2 = \|a\|^2 + 2t\langle a, b \rangle + t^2\|b\|^2.
\]
It follows that the discriminant of this polynomial is non-positive; i.e.,
\[
|\langle a, b \rangle|^2 - \|a\|^2 \|b\|^2 \leq 0.
\]

**Proposition 1.41.** If \( a, b \in \mathbb{R}^d \), then
\[
\|a + b\| \leq \|a\| + \|b\|.
\]

Proof. From
\[
\|a + b\|^2 = \|a\|^2 + 2\langle a, b \rangle + \|b\|^2
\]
and Proposition 1.40 it follows that
\[
\|a + b\|^2 \leq \|a\|^2 + 2\|a\| \|b\| + \|b\|^2 = (\|a\| + \|b\|)^2.
\]

1.8. **Problems.**

**Problem 1.1.** Let \( \mathbb{Z}_3 = (\{0, 1, 2\}, +, \cdot) \) where
\[
x + y = x + y \mod 3 \quad \text{and} \quad xy = xy \mod 3.
\]
It is easy, but tedious, to verify that \( \mathbb{Z}_3 \) is a field with additive identity 0 and multiplicative identity 1.

Find the additive inverse for 1 and multiplicative inverse for 2.

Show there is no order \(<\) on \( \{0, 1, 2\} \) such that \((\mathbb{Z}_3, <)\) is an ordered field. (Suggestion: Arguing by contradiction, show the additive inverse for 1 would have to be both positive and negative.)

**Problem 1.2.** See the wikipedia page on the field of *complex numbers* \( \mathbb{C} \). Suppose \( z \in \mathbb{C} \) is not zero.

(a) Show if \( z = x + iy \) is the rectangular representation of \( z \), then
\[
z^{-1} = \frac{\overline{z}}{|z|^2},
\]
where \( |z|^2 = x^2 + y^2 \).

(b) Show, if \( z = r(\cos(\theta) + i\sin(\theta)) \) is the polar representation of \( z \), then
\[
z^{-1} = \frac{1}{r}(\cos(\theta) - i\sin(\theta)).
\]
Interpret geometrically.

**Problem 1.3.** Show there is no order \(<\) on \( \mathbb{C} \) such that \((\mathbb{C}, <)\) is an ordered field. (Suggestion: Consider \( i^2 \).)

**Problem 1.4.** The *greatest lower bound* (glb) or *infimum* (inf) is defined by simply reversing the inequalities in the definition of least upper bound.

Prove, using Theorem 1.28, that if \( S \subset \mathbb{R} \) is nonempty and bounded below, then \( S \) has a unique greatest lower bound.
Problem 1.5. Let

\[ S = \left\{ \frac{1}{n} : n \in \mathbb{N}^+ \right\} \subset \mathbb{R}. \]

Show \( S \) has a greatest lower bound and \( \text{glb}(S) = 0. \)

Problem 1.6. Show, if \( S \subset \mathbb{R} \) is nonempty and bounded, then \( \text{inf}(S) \leq \sup(S). \)

Problem 1.7. Suppose \( S \subset T \subset \mathbb{R} \). Show, if \( T \) is bounded above and \( S \) is non-empty, then both \( S \) and \( T \) have least upper bounds and moreover,

\[ \sup(S) \leq \sup(T). \]

Problem 1.8. Suppose \( S \subset \mathbb{R} \) is non-empty and bounded above (and hence has a least upper bound). Given \( a \in \mathbb{R} \), let

\[ T = a + S := \{ a + s : s \in S \}. \]

Prove that \( T \) is non-empty and bounded above and moreover,

\[ \sup(T) = a + \sup(S). \]

Problem 1.9. Show, if \( S \) and \( T \) are both nonempty and bounded above, then so is

\[ S + T = \{ s + t : s \in S, \ t \in T \} \]

and moreover,

\[ \sup(S + T) = \sup(S) + \sup(T). \]

[Suggestion. Given \( s \in S \), note that \( \sup(s + T) \leq \sup(S + T) \). On the other hand, by the previous problem, \( \sup(s + T) = s + \sup(T) \). Proceed.]

Problem 1.10. Given a positive real number \( y \) and positive integers \( m \) and \( n \), show

\[ (y^{\frac{1}{n}})^m = (y^m)^{\frac{1}{n}}. \]

Thus, \( y^{\frac{m}{n}} \) is unambiguously defined.

Problem 1.11. Verify the claims in Example 1.38.

Problem 1.12. Give an example of a set with exactly two accumulation points.

Problem 1.13. Let \( S' \) denote the set of accumulation points of a subset \( S \) of \( \mathbb{R} \). Show, \( (S')' \subset S' \).

Use the set \( S \) from item (ii) of Example 1.38 to show that inclusion can be proper.

Show, if \( \mathbb{Q} \subset S' \), then \( S' = \mathbb{R} \).

As a challenge question: What about \( S'''' = ((S')')' \)?

Problem 1.14. Show, if \( a, b \) are non-negative real numbers, then

(i) \( a < b \) if and only if \( a^2 < b^2 \); and

(ii) \( ab \leq \frac{a^2 + b^2}{2} \).

Problem 1.15 (optional). Show, if \( a, b \in \mathbb{R}^d \), then \( ||a|| - ||b|| \leq ||a - b|| \).

Problem 1.16. Make a guess at the set of limit points for the set

\[ S = \left\{ \frac{1}{n} + \frac{1}{m} : m, n \in \mathbb{N}^+ \right\}. \]

Why do you believe your guess is correct? As a challenge, prove that your guess is correct.

Problem 1.17. Suppose \( S \subset \mathbb{R} \). Show if \( b \in \mathbb{R} \) is both an upper bound and accumulation point for \( S \), then \( S \) has a lub and \( \text{lub}(S) = b. \)
2. Sequences

**Definition 2.1.** A sequence is a function $a$ whose domain is $\mathbb{N}$ or more generally a set of the form $\{n \in \mathbb{Z} : n \geq k\}$ for some $k \in \mathbb{Z}$. It is commonly denoted as $(a_n) = (a_n)_{n=k}^{\infty}$ where $a_n = a(n)$ is the value of $a$ at $n$. In these notes, generally $a$ is assumed to take real values so that each $a_n \in \mathbb{R}$.

**Example 2.2.** Here are a few examples of sequences.

(i) $(a_n = \frac{1}{n})_{n=1}^{\infty}$;
(ii) $(a_n = (-1)^n)$;
(iii) $(a_n = n)$.

2.1. Limits.

**Definition 2.3.** Suppose $A \in \mathbb{R}$ and $(a_n)$ is a sequence of real numbers. The sequence $(a_n)$ converges to $A$ if for every $\epsilon > 0$ there is an $N$ such that if $n \geq N$, then $|a_n - A| < \epsilon$. The notations $(a_n) \to A$ and

$$
\lim_{n \to \infty} a_n = \lim_{n \to \infty} a_n = A
$$

are short hand for the statement $(a_n)$ converges to $A$.

The sequence $(a_n)$ converges if there is an $A \in \mathbb{R}$ such that $(a_n)$ converges to $A$. Otherwise, the sequence diverges.

**Proposition 2.4.** If $(a_n)$ converges to both $A$ and $B$, then $A = B$.

**Definition 2.5.** Suppose $(a_n)$ is a sequence from $\mathbb{R}$ and $A \in \mathbb{R}$. If $(a_n)$ converges to $A$, then $A$ is the limit of the sequence.

**Example 2.6.** Show that $(\frac{1}{n})$ converges to 0.

**Solution.** Given $\epsilon > 0$ choose, by the Archimedean property of $\mathbb{R}$ (Theorem 1.32), an $N \in \mathbb{N}^+$ such that $\frac{1}{N} < \epsilon$. Now, if $n \geq N$, then,

$$
|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.
$$

\[ \square \]

**Example 2.7.** Show that the sequence $(a_n)$ defined by

$$
a_n = \frac{n}{n + 2}
$$

converges to 1.

**Solution.** Given $\epsilon > 0$ choose, by the Archimedean property, $N \in \mathbb{N}^+$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. Now, if $n \geq N$, then

$$
|\frac{n}{n + 2} - 1| = \frac{2}{n + 2} \leq \frac{2}{N} < \epsilon. \ \square
$$

**Example 2.8.** Show that the sequence $(b_n)$ defined by

$$
b_n = \frac{n^2 + 2}{2n^3 - n - 2}
$$

converges to 0.
Solution. Given \( \epsilon > 0 \) choose, by the Archimedean property, \( N \) so that \( N \geq \max\{2, \frac{2}{\epsilon}\} \). Now, if \( n \geq N \), then
\[
|b_n - 0| = \frac{n^2 + 2}{2n^3 - n - 2} \leq \frac{2n^2}{2n^3 - n - 2} \leq \frac{2n^2}{n^3} \leq \frac{2}{N} < \epsilon.
\]
Here we have used, for \( n \geq 2 \),
\[
2n^3 - n - 2 = n^3 + (n^3 - n - 2) \geq n^3 + (n^2 - n - 2) = n^3 + (n - 2)(n + 1) \geq n^3.
\]

Example 2.9. Fix \( 0 \leq a < 1 \) and let \( a_n = a^n \) (for \( n \geq 0 \)). Show the sequence \((a_n)\) converges to 0.

Solution. Recall, from Example 1.34, that the greatest lower bound of the set \( A = \{a^n : n \in \mathbb{N}\} \) is 0. In particular, given \( \epsilon > 0 \), there is a \( b \in A \) such that \( 0 \leq b < \epsilon \). There is an \( N \) such that \( b = a^N \). If \( n \geq N \), then \( 0 \leq a^n \leq a^N = b < \epsilon \). Hence, if \( n \geq N \), then \( |0 - a^n| < \epsilon \) and thus \((a^n)\) converges to 0.

Another proof that \((a^n)\) converges to 0 is given in Example 2.20.

Here is a list of simple properties of limits.

**Proposition 2.10.** Let \((a_n)\) be a sequence from \( \mathbb{R} \) and suppose \( A \in \mathbb{R} \).

(a) The sequence \((a_n)\) converges if and only if for each \( \ell > k \) the sequence \((a_n)\) converges;
(b) if there is an \( M \in \mathbb{N} \) and a \( c \in \mathbb{R} \) such that \( a_n = c \) for \( n \geq M \), then the sequence \((a_n)\) converges to \( c \);
(c) if there is an \( N \) and an \( \ell \) such that for \( n \geq N \), \( b_n = a_{n+\ell} \), then \((a_n)\) converges if and only if \((b_n)\) converges and in that case they converge to the same value;
(d) if \((a_n)\) converges to \( A \) and \( \epsilon \in \mathbb{R} \), then \((ca_n)\) converges to \( cA \);
(e) The sequence \((a_n)\) converges to \( A \) if and only if the sequence \((a_n - A)\) converges to 0.

### 2.2. Cauchy Sequences.

**Definition 2.11.** A sequence \((a_n)\) from \( \mathbb{R} \) is Cauchy if for every \( \epsilon > 0 \) there is an \( N \) so that if \( m, n \geq N \), then \( |a_n - a_m| < \epsilon \).

**Proposition 2.12.** If \((a_n)\) converges, then \((a_n)\) is Cauchy.

**Proof.** Let \( A \) denote the limit of the sequence \((a_n)\). Let \( \epsilon > 0 \) be given. There is an \( N \) so that if \( n \geq N \), then \( |a_n - A| < \frac{\epsilon}{2} \). Hence, if both \( m, n \geq N \), then
\[
|a_n - a_m| \leq |a_n - A| + |A - a_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
Example 2.13. The sequence \((-1)^n\) diverges.

Solution. Using the contra-positive of Proposition 2.12 it suffices to show there exists an \(\epsilon_0 > 0\) such that for every \(N\) there exists \(m, n \geq N\) such that \(|a_n - a_m| \geq \epsilon_0\) (with \(a_n = (-1)^n\)).

Choose \(\epsilon_0 = 1\). Given \(N\), let \(n = N\) and \(m = N + 1\). Since \(m, n\) have different parities, \(|(-1)^n - (-1)^m| = 2 \geq \epsilon_0 = 1\). Note that this argument shows if \(a \leq -1\), then \((a^n)\) does not converge.

\(\square\)

Definition 2.14. A sequence \((a_n)\) is bounded above if the set \(S = \{a_n : n \in \mathbb{N}\}\) is bounded above; that is, if there is an \(M\) so that \(a_n \leq M\) for all \(n\). It is bounded if it is bounded above and below; i.e., if there is an \(M\) so that \(|a_n| \leq M\) for all \(n\).

Proposition 2.15. If \((a_n)\) is Cauchy, then \((a_n)\) is bounded. In particular, convergent sequences are bounded.

The proof does not use anywhere near the full strength of the Cauchy condition.

Proof. With \(\epsilon = 1\) there is an \(N\) so that if \(n, m \geq N\), then \(|a_n - a_m| < 1\). Hence, \(|a_n - a_N| < 1\) for all \(n \geq N\) and thus \(|a_n| \leq |a_N| + 1\) for \(n \geq N\). There is a \(C\) such that \(|a_n| \leq C\) for all \(n \leq N - 1\). Hence, \(|a_n| \leq |a_N| + 1 + C\) for all \(n\).

\(\square\)

Example 2.16. The sequence \((a_n = n)\) diverges, since the set \(\mathbb{N}\) is not bounded above by Theorem 1.32(i).

Theorem 2.17. If \((a_n)\) is Cauchy, then \((a_n)\) converges.

Proof. Let \(S\) denote the range of the sequence \((a_n)_{n=0}^{\infty}\). Thus \(S = \{a_n : n \in \mathbb{N}\}\). By Proposition 2.15, the set \(S\) is bounded. If \(S\) is finite, then there is an \(A\) such that \(a_n = A\) for infinitely many \(n\). In particular, the set \(I = \{m : a_m = A\}\) is infinite. To prove \((a_n)\) converges to \(A\), let \(\epsilon > 0\) be given. There is an \(N\) such that for \(m, n \geq N\), \(|a_n - a_m| < \epsilon\). Thus, if \(n \geq N\), then, choosing \(m \in I\) and \(m \geq N\), it follows that \(|A - a_n| = |a_m - a_n| < \epsilon\).

Now suppose \(S\) is infinite. Then, by Theorem 1.39, \(S\) has an accumulation point \(A\). To prove that \((a_n)\) converges to \(A\), let \(\epsilon > 0\) be given. There is an \(N\) such that if \(m, n \geq N\), then \(|a_n - a_m| < \frac{1}{2}\epsilon\). On the other hand, the set \((A - \epsilon, A + \epsilon) \cap S\) is infinite by Lemma 1.37 item (iii), so there is an \(m \geq N\) such that \(a_m \in (A - \epsilon, A + \epsilon)\); that is, \(|A - a_m| < \frac{1}{2}\epsilon\). Hence, if \(n \geq N\), then

\(|A - a_n| \leq |A - a_m| + |a_m - a_n| < \epsilon.\)

\(\square\)

Combining Theorem 2.17 and Proposition 2.15 shows, if \((a_n)\) converges, then \((a_n)\) is bounded.

2.3. Monotone Sequences.

Definition 2.18. A sequence \((a_n)\) from \(\mathbb{R}\) is increasing if \(a_{n+1} \geq a_n\) for all \(n\) and it is strictly increasing if \(a_{n+1} > a_n\) for all \(n\). The notions of decreasing and strictly decreasing are defined analogously. A monotone sequence is one that is either increasing or decreasing.

Similarly, a sequence \((a_n)_{k}^\infty\) is eventually monotone if there is an \(M\) such that the sequence \((a_n)_{M}^\infty\) is monotone.
**Theorem 2.19.** If \( (a_n)_{n=M}^{\infty} \) is increasing and bounded above, then it converges with limit sup\( \{a_n : n \geq M\} \).

If \( (a_n)_{n=k}^{\infty} \) is eventually increasing and bounded above, then it converges.

**Proof.** The set \( S = \{a_n : n \geq M\} \) is non-empty and, by hypothesis, bounded above. Hence \( S \) has a supremum \( A \). To show that \( (a_n)_{n=M}^{\infty} \) converges to \( A \), let \( \epsilon > 0 \) be given. There is \( s \in S \) such that \( A - \epsilon < s \). There is an \( N \) so that \( s = a_N \). Now, if \( n \geq N \), then

\[
0 \leq A - a_n \leq A - a_N = A - s < \epsilon.
\]

By Proposition 2.10(d), the second part of the Theorem follows immediately from the first. \hfill \square

**Example 2.20.** Fix \( a \in [0, \infty) \). Show that the sequence \( (a_n = a^n) \) converges to 0 if \( 0 \leq a < 1 \); converges to 1 if \( a = 1 \) and does not converge if \( a > 1 \).

**Solution.** First suppose \( 0 \leq a < 1 \). Observe \( a_n - a_{n+1} = a^n(1 - a) \geq 0 \) since both \( a^n \) and \( 1 - a \) are nonnegative. Hence \( (a_n) \) is decreasing. Since \( a \geq 0 \), it follows that \( 0 \leq a_n = a^n \) for all \( n \). Hence \( (a_n) \) is decreasing and bounded below and therefore converges to inf\( (S) \) where \( S = \{a^n : n \in \mathbb{N}\} \) by Theorem 2.19. By Example 1.34, inf\( (S) \) = 0. (As an alternate proof, note that Theorem 2.19 implies \( (a_n) \) converges to some \( L \). Thus \( (a_{n+1}) \) converges to \( L \) too.)

On the other hand, \( (a_{n+1}) = (a a_n) \) also converges to \( a L \). Thus \( L = a L \) and since \( 0 \leq a < 1 \), it follows that \( L = 0 \).

Evidently, if \( a = 1 \), then the sequence \( (a_n = a^n = 1) \) converges to 1.

If \( a > 1 \), then, for \( n \geq 0 \) we have \( |a^{n+1} - a^n| = a^n(a - 1) \geq a - 1 > 0 \). Thus, in this case \( (a^n) \) is not Cauchy and therefore doesn’t converge by Proposition 2.12. \hfill \square

**Example 2.21.** Let \( a_1 = \sqrt{2} \) and define, recursively, \( a_{n+1} = \sqrt{2 + \sqrt{a_n}} \). The following induction argument shows that \( (a_n) \) is increasing (and simultaneously that \( a_n \geq 0 \) so that \( \sqrt{a_n} \) is defined for all \( n \)).

First, note that \( a_2 = \sqrt{2 + \sqrt{2}} \geq \sqrt{2} = a_1 \). Now suppose that \( a_n \geq a_{n-1} \). In this case, \( \sqrt{a_n} \geq \sqrt{a_{n-1}} \) and hence,

\[
a_{n+1} = \sqrt{2 + \sqrt{a_n}} \geq \sqrt{2 + \sqrt{a_{n-1}}} = a_n,
\]

and the induction argument is complete.

Next, note that \( a_1 \leq 2 \). Further, if \( a_n \leq 2 \), then

\[
a_{n+1} = \sqrt{a_n + 2} \leq \sqrt{2 + 2} = 2.
\]

Hence, by induction, \( a_n \leq 2 \) for all \( n \); that is \( (a_n) \) is bounded above by 2. Hence, by Theorem 2.19, the sequence \( (a_n) \) converges to some \( A \) and moreover \( a_1 = \sqrt{2} \leq A \leq 2 \). Later, after more machinery has been developed, we identify \( A \) as a solution to a quartic equation. (See example 2.30).

**Definition 2.22.** The sequence \( (a_n) \) diverges to \( \infty \) if for each \( C > 0 \) there is an \( N \) so that if \( n \geq N \), then \( a_n > C \).

**Example 2.23.** The sequence \( (a_n = n) \) diverges to \( \infty \).

Show the sequence \( (b_n = \sqrt{n}) \) diverges to \( \infty \).
Solution. Let $C > 0$ be given. Choose, by Theorem 1.32, an $N$ so that $N > C^2$. If $n \geq N$, then
\[
\sqrt{n} \geq \sqrt{N} > C. \quad \Box
\]

Example 2.24. Show the sequence $(a_n = \frac{n^2 - 1}{n + 2})_{n=0}^{\infty}$ diverges to $\infty$.

Solution. Observe, for $n \geq 2$, that $n^2 - 1 \geq \frac{1}{2}n^2$ and at the same time $n + 2 \leq 2n$. Thus, for $n \geq 2$,
\[
\frac{n^2 - 1}{n + 2} \geq \frac{1}{4} \frac{n^2}{n} = \frac{n}{4}.
\]
Given $C > 0$ choose $N$ such that $N > \max\{2, 4C\}$. With this choice of $N$, if $n \geq N$, then
\[
\frac{n^2 - 1}{n + 2} \geq \frac{n}{4} \geq \frac{N}{4} > C. \quad \Box
\]

Note, if $(a_n)$ is eventually increasing, then it is bounded if and only if it is bounded above.

Theorem 2.25. An eventually increasing sequence $(a_n)$ converges if and only if it is bounded above.

If $(a_n)$ is eventually increasing, but not bounded above (equivalently diverges), then $(a_n)$ diverges to $\infty$.

Thus, if $(a_n)$ is eventually increasing, then either $(a_n)$ converges or diverges to $\infty$ depending on whether it is bounded above or not.

Proof. By Proposition 2.10(c), we may (and do) assume $(a_n)$ is increasing.

That a bounded increasing sequence converges has already been established (Theorem 2.19). On the other hand, convergent sequences are bounded. Thus, assuming $(a_n)$ is increasing, convergence and boundedness are equivalent.

To prove the second statement, suppose $(a_n)$ is increasing, but not bounded and let $C > 0$ be given. Since $(a_n)$ is not bounded, there is an $N$ so that $a_N > C$. Now, if $n \geq N$, then $a_n \geq a_N > C$ and so $(a_n)$ diverges to $\infty$. \hfill \Box

2.3.1. At most countable and uncountable sets.

Definition 2.26. A set $A$ is at most countable if there is an onto mapping $f : \mathbb{N} \to A$. Otherwise $A$ is uncountable.

Observe that a set $A$ is countable if and only if $A$ is the range of a sequence.

Proposition. Here are some facts and examples about at most countable sets (that we will not prove).

(i) $\mathbb{N} \times \mathbb{N}$ is at most countable;
(ii) $\mathbb{Z}$ is at most countable;
(iii) If $A$ and $B$ are at most countable, then $A \times B$ is at most countable;
(iv) $\mathbb{Q}$ is at most countable.

Sketch. The picture below can be turned into a proof that $\mathbb{N} \times \mathbb{N}$ is countable. That is, that there is an onto map $F : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.
Define $G : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ by $F((m,n)) = n - m$. Evidently $G$ is onto. Thus $f = G \circ F : \mathbb{N} \to \mathbb{Z}$ is onto.

Assuming $A$ and $B$ are at most countable, there exist onto maps $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to A$. Define $h : \mathbb{N} \times \mathbb{N} \to A \times B$ by $h((m,n)) = (g(m), f(n))$. Since $h$ is onto so is $h \circ F$ and thus $A \times B$ is at most countable.

To prove that $\mathbb{Q}$ is at most countable, consider the map $R : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ defined by $R(m,n) = \frac{m}{n+1}$.

**Theorem 2.27.** The set $\mathbb{R}$ is uncountable.

**Proof.** It suffices to show if $f : \mathbb{N} \to \mathbb{R}$, then $f$ is not onto. For notational ease, let $x_j = f(j)$.

Choose $b_0 > a_0$ such that $x_0 \notin I_0 := [a_0, b_0]$. Next choose $a_1 < b_1$ such that $a_0 \leq a_1 < b_1 \leq b_0$ and $x_1 \notin I_1 = [a_1, b_1]$. Continuing in this fashion, construct, by the principle of recursion, a sequences $(a_j)_{j=0}^{\infty}$ and $(b_j)_{j=0}^{\infty}$ such that,

(i) $(a_j)_{j=0}^{\infty}$ is increasing and $(b_j)_{j=0}^{\infty}$ is decreasing;
(ii) $b_j - a_j > 0$ for each $j \in \mathbb{N}$ and
(iii) $x_m \notin I_m := [a_m, b_m]$ for $m \in \mathbb{N}$.

In particular, given $m \in \mathbb{N}$, the sequence $(a_j)$ is increasing and is bounded above by $b_m$. By Theorem 2.19 $(a_j)$ converges to

$y = \sup \{a_j : j \in \mathbb{N}\}$.

In particular, $a_m \leq y \leq b_m$. Thus, $y \in I_m$ for all $m \in \mathbb{N}$. On the other hand, for each $m \in \mathbb{N}$,

$x_m \notin I_m$.
and so \( y \neq x_m \). Hence \( y \) is not in the set \( \{ x_k : k \in \mathbb{N} \} \) which is the range of \( f \). Thus \( f \) is not onto.

Note that the proof of Theorem 2.27 actually shows, if \( a, b \in \mathbb{R} \) and \( a < b \), then the open interval \((a, b) \subset \mathbb{R}\) is uncountable.

2.4. Limit Theorems.

**Theorem 2.28.** Let \((a_n)\) and \((b_n)\) be sequences from \(\mathbb{R}\) that converges to \(A\) and \(B\) respectively.

(i) The sequence \((a_n + b_n)\) converges to \(A + B\);  
(ii) The sequence \((a_n b_n)\) converges to \(AB\);  
(iii) If \(b_n \neq 0\) for all \(n\) and \(B \neq 0\), then \((\frac{1}{b_n})\) converges to \(\frac{1}{B}\); and  
(iv) if there is a \(K\) so that \(a_n \leq b_n\) for all \(n \geq K\), then \(A \leq B\).

**Proof.** Item (a) is left as an exercise.

To prove item (b) first observe that \((a_n)\) is a bounded sequence (since it converges) and hence there is a \(C > 0\) such that \(|a_n| \leq C\) for all \(n\). Now, given \(\epsilon > 0\) there is an \(N_a\) such that if \(n \geq N_a\), then  
\[ |a_n - A| < \frac{\epsilon}{2(|B| + 1)}. \]

Similarly, there is a \(N_b\) such that, for \(n \geq N_b\),  
\[ |b_n - B| < \frac{\epsilon}{2C}. \]

Let \(N = \max\{N_a, N_b\}\). If \(n \geq N\), then  
\[ |a_n b_n - AB| \leq |a_n (b_n - B)| + |(a_n - A)B| \]
\[ \leq C|b_n - B| + |B||a_n - A| \]
\[ \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}. \]

To prove item (c), first note that, with \(\epsilon = \frac{|B|}{2}\), there is a \(K\) so that \(|b_n - B| \leq \frac{|B|}{2}\) for \(n \geq K\). Thus, \(|b_n| \geq \frac{|B|}{2}\) for \(n \geq K\). Now, given \(\epsilon > 0\) there is an \(M\) such that for \(n \geq M\),  
\[ |b_n - B| < \frac{\epsilon|B|^2}{2}. \]

Choose \(N = \max\{K, M\}\). If \(n \geq N\), then  
\[ \left| \frac{1}{b_n} - \frac{1}{B} \right| = \frac{|b_n - B|}{|b_nB|} \leq \frac{2|b_n - B|}{|B|^2} < \epsilon. \]

To prove item (iv), let \(\epsilon > 0\) be given. There exists \(N_a\) and \(N_b\) so that \(a_n > A - \epsilon \) and \(b_n < B - \epsilon\) for \(n \geq N_a\) and \(n \geq N_b\) respectively. Hence, for any \(m \geq \max\{N_a, N_b\}\)  
\[ A - \epsilon < a_m \leq b_m < B + \epsilon. \]

Hence \(A - B < 2\epsilon\). Since \(\epsilon > 0\) is arbitrary, it follows that \(A - B \leq 0\). □

**Proposition 2.29.** Suppose \((a_n)\) is a sequence of non-negative numbers and \(q \in \mathbb{Q}^{+}\) (\(q\) is a positive rational number). If \((a_n)\) converges to \(A\), then \((a_n^q)\) converges to \(A^q\).
The proof exploits the identity, for \( m \in \mathbb{N}^+ \),

\[
(1) \quad (y - x) \sum_{j=0}^{m-1} y^j x^{m-j-1} = y^m - x^m.
\]

**Sketch of proof.** For \( k \in \mathbb{N}^+ \), an induction argument based on Theorem 2.28 shows \((a_n^k)\) converges to \(A^k\).

Now fix \( m \in \mathbb{N}^+ \). Suppose \( A = 0 \); that is, \((a_n)\) converges to 0. In this case, to see \((a_n^m)\) converges to 0, given \( \epsilon > 0 \), note that there is an \( N \) such that

\[
0 \leq a_n < \epsilon^m
\]

for \( n \geq N \). Thus, for \( n \geq N \),

\[
|a_n^m| \leq \epsilon
\]

and we have shown \((a_n^m)\) converges to 0.

Now suppose \( A > 0 \). Replace \( y \) by \( A^\frac{1}{m} \) and \( x \) by \( a_n^\frac{1}{m} \) gives, using (1),

\[
|A - a_n| = |A^{\frac{1}{m}} - a_n^{\frac{1}{m}}| \left( \sum_{j=0}^{m-1} A^{\frac{m-j-1}{m}} a_n^{\frac{m-j-1}{m}} \right) \geq |A^{\frac{1}{m}} - a_n^{\frac{1}{m}}| A^{\frac{m-1}{m}}.
\]

From here it is easy to show that \((a_n^m)\) converges to \(A^\frac{1}{m}\). Finally, given the rational number \( q = \frac{k}{m} \), note that, from what has already been proved, \( b_n^k = a_n^\frac{1}{m} \) converges to \(B = A^k\). Thus, again by what has already been proved, \((b_n^k = (b_n^k)^{\frac{1}{m}})\) converges to \(B^\frac{m}{k}\) and the proof is complete. \(\square\)

**Example 2.30.** Recall that the sequence \((a_n)\) defined recursively in Example 2.21 converges to some \(A\) and \(\sqrt{2} = a_1 \leq A \leq 2\). By Proposition 2.29, \((\sqrt{a_n})\) converges to \(\sqrt{A}\). Hence, \((2 + \sqrt{a_n})\) converges to \((2 + \sqrt{A})\) by Theorem 2.28. Another application of Proposition 2.29 implies that

\[
\lim_{n \to \infty} \sqrt{2 + \sqrt{a_n}} = \sqrt{2 + \sqrt{A}}.
\]

Now, the recursive definition of \(a_{n+1}\) and Proposition 2.10, imply

\[
A = \sqrt{2 + \sqrt{A}}.
\]

Hence \(\sqrt{2} \leq A \leq 2\) is a solution to

\[
p(x) = x^4 - 4x^2 - x + 4 = 0.
\]

It is easy to see that 1 is a root of \(p\). From there, one shows that \(p\) has only one other real root (the remaining two being complex) and this root is the limit \(A\). Numerically \(A\) is approximately 1.83.

**Example 2.31.** For \( q \in \mathbb{Q}^+ \), we have

\[
\lim_{n \to \infty} \frac{1}{n^q} = 0,
\]

meaning, (for \( k \in \mathbb{N}^+ \)), the sequence \((\frac{1}{n^q})\) converges to 0.
Example 2.32. Recall Example 2.8 in which
\[ b_n = \frac{n^2 + 2}{2n^3 - n - 2}. \]
Show \((b_n)\) converges to 0.

Solution. Observe \(b_n = \frac{\frac{1}{n} + \frac{2}{n^3}}{2 - \frac{1}{n^2} - \frac{2}{n}}\).

Let \(c_n\) and \(d_n\) denote the numerator and denominator of \(b_n\). By Example (2.31), both sequences \((\frac{1}{n})\) and \((\frac{1}{n^2})\) converge to 0. Hence, by Theorem 2.28, \((c_n)\) converges to 0. Similar reasoning shows \((d_n)\) converges to 2. Hence another application of Theorem 2.28 shows \((b_n)\) converges to 0. □

Example 2.33. Let \((a_n)_{n=0}^\infty\) denote the sequence
\[ a_n = \frac{2n^2 + 5}{n^2 - 3n + 2}. \]
Show \((a_n)\) converges to 2.

Proposition 2.34. If \((b_n)\) diverges to \(\infty\) and if \(\frac{a_n}{b_n}\) converges to some \(L > 0\), then \((a_n)\) diverges to \(\infty\).

Proof. For notational ease, let \(c_n = \frac{a_n}{b_n}\). Let \(C > 0\) be given. Since \((c_n)\) converges to \(L\) and \(L > 0\), there is an \(M\) such that if \(n \geq M\), then \(|c_n - L| < \frac{L}{2}\) since \((c_n)\) converges to \(L > 0\). In particular, \(c_n > \frac{L}{2}\) for such \(n\). Since \((b_n)\) diverges to \(\infty\), there is a \(K\) such that if \(n \geq K\), then \(b_n > \frac{2C}{L}\). Letting \(N = \max\{M, K\}\), it now follows that if \(n \geq N\), then \(a_n = b_n c_n > C\). □

Example 2.35. Revisiting Example 2.21, show \((a_n) = \frac{n^2 - 1}{n+1}\) diverges to \(\infty\).

Let \(b_n = n\) and
\[ c_n := \frac{a_n}{b_n} = \frac{1 - \frac{n^2}{1 + \frac{n}{2}}}{1 + \frac{n}{2}}. \]

Using various limits theorems, \((c_n)\) converges to 1. Since \((b_n)\) diverges to \(\infty\) by Example 2.23, it follows that \((a_n)\) does too by Proposition 2.34.

This section closes with a couple of concrete limits.

Proposition 2.36. Fix a positive number \(c\). Both sequence \((c^\frac{1}{x})_{n=1}^\infty\) and \((n^\frac{1}{n})_{n=1}^\infty\) converge to 1.

Proof. For a real number \(x\), the Binomial Theorem gives,
\[ (1 + x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j. \]

For \(x > 0\) it follows that
\[ (1 + x)^n \geq 1 + nx. \]
Thus, if \(c > 1\) and \(x = c^n - 1\), then
\[ c - 1 \geq n(c^{\frac{1}{n}} - 1) > 0. \]
Dividing by \( n \) and using the fact that the sequence \( \left( \frac{1}{n} \right) \) converges to 0 shows \( (c^{\frac{1}{n}}) \) converges to 0.

If \( 0 < c < 1 \), then \( \frac{1}{c} > 1 \) and from what has already been proved, \( (\frac{1}{c})^{\frac{1}{n}} \) converges to 1. Hence, by Theorem 2.28, \( (c^{\frac{1}{n}}) \) converges to \( \frac{1}{c} = 1 \) too.

To prove the second part of the Proposition, note that the Binomial Theorem gives, for \( x > 0 \) and \( n \geq 2 \),

\[
(1 + x)^n \geq \frac{n(n-1)}{2}x^2.
\]

Thus, with \( x = n^{\frac{1}{n}} - 1 \),

\[
n \geq \frac{n(n-1)}{2}x^2.
\]

Hence, for \( n \geq 2 \),

\[
\frac{2}{n-1} \geq (n^{\frac{1}{n}} - 1)^2 \geq 0,
\]

from which it follows that \( (n^{\frac{1}{n}}) \) converges to 1. \( \Box \)

2.5. Super Cauchy sequences and the contraction principle.

**Definition 2.37.** A real sequence \( (a_n)_{n=k}^{\infty} \) is **super Cauchy** if there is a \( C \in \mathbb{R} \) such that

\[
\sum_{j=k}^{n} |a_{j+1} - a_j| \leq C
\]

for all \( n \).

**Lemma 2.38.** Suppose \( (a_n) \) is a sequence of real numbers. If \( (a_n) \) is super Cauchy, then \( (a_n) \) is Cauchy (and therefore converges).

**Proof.** The sequence, \( (s_n)_{k+1}^{\infty} \) defined by

\[
s_n = \sum_{j=k}^{n-1} |a_{j+1} - a_j|
\]

is increasing and bounded above by \( C \). Hence \( (s_n) \) is convergent by Theorem 2.19. In particular it is Cauchy by Proposition 2.12. Thus, given \( \epsilon > 0 \) there is an \( N \) so that if \( n, m \geq N \), then \( |s_m - s_n| < \epsilon \). Hence, for \( m \geq n \geq N \),

\[
|a_m - a_n| \leq \sum_{j=n}^{m-1} |a_{j+1} - a_j| = |s_m - s_n| < \epsilon.
\]

Thus \( (a_n) \) is Cauchy (and therefore converges by Theorem 2.17). \( \Box \)

It is easy to see that the sequence \( \left( (-1)^n \right)_{n=1}^{\infty} \) is Cauchy (it converges to 0), but not super Cauchy.

\(^3\)Non-standard terminology.
Proposition 2.39 (Contraction Principle). Suppose \((a_n)_{n=k}^\infty\) is a sequence from \(\mathbb{R}\). If there is a positive integer \(N\) and an \(0 \leq r < 1\) such that
\[
|a_{n+2} - a_{n+1}| \leq r|a_{n+1} - a_n|
\]
for all \(n \geq N\), then \((a_n)_{n=k}^\infty\) is super Cauchy and hence converges.

Proof. It can be assumed that equation (2) holds for all \(n\) and \(k = 0\). In that case, an induction argument shows,
\[
|a_{j+1} - a_j| \leq r^j|a_1 - a_0|
\]
Summing over \(j\) gives,
\[
\sum_{j=1}^n |a_{j+1} - a_j| \leq |a_1 - a_0| \sum_{j=1}^n r^j.
\]
On the other hand, by equation (1) (with \(y = 1\) and \(x = r\))
\[
\sum_{j=1}^n r^j = r \frac{1 - r^n}{1 - r} \leq \frac{r}{1 - r}.
\]
\(\Box\)

Example 2.40. Define a sequence of real numbers recursively as follows. Let \(a_1 = 1\) and
\[
a_{n+1} = 1 + \frac{1}{1 + a_n}.\]
Show \(a_n \geq 1\) for each \(n\) and \((a_n)\) is not eventually monotonic (that is neither increasing or decreasing), but does converge.

Solution. First note that \(a_1 \geq 1\). Now suppose \(n \in \mathbb{N}^+\) and \(a_n \geq 1\). It follows that \(\frac{1}{a_n} > 0\) and hence \(a_{n+1} = 1 + \frac{1}{1 + a_n} > 1\). Hence we have shown, by induction, that \(a_n \geq 1\) for all \(n \in \mathbb{N}^+\).

Next observe that
\[
a_{n+2} - a_{n+1} = \frac{(a_n - a_{n+1})(1 + a_{n+1})(1 + a_n)}{(1 + a_{n+1})(1 + a_n)}.
\]
An induction argument based the identity of equation (3) shows, for \(n \in \mathbb{N}^+\), that \(b_n = a_{n+1} - a_n \neq 0\) and \(a_n\) and \(a_{n+1}\) have opposite signs. Hence \((a_n)\) is not eventually monotonic.

From equation (3) and the fact that \(a_n \geq 1\) for all \(n\),
\[
|a_{n+2} - a_{n+1}| \leq \frac{1}{4}|a_{n+1} - a_n|.
\]
Hence, by Proposition 2.39, \((a_n)\) converges. It is now easy to see that it converges to \(\sqrt{2}\). \(\Box\)

2.6. Subsequences.

Definition 2.41. Suppose \((a_n)\) is a sequence from \(\mathbb{R}\). If \(n_1 < n_2 < \cdots\) is an increasing sequence of integers, then the sequence \((a_{n_j})_j\) is subsequence of \((a_n)\).

Example 2.42. For \((a_n) = (-1)^n\) both \((b_j = a_{2j} = 1)\) and \((c_j = a_{2j+1} = -1)\) are subsequences of \((a_n)\).

Similarly, choosing \(n_j = j^2\), the sequence \((\frac{1}{2^j})\) is a subsequence of \((\frac{1}{n})\).

Definition 2.43. A point \(A\) is a subsequential limit of the sequence \((a_n)\) if there is a subsequence \((a_{n_j})_j\) of \((a_n)_n\) that converges to \(A\).
Example 2.44. The points 1 and −1 are both subsequential limits of the sequence $((-1)^n)$.

Lemma 2.45. Suppose $(a_n)_{n=0}^\infty$ is a sequence from $\mathbb{R}$. If $A$ is an accumulation point of $S = \{a_n : n \in \mathbb{N}\}$, then $A$ is a subsequential limit of $(a_n)$.

Proof. The set $(A - 1, A + 1) \cap S$ is infinite by item (iii) of Lemma 1.37. Hence, there exists an $n_1$ such that $|a_{n_1} - A| < 1$. The set $S \cap (A - \frac{1}{2}, A + \frac{1}{2})$ is infinite. Hence, there is an $n_2 > n_1$ such that $|a_{n_2} - A| < \frac{1}{2}$. Continuing in this fashion (recursively), constructs $n_1 < n_2 < n_3 < \ldots$ such that $|a_{n_j} - A| < \frac{1}{j}$. Thus $(a_{n_j})$ is a subsequence of $(a_n)$ that converges to $A$. □

Theorem 2.46. A bounded sequence from $\mathbb{R}$ has a convergent subsequence.

Proof. Suppose $(a_n)_{n=k}^\infty$ is a bounded sequence of real numbers. Thus, there is a $C$ such that $|a_n| \leq C$ for all $n$. Let $S = \{a_n : n \geq k\}$ denote the range of the sequence. Suppose $S$ is infinite. In this case $S$ has an accumulation point $A$ by Theorem 1.39. By Lemma 2.45, a subsequence of $(a_n)$ converges to $A$; that is, $(a_n)$ has a convergent subsequence.

If $S$ is finite, then there is an $A$ such that $A = a_n$ for infinitely many $n$. It is an easy exercise, left to the reader, to show that there is a subsequence of $(a_n)$ that converges to $A$. □

Proposition 2.47. Suppose $(a_n)$ is a sequence from $\mathbb{R}$ and $A \in \mathbb{R}$. If $(a_n)$ converges to $A$ and $(a_{n_j})_j$ is a subsequence of $(a_n)_n$, then $(a_{n_j})$ converges to $A$.

Example 2.48. The sequence $((-1)^n)$ diverges.

2.6.1. The limits superior and inferior. 4

Definition 2.49. Suppose $(a_n)$ is a bounded sequence. Let, for $m \in \mathbb{N}$,

$$b_m = \sup\{a_n : n \geq m\}.$$ 

The sequence $(b_m)$ is decreasing and bounded below (by any lower bound for $(a_n)$). Hence $(b_m)$ converges to some $L$ which is called the limit superior or limsup of $(a_n)$ and is denoted by $\limsup a_n$ or $\lim a_n$.

The liminf, denoted $\lim$ or liminf, is defined analogously.

Example 2.50. Find the limsup of the sequence $(a_n = \sin(n \frac{\pi}{2}))$.

First observe that the range of the sequence is the bounded set $S = \{0, 1, -1\}$. Hence the sequence has both a limsup and a liminf. Further, given an $m$, the set $\{a_n : n \geq m\} = S$. Hence, in the notation above, $b_m = 1$ for all $m$. It follows that $\limsup a_n = \lim b_m = 1$. Similarly, $\liminf a_n = -1$.

The proofs Propositions 2.51 and 2.52 below are left to the interested reader.

Proposition 2.51. A sequence $(a_n)$ converges if and only if it is bounded and $\limsup a_n = \liminf a_n$.

The following proposition says that limsup $a_n$ is the largest subsequential limit of the sequence $(a_n)$ (and in particular asserts that a largest exists). This gives another rationale for the name limsup.

4This section is optional.
Proposition 2.52. Suppose \((a_n)\) is bounded. There is a subsequence \((a_{n_j})\) of \((a_n)\) that converges to \(\limsup a_n\). Moreover, if \((a_{n_k})\) is any convergent subsequence, then \(\lim k a_{n_k} \leq \limsup a_n\).

Example 2.53. Find the limsup of the sequence \(a_n = (-1)^n(1 + \frac{1}{n})\).

Let \(c_n = 1 + \frac{1}{n}\) and observe that \(c_n\) converges to 1. Suppose \(A\) is a subsequential limit of \((a_n)\). Hence, there is a subsequence \((a_{n_j})\) of \((a_n)\) that converges to \(A\). In this case \(a_{n_j} \leq |a_{n_j}| = (1 + \frac{1}{n_j}) = c_{n_j}\). It follows that \(A = \lim j a_{n_j} \leq \lim j c_{n_j} = 1\). On the other hand, the subsequence \(a_{2n} = 1 + \frac{1}{2n}\) of \((a_n)\) converges to 1. Thus, 1 is a subsequential limit of \((a_n)\). Hence \(1 = \limsup a_n\).

2.7. Problems.

Problem 2.1. Let \(a_n = \frac{n-2}{2n+3}\). Show, directly from the definition of limit, that \((a_n)\) converges to \(\frac{1}{2}\).

Problem 2.2. Let \(a_n = \frac{2n^2 - n + 1}{n^2 + n + 3}\). Show \((a_n)\) converges.

Problem 2.3. Let \(b_n = \frac{n+3}{n^2 + n + 3}\). Show \((b_n)\) converges to 0.

Problem 2.4. For \(n \geq 2\), let \(b_n = \frac{n+3}{n^2 - n - 1}\). Prove, directly from the definition of limit, that \((b_n)\) converges.

Problem 2.5. In Problem 2.4, rewrite

\[ b_n = \frac{\frac{1}{n} - \frac{2}{n^2}}{1 - \frac{1}{n} - \frac{1}{n^2}} \]

and use both known limits and limit theorems to show \((b_n)\) converges to 0. Carry out a similar program with Problem 2.1.

Problem 2.6. Let \(a_0 = 1\) and define, recursively, \(a_{n+1} = \sqrt{2 + a_n}\). Prove, by induction, that \(a_n \leq 2\) for all \(n\) and that \((a_n)\) is increasing. Conclude that \((a_n)\) converges. Identify the limit.

Problem 2.7. Fix \(r > 1\). Let \(a_1 = 1\) and define recursively,

\[ a_{n+1} = \frac{1}{r}(a_n + r + 1). \]

Show, that \((a_n)\) is increasing. Show by induction that \((a_n)\) is bounded above by \(\frac{r+1}{r-1}\). Does the sequence converge? If so, identify the limit.

Problem 2.8. Fix \(a > 1\). Show that the sequence \((a^n)\) diverges to \(\infty\). (Suggestion: use the fact that \((\frac{1}{a})^n\) converges to 0.)

Problem 2.9. Suppose \((a_n)\) and \((b_n)\) are sequences from \(\mathbb{R}\). Show, if \((a_n)\) converges to 0 and \((b_n)\) is bounded, then \((a_n b_n)\) converges to 0.

Problem 2.10. Let \(a_n = \sin(\frac{3\pi}{n})\). Show \((a_n)\) is not Cauchy. Conclude that \((a_n)\) doesn’t converge.

Problem 2.11. Show that \(a\) is an accumulation point of a set \(D\) if and only if there is a sequence \((a_n)\) from \(D \setminus \{a\}\) that converges to \(a\). Perhaps this result explains the reason that limit point is a synonym for accumulation point.
Problem 2.12. Let \( F_0 = 0 \) and \( F_1 = 1 \) and define, recursively,
\[
F_{n+1} = F_n + F_{n-1}
\]
(the Fibonacci sequence). Let \( a_n = \frac{F_{n+1}}{F_n} \). Is the sequence \((a_n)\) monotone?
Show \( a_{n+1}a_n \geq 2 \). Show,
\[
\left| a_{n+1} - a_n \right| = \frac{|a_{n-1} - a_n|}{a_na_{n-1}}.
\]
Conclude that \((a_n)\) converges. Identify the limit.

Problem 2.13. Let \( a_0 = 1 \) and define, recursively,
\[
a_{n+1} = \frac{a_n + 2}{a_n}.
\]
Is the sequence \((a_n)\) monotone?

Show, by induction, that

(i) \( a_n^2 \geq 1 \) for all \( n \);
(ii) \( a_n^2 \leq 3 \); and
(iii) \( \frac{5}{2} \geq a_{n+1}a_n \geq \frac{3}{2} \).

Conclude that
\[
\left| \frac{1}{2} - \frac{1}{a_n} \right| \leq \frac{1}{2}.
\]
Show
\[
|a_{n+1} - a_n| \leq \frac{1}{2} |a_n - a_{n-1}|.
\]

Explain why \((a_n)\) converges and find, if possible, its limit.

Problem 2.14. Let \((a_n)\) be a sequence of real numbers. If there is an \( A \) such that every subsequence of \((a_n)\) has a further subsequence that converges to \( A \), then \((a_n)\) converges to \( A \).

Problem 2.15. Define a real sequence recursively as follows. Let \( a_1 = 1 \) and \( a_{n+1} = \sqrt{1 + a_n} \) for \( n \in \mathbb{N}^+ \). Show that \((a_n)\) converges to the golden ratio \( \frac{\sqrt{5}+1}{2} \).

Problem 2.16. Given \( 0 < b_1 < a_1 \), define sequences \((a_n)\) and \((b_n)\) recursively by
\[
a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_nb_n}.
\]

Show,

(1) \( a_n^2 - b_n^2 > 0 \) and hence \( a_n > b_n \);
(2) \((a_n)\) is decreasing and \((b_n)\) is increasing;
(3) explain why both sequences converge and then deduce that they have a common limit \( L \). (Suggestion: estimate \( a_n - b_n \) in terms of \( a_1 - b_1 \).)

Problem 2.17. Define, for positive integers \( n \),
\[
a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = \sum_{j=1}^{n} \frac{1}{j}.
\]

Show, for each \( m \in \mathbb{N} \),
\[
a_{2m} \geq 1 + \frac{m}{2}.
\]
Deduce \((a_n)\) diverges (to \(\infty\)).

**Problem 2.18.** Let for \(n \in \mathbb{N}\), let \(a_n = \frac{2n^2 + 7}{3n^2 - 2}\). Show, directly from the definition of limit, that \((a_n)\) diverges to \(\infty\).

**Problem 2.19.** Revisit Problem 2.18, but now use Proposition 2.34 to deduce that the sequence diverges to \(\infty\).

**Problem 2.20** (Squeeze Theorem). Let \((a_n), (b_n), (c_n)\) be given sequences from \(\mathbb{R}\).

If there is a \(K\) so that \(a_n \leq b_n \leq c_n\) for \(n \geq K\) and if there is an \(L\) so that both \((a_n)\) and \((c_n)\) converge to \(L\), then \((b_n)\) converges to \(L\).

If there is a \(K\) so that \(a_n \leq b_n\) for \(n \geq K\) and if \((a_n)\) diverges to \(\infty\), then so does \((b_n)\).

**Problem 2.21.** Define a real sequence recursively by letting \(b_1 = \frac{1}{2}\) and \(b_{n+1} = \sqrt{1 - b_n}\). (In particular, it needs to be shown that \(1 - b_n \geq 0\) for all positive integers \(n\).) Show, by filling in the following outline or otherwise, that \((b_n)\) converges to \(\sqrt{\frac{5 - 1}{2}}\).

(a) Show \(\frac{1}{2} \leq b_j \leq \frac{1}{\sqrt{2}}\) for all \(j\);
(b) the \((b_{2j+1})_j\) is an increasing sequence;
(c) there is an \(\alpha > 1\) such that \(b_{j+1} + b_j \geq \alpha\) for all \(j\);
(d) \(|b_{j+2} - b_j| \leq \frac{\alpha}{\alpha} |b_{j+1} - b_j|\);
(e) Conclude the sequence \((b_j)\) converges and determine the limit.

**Problem 2.22.** Suppose \((a_n)\) is a decreasing sequence from \(\mathbb{R}\). Show, if \((a_n)\) is Cauchy, then \((a_n)\) is super Cauchy.

**Problem 2.23.** Fix \(0 < c\) and fill in the following alternate proof that \((a_n = c^{\frac{1}{n}})_{n=1}^{\infty}\) converges to 0.

(i) Suppose \(c > 1\) and show \((a_n)\) is decreasing and bounded below (by 1) and thus converges to some \(A\).
(ii) Still assuming \(c > 1\), argue that \((a_{2n})\) converges to both \(A\) and \(\sqrt{A}\) and conclude \(A = 1\).
(iii) Now treat the cases \(c = 1\) and \(0 < c < 1\).

3. Limits

**3.1. Definitions and examples.**

**Definition 3.1.** Suppose \(D \subset \mathbb{R}\), the real number \(a\) is an accumulation point of \(D\), and \(f : D \to \mathbb{R}\). We say that \(f\) has a limit at \(a\) if there exists a real number \(L\) such that for every \(\epsilon > 0\) there is a \(\delta > 0\) such that if \(s \in D\) and \(0 < |s - a| < \delta\), then \(|f(s) - L| < \epsilon\). In this case we say \(f\) has limit \(L\) as \(x\) approaches \(a\) and write,

\[L = \lim_{x \to a} f(x)\]

and call \(L\) the limit of \(f\) at \(a\).

It is an exercise to show \(L\), if it exists, is unique and hence can deserves the title of the limit. It is also an exercise to show \(L = \lim_{x \to a} f(x) = L\) if and only if \(\lim_{x \to a}(f(x) - L) = 0\).

**Example 3.2.** Let \(D = \mathbb{R}\) and \(f(x) = x^2\). Show \(\lim_{x \to 1} f(x) = 1\).
Solution. Given \( \epsilon > 0 \) choose \( \delta = \min\{1, \frac{\epsilon}{3}\} \). If \( |x - 1| < \delta \), then
\[
|f(x) - 1| = |x + 1||x - 1| \leq 3|x - 1| < \epsilon.
\]
\[\square\]

Notice that, in Definition 3.1, \( f \) may, or may not, being defined at \( a \) and if it is defined at \( 1 \), the limit doesn’t depend upon the value of \( f \) at \( a \) even if \( a \) is in the domain of \( f \). The following examples illustrates this point and generally that the notion of limit of a function at a point depends critically on the domain.

Example 3.3. Let \( D = (-\infty, 1) \cup (1, \infty) = \mathbb{R} \setminus \{1\} \) and define \( g : D \to \mathbb{R} \) by \( g(x) = x^2 \). Show, \( \lim_{x \to 1} g(x) = 1. \)

Let \( D = \mathbb{R} \) and define \( h : D \to \mathbb{R} \) by \( h(x) = x^2 \) for \( x \neq 1 \) and \( h(1) = 0 \). Show, \( \lim_{x \to 1} h(x) = 1. \)

Example 3.4. Define \( F : (0, \infty) \setminus \{1\} \to \mathbb{R} \) by \( F(x) = 1 - \frac{1}{\sqrt{x}} \). Show \( F \) has a limit at \( 1. \)

Solution. First, observe that, for \( x > 0 \),
\[
|f(x) - 2| = \left| \frac{1 - x}{1 - \sqrt{x}} - 2 \right| = \frac{1}{1 - \sqrt{x}} \left| 1 - 2\sqrt{x} + x \right| = \frac{1}{1 - \sqrt{x}} \left| 1 - \sqrt{x} \right| = \frac{1 - x}{1 + \sqrt{x}} \leq |1 - x|.
\]

Now given \( \epsilon > 0 \), choose \( \delta = \epsilon \). If \( x \in D \) and \( |x - 1| < \delta \), then \( |f(x) - 2| < \epsilon \). Hence \( f \) has a limit at \( 1 \) and this limit is \( 2. \) \[\square\]

Example 3.5. Define \( f : (0, 1) \to \mathbb{R} \) by
\[
f(x) = \begin{cases} 
0 & \text{if } x \notin \mathbb{Q} \\
\frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p, q > 0 \text{ and } \gcd(p, q) = 1.
\end{cases}
\]

Show, for each \( a \in (0, 1) \), that \( \lim_{x \to a} f(x) = 0. \)

Solution. Let \( \epsilon > 0 \) be given. Choose \( N \in \mathbb{N}^+ \) so that \( \frac{1}{N} < \epsilon \). Consider the set
\[S_N = \{ \frac{m}{n} : m, n \in \mathbb{N}^+, \ n \leq N, \ m \leq n \}.\]

Since \( S_N \) is finite,
\[
\delta = \min\{|s - a| : s \in S, \ s \neq a\} > 0.
\]
In particular, \( (a - \delta, a + \delta) \cap S_N \subset \{a\} \). Hence, if \( 0 < |a - x| < \delta \), then either \( x \notin \mathbb{Q} \) in which case \( |f(x) - 0| = |0 - 0| < \epsilon \); or \( x \in \mathbb{Q} \) and \( x = \frac{p}{q} \) where \( q > N \), in which case \( |f(x) - 0| = |\frac{1}{q}| < \frac{1}{N} < \epsilon. \) \[\square\]

Example 3.6. Let \( D = \mathbb{R} \setminus \{0\} \) and define \( g : D \to \mathbb{R} \) by \( g(x) = \frac{x}{|x|} \). Show, \( \lim_{x \to 0} g(x) \) doesn’t exist.
Solution. Let $L \in \mathbb{R}$ be given. To show that $f$ does not have limit $L$ at $0$, choose $\epsilon_0 = 1$ and let $\delta > 0$ be given. If $L < 0$, choosing $x = \frac{\delta}{2}$ and if $L \leq 0$ choose $x = -\frac{\delta}{2}$. In either case $0 < |x - 0| < \delta$, but
\[ |f(x) - L| \geq 1 = \epsilon_0. \]
Hence $f$ does not have a limit at 0. \hfill \Box

Proposition 3.7. Suppose $f : D \to \mathbb{R}$ and $a$ is an accumulation point of $D$. If $f$ has a limit as $x$ approaches $a$, then for every $\epsilon > 0$ there is an $\eta > 0$ such that if $x,y \in D$ and both $|x-a|,|y-a| < \eta$, then $|f(x) - f(y)| < \epsilon$.

Proposition 3.7 is an analog of the fact that convergent sequences are Cauchy (Proposition 2.12). Its proof is left as an exercise.


Example 3.9. Define $f : (0, \infty) \to \mathbb{R}$ by
\[ f(x) = \sin\left(\frac{1}{x}\right). \]
Show $f$ does not have a limit at 0.

Solution. Choose $\epsilon_0 = 1$. Given $\eta > 0$ there exists $n \in \mathbb{N}^+$ such that $\frac{1}{2n\pi} < \eta$. With $x = \frac{1}{2n\pi}$ and $y = \frac{1}{(2n+\frac{1}{2})\pi}$, we have $0 < x,y < \eta$, but $|f(x) - f(y)| = 1 \geq \epsilon_0$. Hence, by (the contrapositive of) Proposition 3.7, $f$ does not have a limit at 0. \hfill \Box

This subsection closes with a simple observation that will be used repeatedly. Given sets $X,Y$ and $E \subset X$ and a function $f : X \to Y$, the restriction of $f$ to $E$, $f|_E : E \to Y$ is the function (with domain $E$) defined, for $x \in E$, by $f|_E(x) = f(x)$. Now suppose $E \subset D \subset \mathbb{R}$, a limit point $a$ of $E$ (and hence of $D$) and $f : D \to \mathbb{R}$. If $f$ has limit $L$ at $a$, then so does $f|_E$ (Problem 3.14).

3.2. The sequential formulation of limit of a function.

Proposition 3.10. Suppose $f : D \to \mathbb{R}$, the point $a \in \mathbb{R}$ is an accumulation point of $D$ and $L \in \mathbb{R}$.

If $f$ has limit $L$ as $x$ approaches $a$ and if $(a_n)$ is a sequence from $D \setminus \{a\}$ that converges to $a$, then $(f(a_n))$ converges to $L$.

Conversely, if for every sequence $(a_n)$ from $D \setminus \{a\}$ that converges to $a$, the sequence $(f(a_n))$ converges to $L$, then $f$ has limit $L$ at $a$.

Proof. First suppose $\lim_{x \to a} f(x) = L$ (meaning $f$ has limit $L$ at $a$) and that $(a_n)$ is a sequence from $D \setminus \{a\}$ that converges to $a$. To prove $(f(a_n))$ converges to $L$, let $\epsilon > 0$ be given. There is a $\delta > 0$ such that if $0 < |x-a| < \delta$ and $x \in D$, then $|f(x) - L| < \epsilon$. Since $(a_n)$ converges to $a$ and each $a_n \in D \setminus \{a\}$, there is an $N$ so that if $n \geq N$, then $0 < |a_n - a| < \delta$. Hence, if $n \geq N$, then $|f(a_n) - L| < \epsilon$.

To prove the second statement, suppose $\lim_{x \to a} f(x) \neq L$. Thus, there is an $\epsilon_0 > 0$ such that for every $\delta > 0$ there is a point $x \in D$ such that $0 < |x-a| < \delta$, but $|f(x) - L| \geq \epsilon_0$.

Thus, with $\delta_n = \frac{1}{n}$, there exists $a_n \in D$ such that $0 < |a_n - a| < \delta_n$ and $|f(a_n) - L| \geq \epsilon_0$. It follows that $(a_n)$ converges to $a$, but $(f(a_n))$ does not converge to $L$. \hfill \Box
Corollary 3.11. Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$ and $a$ is an accumulation point of $D$. If there exists a sequence $(a_n)$ from $D \setminus \{a\}$ such that $(a_n)$ converges to $a$, but $(f(a_n))$ diverges, then $f$ does not have a limit at $x$ tends to $a$.

Similarly, if there exists sequences $(a_n)$ and $(b_n)$ from $D \setminus \{a\}$ such that both converge to $a$, but $(f(a_n))$ and $(f(b_n))$ don’t converge to the same value (which includes the case that one or both diverges), then $f$ does not have a limit as $x$ approaches $a$.

Example 3.12. Let $D = \mathbb{R} \setminus \{0\}$ and define $f : D \to \mathbb{R}$ by $f(x) = \sin(\frac{1}{x})$. Show that $f$ does not have a limit at $0$.

Solution. Choose $a_n = \frac{1}{(n+\frac{1}{2})\pi}$ for $n \in \mathbb{N}$. Note that $(a_n)$ converges to 0, but the sequence $(f(a_n)) = (-1)^n$ diverges (see Example 2.48). Now apply Corollary 3.11.

Example 3.13. Let $f : [0, 1] \to \mathbb{R}$ denote the indicator function (synonymously characteristic function) of $\mathbb{Q} \cap [0, 1]$. It is defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Show, for each $a \in (0, 1)$ that $\lim_{x \to a} f(x)$ does not exists.

Solution. Fix $a \in (0, 1)$. Since, by Example 1.38(v) $a$ is a limit point of $\mathbb{Q}$, there is a sequence $(a_n)$ from $\mathbb{Q} \cap (0, 1)$ that converges to $a$ and such that for each $n$, $a_n \neq a$. Since $a_n \in \mathbb{Q}$, we have $(f(a_n)) = (1)$ converges to 1. There is also a sequence $b_n$ from $[0, 1] \setminus \mathbb{Q}$ such that $(b_n)$ converges $a$ and for each $n$, $b_n \neq a$ (it is an exercise to verify this statement). We have $(f(b_n)) = (0)$ converges to 0. Thus, by Corollary 3.11, $f$ does not have a limit at $a$.

Proposition 3.14. Fix $q \in \mathbb{Q}^+$. Define $f : [0, \infty) \to \mathbb{R}$ by $f(x) = x^q$. If $a \in [0, \infty)$ (that is, $a$ is an accumulation point of $(0, \infty)$), then

$$\lim_{x \to a} f(x) = a^q = f(a).$$

Likewise, for $g : (0, \infty) \to \mathbb{R}$ defined by $g(x) = x^{-q}$ and $a \in (0, \infty)$,

$$\lim_{x \to a} g(x) = g(a).$$

For $n \in \mathbb{N}^+$ and $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^n$ and $a \in \mathbb{R}$,

$$\lim_{x \to a} h(x) = a^n = h(a).$$

Finally, for $c \in \mathbb{R}$ and $k : \mathbb{R} \to \mathbb{R}$ defined by $k(x) = c$

$$\lim_{x \to a} k(x) = c.$$

for each $a \in \mathbb{R}$.

Proof. Suppose $(a_n)$ is a sequence from $[0, \infty)$ and $(a_n)$ converges to $a$. By Proposition 2.29, $(a_n^q = f(a_n))$ converges to $a^q$. An application of Proposition 3.10 completes the proof of the first part of the proposition. The second part follows from Theorem 2.28(iii). The proofs of the remaining items are similar.
3.3. Infinite limits and limits at infinity.

**Definition 3.15.** Suppose $f : D \to \mathbb{R}$ and $a$ is an accumulation point of $D$. The limit of $f$ as $x$ approaches $a$ is $\infty$ if for every $C > 0$ there is a $\delta$ such that if $x \in D$ and $0 < |x - a| < \delta$, then $f(x) > C$, denoted

$$\lim_{x \to a} f(x) = \infty.$$  

Given $D \subset \mathbb{R}$, we say $\infty$ is a limit point of $D$ if for every $C > 0$ there is an $x \in D$ such that $x > C$.

Suppose $f : D \to \mathbb{R}$ and $\infty$ is a limit point of $D$ and $A \in \mathbb{R}$. The limit of $f$ as $x$ approaches $\infty$ is $A$ if for every $\epsilon > 0$ there is a $C > 0$ such that if $x \in D$ and $x > C$, then $|f(x) - A| < \epsilon$, denoted

$$\lim_{x \to \infty} f(x) = A.$$  

The expression, 

$$\lim_{x \to \infty} f(x) = \infty$$

is defined similarly.

**Example 3.16.** Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = x^{-2}$. Show

$$\lim_{x \to \infty} f(x) = 0.$$  

**Solution.** Given $\epsilon > 0$ choose $C = \epsilon^{-\frac{1}{2}}$. Now, if $x > C$, then $0 < x^{-2} - 0 < C^{-2} = \epsilon$. □

**Example 3.17.** Let $D = (0, \infty)$ and define $g : D \to \mathbb{R}$ by $g(x) = x^{-1}$. Show

$$\lim_{x \to 0} g(x) = \infty.$$  

Note, this fact is sometimes expressed as

$$\lim_{x \to 0^+} \frac{1}{x} = \infty.$$  

**Solution.** Given $C > 0$, choose $\epsilon = \frac{1}{C} > 0$. If $x \in D$ and $|x - 0| < \epsilon$, then $0 < x < \frac{1}{C}$ and hence $0 < g(x) = x^{-1} > C$. □

**Example 3.18.** Assuming knowledge of the log function, show,

$$\lim_{x \to \infty} \log(x) = \infty.$$  

**Solution.** Recall,

(i) $\log(2^k) = k \log(2)$ for $k \in \mathbb{N}$;
(ii) $\log(2) > 0$; and
(iii) if $y > x > 0$, then $\log(y) > \log(x)$.

Given $C > 0$ choose $k \in \mathbb{N}^+$ such that $k \log(2) > C$ (equivalently $k > \frac{C}{\log(2)}$). Choose $K = 2^k$. If $x > K$, then $\log(x) > \log(K) = k \log(2) > C$. □

There are sequential formulations of limits at infinity and infinite limits. Rather than state all the variations, we offer the following proposition as a sample result.

**Proposition 3.19.** Suppose $f : D \to \mathbb{R}$ and $a$ is an accumulation point of $D$. If $\lim_{x \to a} f(x) = \infty$ and if $(a_n)$ is a sequence from $D \setminus \{a\}$ that converges to $a$, then $\lim_{n \to \infty} f(a_n) = \infty$. 
Example 3.20. Show \( \lim_{x \to 0} \frac{1}{x} \) does not exists and is not either \( \pm \infty \). Implicitly, here \( D = \mathbb{R} \setminus \{0\} \) and \( f(x) = \frac{1}{x} \).

Solution. Consider the sequence \( (a_n = \frac{1}{n})_{n=1}^{\infty} \). The sequence \( (b_n = f(a_n)) = (n) \) diverges to \( \infty \). Hence, no real number or \( -\infty \) can be the limit. On the other hand, with \( (c_n = -a_n) \) the sequence \( (f(c_n)) = (-n) \) diverges to \( -\infty \) and thus \( \infty \) can not be the limit. Hence the limit fails to exist even in the sense of \( \pm \infty \).

□

Proposition 3.21. If \( q \in \mathbb{Q}^+ \) and \( f : (0, \infty) \to \mathbb{R} \) is defined by \( f(x) = x^{-q} \), then \( \lim_{x \to \infty} f(x) = 0 \) and \( \lim_{x \to 0} f(x) = \infty \). Similarly, for \( g : (0, \infty) \to \mathbb{R} \) defined by \( g(x) = x^q \), \( \lim_{x \to \infty} g(x) = \infty \).

3.3.1. A change of variable. 5

Proposition 3.22. Suppose \( D \subset (0, \infty) \) and let \( E = \{ \frac{1}{x} : x \in D \} \subset (0, \infty) \). Then \( \infty \) is a limit point of \( D \) if and only if \( 0 \) is a limit point of \( E \).

Suppose \( f : D \to \mathbb{R} \) and \( g : E \to \mathbb{R} \) is defined by \( g(x) = f(\frac{1}{x}) \). Then \( f \) has a limit at 0 if and only if \( g \) has a limit at 0 and in this case

\[
\lim_{x \to \infty} f(x) = \lim_{x \to 0} g(x).
\]

Remark 3.23. The conclusion is that either both limits exist and are equal or they both fail to exist. Proposition 3.22 is a variant of Proposition 3.33.

Example 3.24. Redo Example 3.16.

Solution. In that example \( D = (0, \infty) \) and \( f : D \to \mathbb{R} \) is defined by \( f(x) = x^{-2} \) and we are to show \( \lim_{x \to \infty} f(x) = 0 \). To this end, let \( E = D \) and define \( g : E \to \mathbb{R} \) by \( g(x) = f(\frac{1}{x}) = x^2 \).

To see that \( \lim_{x \to \infty} f(x) = 0 \) it suffices, by Proposition 3.22, to show

\[
\lim_{x \to 0} g(x) = 0.
\]

To prove this statement, let \( \epsilon > 0 \) be given. Choose \( \delta = \min\{1, \epsilon\} \). Now, if \( x \in D \) and \( |x - 0| < \delta \leq 1 \), then

\[
|g(x) - 0| = x^2 < |x| < \delta \leq \epsilon,
\]

where we have used \( 0 < x < 1 \) in the first inequality. □

3.4. Limit Theorems. The following theorem says, like in the case of limits of sequences, limits of functions are compatible with the algebraic operations on \( \mathbb{R} \).

Proposition 3.25. Suppose \( f, g, h : D \to \mathbb{R} \) and \( a \) is an accumulation point of \( D \). If the limits \( \lim_{x \to a} f(x) \), \( \lim_{x \to a} g(x) \) and \( \lim_{x \to a} h(x) \) exist and equal \( A, B, C \) respectively, then

(i) \( \lim_{x \to a} (f + g)(x) = A + B \);
(ii) \( \lim_{x \to a} fg(x) = AB \); and
(iii) if \( C \neq 0 \) and \( h \) is never zero, then \( \lim_{x \to a} \frac{1}{h(x)} = \frac{1}{C} \).

Proof. Suppose \( (a_n) \) is a sequence from \( D \setminus \{a\} \) that converges to \( a \). By Proposition 3.10, \( \lim_{n \to \infty} f(a_n) = A \) and similarly, \( \lim_{n \to \infty} g(a_n) = B \). It follows, from Theorem 2.28, that \( \lim_{n \to a} fg(a_n) = AB \) and \( \lim_{n \to a} (f + g)(a_n) = A + B \). Hence another application of Proposition 3.10 proves both items (i) and (ii).

The proof of item (iii) is similar. The details are left as an exercise. □

5optional
Remark 3.26. Analogous results hold for limits at infinity and infinite limits, with, in the latter case, the obvious caveats.

Example 3.27. Find
\[ \lim_{x \to \infty} \frac{2x - 1}{x + 3} , \]
if it exists.

Solution. Here the function (and hence domain) and are not explicitly given. Implicitly, \( D = \mathbb{R} \setminus \{-3\} \) and \( f : D \to \mathbb{R} \) is defined by
\[ f(x) = \frac{2x - 1}{x + 3} \]
and we are to find the limit of \( f \) as \( x \) approaches 3.

Solution. By Proposition 3.21, \( \lim_{x \to \infty} \frac{1}{x} = 0 \). Using this fact, \( \lim_{x \to \infty} \frac{3}{x} = 0 \).

Define \( g, h : D \to \mathbb{R} \) by \( g(x) = 2 - \frac{1}{x} \) and \( h(x) = 1 + \frac{3}{x} \). It follows that \( \lim_{x \to \infty} g(x) = 2 \) and \( \lim_{x \to \infty} h(x) = 1 \). Since \( f = \frac{g}{h} \) and \( h(x) \) is never 0 and \( \lim_{x \to \infty} h(x) \) exists and is not 0, using Proposition 3.25 (versions for limits at infinity),
\[ \frac{2}{1} = \lim_{x \to \infty} \frac{g(x)}{h(x)} = \lim_{x \to \infty} f(x) . \]

Definition 3.28. Given polynomials \( p \) and \( q \), let \( D = \{ x \in \mathbb{R} : q(x) \neq 0 \} \). In particular, \( D = \mathbb{R} \setminus Z(q) \), where \( Z(q) \) is the zero set of \( q \),
\[ Z(q) = \{ x \in \mathbb{R} : q(x) = 0 \} . \]

The function \( r : D \to \mathbb{R} \) defined by \( r(x) = \frac{p(x)}{q(x)} \) is a rational function.

Proposition 3.29. Suppose \( r \) is a rational function with domain \( D \). Thus \( r = \frac{p}{q} \) and \( D = \mathbb{R} \setminus Z(q) \). For \( a \in D \),
\[ \lim_{x \to a} r(x) = r(a) . \]

Proof sketch. To prove the first part of the proposition, fix \( a \in \mathbb{R} \). For \( n \in \mathbb{N} \), Proposition 3.14 shows \( \lim_{x \to a} x^n = a^n \). A further application of Propositions 3.25 and 3.14 shows for \( c \in \mathbb{R} \) and \( n \in \mathbb{N} \) that \( \lim_{x \to a} cx^n = ca^n \). Thus, if \( p \) is a polynomial,
\[ p(x) = \sum_{n=0}^{d} p_n x^n , \]
then repeated applications of what has already been proved and Proposition 3.25 shows \( \lim_{x \to a} p(x) = p(a) \).

It now follows from Proposition 3.25, for \( x \in D \), that
\[ \frac{1}{q(a)} = \lim_{x \to a} \frac{1}{q(x)} = \lim_{x \to a} \frac{1}{q(x)} . \]

Yet one more application of Proposition 3.25 - this time to \( \frac{1}{q} \) - completes the proof. \( \square \)
3.4.1. Order and limits.

**Proposition 3.30.** Suppose \( f, g : D \to \mathbb{R} \) and \( a \) is a limit point of \( D \). If there is an \( \eta > 0 \) such that if \( x \in D \) and \( 0 < |x - a| < \eta \), then \( f(x) \leq g(x) \) and if both \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exists, then

\[
\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).
\]

There is a version of Proposition 3.30 for limits at \( \infty \) and infinite limits.

**Proof.** For notational ease, let \( A = \lim_{x \to a} f(x) \) and \( B = \lim_{x \to a} g(x) \). Since \( a \) is a limit point of \( D \), there exists a sequence \( (a_n) \) from \( D \) such that \( 0 < |a_n - a| < \eta \) for all \( n \) and \( (a_n) \) converges to \( a \). It follows that \( f(a_n) \leq g(a_n) \) and the sequences \( (f(a_n)) \) and \( (g(a_n)) \) converge to \( A \) and \( B \) respectively. Thus, by Theorem 2.28, \( A \leq B \). □

**Proposition 3.31.** Suppose \( f, g, h : D \to \mathbb{R} \) and \( a \) is an accumulation point of \( D \). If \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} h(x) \) exist and equal \( L \) and if there is an \( \eta > 0 \) such that \( f(x) \leq g(x) \) for \( x \in D \) and \( 0 < |x - a| < \eta \), then \( \lim_{x \to a} g(x) = L \).

**Proof.** Let \( \epsilon > 0 \) be given. There exists a \( 0 < \delta < \eta \) such that if \( x \in D \) and \( 0 < |x - a| < \delta \), then \( L - f(x) < \epsilon \) and \( h(x) - L < \epsilon \). Hence, for such \( x \),

\[-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon \]

and the conclusion follows. □

**Example 3.32.** Let \( D = (0, \infty) \) and define \( g : (0, \infty) \to \mathbb{R} \) by

\[ g(x) = \frac{x}{\sqrt{x^2 + 1}}. \]

Show,

\[ \lim_{x \to \infty} g(x) = 1. \]

**Solution.** Define \( f, h : (0, \infty) \to \mathbb{R} \) by \( h(x) = 1 \) and

\[ f(x) = 1 - \frac{1}{x}. \]

Verify that the inequalities \( f(x) \leq g(x) \leq h(x) \) hold for \( x > 1 \). Since both \( f \) and \( h \) approach 1 as \( x \) tends to \( \infty \),

\[ \lim_{x \to \infty} g(x) = 1 \]

too. □

3.4.2. Compositions.

**Proposition 3.33.** Suppose \( D, E \subset \mathbb{R} \) and \( a, b \in \mathbb{R} \) are limit points of \( D \) and \( E \) respectively and \( g : D \to E \) and \( f : E \to \mathbb{R} \). If

(i) \( \lim_{x \to a} g(x) \) exists and is equal \( b \);
(ii) \( \lim_{y \to b} f(y) \) exists and is say \( L \);
(iii) either there is a \( \tau > 0 \) such that \( b \notin g(\{x \in D : 0 < |x - a| < \tau\}) \) or \( f(b) = L \),

then \( f \circ g : D \to \mathbb{R} \) has limit \( L \) at \( a \).

Item (iii) holds if \( b \notin E \).
Proof. Let $\epsilon > 0$ be given. By item (ii), there is a $\gamma > 0$ such that if $0 < |y - b| < \gamma$, and $y \in E$, then $|f(y) - L| < \epsilon$. With this $\gamma > 0$, by item (i), there is an $\eta > 0$ such that if $0 < |x - a| < \eta$ and $x \in D$, then $|g(x) - b| < \gamma$. Now suppose $f(b) = L$. In this case, if $0 < |x - a| < \eta$ and $x \in D$, then $g(x) \in E$ and either $0 < |g(x) - b| < \gamma$ or $g(x) = b$. In either case $|f(g(x)) - L| < \epsilon$. Finally suppose there is a $\tau > 0$ such that $b \notin g(D)$. In this case, if $0 < |x - a| < \tau$ and $x \in D$, then $g(x) \in E$ and either $0 < |g(x) - b| < \gamma$ or $g(x) = b$. In either case $|f(g(x)) - L| < \epsilon$. \Box

Example 3.34. In this example the basic properties of the sin function are assumed. Let $D = (-\frac{\pi}{5}, \frac{\pi}{5}) \setminus \{0\}$ and let $f : D \to \mathbb{R}$ denote the function

$$h(x) = \frac{\sin(4x)}{\sin(5x)}.$$ 

Show $\lim_{x \to 0} h(x)$ exists and is $\frac{4}{5}$.

Solution. It is a basic property of the sin function that, with $D = \mathbb{R} \setminus \{0\}$ and $f : D \to \mathbb{R}$ defined by $f(x) = \frac{\sin(x)}{x}$,

$$\lim_{x \to 0} f(x) = 1.$$

Let $g(x) = 4x$ defined on $\mathbb{R} \setminus \{0\}$. Thus $g$ maps into the domain $D$ of $f$ and and $\lim_{x \to 0} g(x) = 0$. Thus the hypotheses of the Proposition 3.33 are satisfied with $a = 0 = b$. Hence,

$$\lim_{x \to 0} f(g(x)) = \lim_{t \to 0} f(t) = 1.$$ 

Thus,

$$\lim_{x \to 0} \frac{\sin(4x)}{4x} = 1.$$ 

Similarly,

$$\lim_{x \to 0} \frac{\sin(5x)}{5x} = 1.$$ 

Finally, for $x \in D$,

$$f(x) = \frac{\sin(4x)}{\sin(5x)} = \frac{4 \sin(4x)}{5 \sin(5x)}.$$ 

Using rules of limits (mostly notably the limit of a quotient is the quotient of the limits provided both limits exist and the limit in the denominator is not zero), we find,

$$\lim_{x \to 0} f(x) = \frac{4}{5}.$$

\Box

Combining Proposition 3.14 with Proposition 3.33 produces the following result.

Proposition 3.35. Suppose $D \subset \mathbb{R}$, $a$ is an accumulation point of $D$, $q \in \mathbb{Q}^+$ and $f : D \to [0, \infty)$. If $\lim_{x \to a} f(x)$ exists an equals $L$, then,

$$\lim_{x \to a} f(x)^q = L^q.$$ 

If $q = \frac{m}{n}$ with $m, n \in \mathbb{N}^+$ and $m$ odd, then one can take the codomain of $f$ to be $\mathbb{R}$ in Proposition 3.35. A version holds for limits at infinity too.
Example 3.36. Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = (\sqrt{x} + 1) (x + 1)^{-\frac{1}{2}}$. Find, if it exists, $\lim_{x \to 4} f(x)$.

Example 3.37. We revisit Example 3.32 computing the limit using Proposition 3.35. Let $D = (0, \infty)$ and define $g : (0, \infty) \to \mathbb{R}$ by

$$g(x) = \frac{x}{\sqrt{x^2 + 1}}.$$

Show,

$$\lim_{x \to \infty} g(x) = 1.$$

Solution. First, rewrite $g$ as

$$g(x) = \frac{1}{\sqrt{1 + \frac{1}{x^2}}}.$$

and let $h : D \to \mathbb{R}$ denote the function

$$h(x) = \sqrt{1 + \frac{1}{x^2}}.$$

Thus $g = \frac{1}{h}$. Now, by Proposition 3.21 $\lim_{x \to \infty} \frac{1}{x^2} = 0$ and thus $\lim_{x \to \infty} 1 + \frac{1}{x^2} = 1$. It follows that

$$\lim_{x \to \infty} h(x) = 1$$

and thus,

$$1 = \frac{1}{1} = \frac{1}{\lim_{x \to \infty} h(x)} = \lim_{x \to \infty} g(x).$$

\[\square\]

3.5. One sided limits.

Definition 3.38. Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Given a point $a \in \mathbb{R}$, let

$$D_{a+} = \{ x \in D : x > a \}.$$

Fix $L \in \mathbb{R}$. If $a$ is a limit point of $D_{a+}$, then $f$ has limit $L$ at $a$ from the right (or above) if

$$\lim_{x \to a} f|_{D_{a+}} = L.$$

In this case we write

$$L = \lim_{x \to a+} f(x).$$

The notion of the limit of $f$ at $a$ from the left is defined similarly. These limits, to the extent they exist, are the one-sided limits of $f$ at $a$.

Example 3.39. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Show both limits $\lim_{x \to 0+} h(x)$ and $\lim_{x \to 0-} h(x)$ exist.

Proposition 3.40. Suppose $f : D \to \mathbb{R}$ and $a \in \mathbb{R}$. If $a$ is a limit point of both $D_{a+}$ and $D_{a-}$, then $\lim_{x \to a} f(x)$ exists if and only if both one sided limits at $a$ exists and are equal. In this case,

$$\lim_{x \to a-} f(x) = \lim_{x \to a} f(x) = \lim_{x \to a+} f(x).$$
Remark 3.41. The proof is left as an easy exercise. 

A similar result holds for infinite limits. \hfill \Box

Example 3.42. The function $h$ in Example 3.39 does not have a limit at 0.

Example 3.43. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} x^3 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0. \end{cases}$$

Show $g$ has limit 0 at 0.

Solution. Let $p$ denote the function $p : \mathbb{R} \to \mathbb{R}$ defined by $p(x) = x^2$. Since $p$ is a polynomial, $\lim_{x \to 0} p(x) = p(0) = 0$. Thus $\lim_{x \to 0 \pm} = 0$ by Proposition 3.40. Since $p|_{(0, \infty)} = g|_{(0, \infty)}$, it follows that $\lim_{x \to 0} g(x) = \lim_{x \to 0^+} p(x) = 0$. Similarly, $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} = 0$. Thus, by Proposition 3.40, $\lim_{x \to 0} g(x) = 0$. \hfill \Box

3.6. Monotone functions.

Definition 3.44. A function $f : D \to \mathbb{R}$ is increasing if $x, y \in D$ and $x < y$ implies $f(x) \leq f(y)$. If is bounded above if there is a $C$ such that $f(x) \leq C$ for all $x \in D$.

Proposition 3.45. If $f : D \to \mathbb{R}$ is increasing and $a \in \mathbb{R}$ is an accumulation point of $D_{a-}$ and $S = \{f(x) : x \in D_{a-}\}$ is bounded above, then $\lim_{x \to a^-} f(x)$ exists.

Informally, Proposition 3.45 says if $f$ is monotone, then $f$ has one-sided limits. If there is an $x \in D$ such that $x \geq a$, then $f(x)$ is an upper bound for the set $S$.

Proof. By hypothesis

$$S = \{f(x) : x \in D, \ x < a\}$$

is bounded above. Since $a$ is an accumulation point of $D_{a-}$ the set $D_{a-}$, and hence $S$, is nonempty. Therefore $S$ has a least upper bound $L$. To see that $\lim_{x \to a^-} f(x) = L$, let $\epsilon > 0$ be given. By the least property of $L$, there exists a $z \in D$ with $z < a$ and $f(z) > L - \epsilon$. Let $\delta = a - z$. If $0 < a - x < \delta$ and $x \in D$, then $z < x < a$ so that $f(z) \leq f(x) \leq f(a)$. Thus $L - \epsilon < f(z) \leq f(x) \leq L$ so that

$$|f(x) - L| < \epsilon.$$ \hfill \Box

Remark 3.46. By a similar argument, if $a$ is also a limit point of $D_{a^+}$, then $\lim_{x \to a^+} f(x)$ exists and moreover,

$$\lim_{x \to a^-} f(x) \leq f(a) \leq \lim_{x \to a^+} f(x).$$

3.7. Problems.

Problem 3.1. In each case show, directly from the definition of limit, that, for $f : \mathbb{R} \to \mathbb{R}$,

(a) if $f(x) = x^2$, then $\lim_{x \to 2} f(x) = 4$;
(b) if $f(x) = x^3$, then $\lim_{x \to 1} f(x) = 1$;
(c) if $f(x) = \frac{x^2 - 4}{x - 2}$ for $x \neq 2$ and $f(2) = 0$, then $\lim_{x \to 2} f(x) = 4$.

Problem 3.2. Define $f : (0, \infty) \setminus \{4\} \to \mathbb{R}$ by $f(x) = \frac{x - 4}{\sqrt{x - 2}}$. Show $\lim_{x \to 4} f(x) = 4$. 


Problem 3.3. Define \( f : \mathbb{R} \to \mathbb{R} \) by
\[
f(x) = \begin{cases} 
  x & \text{if } x \in \mathbb{Q} \\
  0 & \text{if } x \notin \mathbb{Q}.
\end{cases}
\]
For which \( a \in \mathbb{R} \) does \( \lim_{x \to a} f(x) \) exist? Prove your answer.

Problem 3.4. Suppose \( f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \) is bounded (that is, there is a \( C > 0 \) such that \( |f(x)| \leq C \) for all \( x \in \mathbb{R} \setminus \{0\} \)). Show
\[
\lim_{x \to 0} x f(x) = 0.
\]

Problem 3.5. In each case, interpret carefully and show (assuming the usual properties of the \( \sin \) function),
\begin{enumerate}
  \item \( \lim_{x \to 0} x \sin \left( \frac{1}{x} \right) = 0; \)
  \item \( \lim_{x \to \infty} \sin \left( \frac{1}{x} \right) = 0 \) (here take the domain as \( (0, \infty) \));
  \item \( \lim_{x \to \infty} \sin(x) \) does not exist.
\end{enumerate}

Problem 3.6. In each establish the limit directly from the definitions.
\begin{enumerate}
  \item \( \lim_{x \to \infty} f(x) = 2 \), where \( f : \mathbb{R} \to \mathbb{R} \) is defined by \( f(x) = \frac{2x^2 + 3x + 1}{x^2 + 2} \).
  \item \( \lim_{x \to \infty} f(x) = \infty \), where \( f : \mathbb{R} \to \mathbb{R} \) is defined by \( f(x) = \frac{2x^2 - 3x + 1}{x^2 + 2} \).
\end{enumerate}

Problem 3.7. Suppose \( D \subset \mathbb{R} \) and \( \infty \) is a limit point of \( D \) and \( f : D \to \mathbb{R} \). Show, if there is an \( R \) such that if \( R < x < y \) and \( x, y \in D \), then \( f(x) \leq f(y) \), then \( \lim_{x \to \infty} f(x) \) exists or \( \lim_{x \to \infty} f(x) = \infty \). Informally, the problem says, if \( f \) is \textit{eventually increasing}, then \( \lim_{x \to \infty} f(x) \) exists as an extended real number.

Problem 3.8. Fill in the following outline that if \( f : [0, 1] \to \mathbb{R} \) is monotone, then \( f \) has a limit at all but a most countably many points in \([0, 1]\); precisely, there is a set \( J \subset [0, 1] \) that is a countable union of finite sets such that if \( y \in [0, 1] \setminus J \), then \( f \) has a limit at \( y \).
\begin{enumerate}
  \item Given \( a \in (0, 1) \), \( f \) does not have a limit at \( a \) if and only if
  \[
  \lim_{x \to a^-} f(x) < \lim_{x \to a^+} f(x) =: f(a+).
  \]
  Let \( J = \{ a \in (0, 1) : f(a+) - f(a-) > 0 \} \cup \{0, 1\} \) and, for positive integers \( n \), let \( J_n = \{ a \in (0, 1) : f(a+) - f(a-) > \frac{1}{n} \} \). Thus
  \[ J = \bigcup_{n=1}^{\infty} J_n \cup \{0, 1\} \]
  and if \( y \in [0, 1] \setminus J \), then \( f \) has a limit at \( y \). Thus it suffices to show each \( J_n \) is finite. To this end, let \( T = f(1) - f(0) \).
  \item Show \( |J_n| \leq nT \). Here \(|J_n|\) is the cardinality (number of points in) of \( J_n \). (Suggestion: suppose \( N \in \mathbb{N}^+ \), \( a_1, \ldots, a_N \in J_n \) are distinct and show
  \[
  f(1) - f(0) \geq \sum_{j=1}^{N} |f(a_j^+) - f(a_j^-)| \geq N \frac{1}{n}.
  \]
\end{enumerate}

Problem 3.9. Given an example of (nonempty) sets \( D, E \subset \mathbb{R} \) with accumulation points \( a \) and \( b \) respectively and functions \( g : D \to E \) and \( f : E \to \mathbb{R} \) such that \( g \) has limit \( b \) at \( a \) and \( f \) has a limit at \( b \), but \( f \circ g : D \to \mathbb{R} \) does not have a limit at \( a \). Explain why Proposition 3.33 does not apply to the example given.
Problem 3.10. Provide a careful interpretation of, and compute, the limits

1. \( \lim_{x \to 1} \sqrt[6]{x^7 + 7} \); \\
2. \( \lim_{x \to \infty} \frac{\sqrt[3]{x} + 7}{\sin(3x)} \); \\
3. \( \lim_{x \to 0} \frac{\sin(5x)}{\sin(3x)} \);

Problem 3.11. A geometric argument shows, for \( |x| < \frac{\pi}{2} \), that \( |\sin(x)| \leq |x| \). Use this inequality to prove \( \lim_{x \to 0} \sin(x) = 0 \). Use the identity \( \cos(x)^2 = 1 - \sin(x)^2 \) to prove \( \lim_{x \to 0} \cos(x) = 1 \). Now use the identities

\[
\cos(x + h) = \cos(x) \cos(h) - \sin(x) \sin(h) \\
\sin(x + h) = \cos(x) \sin(h) + \sin(x) \cos(h)
\]

to prove, for each \( a \in \mathbb{R} \), that

\[
\lim_{x \to a} \cos(x) = \cos(a) \\
\lim_{x \to a} \sin(x) = \sin(a).
\]

A geometric argument shows there is an \( \eta > 0 \) such that for \( 0 < |x| < \eta \), that

\[
\cos(x) \leq \frac{\sin(x)}{x} \leq 1.
\]

See for instance [https://people.clas.ufl.edu/sam/files/SqueezeTheorem.pdf](https://people.clas.ufl.edu/sam/files/SqueezeTheorem.pdf)

Use these inequalities to show

\[
\lim_{h \to 0} \frac{\sin(h)}{h} = 1.
\]

Show

\[
\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0.
\]

Now, letting \( f : \mathbb{R} \to \mathbb{R} \) denote the sin function, show, for each \( a \in \mathbb{R} \),

\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \cos(a).
\]

Problem 3.12. There are several things wrong with the statement: if \( f \) is bounded in some neighborhood of \( x = a \), then \( \lim_{x \to a} f(x) \) is bounded. Name a few. [Note: This problem (in italics) actually appears as a problem in a book on advanced calculus!]

Problem 3.13. Given a subset \( S \) of a set \( X \), the indicator function of \( S \), denoted \( 1_S \), is the function \( 1_S : X \to \mathbb{R} \) defined by

\[
1_S(x) = \begin{cases} 
1 & \text{if } x \in S \\
0 & \text{if } x \notin S.
\end{cases}
\]

Now suppose \( S \subset \mathbb{R} \) and \( a \in \mathbb{R} \). Prove \( 1_S \) has a limit at \( a \) if and only if \( a \) is not a limit point of both \( S \) and \( \tilde{S} \). Here \( \tilde{S} = \mathbb{R} \setminus S \). In general, if \( S \subset X \), then \( \tilde{S} = \mathbb{R} \setminus S \) is the complement of \( S \) (in \( X \)).
Problem 3.14. Suppose \( f : D \to \mathbb{R} \) and \( E \subset D \). The function \( f|_E : E \to \mathbb{R} \) defined by \( f|_E(x) = f(x) \) (for \( x \in E \)) is the \textit{restriction} of \( f \) to \( E \).

Suppose \( f : D \to \mathbb{R} \) and \( E \subset D \) and \( a \) is a limit point of \( E \). If \( f \) has a limit \( L \) at \( a \), then so does \( f|_E \).

Problem 3.15. Define \( f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \) by \( f(x) = x \sin(\frac{1}{x}) \) and define \( g : \mathbb{R} \to \mathbb{R} \) by
\[
g(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{if } x \neq 0.
\end{cases}
\]

Show
\[
\lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} g(x),
\]
but \( g \circ f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \) does not have a limit at 0. Discuss carefully the relation of this example to Proposition 3.33.

Problem 3.16. Suppose \( D \subset \mathbb{R} \), \( f : D \to \mathbb{R} \), \( a \) is an accumulation point of \( D \) and \( L \in \mathbb{R} \). Define \( g : D \to \mathbb{R} \) by \( g(x) = f(x) - L \). Show, \( L = \lim_{x \to a} f(x) \) if and only if \( 0 = \lim_{x \to a} g(x) \).

4. Continuity

Definition 4.1. Given \( D \subset \mathbb{R} \) and \( a \in D \), a function \( f : D \to \mathbb{R} \) is \textit{continuous at} \( a \) if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( x \in D \) and \( |x - a| < \delta \), then \( |f(x) - f(a)| < \epsilon \).

The function \( f \) is \textit{continuous} if it is continuous at each point \( a \in D \).

Remark 4.2. If \( a \) is an accumulation point of \( D \), then \( f \) is continuous at \( a \) if and only if
\[
\lim_{x \to a} f(x) = f(a).
\]

If \( a \) is not an accumulation point of \( D \), then \( f \) is continuous at \( a \). In particular, and as an example, every function \( f : \mathbb{N} \to \mathbb{R} \) is continuous in view of Example 1.38(iii).

Example 4.3. Show that the function \( h \) from Example 1.3 (see also example 3.13) is nowhere continuous.

Example 4.4. The function from Example 3.5 is continuous at the irrational points in \((0,1)\) and discontinuous at the rational points in \((0,1)\).

Many of our facts about limits can be interpreted in terms of continuity.

Proposition 4.5. \textit{Polynomials and rational functions are continuous (on their domains).}

Proof. An immediate consequence of Proposition 3.29. \( \square \)

Proposition 4.6. Given \( q \in \mathbb{Q}^+ \), then function \( s : [0, \infty) \to [0, \infty) \) defined by \( s(x) = x^q \) is continuous by Proposition 3.14.

Likewise the function \( t : (0, \infty) \to (0, \infty) \) defined by \( t(x) = x^{-q} \) is continuous.

Remark 4.7. We will accept, without proof, that the functions \( e^x \), \( \cos(x) \), \( \sin(x) \), \( \log(x) \) and other standard functions from calculus are continuous on their domains. See for instance Problem 3.11. A suitable definition for the log function must wait until after the Riemann integral.
Continuity behaves well when restricting the domain of a function. The proof follows readily from the definitions and prior facts about limits.

**Proposition 4.8.** Suppose \( E \subset D \subset \mathbb{R}, f : D \to \mathbb{R} \) and \( a \in E \). If \( f \) is continuous at \( a \), then so is \( f|_E \). If \( f \) is continuous, then \( g = f|_E \) is continuous.

Conversely, if there is an \( \eta > 0 \) such that, with \( G = D \cap (a - \eta, a + \eta) \), the function \( f|_G \) is continuous at \( a \), then \( f \) is continuous at \( a \).

That continuity behaves well with respect to the algebraic operations on \( \mathbb{R} \) again follows from the corresponding facts about limits.

**Proposition 4.9.** Suppose \( D \subset \mathbb{R}, f, g : D \to \mathbb{R} \) and \( a \in D \). If both \( f \) and \( g \) are continuous at \( a \), then so are

(i) \( f + g \);
(ii) \( fg \); and
(iii) \( \frac{1}{g} \), assuming that \( g \) is never 0.

Moreover, if \( f \) takes non-negative values and \( q \) is a positive rational number, then \( h : D \to \mathbb{R} \) defined by \( h(x) = f(x)^q \) is continuous at \( a \).

**Proof.** To prove item (ii), first observe if \( a \) is not an accumulation point of \( D \), then \( fg \) is continuous at \( a \). Now suppose \( a \) is an accumulation point of \( D \). By assumption both limits

\[
\lim_{x \to a} f(x) = f(a), \quad \lim_{x \to a} g(x) = g(a).
\]

exist and equal the indicated values. By Theorem 2.28,

\[
\lim_{x \to a} fg(x) = f(a)g(a) = fg(a).
\]

Hence \( fg \) is continuous at \( a \).

The proofs of the other items are similar. \( \blacksquare \)

4.1. Compositions of continuous functions.

**Proposition 4.10.** Suppose \( D, E \subset \mathbb{R}, f : D \to E \) and \( g : E \to \mathbb{R} \).

If \( f \) is continuous at \( a \in D \) and \( g \) is continuous at \( b = f(a) \in E \), then \( g \circ f : D \to \mathbb{R} \) is continuous at \( a \).

If both \( f \) and \( g \) are continuous, then \( g \circ f \) is continuous.

**Proof.** Let \( \epsilon > 0 \) be given. There is a \( \eta > 0 \) such that if \( y \in E \) and \( |y - b| < \eta \), then \( |g(y) - g(b)| < \epsilon \). There is a \( \delta > 0 \) such that if \( x \in D \) and \( |a - x| < \delta \), then \( |f(x) - f(a)| < \eta \). Hence if \( x \in D \) and \( |a - x| < \delta \), then \( |g(f(x)) - g(f(a))| < \epsilon \).

The second part follows immediately from the first. \( \blacksquare \)

Note that an alternate proof of Proposition 4.10 can be constructed using Proposition 3.33.

**Example 4.11.** The function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = \sqrt{x^2 + 1} \) is continuous.

Continuity at a point also has a sequential characterization.
Proposition 4.12. Suppose \( D \subset \mathbb{R} \), \( f : D \to \mathbb{R} \) and \( a \in D \). If \( f \) is continuous at \( a \) and if \((a_n)\) is a sequence from \( D \) that converges to \( a \), then \((f(a_n))\) converges to \( f(a) \).

Conversely, if for every sequence \((a_n)\) from \( D \) that converges to \( a \), the sequence \((f(a_n))\) converges to \( f(a) \), then \( f \) is continuous at \( a \).

Proof. First suppose \( f \) is continuous at \( a \) and \((a_n)\) is a sequence from \( D \) that converges to \( a \). In this case, given \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( x \in D \) and \(|a - x| < \delta\), then \(|f(x) - f(a)| < \epsilon\). Since \((a_n)\) converges to \( a \), there is an \( N \in \mathbb{N}^+ \) such that if \( n \geq N \), then \(|a - a_n| < \delta\). Hence, if \( n \geq N \), then \(|f(a_n) - f(a)| < \epsilon\) and it follows that \((f(a_n))\) converges to \( f(a) \).

To prove the converse, suppose \( f \) is not continuous at \( a \). In this case there exists an \( \eta > 0 \) such that for every \( \delta > 0 \) there is an \( x \) such that \( x \in D \) and \(|x - a| < \delta\), but \(|f(x) - f(a)| \geq \eta\). Choosing, for \( n \in \mathbb{N}^+ \), \( \delta = \delta_n = \frac{1}{n} \), there is an \( x_n \in D \) such that \(|x_n - a| < \frac{1}{n}\), but \(|f(x_n) - f(a)| \geq \eta\). By construction \((x_n)\) is a sequence from \( D \) that converges to \( a \), but \((f(x_n))\) does not converge to \( f(a) \).

4.2. The extreme and intermediate value theorems.

Lemma 4.13. If \( f : [a, b] \to \mathbb{R} \) is continuous, then \( f \) is bounded; that is the set \( f([a, b]) = \{f(x) : a \leq x \leq b\} \) is bounded both above and below.

Proof. We will argue by contradiction to show that \( f \) is bounded above. Accordingly, suppose \( f \) is not bounded above. In this case, for each \( n \in \mathbb{N} \), there is an \( x_n \in [a, b] \) such that \( f(x_n) \geq n \). There is a subsequence \((x_{n_k})\) of \((x_n)\) that converges to some \( y \) by Theorem 2.46. Since \( a \leq x_{n_k} \leq b \) for each \( k \), it follows that \( y \in [a, b] \) (see Theorem 2.28(iv)). By Proposition 4.12, \((f(x_{n_k}))_k\) converges to \( f(y) \), contradicting the fact that \((f(x_{n_k}))\) is unbounded (as convergent sequences are bounded). Hence \( f \) is bounded above. To see that \( f \) is bounded below, replace \( f \) by \(-f\) and use that \(-f \) is bounded above.

Theorem 4.14 (Extreme Value Theorem (EVT)). If \( f : [a, b] \to \mathbb{R} \) is continuous, then there exists \( a \leq y \leq b \) such that \( f(y) \geq f(x) \) for all \( a \leq x \leq b \).

Remark 4.15. Under the hypotheses of the theorem, the conclusion says that the range of \( f \), namely the set \( S = \{f(x) : a \leq x \leq b\} \) has a largest value. Thus the set \( S \) has a maximum and of course this maximum is the lub of \( S \). Informally, the theorem is stated as: a continuous function on a closed bounded interval attains its maximum. Of course, replacing \( f \) by \(-f\), we also see that \( f \) attains its minimum. The maximum and minimum of \( f \) are the extrema or extreme values of \( f \). Finally, Corollary 4.34 generalizes Theorem 4.14.

Proof. The set \( S = \{f(x) : x \in [a, b]\} \) is bounded by Lemma 4.13 and contains \( f(a) \). It thus has a least upper bound, say \( M \). For each \( n \in \mathbb{N}^+ \) there is an \( y_n \in S \) such that \( M - \frac{1}{n} < y_n \leq M \), by the least property of \( M \). For each \( n \) there is an \( x_n \in [a, b] \) so that \( f(x_n) = y_n \). By Proposition 2.46, there is a \( z \) and a subsequence \((x_{n_k})\) of \((x_n)\) that converges to \( z \). In particular, \( a \leq z \leq b \) because \( a \leq x_{n_k} \leq b \) for each \( k \) (see Theorem 2.28(iv)). Since \( f \) is continuous, \((f(x_{n_k}))_k\) converges to \( f(z) \) by Proposition 4.12. On the other hand, from the construction \((y_n = f(x_n))\) converges to \( M \) and thus so does \((y_{n_k} = f(x_{n_k}))\) by Proposition 2.47. Hence \( f(z) = M \) and the proof is complete.

Theorem 4.16 (Intermediate Value Theorem (IVT)). Suppose \( f : [a, b] \to \mathbb{R} \) is continuous and \( f(a) < f(b) \). If \( f(a) < k < f(b) \), then there exists \( a < c < b \) such that \( f(c) = k \).
Proof. Let

\[ E = \{x \in [a, b] : f(x) \leq k\}. \]

Note that \( a \in E \) and \( E \) is bounded above by \( b \). Thus \( E \) has a least upper bound \( a < c \leq b \). For each \( n \in \mathbb{N}^+ \) there is a point \( x_n \in E \) such that \( c - \frac{1}{n} < x_n \leq c \). In particular, \( (x_n) \) converges to \( c \) and \( f(x_n) \leq k \). By continuity of \( f \), the sequence \( (f(x_n)) \) converges to \( f(c) \) and moreover, \( f(c) \leq k \) (Theorem 2.28(iv)). In particular, \( c < b \). On the other hand, if \( t > c \), then \( f(t) > k \) as otherwise \( t \in E \). Choosing any sequence \( (t_n) \) from \( (c, b] \) that converges to \( c \), it follows that \((f(t_n)) \) converges to \( f(c) \) (by continuity of \( f \) again) and further \( f(c) \geq k \) (another application of Theorem 2.28(iv)). \( \square \)

**Corollary 4.17** (Brower’s fixed point theorem). If \( f : [a, b] \to [a, b] \) is continuous, then there exists a point \( p \in [a, b] \) such that \( f(p) = p \).

**Proof.** Define \( g : [a, b] \to \mathbb{R} \) by \( g(x) = x - f(x) \). Observe that \( g \) is continuous, \( g(a) = a - f(a) \leq 0 \) and \( g(b) = b - f(b) \geq 0 \). Hence, by Theorem 4.16, there is a point \( p \) such that \( g(p) = 0 \); that is \( p - f(p) = 0 \). \( \square \)

**Corollary 4.18.** If \( f : [a, b] \to \mathbb{R} \) is continuous, then there exists \( c \leq d \) such that \( f([a, b]) = [c, d] \).

**Proof.** Let \( c \) and \( d \) denote the minimum and maximum values of \( f \) that exist by Theorem 4.14. In particular, \( f([a, b]) \subset [c, d] \). Moreover, there exists points \( u \) and \( v \) in \( [a, b] \) such that \( f(u) = c \) and \( f(v) = d \). Assuming \( u < v \), given any \( c \leq k \leq d \), there is a point \( u \leq z \leq v \) such that \( f(z) = k \). Hence \( f(C) \supset [c, d] \). The case \( u \geq v \) is similar. \( \square \)

### 4.3. Open and closed sets - elementary topology of the real line.

**Definition 4.19.** The **complement** of a subset \( C \) of \( \mathbb{R} \) is the set

\[ \hat{C} = \{x \in \mathbb{R} : x \notin C\} \]

If \( B \) is also a subset of \( \mathbb{R} \), then

\[ B \setminus C = B \cap \hat{C} \]

In particular,

\[ \hat{C} = \mathbb{R} \setminus C \]

**Definition 4.20.** Given \( \delta > 0 \), the **\( \delta \)-neighborhood** of a point \( a \in \mathbb{R} \) is the set

\[ N_\delta(a) = (a - \delta, a + \delta) \]

A subset \( O \) of \( \mathbb{R} \) is **open** if for each \( p \in O \) there exists a \( \delta > 0 \) such that \( N_\delta(p) \subset O \).

A subset \( C \) of \( \mathbb{R} \) is **closed** if \( \hat{C} \) is open.

**Proposition 4.21.** Given \( r > 0 \) and \( a \in \mathbb{R} \) the set \( N_r(a) \) is open.

**Proof.** To prove that \( N_r(a) \) is open, let \( b \in N_r(a) \) be given. Choose \( \delta = r - |a - b| > 0 \). Now, if \( x \in N_\delta(b) \), then \( |x - a| \leq |x - b| + |b - a| < \delta + |a - b| = r \). Thus, \( N_\delta(b) \subset N_r(a) \). \( \square \)
Example 4.22. Show that the set $C = [0, 1) \subset \mathbb{R}$ is neither open nor closed.

Solution. Observe that $0 \in [0, 1)$, but for each $\delta > 0$ we have $N_\delta(0) \not\subset [0, 1)$ since $-\frac{\delta}{2} \notin N_\delta(0)$, but $-\frac{\delta}{2} \notin C$. Hence $C$ is not open.

Now $\tilde{C} = (-\infty, 0) \cup [1, \infty)$. An argument similar to that above shows, for each $\delta > 0$, that $N_\delta(1) \not\subset [1, \infty)$ and hence $\tilde{C}$ is not open and therefore $C$ is not closed.

□

Proposition 4.23. Suppose $I$ is a set. If for each $i \in I$ the set $O_i \subset \mathbb{R}$ is open, then

$$O = \bigcup_{i \in I} O_i$$

is open; that is, an arbitrary union of open sets is open.

Proof. Let $x \in O$ be given. There is a $j \in I$ such that $x \in O_j$. Since $O_j$ is open and $x \in O_j$, there is a $\delta > 0$ such that $N_\delta(x) \subset O_j$. Since $O_j \subset O$, it follows that $N_\delta(x) \subset O$ and hence $O$ is open.

□

Example 4.24. For $a \in \mathbb{R}$, the set $(a, \infty) = \bigcup_{i=0}^{\infty} (a + i, a + i + 2)$ is open, since each $(a + i, a + i + 2) = N_1(a + i + 1)$ is open by Proposition 4.21.

Example 4.25. To see that the set $C = [0, 1)$ is closed, simply observe that $\tilde{C} = (-\infty, 0) \cup (1, \infty)$ is the union of open sets and is hence open.

Proposition 4.26. Suppose $n \in \mathbb{N}^+$. If $O_1, \ldots, O_n \subset \mathbb{R}$ are open sets, then so is

$$O = \bigcap_{j=1}^{n} O_j.$$

Proof. Let $a \in O$ be given. Thus $a \in O_j$ for each $1 \leq j \leq N$. Since each $O_j$ is open, there $\delta_j$ such that

$$N_{\delta_j}(a) \subset O_j, \quad 1 \leq j \leq N.$$

Let $\delta = \min\{\{\delta_1, \ldots, \delta_N\}\} > 0$ and note that $N_{\delta}(a) \subset \cap N_{\delta_j}(a) \subset O$.

□

Example 4.27. Let $U_n = (0, 1 + \frac{1}{n})$. Then each $U_n$ is open, but

$$(0, 1] = \bigcap_{n=1}^{\infty} U_n$$

is not. Thus an infinite intersection of open sets need not be open.

Note that a finite union of closed sets is closed and an arbitrary intersection of closed sets is closed by the definition of closed set and Propositions 4.26 and 4.23 respectively. See Problem 4.16.

Suppose $f : X \to Y$ is a function and $T \subset Y$. The inverse image of $T$ under $f$, is

$$f^{-1}(T) = \{x \in X : f(x) \in Y\}.$$

Theorem 4.28. Suppose $D \subset \mathbb{R}$ is open and $f : D \to \mathbb{R}$. The function $f$ is continuous if and only if $f^{-1}(U)$ is open for every open set $U \subset \mathbb{R}$.

Proof. Suppose $f$ is continuous and $U \subset \mathbb{R}$ is open. To prove that $V = f^{-1}(U)$ is open, let $x \in V$ be given. Thus $y = f(x) \in U$. Since $U$ is open, there is an $\epsilon > 0$ such that if $|z - y| < \epsilon$, then $z \in U$. Since $f$ is continuous (at $x$) and $D \subset \mathbb{R}$ is open, there is a $\delta > 0$ such that if $|s - x| < \delta$, then $|f(s) - f(x)| < \epsilon$. Hence, $f(s) \in U$ and therefore $s \in V$. Thus $(x - \delta, x + \delta) \subset V$ and $V$ is open.

The converse is left as an exercise. See Problem 4.3. □
4.4. Closed sets.

Proposition 4.29. A subset \( C \) of \( \mathbb{R} \) is closed if and only if \( C \) contains all its accumulation points.

Proof. First suppose \( a \) is an accumulation point of \( C \), but \( a \notin C \); that is \( a \in \bar{C} \). Because \( a \) is an accumulation point of \( C \), given \( \delta > 0 \) the set

\[ N_\delta(a) \cap C \neq \emptyset. \]

Hence, for every \( \delta > 0 \), \( N_\delta(a) \not\subset \bar{C} \). Thus \( \bar{C} \) is not open and thus \( C \) is not closed.

Conversely, suppose \( C \) is not closed; that is, \( \bar{C} \) is not open. Hence, there exists a point \( a \in \bar{C} \) such that for every \( \delta > 0 \), \( N_\delta(a) \not\subset \bar{C} \).

Thus, for every \( \delta > 0 \), \( N_\delta(a) \cap C \neq \emptyset \).

Since \( a \notin C \), it follows that \( a \) is an accumulation point of \( C \). \qed

Proposition 4.30. If \( C \subset \mathbb{R} \) is nonempty and bounded above, then either \( \sup(C) \in C \) or \( \sup(C) \) is an accumulation point of \( C \). If \( C \) is also closed, then \( C \) contains its least upper bound; that is \( \sup(C) \in C \).

Proof. The hypotheses imply that \( C \) has a least upper bound, say \( b \). Given \( \delta > 0 \), there is a \( c \in C \) such that \( b - \delta < c \leq b \). Thus, \( b \) is either in \( C \) or \( b \) an accumulation point of \( C \). If in addition \( C \) is closed, then, by Proposition 4.29, \( b \in C \). \qed

Proposition 4.31. If \( C \subset \mathbb{R} \) is closed and \( (a_n) \) is a sequence from \( C \) that converges to \( A \in \mathbb{R} \), then \( A \in C \).

Proof. Let \( A = \lim a_n \). If \( A = a_n \) for some \( n \), then \( A \in C \). Otherwise, by Problem 2.11, \( A \) is a limit point of \( C \). Since \( C \) is closed, Proposition 4.29 implies that \( A \in C \). \qed

Proposition 4.32. Suppose \( C \subset \mathbb{R} \) is closed and bounded. If \( (a_n) \) is a sequence from \( C \), then \( (a_n) \) has a subsequence that converges to some point of \( C \).

Proof. Since \( C \) is bounded, so is \( (a_n) \). Hence, by Theorem 2.46, there is a subsequence \( (a_{n_k}) \) of \( (a_n) \) that converges to some \( A \in \mathbb{R} \). By Proposition 4.31, \( A \in C \). \qed

4.5. Continuous functions on closed bounded sets.

Proposition 4.33. Suppose \( C \) is nonempty, closed and bounded. If \( f : C \to \mathbb{R} \) is continuous, then \( f(C) \) is closed and bounded too.

Proof. Suppose \( f(C) \) is not bounded above. In this case, for each \( n \) there exists \( y_n \in C \) such \( y_n \geq n \). For each \( n \) there is an \( x_n \in C \) such that \( f(x_n) = y_n \). There is a subsequence \( (x_{n_k}) \) of \( (x_n) \) converging to some \( z \in C \) by Proposition 4.32. By continuity of \( f \), the sequence \( (f(x_{n_k}) = y_{n_k}) \) is convergent and thus bounded, a contradiction which shows \( f(C) \) is in fact bounded above. Replacing \( f \) by \(-f\) and applying what has already been proved shows \( f(C) \) is bounded below too.
By Proposition 4.29, to see that \( f(C) \) is closed it suffices to show that it contains all its accumulation points. Accordingly, suppose \( p \) is an accumulation point of \( f(C) \). In particular, there is a sequence \( (y_n) \) from \( f(C) \) that converges to \( p \). For each \( n \) there is an \( x_n \in C \) such that \( y_n = f(x_n) \). By Proposition 4.32, there is a subsequence \( (x_{nk}) \) that converges to some \( z \in C \). By continuity of \( f \), the sequence \( (y_{nk} = f(x_{nk})) \) converges to \( f(z) \). But \( (y_{nk}) \) converges to \( p \). Hence \( f(z) = p \) and thus \( p \in f(C) \).

**Corollary 4.34 (EVT II).** If \( C \) is nonempty, closed and bounded and if \( f : C \rightarrow \mathbb{R} \) is continuous, then there exists a point \( z \in C \) such that \( f(z) \geq f(x) \) for every \( x \in C \); that is \( f \) attains its extrema on \( C \).

**Proof.** By Proposition 4.33, \( f(C) \) is closed and bounded. Hence, by Proposition 4.31, sup\((f(C)) \in f(C) \).

### 4.6. Inverse functions.

**Definition 4.35.** Given a set \( A \), the function \( id_A : A \rightarrow A \) defined by \( id_A(a) = a \) is called the identity function.

**Proposition 4.36.** Given a function \( f : A \rightarrow B \), there exists a function \( g : B \rightarrow A \) such that \( f \circ g = id_B \) and \( g \circ f = id_A \) if and only if \( f \) is one-one and onto. Moreover, in this case, \( g \) is unique.

**Proof.** First suppose that \( f \) is one-one and onto. Define \( g : B \rightarrow A \) as follows. Given \( b \in B \) there is a unique \( a \in A \) such that \( b = f(a) \) (because \( f \) is both one-one and onto). Let \( g(b) = a \). Then \( g(f(a)) = g(b) = a \) and \( f(g(b)) = f(a) = b \).

Conversely, suppose there is a \( g \) such that both \( f \circ g = id_B \) and \( g \circ f = id_A \). To prove that \( f \) is one-one, suppose \( f(x) = f(y) \). Then \( x = g(f(x)) = g(f(y)) = y \). To prove that \( f \) is onto, let \( b \in B \) be given and observe that \( f(g(b)) = b \).

Finally, to see that \( g \) is unique suppose also that \( f \circ h = id_B \). It follows that \( g \circ f \circ h = f \circ id_B = f \) and also \( g \circ f \circ h = g \circ id_B = g \).

**Definition 4.37.** The function \( g \) in Proposition 4.36 (assuming it exists) is called the inverse of \( f \) and is denoted \( f^{-1} \).

**Remark 4.38.** Of course if \( f \) has an inverse \( g \), then by Proposition 4.36, \( g \) has an inverse and \( g^{-1} = f \).

Do not confuse inverse image with inverse function. Given \( f : X \rightarrow Y \), the inverse image \( f^{-1}(B) \) is defined for any \( B \subset Y \) whether or not \( f \) is invertible. If \( f \) does have an inverse \( g \), then \( f^{-1}(B) = g(B) \). Also note, in the case \( f^{-1} \{ \{y\} \} = \{f^{-1}(y)\} \) for \( y \in Y \).

**Example 4.39.** Define \( f : [0, \infty) \rightarrow [0, \infty) \) by \( f(x) = x^2 \). From Problem 1.14 it follows that \( f \) is one-one. On the other hand, for each \( b > 0 \) the Intermediate Value Theorem (Theorem 4.16) implies that \( f([0, b]) = [0, b^2] \). Consequently, \( f \) is onto and thus has an inverse. Of course, the inverse of \( f \) is the square root function.

The functions \( \log \) and \( \exp \) are inverses of each other.

The function \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = \sin(x) \) does not have an inverse. However, the function \( F : [-\pi/2, \pi/2] \rightarrow [-1, 1] \) defined by \( F(x) = \sin(x) \) is one-one and, using the Intermediate Value Theorem (and continuity of \( \sin \)) it is onto. Thus \( F \) has an inverse \( G \) we call the arcsin. Thus \( G : [-1, 1] \rightarrow [-\pi/2, \pi/2] \) and for \( x \in [-\pi/2, \pi/2] \), \( G(F(x)) = x \) and \( F(G(y)) = y \) for \(-1 \leq y \leq 1 \).
**Theorem 4.40.** If \( C \subset \mathbb{R} \) is nonempty, closed and bounded, \( E \subset \mathbb{R} \) and if \( f : C \to E \) is one-one and onto, then \( f^{-1} : E \to C \) is continuous.

**Proof.** For notational ease, let \( g = f^{-1} \). Fix \( w \in E \) and, arguing by contradiction, suppose \( g \) is not continuous at \( w \). In this case, there exists an \( \eta > 0 \) such that for each \( n \in \mathbb{N}^+ \) there exists \( y_n \in E \) such that \( |y_n - w| < \frac{1}{n} \), but

\[
|g(y_n) - g(w)| \geq \eta.
\]  

(4)

Let \( x_n = g(y_n) \). Since \((x_n)\) is a sequence from the closed and bounded set \( C \), it has a subsequence \((x_{n_k})\) that converges to some \( z \in C \) by Proposition 4.32. By continuity of \( f \), the sequence \((f(x_{n_k}) = y_{n_k})\) converges to \( f(z) \). But \((y_{n_k})\) converges to \( w \) by construction. Thus \( w = f(z) \) so that \( z = g(w) \). Hence \((g(y_{n_k}) = x_{n_k})\) converges to \( z = g(w) \), contradicting (4). \( \square \)

There is also a simple proof using Problem 2.14.

### 4.7. Uniform continuity.

**Definition 4.41.** Given \( D \subset \mathbb{R} \), a function \( f : D \to \mathbb{R} \) is uniformly continuous if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( x, y \in D \) and \( |x - y| < \delta \), then \( |f(x) - f(y)| < \epsilon \).

**Example 4.42.** The function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x \) is uniformly continuous.

**Example 4.43.** The function \( g : \mathbb{R} \to \mathbb{R} \) defined by \( g(x) = x^2 \) is not uniformly continuous.

**Solution.** To show \( g \) is not uniformly continuous, choose \( \epsilon_0 = 1 \). Given \( \delta > 0 \) choose \( x = \frac{1}{\delta} \) and \( y = x + \frac{\delta}{2} \). Then \( |x - y| < \delta \), but

\[
|f(y) - f(x)| = 1 + \frac{\delta^2}{4} \geq \epsilon_0.
\]

(4)

**Example 4.44.** Define \( h : [1, \infty) \to \mathbb{R} \) by \( h(x) = \sqrt{x} \).

**Solution.** To see that \( h \) is uniformly continuous, let \( \epsilon > 0 \) be given. Choose \( \delta = 2\epsilon \). If \( 1 \leq x, y \) and \( |x - y| < \delta \), then

\[
|h(x) - h(y)| = |\sqrt{x} - \sqrt{y}|
\]

\[
= \frac{|x - y|}{\sqrt{x} + \sqrt{y}}
\]

\[
\leq \frac{|x - y|}{2}
\]

\[
< \frac{\delta}{2} = \epsilon.
\]

(5)

**Theorem 4.45.** If \( C \subset \mathbb{R} \) is nonempty, closed and bounded and if \( f : C \to \mathbb{R} \) is continuous, then \( f \) is uniformly continuous.

**Proof.** Arguing by contradiction, suppose \( f \) is not uniformly continuous. In this case there is an \( \epsilon_0 > 0 \) such that for every \( n \in \mathbb{N}^+ \) there exists points \( x_n, y_n \in C \) such that

\[
|x_n - y_n| < \frac{1}{n},
\]

(5)
but
\[ |f(x_n) - f(y_n)| \geq \epsilon_0. \]
By Proposition 4.32 there is a point \( z \in C \) and a subsequence \((x_{n_k})_k\) of \((x_n)_n\) that converges to \( z \). As an exercise for the reader, use (5) to show that \((y_{n_k})_k\) converges to \( z \) also.

By continuity of \( f \), the sequences \((f(x_{n_k}))_k\) and \((f(y_{n_k}))_k\) both converge to \( f(z) \), contradicting (6). \( \square \)

Example 4.46. Given \( a < b \), the function \( f : [a, b] \to \mathbb{R} \) defined by \( f(x) = x^2 \) is uniformly continuous, since \( f \) is continuous and its domain is closed and bounded.

Proposition 4.47. If \( D \subset \mathbb{R} \), \( f : D \to \mathbb{R} \) is uniformly continuous and if \((x_n)\) is a Cauchy sequence from \( D \), then \((f(x_n))\) is a Cauchy sequence.

The proof is left as an exercise for the gentle reader (Problem 4.9)

4.8. Lipschitz continuity. \(^6\)

Definition 4.48. For \( D \subset \mathbb{R} \), a function \( f : D \to \mathbb{R} \) is Lipschitz continuous if there exists a \( C \) such that
\[ |f(x) - f(y)| \leq C|x - y| \]
for all \( x, y \in D \).

If \( C \) can be chosen such that \( 0 \leq C < 1 \), then \( f \) is a contraction or contraction mapping.

Remark 4.49. If \( f \) is Lipschitz continuous, then \( f \) is uniformly continuous.

Example 4.50. The function \( f : [0, 1] \to \mathbb{R} \) defined by \( f(x) = \sqrt{x} \) is uniformly continuous, but not Lipschitz continuous.

Because \( f \) is a continuous function on a closed and bounded set it is uniformly continuous. To see that \( f \) is not Lipschitz continuous, let \( C > 0 \) be given. With \( 0 < x < C^{-\frac{1}{2}} \), we have \( C < \frac{1}{\sqrt{x}} \) and hence
\[ C|0 - x| = Cx < \sqrt{x} = |f(0) - f(x)|. \]


Problem 4.2. Prove finite sets are closed.

Problem 4.3. Prove Proposition 4.28.

Problem 4.4. Show, if \( f : [a, b] \to \mathbb{R} \) is an increasing function and the range of \( f \) is an interval, then \( f \) is continuous.

Problem 4.5. Prove the converse of Proposition 4.31.

Problem 4.6. Define \( f : (0, 1) \to \mathbb{R} \) by \( f(x) = \frac{1}{x} \). Show \( f \) is not uniformly continuous.

Problem 4.7. Suppose \( f : [0, \infty) \to \mathbb{R} \) is continuous. Show, if there is a \( b > 0 \) such that \( f|_{(b, \infty)} \) is uniformly continuous, then \( f \) is uniformly continuous.

Show that \( f : [0, \infty) \to \mathbb{R} \) defined by \( f(x) = \sqrt{x} \) is uniformly continuous. (Compare with Example 4.44.)

\(^6\)This section is optional.
Problem 4.8. Show, if \( f : [a, b] \to \mathbb{R} \) is continuous and \( \lim_{x \to b} f(x) \) exists, then \( f \) is uniformly continuous and bounded. (Suggestion: Let \( L \) denote the limit and define \( g : [a, b] \to \mathbb{R} \) by \( g(x) = f(x) \) if \( a \leq x < b \) and \( g(b) = L \) and prove \( g \) is continuous.)


Problem 4.10. Show if \( f : [a, b] \to \mathbb{R} \) is uniformly continuous, then \( \lim_{x \to b} f(x) \) exists. Conclude that \( f \) is bounded.

Problem 4.11. Suppose \( S \subset \mathbb{R} \). Show that the set \( S' \) of limit points of \( S \) is closed.

Problem 4.12. Use the IVP to show that there is a real number \( s > 0 \) such that \( s^2 = 2 \).

Problem 4.13. Show, if \( p \) is a polynomial of odd degree, then \( p \) has a zero.

Problem 4.14. Show the equation \( \exp(x) = 3x \) has a solution in \([0, 1]\).

Problem 4.15. Suppose \( a, b \in \mathbb{R}, a < b \). Show, if \( f : [a, b] \to \mathbb{R} \) is continuous and one-one, then \( f \) is strictly monotone. You may wish, or not, to use one of the two proof outlines below.

1) Suppose \( f(a) < f(b) \).
   (i) Show, \( f(a) < f(x) < f(b) \) for all \( a < x < b \);
   (ii) Show, if \( a < x < y < b \), then \( f(a) < f(x) < f(y) \) by arguing by contradiction;
   (iii) Complete the proof.

2) Suppose \( f(a) < f(b) \). Given \( a \leq x < y \leq b \), define \( u, v, h : [0, 1] \to \mathbb{R} \) by \( u(t) = tx + (1 - t)a, v(t) = ty + (1 - t)b \) and \( h(t) = f(v(t)) - f(u(t)) \).
   (a) Explain why \( h \) is continuous;
   (b) Show \( u(t) < v(t) \) for all \( 0 \leq t \leq 1 \);
   (c) Show \( h \) is never 0;
   (d) Show \( h(0) > 0 \);
   (e) Conclude \( h(1) > 0 \);
   (f) Finish the proof.

Problem 4.16. Let \( X \) be a set and suppose \( A, B \subset X \). Show,

(i) \( \overline{A \cup B} = \overline{A} \cap \overline{B} \); and
(ii) \( A \cap B = \overline{A} \cup \overline{B} \).

More generally, let \( P(X) \) denote its power set; that is \( P(X) \) is the set of all subsets of \( X \). Let \( \emptyset \neq \mathcal{F} \subset P(X) \) and show,

(i) \( \overline{\bigcup_{F \in \mathcal{F}} F} = \bigcap_{F \in \mathcal{F}} \overline{F} \); and
(ii) \( \overline{\bigcap_{F \in \mathcal{F}} F} = \bigcup_{F \in \mathcal{F}} \overline{F} \).

Show, if \( \mathcal{F} \subset P(\mathbb{R}) \) is a non-empty collection of closed subsets of \( \mathbb{R} \), then \( C = \bigcup_{F \in \mathcal{F}} F \) is closed.

Likewise, show if \( N \in \mathbb{N}^+ \) and \( C_1, \ldots, C_N \subset \mathbb{R} \) are closed, then \( C = \bigcup_{n=1}^{N} C_n \) is closed.

Problem 4.17. Suppose \( A, B, D \subset \mathbb{R} \) and \( f : D \to \mathbb{R} \). Show

1) \( f^{-1}(\overline{A}) = \overline{f^{-1}(A)} \);
2) \( f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \);
3) \( f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \).

Problem 4.18. Find the indicated inverse images, \( f^{-1}(S) \).
(i) Define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = x^2 \) and let
   (a) \( S = [1, 4) \);
   (b) \( S = (-4, -1] \);
(ii) Define \( f : [0, \infty) \to \mathbb{R} \) by \( f(x) = x^2 \) and let \( S = [1, 4) \);
(iii) Define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = 1 \) if \( x \in \mathbb{Q} \) and \( f(x) = 0 \) if \( x \notin \mathbb{Q} \) and let
   (a) \( S = [0, 1) \);
   (b) \( S = [1, \infty) \);
   (c) \( S = (0, 1) \);
   (d) \( S = [0, 1] \);

**Problem 4.19.** Suppose \( f : \mathbb{R} \to \mathbb{R} \) and \( f(Q) \subset \tilde{Q} \) and \( f(\tilde{\mathbb{Q}}) \subset \mathbb{Q} \). Show the range of \( h = f \circ f \) is at most countable and contains at least two points. Conclude that \( f \) can not be continuous.

**Problem 4.20.** Suppose \( f : \mathbb{R} \to \mathbb{R} \). Show \( f \) is continuous if and only if \( f^{-1}(C) \) is closed for every closed set \( C \subset \mathbb{R} \).

**Problem 4.21.** Use Theorem 4.28 to give a (comic book) proof that if \( f, g : \mathbb{R} \to \mathbb{R} \) are continuous, then so is the composition \( f \circ g \).

**Problem 4.22.** Give an alternate proof of Theorem 4.40 based on Problem 2.14.

5. **Differentiation**

5.1. **Definitions and examples.**

**Definition 5.1.** Suppose \( D \subset \mathbb{R} \), \( f : D \to \mathbb{R} \) and \( a \in D \) is an accumulation point of \( D \). Define \( g : D \setminus \{a\} \to \mathbb{R} \) by
   \[
g(x) = \frac{f(x) - f(a)}{x - a}.
   \]
   If \( \lim_{x \to a} g(x) \) exists, then \( f \) is differentiable at \( a \) and the limit is the derivative of \( f \) at \( a \), denoted \( f'(a) \).

**Example 5.2.** Show \( f : \mathbb{R} \to \mathbb{R} \) defined by
   \[
f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}
   \]
is differentiable at 0.

**Solution.** To see that \( f \) is differentiable at 0, consider \( g : \mathbb{R} \setminus \{0\} \to \mathbb{R} \) defined by
   \[
g(x) = \frac{f(x) - f(0)}{x - 0} = x \sin\left(\frac{1}{x}\right).
   \]
   A routine argument (Problem 3.5) shows \( \lim_{x \to 0} g(x) = 0 \). Thus \( f \) is differentiable at 0 and \( f'(0) = 0 \). \( \square \)

**Example 5.3.** Show \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = |x| \) is not differentiable at 0.

**Proposition 5.4.** Suppose \( D \subset \mathbb{R} \), \( f : D \to \mathbb{R} \) and \( a \) is an accumulation point of \( D \). If \( f \) is differentiable at \( a \), then \( f \) is continuous at \( a \).

**Remark 5.5.** Example 5.3 shows the converse of the proposition is false.
Proof. Note that
\[
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)
\]
exists by hypothesis. Thus,
\[
0 = f'(a) \lim_{x \to a} (x - a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a} (x - a) = \lim_{x \to a} [f(x) - f(a)].
\]
Hence, \( \lim_{x \to a} f(x) = f(a) \) (Problem 3.16).

**Definition 5.6.** Suppose \( D \subset \mathbb{R} \), \( f : D \to \mathbb{R} \) and every point of \( D \) is an accumulation point of \( D \). If \( f \) is differentiable at each \( a \in D \), then \( f \) is **differentiable**.

**Remark 5.7.** Often, when discussing differentiation, \( D \) is an open interval.

If \( f \) is differentiable, then we obtain a function \( f' : D \to \mathbb{R} \).

**Example 5.8.** Show, if \( c \) is a constant, then \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = c \) is differentiable and \( f'(x) = 0 \).

Show \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x \) is differentiable and \( f'(a) = 1 \) (for each \( a \)).

Show \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x^2 \) is differentiable.

### 5.2. Properties of the derivative.

**Theorem 5.9.** Suppose \( D \subset \mathbb{R} \), \( f, g : D \to \mathbb{R} \) and \( a \in D \) is an accumulation point of \( D \). If both \( f \) and \( g \) are differentiable at \( a \), then

(i) \( f + g \) is differentiable at \( a \) and \( (f + g)'(a) = f'(a) + g'(a) \);

(ii) \( fg \) is differentiable at \( a \) and \( (fg)'(a) = f'(a)g(a) + f(a)g'(a) \); and

(iii) if \( g \) is never 0 and \( g'(a) \neq 0 \), then \( h = \frac{1}{g} \) is differentiable at \( a \) and \( h'(a) = -\frac{g'(a)}{g^2(a)} \).

**Proof.** We prove item (ii). The proofs of the other items are similar and omitted. Using properties of limits and Proposition 5.4,
\[
f'(a)g(a) + f(a)g'(a) = g(a) \lim_{x \to a} \frac{f(x) - f(a)}{x - a} + f(a) \lim_{x \to a} \frac{g(x) - g(a)}{x - a}
\]
\[
= g(a) \lim_{x \to a} \frac{f(x) - f(a)}{x - a} + f(a) \lim_{x \to a} \frac{g(x) - g(a)}{x - a}
\]
\[
= g(x) \lim_{x \to a} \frac{(f(x) - f(a))}{x - a} + f(a) \lim_{x \to a} \frac{g(x) - g(a)}{x - a}
\]
\[
= \lim_{x \to a} (f(x) - f(a))g(x) + (g(x) - g(a))f(a)
\]
\[
= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a}.
\]
Hence \( fg \) is differentiable at \( a \) and \( (fg)'(a) = f'(a)g(a) + f(a)g'(a) \). □

**Remark 5.10.** It follows from Theorem 5.9 and Example 5.8 that a rational function is differentiable (on its domain).
Theorem 5.11 (Chain Rule). Suppose $f : D \to E$, $g : E \to \mathbb{R}$ and $a \in D$ and $b = f(a) \in E$ are accumulation points of $D$ and $E$ respectively. If $f$ is differentiable at $a$ and $g$ is differentiable at $b$, then $h = g \circ f$ is differentiable at $a$ and $h'(a) = g'(f(a))f'(a)$.

The proof will use the fact that the assumption that $g$ is differentiable at $b$ implies continuity at $b$ of the function $F : E \to \mathbb{R}$

$$F(y) = \begin{cases} \frac{g(y)-g(b)}{y-b} & y \neq b \\ g'(b) & y = b. \end{cases}$$

Proof. Since $F$ (above) is continuous at $b$, Proposition 4.10 gives

$$\lim_{x \to a} F(f(x)) = F(b) = g'(b).$$

Note that

$$F(f(x))\frac{f(x) - f(a)}{x - a} = \frac{h(x) - h(a)}{x - a}, \quad a \neq x \in D,$$

even if $f(x) = f(a) = b$. Thus, routine properties of limits gives,

$$\lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{g(f(x)) - g(b)}{x - a} = \lim_{x \to a} F(f(x))\frac{f(x) - f(a)}{x - a} = \lim_{x \to a} F(f(x)) \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = g'(b)f'(a).$$

□

In the statement of Proposition 5.12 immediately below, by open interval $I \subset \mathbb{R}$ we mean a set of one of the forms $(-\infty, \infty) = \mathbb{R}$, $(-\infty, d)$, $(c, d)$, or $(c, \infty)$ where, when appropriate $c < d$. Observe that if $a < c < b$ are real numbers and $f : (a, b) \to \mathbb{R}$ is continuous and (strictly) increasing, the IVT implies the range of $f$ is an open interval (Problem 5.11).

Proposition 5.12 (The derivative of an inverse). Suppose $I \subset \mathbb{R}$ be an open interval, $f : (a, b) \to I$ is continuous, (strictly) increasing and onto. If $f$ is differentiable at $a < c < b$ and $f'(c) \neq 0$, then $f^{-1} : I \to (a, b)$ is differentiable at $f(c)$ and

$$(f^{-1})'(f(c)) = \frac{1}{f'(c)}.$$

Proof. For notational ease, let $g = f^{-1}$ and $d = f(c)$. The function $F : (a, b) \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} \frac{x-c}{f(x)-f(c)} & x \neq c \\ \frac{1}{f'(c)} & x = c \end{cases}$$

is continuous, including at $c$. Since also $g$ is continuous (Theorem 4.40) and the composition of continuous functions is continuous (Proposition 4.10), it follows that

$$\lim_{y \to d} F(g(y)) = F(g(d)) = F(c).$$

Noting that $F(g(y)) = \frac{g(y) - d}{y - d}$ completes the proof. □
5.3. The Mean Value Theorem.

**Definition 5.13.** Suppose $D \subseteq \mathbb{R}$. A function $f : D \rightarrow \mathbb{R}$ has a local (relative) minimum at $c \in D$ if there is a $\delta > 0$ such that if $y \in D$ and $|c - y| < \delta$, then $f(c) \leq f(y)$.

**Lemma 5.14.** Suppose $c \in D \subseteq \mathbb{R}$ and there is an $\eta > 0$ such that $(c - \eta, c + \eta) \subseteq D$. If $f : D \rightarrow \mathbb{R}$ has a local minimum at $c$ and if $f$ is differentiable at $c$, then $f'(c) = 0$.

**Lemma 5.15** (Rolle’s Theorem). Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous. If $f(a) = f(b)$ and if $f$ is differentiable on the open interval $(a, b)$, then there is a point $a < c < b$ such that $f'(c) = 0$.

**Proof.** Without loss of generality, it can be assumed that $f$ is not constant. Since $f$ is continuous on the closed bounded interval $[a, b]$, it attains its extrema. Since $f(a) = f(b)$ and $f$ is not constant, $f$ attains either its maximum or minimum at some point $a < c < b$. From Lemma 5.14 it follows that $f'(c) = 0$. □

**Theorem 5.16** (Cauchy Mean Value Theorem). If $f, g : [a, b] \rightarrow \mathbb{R}$ are continuous and differentiable at each point in $(a, b)$, then there is a $c$ with $a < c < b$ so that $(f(b) - f(a))g'(c) = f'(c)(g(b) - g(a))$.

**Proof.** Define $F : [a, b] \rightarrow \mathbb{R}$ by $F(x) = (f(x) - f(a))(g(b) - g(a)) - (f(b) - f(a))(g(x) - g(a))$. Then $F(a) = F(b) = 0$ and $F$ satisfies the hypotheses of Rolle’s Theorem. Hence there is a $a < c < b$ such that $F'(c) = 0$; that is $f'(c)(g(b) - g(a)) = f'(c)(g(b) - g(a))$. □

Choosing $g(x) = x$ in the Cauchy Mean Value Theorem captures the usual Mean Value Theorem.

**Corollary 5.17** (Mean Value Theorem). If $f : [a, b] \rightarrow \mathbb{R}$ is continuous and differentiable at each point in $(a, b)$, then there exists $a < c < b$ so that $f(b) - f(a) = f'(c)(b-a)$.

**Corollary 5.18.** Suppose $f : (u, v) \rightarrow \mathbb{R}$ is differentiable.

The function $f$ is increasing if and only if $f' \geq 0$ (meaning $f'(x) \geq 0$ for all $x \in (u,v)$).

The function $f$ is constant if and only if $f' = 0$ (meaning $f'(x) = 0$ for each $u < x < v$).

**Example 5.19.** Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Show $f$ is differentiable and $f'(0) > 0$, but there is no interval properly containing 0 on which $f$ is increasing.

**Solution.** From Example 5.2 and Theorem 5.9, $f'(0) = 1 > 0$. That $f$ is differentiable at each point $x \neq 0$ follows from standard facts about differentiation and the derivative of the sin function in particular. For $n \in \mathbb{Z} \setminus \{0\}$, let

$$x_n = \frac{1}{2n\pi}$$

and note that $f'(x_n) = -1 < 0$. Since any interval $0 \in I \subseteq \mathbb{R}$ that properly contains 0 contains $x_n$ for some $n$, we conclude, from Corollary 5.18, that $f$ is not increasing on $I$. □
5.4. Further topics - applications of the MVT.

**Theorem 5.20** (Taylor’s Theorem). Let $I = (u, v) \subset \mathbb{R}$ be an open interval, $n \in \mathbb{N}$, and suppose $f : I \to \mathbb{R}$ is $(n + 1)$ times differentiable. If $u < a < b < v$, then there is a $c$ such that $a < c < b$ and

$$f(b) = \sum_{j=0}^{n} \frac{f^{(j)}(a)(b-a)^j}{j!} + \frac{f^{(n+1)}(c)(b-a)^{n+1}}{(n+1)!}.$$

**Proof.** Define $R_n : I \to \mathbb{R}$ by

$$R_n(x) = f(b) - \sum_{j=0}^{n} \frac{f^{(j)}(x)(b-x)^j}{j!}.$$

There is a $K$ so that $R_n(a) = K \frac{(b-a)^{n+1}}{(n+1)!}$ and the goal is to prove there is a $a < c < b$ such that $K = f^{(n+1)}(c)$.

Let

$$\varphi(x) = R_n(x) - K \frac{(b-x)^{n+1}}{(n+1)!}.$$

Note that $\varphi : [a, b] \to \mathbb{R}$ is continuous and differentiable on $(a, b)$. Moreover, $\varphi(a) = 0 = \varphi(b)$. Thus, by the MVT, there is a $a < c < b$ such that $\varphi'(c) = 0$. Since,

$$\varphi'(x) = -f^{(n+1)}(x) \frac{(b-x)^n}{n!} + K \frac{(b-x)^n}{n!},$$

it follows that

$$0 = (-f^{(n+1)}(c) + K) \frac{(b-c)^n}{n!}.$$

The conclusion of the theorem follows. \qed

**Proposition 5.21** (A version of L’hopitals rule). Suppose $f, g : (a, b) \to \mathbb{R}$ and

(i) both $f$ and $g$ are differentiable;
(ii) $\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$; and;
(iii) both $g$ and $g'$ are never 0.

If

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

exists, then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$$

**Proof.** The functions $f$ and $g$ extend to be continuous on $[a, b]$ by defining $f(a) = g(a) = 0$.

Let

$$L = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$$

Given $\epsilon > 0$ there is a $\delta > 0$ such that if $a < y < a + \delta$, then

$$|L - \frac{f'(y)}{g'(y)}| < \epsilon.$$
From the Cauchy mean value theorem and hypothesis (iii), given \( a < x < a + \delta \) there is an \( a < c < x \) such that

\[
\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}.
\]

Thus, if \( a < x < a + \delta \), then,

\[
|L - \frac{f(x)}{g(x)}| = |L - \frac{f'(c)}{g'(c)}| < \epsilon.
\]

\( \square \)

5.5. Problems.

**Problem 5.1.** Determine which of the following functions \( f : \mathbb{R} \to \mathbb{R} \) are differentiable at 0.

(i) \( f(x) = x \sin\left(\frac{x}{2}\right) \) for \( x \neq 0 \) and \( f(0) = 0 \).

(ii) \( f(x) = x^3 \sin\left(\frac{x}{2}\right) \) for \( x \neq 0 \) and \( f(0) = 0 \).

(iii) \( f(x) = x^2 \) for \( x \leq 0 \) and \( f(x) = x^3 \) for \( x > 0 \).

(iv) \( f(x) = x^2 \) for \( x \leq 0 \) and \( f(x) = x \) for \( x > 0 \).

**Problem 5.2.** Suppose \( f : \mathbb{R} \to \mathbb{R} \) and \( 0 \neq a \in \mathbb{R} \). Let \( h : \mathbb{R} \to \mathbb{R} \) denote the function \( h(x) = xf(x) \). Show, if \( h \) is differentiable at \( a \), then so is \( f \).

**Problem 5.3.** Define \( f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1] \) by \( f(x) = \sin(x) \). Assuming we know that \( f \) is differentiable (and its derivative is \( \cos(x) \)), the hypotheses of the Inverse Function Theorem are satisfied for \( f \) and any point \( c \in (-\frac{\pi}{2}, \frac{\pi}{2}) \). Let \( g = f^{-1} \) and find \( g'(f(c)) \).

**Problem 5.4.** Prove Corollary 5.18. Show, if \( f, g : (u,v) \to \mathbb{R} \) are both differentiable and \( f' = g' \), then \( f - g \) is constant.

**Problem 5.5.** Suppose \( f : [a,b] \to \mathbb{R} \) is differentiable. Show, if \( f' > 0 \) (so \( f'(x) > 0 \) for all \( x \in [a,b] \)), then \( f \) is strictly increasing. (In particular, the hypotheses of the Inverse Function Theorem are satisfied.)

Show the result remains true if \( f'(x) > 0 \) for all except possibly one \( x \in [a,b] \).

**Problem 5.6.** Show, if \( f : \mathbb{R} \to \mathbb{R} \) is differentiable and \( f' : \mathbb{R} \to \mathbb{R} \) is bounded, then \( f \) is uniformly continuous.

**Problem 5.7.** Suppose \( f : \mathbb{R} \to \mathbb{R} \). A point \( p \in \mathbb{R} \) is a fixed point of \( f \) if \( f(p) = p \). Show, if \( f \) is differentiable and \( |f'(t)| < 1 \) for all \( t \in \mathbb{R} \), then \( f \) has at most one fixed point.

**Problem 5.8.** Suppose \( f : (0, \infty) \to \mathbb{R} \) is differentiable, \( f(0) = 0 \) and \( f' \) is increasing. Define \( g : (0, \infty) \to \mathbb{R} \) by \( g(x) = \frac{f(x)}{x} \). Prove \( g \) is increasing. (It may help to observe

\[
f'(x) - g(x) = f'(x) - \frac{f(x) - f(0)}{x - 0}.
\]

**Problem 5.9.** Prove the inequalities (assuming the usual properties of \( \sin(x) \) and \( \exp(x) \)).

1. \( |\sin(a) - \sin(b)| \leq |a - b| \), for \( a, b \in \mathbb{R} \);
2. \( 1 + a \leq \exp(a) \) for \( a > 0 \).

**Problem 5.10.** Suppose \( f : \mathbb{R} \to \mathbb{R} \) is differentiable and \( f' \) is differentiable at 0. Show

\[
f''(0) = \lim_{h \to 0} \frac{f(h) + f(-h) - 2f(0)}{h^2}.
\]
Problem 5.11. Show if \( f : (a, b) \to \mathbb{R} \) is continuous and (strictly) increasing, then its range is an open interval (as defined before Proposition 5.12).

6. Riemann Integration

This chapter develops the theory of the Riemann integral of a bounded real-valued function \( f \) on an interval \([a, b] \subset \mathbb{R}\). The approach used, approximating from above and below, is very efficient and intuitive, though a bit limited because it relies on the order structure of \( \mathbb{R} \).

6.1. Definition of the Integral.

Definition 6.1. A partition \( P \) of the interval \([a, b]\) consists of a finite set of points \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \).

Given the partition \( P \), let \( \Delta_j = x_j - x_{j-1} \). Given a bounded function \( f : [a, b] \to \mathbb{R} \), let
\[
m_j = \inf\{f(x) : x_{j-1} \leq x \leq x_j\}
\]
\[
M_j = \sup\{f(x) : x_{j-1} \leq x \leq x_j\};
\]
define the lower and upper sums of \( f \) with respect to \( P \) by
\[
L(P, f) = \sum_{j=1}^{n} m_j \Delta_j
\]
\[
U(P, f) = \sum_{j=1}^{n} M_j \Delta_j;
\]
define the lower and upper Riemann integrals of \( f \) (on \([a, b]\)) by
\[
\int_{a}^{b} f \, dx = \sup\{L(P, f) : P\}
\]
\[
\int_{a}^{b} f \, dx = \inf\{U(P, f) : P\}.
\]
Finally, say \( f \) is Riemann integrable on \([a, b]\) if the upper and lower integrals agree. The set of Riemann integrable functions on \([a, b]\) is denoted by \( \mathcal{R}([a, b]) \). In this case, the common value of the upper and lower integrals is the Riemann integral of \( f \) on \([a, b]\), denoted
\[
\int_{a}^{b} f \, dx.
\]

Example 6.2. For the function \( f : [0, 1] \to [0, 1] \) defined by \( f(x) = 1 \) it is evident that \( U(P, f) = 1 = L(P, f) \) for every \( P \). Hence \( f \) is Riemann integrable and
\[
\int_{0}^{1} 1 \, dx = 1.
\]

Do Problem 6.3.

Example 6.3. Let \( f : [0, 1] \to \mathbb{R} \) denote the indicator function of \([0, 1] \cap \mathbb{Q}\). Thus \( f(x) = 1 \) if \( x \in \mathbb{Q} \) and \( f(x) = 0 \) otherwise. Verify, for any partition \( P \) of \([0, 1]\), that \( L(P, f) = 0 \) and \( U(P, f) = 1 \). Thus
\[
\int_{0}^{1} f \, dx = 0 < 1 = \int_{0}^{1} f \, dx
\]
and so \( f \) is not Riemann integrable (on \([0, 1]\)).

**Remark 6.4.** If \( f : [a, b] \to \mathbb{R} \) and \( P \) is a partition of \([a, b]\), then
\[
L(P, f) \leq U(P, f).
\]

**Definition 6.5.** Let \( P \) and \( Q \) denote partitions of \([a, b]\). We say \( Q \) is a **refinement** of \( P \) if \( P \subset Q \). The **common refinement** of \( P \) and \( Q \) is \( P \cup Q \).

**Lemma 6.6.** Suppose \( f : [a, b] \to \mathbb{R} \) is bounded and \( P \) and \( Q \) are partitions of \([a, b]\).

(i) If \( Q \) is a refinement of \( P \), then
\[
L(P, f) \leq L(Q, f) \leq U(Q, f) \leq U(P, f).
\]

(ii) If \( P \) and \( Q \) are any partitions of \([a, b]\), then
\[
L(P, f) \leq U(Q, f).
\]

(iii) In particular,
\[
\int_a^b f \, dx \leq \int_a^b f \, dx.
\]

**Sketch of proof.** The middle inequality in item (i) has is evident from the definitions (as already noted). The first and third inequalities can be reduced to the following situation: \( P = \{a < b\} \) and \( Q = \{a < t < b\} \) where the result is evident.

To prove item (ii), let \( R \) denote the common refinement of \( P \) and \( Q \) and apply (i) (twice) to obtain
\[
L(P, f) \leq L(R, f) \leq U(R, f) \leq U(Q, f).
\]

To prove (iii), fix a partition \( Q \). Since \( L(P, f) \leq U(Q, f) \), for all partitions \( P \), it follows that
\[
\int_a^b f \, dx \leq U(Q, f).
\]

Since this inequality holds for all \( Q \), the result follows. \( \square \)

**Example 6.7.** Let \( g : [0, 1] \to [0, 1] \) denote the identity function, \( g(x) = x \). Given a positive integer \( n \), let \( P_n \) denote the partition
\[
P_n = \{x_j = \frac{j}{n} : j = 0, \ldots, n\}.
\]

The corresponding upper and lower sums are easily seen to be
\[
U(P_n, g) = \sum_{j=1}^n \frac{j}{n} \frac{1}{n} = \frac{n+1}{2n}
\]
and
\[
L(P_n, g) = \sum_{j=0}^{n-1} \frac{j}{n} \frac{1}{n} = \frac{n-1}{2n}.
\]

It follows that
\[
\int_0^1 x \, dx \leq \frac{1}{2}
\]
and 
\[ \int_{0}^{1} x \, dx \geq \frac{1}{2}. \]
Thus the upper and lower integrals are both \( \frac{1}{2} \). Consequently \( g \) is integrable and its integral is \( \frac{1}{2} \).

Do Problem 6.4.

6.2. Sufficient Conditions for Integrability.

Proposition 6.8. Suppose \( f : [a, b] \to \mathbb{R} \) is bounded.

\( f \in \mathcal{R}([a, b]) \) if and only if for each \( \epsilon > 0 \) there is a partition \( P \) of \([a, b]\) such that
\[ U(P, f) - L(P, f) < \epsilon. \]

Proof. First suppose \( f \in \mathcal{R}([a, b]) \) and let \( \epsilon > 0 \) be given. There exists partitions \( Q, S \) such that
\[ \int_{a}^{b} f \, dx < L(Q, f) + \epsilon, \]
\[ \int_{a}^{b} f \, dx > U(S, f) - \epsilon. \]
Since the upper and lower integrals are equal, it follows that
\[ L(Q, f) + \epsilon > U(S, f) - \epsilon. \]
Choosing \( P \) equal to the common refinement of \( Q \) and \( S \) and applying Lemma 6.6 gives,
\[ L(P, f) + \epsilon > U(P, f) - \epsilon. \]
Hence,
\[ 0 \leq U(P, f) - L(P, f) < 2\epsilon \]
and the proof of one direction of the proposition is complete.

The estimate
\[ L(P, f) \leq \int_{a}^{b} f \, dx \leq \int_{a}^{b} f \, dx \leq U(P, f) \]
proves the converse. \( \square \)

Corollary 6.9. Suppose \( f : [a, b] \to \mathbb{R} \) is bounded.

\( f \in \mathcal{R}([a, b]) \) if and only if there is an \( I \in \mathbb{R} \) such that for each \( \epsilon > 0 \), there exists a partition \( P = \{ a = x_0 < x_1 < \cdots < x_n = b \} \) such that for any \( x_{j-1} \leq s_j \leq x_j \),
\[ |I - \sum_{j=1}^{n} f(s_j) \Delta_j| < \epsilon. \]

In this case, \( I \) is the integral of \( f \).

The sum \( S = \sum_{j=1}^{n} f(s_j) \Delta_j \) is a Riemann sum subordinate to the partition \( P \).
\textbf{Proof.} First suppose \( f \in \mathcal{R}([a,b]) \) and let \( I \) denote the Riemann integral of \( f \) on \([a,b]\). By Proposition 6.8, given \( \epsilon \) there is a partition \( P \) such that \( U(P,f) - L(P,f) < \epsilon \).

Given \( x_{j-1} \leq s_j \leq x_j \) it follows that

\[
L(P,f) \leq S = \sum f(s_j)\Delta_j \leq U(P,f)
\]

Hence both \( I, S \in [L(P,f),U(P,f)] \). Since this interval has length less than \( \epsilon \) it follows that \( |I - S| < \epsilon \).

Conversely, suppose there is an \( I \) such that for each \( \epsilon > 0 \) there is a partition \( P \) such that for each Riemann sum \( S \) subordinate to \( P \) we have \( |I - S| < \epsilon \). To prove \( f \) is Riemann integrable, let \( \epsilon > 0 \) be given and choose a partition \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \) such that for any \( x_{j-1} \leq s_j \leq x_j , \) \[
|I - \sum_{j=1}^{n} f(s_j)\Delta_j| < \epsilon
\]

Let \( M_j = \sup \{ f(x) : x_{j-1} \leq x \leq x_j \} \) and \( m_j = \inf \sup \{ f(x) : x_{j-1} \leq x \leq x_j \} \). Choose \( x_{j-1} \leq s_j, t_j \leq x_j \) such that \( M_j - f(s_j) < \epsilon \) and \( f(t_j) - m_j < \epsilon \). Let \( S = \sum f(s_j)\Delta_j \) and \( T = \sum f(t_j)\Delta_j \). It follows that

\[
0 \leq \sum_{j=1}^{n} f(t_j)\Delta_j - \sum_{j=1}^{n} m_j\Delta_j = \sum_{j=1}^{n} (f(t_j) - m_j)\Delta_j \\
\leq \epsilon \sum_{j=1}^{n} \Delta_j = \epsilon(b - a).
\]

Thus, \( 0 \leq T - L(P,f) < \epsilon(b - a) \)

Similarly, \( 0 \leq U(P,f) - S < \epsilon(b - a) \).

Finally, \[
|U(P,f) - L(P,f)| \leq |U(P,f) - S| + |S - I| + |I - T| + |T - L(P,f)| \leq 2(1 + (b-a))\epsilon.
\]

\( \square \)

\textbf{Theorem 6.10.} If \( f \) is continuous on \([a,b]\), then \( f \in \mathcal{R}([a,b]) \).

\textbf{Proof.} Let \( \epsilon > 0 \) be given. Since \( f \) is continuous on the compact set \([a,b]\), \( f \) is uniformly continuous. Hence, there is a \( \delta > 0 \) so that if \( a \leq s, t \leq b \) and \( |s - t| < \delta \), then \( |f(s) - f(t)| < \epsilon \).

Choose a partition \( P \) of \([a,b]\) of width less than \( \delta \); i.e., \( a = x_0 < x_1 \cdots x_n = b \) with \( \Delta_j < \delta \) for all \( j \). As always, let \( M_j \) and \( m_j \) denote the supremum and infimum of the set \( \{ f(x) : x \in [x_{j-1},x_j] \} \). Since \( f \) restricted to the closed bounded set \([x_{j-1},x_j]\) is continuous, there exits, by Corollary 4.34, \( x_{j-1} \leq s_j, t_j \leq x_j \) such that \( f(t_j) = M_j \) and \( f(s_j) = m_j \). It follows that \( M_j - m_j < \epsilon \). Hence

\[
U(P,f) - L(P,f) < \epsilon(b - a).
\]

An appeal to Proposition 6.8 completes the proof. \( \square \)
Do Problem 6.6.

**Theorem 6.11.** Suppose \( f \in \mathcal{R}([a, b]) \) and \( f : [a, b] \to [m, M] \). If \( \varphi : [m, M] \to \mathbb{R} \) is continuous, then \( h = \varphi \circ f \in \mathcal{R}([a, b]) \).

Of course if \( f \) is continuous, then so is \( h \) and hence \( h \in \mathcal{R}([a, b]) \) by Theorem 6.10.

We will prove Theorem 6.11 under the added hypothesis that \( \varphi \) is Lipschitz continuous. Recall, a function \( \varphi : [c, d] \to \mathbb{R} \) is *Lipschitz continuous* if there is a \( C \) so that for all \( x, y \leq b \),

\[
|\varphi(x) - \varphi(y)| \leq C|x - y|.
\]

**Sketch of proof assuming Lipschitz continuity.** There is a \( C \) so that equation (7) holds. In particular, if \( [u, v] \subset [a, b] \) is any interval and \( M_\ast \) and \( m_\ast \) are the supremum and infimum of \( f \) on \( [u, v] \) and \( M' \) and \( m' \) are the supremum and infimum \( h \) on \( [u, v] \), then

\[
(M' - m') \leq C(M_\ast - m_\ast).
\]

To prove this claim, first note that for \( x, y \in [u, v] \) that \( |f(x) - f(y)| \leq M_\ast - m_\ast \) and hence

\[
|h(x) - h(y)| = |\varphi(f(x)) - \varphi(f(y))| \leq C(M_\ast - m_\ast).
\]

Given \( \eta > 0 \), there exists \( x, y \in [u, v] \) such that \( M' - \eta < h(x) \) and \( m' + \eta > h(y) \). Thus

\[
(M' - m') - 2\eta = (M' - \eta) - (m' + \eta) < h(x) - h(y) \leq C(M_\ast - m_\ast)
\]

and the desired conclusion follows.

Let \( \epsilon > 0 \) be given. Since \( f \in \mathcal{R}([a, b]) \), there is a partition \( P \) such that \( U(P, f) - L(P, f) < \epsilon \). From equation (8) it follows that \( U(P, h) - L(P, h) \leq C\epsilon \). Hence \( h \in \mathcal{R}([a, b]) \).

**Corollary 6.12.** If \( f \in \mathcal{R}([a, b]) \), then so are

(i) \( |f|^p \) for \( p \in \mathbb{Q}, p \geq 0 \);

(ii) \( f_\ast = \max\{f, 0\} \);

(iii) \( f_\ast = \min\{f, 0\} \); and

**Proof.** To prove (i), note that the function \( \varphi : \mathbb{R} \to \mathbb{R} \) defined by \( \varphi(t) = |t|^p \) is continuous for \( p \geq 0 \).

To prove (ii), consider \( \varphi : \mathbb{R} \to \mathbb{R} \) given by \( \varphi(t) = \max\{t, 0\} = \frac{1}{2}(|t| + t) \). It is continuous.

To prove (iii), consider \( \varphi : \mathbb{R} \to \mathbb{R} \) defined by \( \varphi(t) = -\frac{|t| - t}{2} \).

As an aside, the functions in (ii) and (iii) are Lipschitz continuous as is, on any bounded interval, the function \( f(t) = |t|^p \) for \( p \geq 1 \). On the other hand but, for \( 0 < p < 1 \), \( f(t) = |t|^p \) is not Lipschitz continuous on intervals of the form \([0, b]\) by Example 4.50.

We state, without proof, two other sufficient conditions for integrability.

**Proposition 6.13.** If \( f : [a, b] \to \mathbb{R} \) is increasing, then \( f \in \mathcal{R}([a, b]) \).

**Proposition 6.14.** Suppose \( f : [a, b] \to \mathbb{R} \) is bounded. If \( f \) is continuous except at finitely many points, then \( f \in \mathcal{R}([a, b]) \).
6.3. Properties of the Integral. Here is a list of properties of the Riemann Integral. The proofs are mostly left to the reader.

**Proposition 6.15.** If \( f_1, f_2 \in \mathcal{R}([a,b]) \) and \( c_1, c_2 \) are real, then \( c_1 f_1 + c_2 f_2 \in \mathcal{R}([a,b]) \) and
\[
\int_a^b (c_1 f_1 + c_2 f_2) \, dx = c_1 \int_a^b f_1 \, dx + c_2 \int_a^b f_2 \, dx.
\]

**Remark 6.16.** The proposition says \( \mathcal{R}([a,b]) \) is a (real) vector space and the mapping \( I : \mathcal{R}([a,b]) \to \mathbb{R} \) determined by the integral is linear.

Do Problem 6.8.

**Corollary 6.17.** If \( f, g \in \mathcal{R}([a,b]) \), then so is \( fg \).

*Proof.* By the previous proposition \( f + g \in \mathcal{R}([a,b]) \). By the corollary to Theorem 6.11 and several more applications of the previous proposition, it then follows that \( fg = \frac{1}{2}((f + g)^2 - f^2 - g^2) \in \mathcal{R}([a,b]) \). \( \square \)

**Proposition 6.18.** Suppose \( f_1, f_2 : [a,b] \to \mathbb{R} \) are bounded. If \( f_1 \leq f_2 \) and \( P \) is a partition of \([a,b]\), then
\[
L(P, f_1) \leq L(P, f_2), \quad U(P, f_1) \leq U(P, f_2),
\]
and
\[
\int_a^b f_1 \, dx \leq \int_a^b f_2 \, dx.
\]

If \( f_1, f_2 \in \mathcal{R}([a,b]) \) and \( f_1 \leq f_2 \), then
\[
\int_a^b f_1 \, dx \leq \int_a^b f_2 \, dx.
\]

**Corollary 6.19.** If \( f \in \mathcal{R}([a,b]) \), then
\[
|\int_a^b f \, dx| \leq \int_a^b |f| \, dx.
\]

*Proof.* Use \(|f| \geq \pm f\), Proposition 6.18 twice and and Proposition 6.15 with \( c_1 = -1, f_1 = f \) and \( c_2 = 0 \). \( \square \)

**Proposition 6.20.** If \( f \in \mathcal{R}([a,b]) \) and \( a < c < b \) then \( f|_{[a,c]} \in \mathcal{R}([a,c]) \) and
\[
\int_a^b f \, dx = \int_a^c f \, dx + \int_c^b f \, dx.
\]

If \( f : (a,b) \to \mathbb{R} \) is continuous and \( [c,d] \subset [a,b] \), then \( f \in \mathcal{R}([c,d]) \); that is \( f|_{[c,d]} \) is Riemann integrable.

If \( f \in \mathcal{R}([a,b]) \), let
\[
\int_a^b f(t) \, dt = -\int_b^a f(t) \, dt.
\]
With this convention, if \( f \in \mathcal{R}([a,b]) \) and \( a \leq x, y, z \leq b \), then
\[
\int_y^x f \, dt = \int_y^z f \, dt + \int_z^x f \, dt.
\]
and hence
(9)
\[
\int_y^x f \, dt = \int_z^x f \, dt - \int_z^y f \, dt.
\]

6.4. Integration and Differentiation.

**Theorem 6.21.** [Second Fundamental Theorem of Calculus] If \( f : (a,b) \to \mathbb{R} \) is continuous and \( a < z < b \), then the function \( F : (a,b) \to \mathbb{R} \) defined by
\[
F(x) = \int_z^x f(t) \, dt
\]
is differentiable and \( F'(s) = f(s) \) for \( a < s < b \). is continuous.

**Proof.** Fix \( x \in (a,b) \). Observe if \( y \in (a,b) \), then, by Proposition 6.20 (see equation (9)),
\[
F(x) - F(y) = \int_y^x f(t) \, dt.
\]
Given \( \epsilon > 0 \), there is a \( 0 < \delta < \min\{x - a, b - x\} \) so that if \( |t - x| < \delta \), then \( |f(t) - f(x)| < \epsilon \). Thus, if \( |y - x| < \delta \), then
\[
\left| \frac{F(x) - F(y)}{x - y} - f(x) \right| = \left| \frac{1}{x - y} \int_y^x f(t) \, dt - f(x) \right|
\]
\[
= \left| \frac{1}{x - y} \right| \left| \int_y^x [f(t) - f(x)] \, dt \right|
\]
\[
\leq \left| \frac{1}{x - y} \right| \left| \int_y^x |f(t) - f(x)| \, dt \right|
\]
\[
\leq \left| \frac{1}{x - y} \right| \int_y^x |f(t) - f(x)| \, dt = \epsilon.
\]
Hence \( \lim_{y \to x} \frac{F(x) - F(y)}{x - y} = f(x) \) and \( F \) is differentiable at \( x \) with \( F'(x) = f(x) \). \( \square \)

**Example 6.22.** Consider \( f : (0, \infty) \to \mathbb{R} \) defined by \( f(t) = \frac{1}{t} \). Define \( \log : (0, \infty) \to \mathbb{R} \) by
\[
F(x) = \log(x) = \int_1^x \frac{1}{t} \, dt.
\]
Thus \( F(1) = 0 \) and \( F'(x) = \frac{1}{x} \), from which the usual properties of the log follow. (See Problems 6.10 and 6.9.) In particular \( \log(\frac{1}{x}) = -\log(x) \). (Proof: With \( G(x) = -F(\frac{1}{x}) \), by the chain rule, \( G'(x) = \frac{1}{x^2} F'(\frac{1}{x}) = \frac{1}{x^2} x = F'(x) \) and \( G(1) = 0 = F(1) \). Hence \( G = F \).)

Note that, by considering appropriate lower sums,
\[
\log(n + 1) \geq \sum_{j=2}^{n+1} \frac{1}{j}.
\]
Since the harmonic series diverges and the log is continuous, it follows that the range of the log contains \([0, \infty)\). Using \(\log\left(\frac{1}{x}\right) = -\log(x)\) it must also be the case that the range of log contains \((-\infty, 0]\). Hence the range of log is all of \(\mathbb{R}\).

Since the derivative of log is strictly positive, log is strictly increasing and in particular one-one. Thus log has an inverse, called the exponential function \(\exp : \mathbb{R} \to (0, \infty)\). It is determined by \(\exp(\log(x)) = x\) and \(\log(\exp(y)) = y\) for all \(x \in (0, \infty)\) and \(y \in \mathbb{R}\). The usual properties of \(\exp\) now follow from those of log. (See Problem 6.11.)

Recall, to this point, for positive real numbers \(x\), the power \(x^a\) has only been defined for \(a\) a rational number. In Problem 6.9 the reader is asked to show \(\log(x^a) = a \log(x)\) for \(x > 0\) and \(a \in \mathbb{Q}\). In view of this fact, we now define, for \(x > 0\) and \(a\) any real number,

\[
x^a = \exp(a \log(x)).
\]

In particular,

\[
\exp(1)^a = \exp(a).
\]

Hence, letting \(e = \exp(1)\), gives \(e^a = \exp(a)\) and it is customary to denote the exponential function by \(e^x\).

**Theorem 6.23.** [First Fundamental Theorem of Calculus] If \(F : [a, b] \to \mathbb{R}\) is differentiable, and \(F'\) is bounded, then, for all partitions \(P\) of \([a, b]\),

\[
L(P, F') \leq F(b) - F(a) \leq U(P, F').
\]

In particular, if \(F' \in \mathcal{R}([a, b])\), then

\[
F(b) - F(a) = \int_a^b F' dx.
\]

**Proof.** For notational ease, let \(f = F'\).

Let \(P = \{x_0 < x_1 < \cdots < x_n = b\}\) denote a given partition of \([a, b]\). For each \(j\) there exists, by the mean value theorem, a \(x_{j-1} < t_j < x_j\) such that

\[
F(x_j) - F(x_{j-1}) = f(t_j)(x_j - x_{j-1}).
\]

Summing (10) over \(j\) gives and using the telescoping nature of the sum on the left hand side gives,

\[
F(b) - F(a) = \sum f(t_j)(x_j - x_{j-1}).
\]

Further, by Exercise 6.1,

\[
L(P, f) \leq \sum f(t_j)(x_j - x_{j-1}) \leq U(P, f).
\]

Combining (11) and (12) gives

\[
L(P, f) \leq F(b) - F(a) \leq U(P, f).
\]

\[\square\]

Do Problem 6.12.
6.5. Problems.

Problem 6.1. Suppose \( f : [a,b] \to \mathbb{R} \) is a bounded function and \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \) is a partition. Show, if \( x_{j-1} \leq t_j \leq x_j \), then

\[
L(P, f) \leq \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) \leq U(P, f).
\]

The sum above is a Riemann sum.

Problem 6.2. Let \( f : [-1,1] \to \mathbb{R} \) denote the function with \( f(x) = 0 \) for \( x \neq 0 \) and \( f(0) = 1 \). Show \( f \in \mathcal{R}([-1,1]) \) and

\[
\int_{-1}^{1} f \, dx = 0.
\]

Compare with Problem 6.6. See also Problem 6.7.

Problem 6.3. Let \( f : [-1,1] \to \mathbb{R} \) denote the function \( f(x) = 1 \) if \( 0 \leq x \leq 1 \) and \( f(x) = 0 \) otherwise. Prove, directly from the definitions, that \( f \in \mathcal{R}([-1,1]) \) and

\[
\int_{-1}^{1} f \, dx = 1.
\]

Problem 6.4. Recall that

\[
\sum_{j=1}^{n} j^2 = \frac{1}{6}n(n+1)(2n+1).
\]

Use this formula, the definition and Lemma 6.6 to show \( h : [0,1] \to \mathbb{R} \) defined by \( h(x) = x^2 \) is Riemann integrable.

Problem 6.5. Suppose \( f : [a,b] \to \mathbb{R} \) is Riemann integrable. Prove,

\[
\lim_{c \to b, c < b} \int_{a}^{c} f \, dx = \int_{a}^{b} f \, dx.
\]

Problem 6.6. Suppose \( f : [-1,1] \to \mathbb{R} \) takes nonnegative values. Show, if \( f \) is integrable, continuous at 0 and if \( f(0) > 0 \), then

\[
\int_{-1}^{1} f \, dx > 0.
\]

Problem 6.7. Suppose \( f, g : [a,b] \to \mathbb{R} \) and \( f \) and \( g \) are equal except possible at a point \( c \) with \( a < c < b \). Show, if \( f \) is Riemann integrable, then so is \( g \) and moreover,

\[
\int_{a}^{b} f \, dx = \int_{a}^{b} g \, dx.
\]

Note that, by induction, the result holds if \( f \) and \( g \) agree except possibly at finitely many points. Compare with Exercise 6.2 and Proposition 6.14.


Problem 6.9. Prove for \( a \in \mathbb{Q} \) and \( x \in \mathbb{R}^+ \) (meaning \( x \) is a positive real number), that \( \log(x^a) = a \log(x) \). Suggestion, consider \( g(x) = \log(x^a) \) and compute \( g'(x) \).

It now makes sense to define \( x^r = \exp(r \log(x)) \) for \( r \in \mathbb{R} \).
Problem 6.10. Let \( f(x) = \log(x) \). Given \( a > 0 \), let \( g(x) = f(ax) \). Prove, \( g'(x) = f'(x) \) and thus there exists a \( c \) so that \( g(x) = f(x) + c \). Prove, \( c = \log(a) \) and thus \( \log(ax) = \log(a) + \log(x) \).

Problem 6.11. Prove \( \exp(a + b) = \exp(a) \exp(b) \) and \( \exp(ab) = \exp(a)^b \).

Problem 6.12. Suppose \( f : [a, b] \to \mathbb{R} \) is continuous and \( \varphi : [\alpha, \beta] \to [a, b] \) is strictly increasing and continuously differentiable. Show, for \( \alpha < A < B < \beta \),

\[
\int_{\varphi(A)}^{\varphi(B)} f \, dx = \int_{A}^{B} f(\varphi(t)) \varphi'(t) \, dt.
\]
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