MAA 4212 Sample Exam I, Spring 2014

1. Do three of four (30 points). Here X and Y are metric spaces.

   (i) Give an example, if possible, of a sequence of functions $f_n : X \to \mathbb{R}$ which converge pointwise, but not uniformly.

   (ii) Give an example, if possible, of a sequence of continuous functions $f_n : X \to Y$ which converge uniformly to a function $f : X \to Y$, but f is not continuous.

   (iii) Give an example, if possible, of a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable, but $f' : \mathbb{R} \to \mathbb{R}$ is not continuous.

   (iv) Give an example, if possible, of a function $f : [0,1] \to \mathbb{R}$ which is bounded, but not Riemann integrable.

2. Do two of three (30 points).

   (i) State precisely one of the fundamental theorems of Calculus.

   (ii) Is the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|^3$ differentiable at 0?

   (iii) Let $C([0,1])$ denote the continuous real valued functions on $[0,1]$ with the supremum metric. Is $C([0,1])$ complete? Is the set $B \subset C([0,1])$,

   $$B = \{f \in C([0,1]) : \max\{|f(t)| : t \in [0,1]\} \leq 1\},$$

   a bounded set, a closed set, a compact set?

3. Do two of three (40 points).

   (i) Prove, if $f : [-1,1] \to \mathbb{R}$ is continuous, $f(x) \geq 0$ for all x, and $f(0) = 1$, then

   $$\int_{-1}^{1} f\, dx > 0.$$

   (ii) Prove, if $f_n : X \to Y$ is a sequence of bounded functions and (f_n) converges uniformly to f, then f is bounded.

   (iii) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Prove, if f' is bounded, then f is uniformly continuous; and $f'(x) \geq 0$ for all x if and only if f is increasing.