.

1. Review of Sets and Functions

It is assumed that the reader is familiar with the most basic set constructions and functions and knows the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$, though we will review carefully the properties which characterize $\mathbb{R}$.

Familiarity with matrices $M_n(\mathbb{F})$ and $M_{m,n}(\mathbb{F})$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, is also assumed.

1.1. **Unions, intersections, complements, and products.**

Definition 1.1. Given sets $X, Y \subset S$, the *union and intersection* of X and Y are

$$\begin{aligned} X \cup Y =& \{z \in S : z \in X \text{ or } z \in Y\} \subset S \\ X \cap Y =& \{z \in S : z \in X \text{ and } z \in Y\} \subset S, \end{aligned}$$

respectively.

The *complement* of X, denoted $\tilde{X}$, is the set

$$\tilde{X} = \{x \in S : x \notin X\}.$$

The *relative complement* of X in Y is

$$Y \setminus X = Y \cap \tilde{X} = \{z \in S : z \in Y \text{ and } z \notin X\}.$$

Note $\tilde{X} = S \setminus X$.

Definition 1.2. Let X and Y be sets. The *Cartesian product* of X and Y is the set

$$X \times Y = \{(x, y) : x \in X, \quad y \in Y\}.$$

Example 1.3. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is known as the Cartesian plane.

$\mathbb{R}^3$ is the 3-dimensional Euclidean space of third semester Calculus. △

Definition 1.4. Given a set S, let $P(S)$ denote the *power set* of S, the set of all subsets of S.

Example 1.5. Let $S = \{0, 1\}$. Then,

$$P(S) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}.$$

As we shall see later, $P(\mathbb{N})$ is a very large set. △

Definition 1.6. Given sets I and S and a function $\alpha : I \to P(S)$, let $A_i = \alpha(i)$. The *union and intersection* of the collection $\alpha(I)$ are

$$\begin{aligned} \cup_{i \in I} A_i =& \{x \in S : \text{ there is a } j \in I \text{ such that } x \in A_j\} \\ \cap_{i \in I} A_i =& \{x \in S : x \in A_j \text{ for every } j \in I\}. \end{aligned}$$

respectively.

For an example, for $n \in \mathbb{N}$, let $A_n = \{m \in \mathbb{Z} : m \geq n\}$ and observe that

$$\cap_{n\in\mathbb{N}} A_n = \emptyset.$$

Remark 1.7. Given $\mathcal{F} \subset P(S)$, letting $\mathcal{F}$ index itself,

$$\cap_{A\in\mathcal{F}} A = \{x \in S : x \in A \text{ for every } A \in \mathcal{F}\}.$$

◇

Do Problem 1.1.

1.2. **Functions.**

Definition 1.8. A *function* f is a triple (f, A, B) where A and B are sets and f is a rule which assigns to each $a \in A$ a unique $b = f(a)$ in B. We write

$$f : A \to B.$$

(a) The set A is the *domain* of f.
(b) The set B is the *codomain* of f
(c) The *range* of f, sometimes denoted $\mathrm{rg}(f)$, is the set $\{f(a) : a \in A\}$.
(d) The function $f : A \to B$ is *one-one* if $x, y \in A$ and $x \neq y$ implies $f(x) \neq f(y)$.
(e) The function $f : A \to B$ is *onto* if for each $b \in B$ there exists an $a \in A$ such that $b = f(a)$; i.e., if $\mathrm{rg}(f) = B$.
(f) The *graph* of f is the set

$$\mathrm{graph}(f) = \{(a, f(a)) : a \in A\} \subset A \times B.$$

(g) If $f : A \to B$ and $Y \subset B$, the *inverse image* of Y under f is the set

$$f^{-1}(Y) = \{x \in A : f(x) \in Y\}.$$

(h) If $f : A \to B$ and $C \subset A$, the set

$$f(C) = \{f(c) : c \in C\} = \{b \in B : \text{ there is an } c \in C \text{ such that } b = f(c)\}$$

is the *image of C under f*.
(i) The *identity function* on a set A is the function $id_A : A \to A$ with rule $id_A(x) = x$.

Example 1.9. Often one sees functions specified by giving the rule only, leaving the domain implicitly understood (and the codomain unspecified), a practice to be avoided. For example, given $f(x) = x^2$ it is left to the reader to guess that the domain is the set of real numbers. But it could also be $\mathbb{C}$ or even $M_n(\mathbb{C})$, the $n \times n$ matrices with entries from $\mathbb{C}$. If the domain is taken to be $\mathbb{R}$, then $\mathbb{R}$ is a reasonable choice of codomain. However, the range of f is $[0, \infty)$ (a fact which will be carefully proved later) and so the codomain could be any set containing $[0, \infty)$. The moral is that it is important to specify both the domain and codomain as well as the rule when defining a function. △

Example 1.10. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Note that f is neither one-one nor onto.

As an illustrations of the notion of inverse image, $f^{-1}((4,\infty)) = (-\infty, 2) \cup (2,\infty)$ and $f^{-1}((-2,-1)) = \emptyset$. △

Example 1.11. The function $g : \mathbb{R} \to [0,\infty)$ defined by $g(x) = x^2$ is not one-one, but it is, as we'll see in Subsection 2.4, onto.

The function $h : [0,\infty) \to [0,\infty)$ is both one-one and onto. Note $h^{-1}((4,\infty)) = (2,\infty)$. △

Do Exercises 1.3 and 1.1.

Definition 1.12. Given sets A, B and X, Y and functions $f : A \to X$ and $g : B \to Y$, define $f \times g : A \times B \to X \times Y$ by $f \times g(a,b) = (f(a), g(b))$.

Example 1.13. For example, if $f : \mathbb{N} \to \mathbb{N}$ is defined by $f(n) = 2n$ and $g : \mathbb{Z} \to \mathbb{N}$ is defined by $g(m) = 3m^2$, then $f \times g : \mathbb{N} \times \mathbb{Z} \to \mathbb{N} \times \mathbb{N}$ is given by $f \times g(n,m) = (2n, 3m^2)$. △

Do Problem 1.2.

Definition 1.14. Given $f : A \to B$ and $C \subset A$, *the restriction of f to C* is the function $f|_C : C \to B$ defined by $f|_C(x) = f(x)$ for $x \in C$.

Definition 1.15. Given $f : X \to Y$ and $g : Y \to Z$ the *composition* of f and g is the function $g \circ f : X \to Z$ with rule $g \circ f(x) = g(f(x))$.

A function $f : X \to Y$ is *invertible* if there is a function $g : Y \to X$ such that

$$\begin{aligned} g \circ f &= id_X \\ f \circ g &= id_Y. \end{aligned}$$

We call g *the* inverse of f (see part (a) of Proposition 1.16 below), written $g = f^{-1}$.

Proposition 1.16. (a) If f is invertible, then the function g in Definition 1.15 is unique.

(b) $f : X \to Y$ is invertible if and only if f is both one-one and onto.

Example 1.17. The function $h : [0,\infty) \to [0,\infty)$ given by $h(x) = x^2$ of Example 1.11 is one-one and onto and thus has an inverse. Of course this inverse $h^{-1} : [0,\infty) \to [0,\infty)$ is commonly denoted as $\sqrt{\ }$ so that $h^{-1}(x) = \sqrt{x}$. △

Proof. Suppose f is invertible so that there exists $g : Y \to X$ satisfying the conditions of Definition 1.15. If $f(x_1) = f(x_2)$, then $x_1 = g \circ f(x_1) = g \circ f(x_2) = x_2$ and hence f is one-one. Similarly, given $y \in Y$, $f \circ g(y) = y$ so that $y = f(g(y))$ is in the range of f. Hence f is onto.

Suppose f is one-one and $g, h : Y \to X$ satisfy $f \circ g = id_Y = f \circ h$. Then, for each $y \in Y$, $f(g(y)) = y = f(h(y))$. Since f is one-one, $g(y) = h(y)$, proving that if f is invertible, then g as in Definition 1.15 is unique.

Finally, suppose f is both one-one and onto. Define $g : Y \to X$ as follows. Given $y \in Y$, there is a unique $x \in X$ so that $f(x) = y$ (why?). Let $g(y) = x$ and note that $f(g(y)) = y$ and $g(f(x)) = x$. $\square$

See Exercise 1.5 Do Problem 1.3.

1.3. finite and countable sets.

Definition 1.18. Two sets A and B are *equivalent*, denoted $A \sim B$ if there is a one-one onto mapping $f : A \to B$.

Observe that $\sim$ is behaves like an equivalence relation; i.e., $A \sim A$; if $A \sim B$, then $B \sim A$; and finally if $A \sim B$ and $B \sim C$, then $A \sim C$.

Given a positive integer n, let J_n denote the set $\{1, 2, \ldots, n\}$. The show that J_n is not equivalent to $\mathbb{N}$ note, if $f : J_n \to \mathbb{N}$, then $f(j) \le \sum_{\ell=1}^{n} f(\ell)$ for each j and so f is not onto.

Definition 1.19. Let A be a set.

(a) A is *finite* if it is either empty or there is an $n \in \mathbb{N}^+$ such that $A \sim J_n$;
(b) A is *infinite* if it is not finite;
(c) A is *countable* if $A \sim \mathbb{N}$;
(d) A is *at most countable* if either A is finite or countable; and
(e) A is *uncountable* if it is not at most countable.

Here $\mathbb{N}^+$ are the positive natural numbers; i.e., $\mathbb{N} \setminus \{0\}$.

Remark 1.20. Note, by the comments preceding the definition, that $\mathbb{N}$ is infinite. $\diamond$

Proposition 1.21. A set A is at most countable if and only if there is an onto mapping $f : \mathbb{N} \to A$.

We will not prove this proposition.

Do Problem 1.4.

Proposition 1.22. The sets $\mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$, and $\mathbb{Q}$ are all at most countable.

Sketch of proof. Define $f : \mathbb{N} \to \mathbb{Z}$ by $f(2m) = m$ and $f(2m+1) = -m-1$. Since f is onto, $\mathbb{Z}$ is at most countable.

To prove $\mathbb{N} \times \mathbb{N}$ is countable, consider $\mathbb{N}$ as an array. Explicitly, define $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by $g(k) = (n - m, m)$ where $\frac{1}{2}n(n+1) \le k < \frac{1}{2}(n+1)(n+2)$ and $k = \frac{1}{2}n(n+1) + m$.

Now the composition $(f \times id_{\mathbb{N}}) \circ g : \mathbb{N} \to \mathbb{Z} \times \mathbb{N}$ is onto. Thus, to prove that $\mathbb{Q}$ is at most countable, it suffices to exhibit an onto mapping $h : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$, since then $h \circ (f \times id_{\mathbb{N}}) \circ g$ maps $\mathbb{N}$ onto $\mathbb{Q}$. Define h by $h(m, n) = \frac{m}{n+1}$. $\square$

Do Problems 1.5 and 1.6

Proposition 1.23. The set $P(\mathbb{N})$ is not countable.

The proof is accomplished using Cantor's diagonalization argument.

Proof. It suffices to prove, if $f : \mathbb{N} \to P(\mathbb{N})$, then f is not onto.

Given such an f, let

$$B = \{n \in \mathbb{N} : n \notin f(n)\}.$$

We claim that B is not in the range of f. Arguing by contradiction, suppose $m \in \mathbb{N}$ and $f(m) = B$. If $m \notin B$, then $m \in f(m) = B$ a contradiction. On the other hand, if $m \in B$, then $m \notin f(m) = B$, also a contradiction. □

Later we will use the proposition to see that $\mathbb{R}$ is uncountable.

Do Problem 1.7.

1.4. **Exercises.**

Exercise 1.1. Define $f : \mathbb{R} \to \mathbb{R}^2$ by

$$f(x) = (\cos(x), \sin(x)).$$

Let

$$\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

and

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 + (y - 1)^2 < 1\}.$$

Identify

(i) $f^{-1}(S)$;
(ii) $f^{-1}(\mathbb{D})$; and
(iii) $f^{-1}(f((-\frac{\pi}{2}, \frac{\pi}{2})))$.

Exercise 1.2. Consider the function $h = f \times g$ of Example 1.13 and let $6\mathbb{N}$ denote the set $\{6k : k \in \mathbb{N}\}$. Find the inverse image of the set $\{(j, k) : j \in \{2, 3, 4\} \quad k \in 6\mathbb{N}\}$. Find the inverse image of the set $\{(j, k) : j \in \{0, 1, 2\} \quad k \text{ is odd}\}$.

Exercise 1.3. Suppose $f : A \to B$. Prove that f is one-one if and only if for each $b \in B$ the set $f^{-1}(\{b\})$ contains at most one element.

Exercise 1.4. Use induction to show, for $n \in \mathbb{N}^+$, that $P(J_n) \sim J_{2^n}$.

Exercise 1.5. If $f : X \to Y$ is invertible, and $B \subset Y$, $f^{-1}(B)$ could refer to either the inverse image of B under f, or the image of B under the function f^{-1}. Show that, happily, these two sets are the same.

1.5. **Problems.**

Problem 1.1. Show

$$\widetilde{\cup_{A \in \mathcal{F}} A} = \cap_{A \in F} \tilde{A}.$$

Problem 1.2. Suppose $f : X \to S$ and $\mathcal{F} \subset P(S)$. Show,

$$f^{-1}(\cup_{A \in \mathcal{F}} A) = \cup_{A \in \mathcal{F}} f^{-1}(A)$$
$$f^{-1}(\cap_{A \in \mathcal{F}} A) = \cap_{A \in \mathcal{F}} f^{-1}(A)$$

Show, if $A, B \subset X$, then $f(A \cap B) \subset f(A) \cap f(B)$. Give an example, if possible, where strict inclusion holds.

Show, if $C \subset X$, then $f^{-1}(f(C)) \supset C$. Give an example, if possible, where strict inclusion holds.

Problem 1.3. If $f : A \to B$, then graph(f) is a subset of $A \times B$. Conversely, show, if $S \subset A \times B$ has the property that for each $a \in A$ there is a unique $b \in B$ such that $(a, b) \in S$, then defining $g(a) = b$ produces a function $g : A \to B$ such that graph$(g) = S$.

Problem 1.4. Let A be a nonempty set. Prove that A is at most countable if and only if there is a one-one mapping $g : A \to \mathbb{N}$.

Problem 1.5. Prove that an at most countable union of at most countable sets is at most countable; i.e., if S is a set, $\alpha : \mathbb{N} \to P(S)$ is a function such that each $A_j = \alpha(j)$ is at most countable, then

$$T = \cup_{j=0}^{\infty} A_j := \cup_{j \in \mathbb{N}} A_j$$

is at most countable.

Suggestion: For each j there is a function $g_j : \mathbb{N} \to A_j$. Define a function $F : \mathbb{N} \times \mathbb{N} \to T$ by $F(j, k) = g_j(k)$. Proceed.

Problem 1.6. Show that the collection $\mathcal{F} \subset P(\mathbb{N})$ of finite subsets of $\mathbb{N}$ is an at most countable set.

Problem 1.7. Suppose A is a non-empty set. Show there does not exist an onto mapping $f : A \to P(A)$; i.e., show $A \not\sim P(A)$.

Problem 1.8. Let A be a given nonempty set. Show, $2^A = \{f : A \to \{0, 1\}\}$ is equivalent to $P(A)$.

2. The Real Numbers

We will take the view that we know what the real numbers are and we will simply *review* some important properties in this section.

Recall the following notations for the *natural numbers*, *integers*, and *rational numbers*, respectively.

$$\begin{aligned} \mathbb{N} =& \{0, 1, 2, \dots\} \\ \mathbb{Z} =& \{0, \pm 1, \pm 2, \dots\} \\ \mathbb{Q} =& \{\frac{m}{n} : m \in \mathbb{Z}, \quad n \in \mathbb{N}^+\}. \end{aligned}$$

Let $\mathbb{N}^+$ denote the positive integers and $\mathbb{R}$ the real numbers.

Example 2.1. The square root of 2 is not rational; i.e., there is no rational number $s > 0$ such that $s^2 = 2$. △

2.1. **Field Axioms.**

Definition 2.2. A *field* $\mathbb{F}$ is a triple, $(\mathbb{F}, +, \cdot)$, where $\mathbb{F}$ is a set and

$$+, \cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$$

are functions, called addition and multiplication respectively and written $x + y = +(x, y)$ and $xy = \cdot(x, y)$, satisfying the following (long list) of axioms

(i) $x + y = y + x$ for every $x, y \in \mathbb{F}$;
(ii) $xy = yx$, for every x, y;
(iii) $(x + y) + z = x + (y + z)$ for every x, y, z;
(iv) $(xy)z = x(yz)$ for every x, y, z;
(v) there is an element $0 \in \mathbb{F}$ such that $0 + w = w$ for every $w \in \mathbb{F}$;
(vi) there is an element $1 \in \mathbb{F}$, distinct from 0, such that $1w = w$ for every $w \in \mathbb{F}$;
(vii) for each $x \in \mathbb{F}$ there is an element $u \in \mathbb{F}$ such that $x + u = 0$;
(viii) for each $x \neq 0$, there is a y such that $xy = 1$; and
(ix) $(x + y)z = xz + yz$ for every x, y, z.

Proposition 2.3. [Cancellation] Given $x, y, z \in \mathbb{F}$, if $x + y = x + z$, then $y = z$.

Proof. There exists $u \in \mathbb{F}$ such that $x + u = 0$. Thus,

$$\begin{aligned} y =& 0 + y \\ =& (u + x) + y \\ =& u + (x + y) \\ =& u + (x + z) \\ =& (u + x) + z \\ =& 0 + z = z. \end{aligned}$$

□

Remark 2.4. It follows that 0 and additive inverses are unique. Hence it makes sense to write $u = -x$ in case $x + u = 0$ so that $x + (-x) = 0$. ◇

Proposition 2.5. Given $x \in \mathbb{F}$, $0x = 0$ and $-x = (-1)x$.

Proof. Since $0 + 0x = 0x = (0 + 0)x = 0x + 0x$, cancellation gives $0 = 0x$.

Using $0x = 0$, we have $x+(-1)x = 1x+(-1)x = (1+(-1))x = 0x = 0$. □

Remark 2.6. From here on we will use freely, without proof or further comment, the many routine properties of fields which follow from the axioms. ◇

Example 2.7. The sets $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields with their usual operations of addition and multiplication. △

Example 2.8. Let $\mathbb{Z}_3 = (\{0,1,2\}, +, \cdot)$ where

$$x + y = x + y \text{ modulo } 3$$
$$xy = xy \text{ modulo } 3$$

Here the + on the left hand side is addition in $\mathbb{Z}_3$, whereas + on the right hand side is addition in $\mathbb{N}$.

The residue modulo 3 is the remainder after dividing by 3.

$\mathbb{Z}_3$ is a field with neutral elements $0, 1$. △

Definition 2.9. Given fields $\mathbb{F}$ and G, a mapping $f : \mathbb{F} \to G$ is a *field isomorphism* provided

(i) f is one-one;
(ii) f is onto;
(iii) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{F}$; and
(iv) $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{F}$.

Remark 2.10. It follows that $f(0_{\mathbb{F}}) = 0_G$ etc. ◇

Do Problem 2.2.

2.2. **Ordered Fields.**

Definition 2.11. An *ordered set* $(S, <)$ consists of a (nonempty) set S and a relation $<$ on S which satisfies

(i) (*trichotomy*) for each $x, y \in S$, exactly one of the following hold,

$$x < y, \quad y < x, \quad x = y;$$

(ii) (*transitivity*) for $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Example 2.12. The usual order on $\mathbb{R}$ (and thus on any subset of $\mathbb{R}$) is an example of an ordered set.

The dictionary order on $\mathbb{R}^2$ produces an ordered set. △

Definition 2.13. An *ordered field* $\mathbb{F} = (\mathbb{F}, +, \cdot, <)$ consists of a field $(\mathbb{F}, +, \cdot)$ which is also an ordered set $(\mathbb{F}, <)$ such that,

(i) if $x, y, z \in \mathbb{F}$ and $x < y$, then $x + z < y + z$;
(ii) if $x, y \in \mathbb{F}$ and $x, y > 0$, then $xy > 0$.

If $x > 0$ we call x *positive*.

Example 2.14. $\mathbb{R}$ and $\mathbb{Q}$ with the usual ordering are ordered fields. △

Proposition 2.15. Suppose $\mathbb{F}$ is an ordered field and $x \in \mathbb{F}$.

(i) If $x < 0$, then $-x > 0$.
(ii) If $x \neq 0$, then $x^2 > 0$.
(iii) In particular, $1 > 0$ in any ordered field.

Proof. If $x < 0$, then $0 = x - x < 0 - x = -x$.

To prove (ii), note, by trichotomy either $x > 0$ or $x < 0$. If $x > 0$, then $x^2 = xx > 0$. On the other hand, if $x < 0$, then $-x > 0$ and thus $x^2 = (-x)^2 > 0$.

□

Remark 2.16. We will not state (much less) prove the usual facts about the order structure in an ordered field, but rather use them without further comment. ◇

Example 2.17. Prove that there is no order on $\mathbb{Z}_3$ which makes it an ordered field.

We argue by contradiction. Accordingly suppose $<$ is an order on $\mathbb{Z}_3$ which makes $\mathbb{Z}_3$ an ordered field. Since $1 = 1^2$, it follows that $1 > 0$ and hence $-1 < 0$. On the other hand, $-1 = 2 = 1 + 1 > 0 + 0 = 0$, a contradiction (of trichotomy). △

Do Problem 2.1.

2.3. **The least upper bound property.**

Definition 2.18. Let S be a subset of an ordered field $\mathbb{F}$.

(i) The set S is *bounded above* if there is a $b \in \mathbb{F}$ such that $b \geq s$ for all $s \in S$.
(ii) Any $b \in \mathbb{F}$ such that $b \geq s$ for all $s \in S$ is an *upper bound* for S.

Example 2.19. Identify the set of upper bounds for the following subsets of the ordered field $\mathbb{R}$.

(a) $[0, 1)$;
(b) $[0, 1]$;
(c) $\mathbb{Q}$;
(d) $\emptyset$.

△

Lemma 2.20. Let S be a subset of an ordered field $\mathbb{F}$ and suppose both b and b' are upper bounds for S. If b and b' both have the property that if $c \in \mathbb{F}$ is an upper bound for S, then $c \geq b$ and $c \geq b'$, then $b = b'$.

Definition 2.21. The *least upper bound* for a subset S of an ordered field $\mathbb{F}$, if it exists, is a $b \in \mathbb{F}$ such that

(i) b is an upper bound for S; and
(ii) if $c \in \mathbb{F}$ is an upper bound for S, then $c \geq b$.

Remark 2.22. Lemma 2.20 justifies the use of *the* (as opposed to *an*) in describing the least upper bound.

The condition (ii) can be replaced with either of the following conditions

(ii)$'$ if $c < b$, then there exists an $s \in S$ such that $c < s$; or
(ii)$''$ for each $\epsilon > 0$ there is an $s \in S$ such that $b - \epsilon < s$.

The notions of *bounded below, lower bound* and *greatest lower bound* are defined analogously.

Least upper bound is often abbreviated lub. The term *supremum*, often abbreviated *sup*, is synonymous with lub. Likewise *glb* and *inf* for greatest lower bound and infimum. ◇

Example 2.23. Here is a list of examples.

(i) The least upper bound of $S = [0, 1) \subset \mathbb{R}$ is 1.
(ii) The least upper bound of $V = [0, 1] \subset \mathbb{R}$ is also 1.
(iii) The set $\mathbb{Q} \subset \mathbb{R}$ has no upper bound and thus no least upper bound;
(iv) Every real number is an upper bound for the set $\emptyset \subset \mathbb{R}$. Thus $\emptyset$ has no least upper bound.
(v) With some effort, it can be shown that if the subset $S = \{x \in \mathbb{Q} : 0 < x, \quad x^2 < 2\}$ of the ordered field $\mathbb{R}$ has a least upper bound s, then $s > 0$ and $s^2 = 2$; i.e., this least upper bound is the square root of two.

△

Example 2.24. Consider the subset $S = \{q \in \mathbb{Q} : 0 < q, \quad q^2 < 2\}$ of the ordered field $\mathbb{Q}$. Arguing by contradiction, one shows, as in Example 2.23 Item (v), that if S has a least upper bound s, then $s^2 = 2$ contradicting Example 2.1. Thus, there are subsets S of $\mathbb{Q}$ which are nonempty and bounded above but yet do not have least upper bounds (in $\mathbb{Q}$). △

Theorem 2.25. Every nonempty subset of $\mathbb{R}$ which is bounded above has a least upper bound.

Thus there is a positive real number s with $s^2 = 2$.

Definition 2.26. Let $\mathbb{F}$ and $\mathbb{G}$ be fields. A mapping $\varphi : \mathbb{F} \to \mathbb{G}$ is an *ordered field isomorphism* if φ is a field isomorphism and $\varphi(x) <_{\mathbb{G}} \varphi(y)$ whenever $x, y \in \mathbb{F}$ and $x <_{\mathbb{F}} y$.

Proposition 2.27. If $\mathbb{F}$ is an ordered field with the property that every nonempty subset S of $\mathbb{F}$ which is bounded above has a least upper bound (in $\mathbb{F}$ of course), then then there is an ordered field isomorphism $\varphi : \mathbb{F} \to \mathbb{R}$.

Hence $\mathbb{R}$ is the essentially unique ordered field with the property that every set which could possibly have a least upper bound in fact does.

Do Problems 2.4 and 2.5.
We will not prove Theorem 2.25 and Theorem 2.27.

Theorem 2.28. [Archimedean properties] Suppose $x, y \in \mathbb{R}$.

(i) There is a natural number n so that $n > x$.
(ii) If $1 < x - y$, then there is an integer m so that $y < m < x$.
(iii) If $y < x$, then there is a $q \in \mathbb{Q}$ such that $y < q < x$.

Remark 2.29. The last part of the theorem is sometimes expressed as saying $\mathbb{Q}$ is dense in $\mathbb{R}$. ◇

Proof. We prove (i) by arguing by contradiction. Accordingly, suppose no such natural number exists. In that case x is an upper bound for $\mathbb{N}$. It follows that $\mathbb{N}$ has a lub α. If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$. Hence $n + 1 \leq \alpha$ and thus $n \leq \alpha - 1$ for all $n \in \mathbb{N}$. Consequently, $\alpha - 1$ is an upper bound for $\mathbb{N}$, contradicting the least property of α. Hence $\mathbb{N}$ is not bounded above and there is an $n > x$, which proves item (i).

To prove (ii), it suffices to assume that $x > 0$ (why). The set $\{k \in \mathbb{N} : k \geq x\}$ is nonempty and does not contain 0. It has a least element $k > 0$. Thus $x - 1 \leq k - 1 < x$ and since $x - y > 1$, it follows that $y < k - 1 < x$.

Item (iii) is Problem 2.3. As a suggestion, note that, by item (i), there is a positive integer n so that $n(x - y) > 1$. Proceed. □

Example 2.30. Suppose $0 < a < 1$. Show the set $A = \{a^n : n \in \mathbb{N}\}$ is bounded below and its infimum is 0.

Since $a \geq 0$ each $a^n \geq 0$. Thus A is bounded below by 0. The set A is not empty. It follows that A has an infimum. Let $\alpha = \inf(A)$ and note $\alpha \geq 0$. Since $\alpha \leq a^n$ for $n = 0, 1, 2, \dots$, $\alpha \leq a^{n+1}$ for $n \in \mathbb{N}$ and therefore $\frac{\alpha}{a} \leq a^n$ for $n \in \mathbb{N}$. Thus, $\frac{\alpha}{a}$ is a lower bound for A. It follows that $\frac{\alpha}{a} \leq \alpha$. Since $a < 1$ and $\alpha \geq 0$, $\alpha = 0$. △

Do Problems 2.6, 2.7, 2.8, 2.9,

2.4. **The existence of n-th roots.** Here is an outline a proof that positive real numbers have n-th roots for positive integers n.

Proposition 2.31. If $y > 0$ and $n \in \mathbb{N}^+$, then there is a unique positive real number s such that $s^n = y$.

Of course, $s = y^{\frac{1}{n}}$ is the notation for this n-th root.

The uniqueness is straightforward based upon the fact that if $0 < a < b$, then $a^n < b^n$. It should not come as a shock that existence depends upon the existence of least upper bounds, Theorem 2.25.

Let

$$S = \{x \in \mathbb{R} : 0 < x \text{ and } x^n < y\}.$$

First show S is non-empty and bounded above. Hence S has a least upper bound, say s.

Show, if $0 < t$ and $y < t^n$, then t is an upper bound for S.

Show if $0 < t$ and $y < t^n$, then there is a v such that $0 < v < t$ such that $y < v^n$. Hence, $v < t$ and v is an upper bound for S. In particular, t does not satisfy the least property of least upper bound. Thus, $s^n \leq y$.

Finally, show if $0 < t$ and $t^n < y$, then there exists a v such that $0 < t < v$ such that $v^n < y$. Hence, t is not an upper bound for S. Thus $s^n \geq y$. Hence $s^n = y$.

It now follows that the mapping $h : [0, \infty) \to [0, \infty)$ defined by $h(x) = x^n$ is both one-one and onto. Its inverse, $h^{-1} : [0, \infty) \to [0, \infty)$ is then the function commonly denoted by $\sqrt[n]{\ }$ or $x^{\frac{1}{n}}$ so that $h^{-1}(x) = x^{\frac{1}{n}}$.

2.5. **Vector spaces.** Recall that $\mathbb{R}^n$ is the vector space of n-tuples of real numbers. Thus an element $x \in \mathbb{R}^n$ has the form,

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Vectors - elements of $\mathbb{R}^n$ - are added and multiplied by *scalars* (elements of $\mathbb{R}$) entrywise.

The set of polynomials $\mathcal{P}$ (in one variable with real coefficients) is a vector space under the usual operations of addition and scalar multiplication.

Definition 2.32. A *norm* on a vector space V over $\mathbb{R}$ is a function $\|\cdot\| : V \to \mathbb{R}$ satisfying

(i) $\|x\| \geq 0$ for all $x \in V$;
(ii) $\|x\| = 0$ if and only if $x = 0$;
(iii) $\|cx\| = |c|\,\|x\|$ for all $c \in \mathbb{R}$ and $x \in V$; and
(iv) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

The last condition is known as the *triangle inequality.*

Example 2.33. The functions $\|\cdot\|_1$ and $\|\cdot\|_\infty$ mapping $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$\|x\|_1 = \sum_{j=1}^{n} |x_j|$$

and

$$\|x\|_\infty = \max\{|x_j| : 1 \leq j \leq n\}$$

respectively are norms on $\mathbb{R}^n$. △

Definition 2.34. Let V be a vector space over $\mathbb{R}$. A function $< \cdot, \cdot >: V \times V \to \mathbb{R}$ is an *inner product* (or *scalar product*) on V if,

(i) $\langle x, x\rangle \geq 0$ for all $x \in V$;
(ii) $\langle x, x\rangle = 0$ if and only if $x = 0$;
(iii) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$;
(iv) $\langle cx + y, z\rangle = c\langle x, z\rangle + \langle y, z\rangle$.

Example 2.35. On $\mathbb{R}^n$, the pairing,

$$\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j$$

is an inner product. In the case of $n = 2, 3$ it is often called the *dot product.*

On $\mathcal{P}$, the space of polynomials, the pairing

$$\langle p, q\rangle = \int_0^1 pq\, dt$$

is an inner product.

△

Proposition 2.36. [Cauchy-Schwartz inequality] Suppose $\langle\cdot,\cdot\rangle$ is an inner product on a vector space V. If $x, y \in V$, then

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\,\langle y, y\rangle.$$

Proof. Given $x, y \in V$ and $t \in \mathbb{R}$,

$$\begin{aligned} 0 \leq & \langle x+ty, x+ty\rangle \\ = & \langle x,x\rangle + 2t\langle x,y\rangle + t^2\langle y,y\rangle. \end{aligned}$$

Thus, the discriminate satisfies

$$|\langle x,y\rangle|^2 - \langle x,x\rangle \langle y,y\rangle \leq 0.$$

□

Proposition 2.37. If $\langle \cdot, \cdot \rangle$ is an inner product on a vector space V, then the function $\|\cdot\| : V \to \mathbb{R}$ defined by $\|x\| = \sqrt{\langle x,x\rangle}$ is a norm on V.

Remark 2.38. In the case that V has an inner product, the norm $\|\cdot\|$ of Proposition 2.37 is, unless otherwise noted, understood to be *the norm* on V and $\|x\|$ the norm of a vector $x \in V$.

With this notation, the Cauchy-Schwartz inequality says

$$|\langle x,y\rangle| \leq \|x\|\,\|y\|.$$

◇

Proof. Verification that $\|\cdot\|$ satisfies the first three axioms of a norm are straightforward and left to the gentle reader.

To prove the triangle inequality, estimate, using the Cauchy-Schwartz inequality,

$$\begin{aligned} \|x+y\|^2 = & \|x\|^2 + 2\langle x,y\rangle + \|y\|^2 \\ \leq & \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 \\ = & (\|x\| + \|y\|)^2. \end{aligned}$$

□

Example 2.39. On $\mathbb{R}^n$ the norm arising from the inner product of Example 2.35 is the usual *Euclidean norm*,

$$\|x\|^2 = \sum_{j=1}^{n} x_j^2.$$

Unless otherwise indicated, we take these as the inner product and norm on $\mathbb{R}^n$ and refer to $\mathbb{R}^n$ as *Euclidean space.* △

2.6. **Exercises.**

Exercise 2.1. Suppose $f : \mathbb{F} \mapsto G$ is a field isomorphism.

(i) Is $f^{-1} : G \to \mathbb{F}$ a field isomorphism?
(ii) Show that $f(0_{\mathbb{F}}) = 0_G$.
(iii) What is $f(1_{\mathbb{F}})$?

Exercise 2.2. Show that the functions in Example 2.33 are both norms on $\mathbb{R}^n$.

Exercise 2.3. Verify the claims made in Example 2.35.

Exercise 2.4. Given a positive real number y and positive integers m and n, show

$$(y^{\frac{1}{n}})^m = (y^m)^{\frac{1}{n}}.$$

Likewise verify

$$(y^m)^n = (y^n)^m \text{ and } (y^{\frac{1}{m}})^{\frac{1}{n}} = (y^{\frac{1}{n}})^{\frac{1}{m}}.$$

Thus, $y^{\frac{m}{n}}$ is unambiguously defined.

Exercise 2.5. Show there is no order on $\mathbb{Z}_2$ which makes $\mathbb{Z}_2$ an ordered field.

Exercise 2.6. Let $\mathcal{Q}(\sqrt{2}) = \{a+b\sqrt{2} : a, b \in \mathbb{Q}\}$. Show that $\mathbb{Q}(\sqrt{2})$ is closed under both addition and multiplication (the operations inherited from $\mathbb{R}$). It can be shown that $\mathbb{Q}(\sqrt{2})$ is a field. For a nonzero $a + b\sqrt{2}$ in this field, identify its multiplicative inverse.

2.7. **Problems.**

Problem 2.1. Show there is no order on $\mathbb{C}$ which makes $\mathbb{C}$ an ordered field.

Problem 2.2. Show, if G is field with (exactly) three elements, then there is a field isomorphism $f : \mathbb{Z}_3 \to G$.

Problem 2.3. Prove item (iii) of Theorem 2.28.

Problem 2.4. Let A be a nonempty set of real numbers which is bounded both above and below. Prove, $\sup(A) \geq \inf(A)$.

Problem 2.5. Let A be a nonempty set of real numbers which is bounded above. Let $-A = \{-a : a \in A\} = \{x \in \mathbb{R} : -x \in A\}$. Show $-A$ is bounded below and $-\inf(-A) = \sup(A)$.

Problem 2.6. Prove, if $A \subset B$ are subsets of $\mathbb{R}$ and A is nonempty and B is bounded above, then A and B have least upper bounds and

$$\sup(A) \leq \sup(B).$$

Problem 2.7. Suppose $A \subset \mathbb{R}$ is nonempty and bounded above and $\beta \in \mathbb{R}$. Let

$$A + \beta = \{a + \beta : a \in A\}$$

Prove that $A + \beta$ has a supremum and

$$\sup(A + \beta) = \sup(A) + \beta.$$

Problem 2.8. Suppose $A \subset [0, \infty) \subset \mathbb{R}$ is nonempty and bounded above and $\beta > 0$. Let

$$\beta A = \{a\beta : a \in A\}.$$

Prove βA is nonempty and bounded above and thus has a supremum and

$$\sup(\beta A) = \beta \sup(A).$$

Problem 2.9. Suppose $A, B \subset [0, \infty)$ are nonempty and bounded above. Let

$$AB = \{ab : a \in A, \quad b \in B\}.$$

Prove that AB is nonempty and bounded above and

$$\sup(AB) = \sup(A)\sup(B).$$

Here is an outline of a proof. The hypotheses on A and B imply that $\alpha = \sup(A)$ and $\beta = \sup(B)$ both exist. Argue that AB is nonempty and bounded above by $\alpha\beta$ and thus

$$\sup(AB) \le \alpha\beta.$$

Fix $a \in A$. From an earlier exercise,

$$\sup(aB) = a\sup(B) = a\beta.$$

On the other hand, $aB \subset AB$ and thus,

$$a\beta \le \sup(AB)$$

for each $a \in A$. It follows that βA is bounded above by $\sup(AB)$ and thus,

$$\alpha\beta = \sup(\beta A) \le \sup(AB).$$

Problem 2.10. Suppose $A, B \subset \mathbb{R}$ are nonempty and bounded above. Let

$$A + B = \{a + b : a \in A, \quad b \in B\}.$$

Show $A + B$ has a supremum and moreover,

$$\sup(A + B) = \sup(A) + \sup(B).$$

Problem 2.11. Show, if V is a vector space with an inner product, then the norm

$$\|v\| = \sqrt{\langle v, v\rangle} \tag{1}$$

satisfies the *parallelogram law* ,

$$\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2).$$

Explain why this is called the parallelogram law.

Recall the norm $\|\cdot\|_1$ on $\mathbb{R}^n$ defined in Example 2.33. Does this norm come from an inner product?

Problem 2.12. Suppose $f : [a, b] \to [\alpha, \beta]$ and $\varphi : [\alpha, \beta] \to \mathbb{R}$. Let $h = \varphi \circ f$. Show, if there is a $C > 0$ such that

$$|\varphi(s) - \varphi(t)| \le C|s - t|$$

for all $s, t \in [\alpha, \beta]$, then

$$\begin{aligned}\sup\{h(x) : a \le x \le b\} - &\inf\{h(x) : a \le x \le b\} \\ &\le C\left[\sup\{f(x) : a \le x \le b\} - \inf\{f(x) : a \le x \le b\}\right].\end{aligned}$$

3. Metric Spaces

3.1. Definitions and Examples.

Definition 3.1. A *metric space* (X, d) consists of a set X and function $d : X \times X \to \mathbb{R}$ such that, for $x, y, z \in X$,

(i) $d(x, y) \geq 0$;
(ii) $d(x, y) = 0$ if and only if $x = y$;
(iii) $d(x, y) = d(y, x)$; and
(iv) $d(x, z) \leq d(x, y) + d(y, z)$.

We usually call the metric space X and d the *metric*, or *distance function*. Item (iv) is the *triangle inequality*. Items (i) and (ii) together are sometimes expressed by saying d is *positive definite*. Evidently (iii) is a symmetry axiom.

Example 3.2. Here are some examples of metric spaces.

(a) Unless otherwise noted, $\mathbb{R}$ is the metric space with distance function $d(x, y) = |x - y|$.
(b) Let X be any nonempty set and define $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. This is the *discrete metric*.
(c) On the vector space $\mathbb{R}^n$ define,
$$d_1(x, y) = \sum |x_j - y_j|.$$
This is the ℓ^1 *metric*.
(d) On $\mathbb{R}^n$, define d_∞ by
$$d_\infty(x, y) = \max\{|x_j - y_j| : 1 \leq j \leq n\}.$$
This metric is the ℓ^∞ *metric* (or worst case metric). In particular $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_\infty)$ are different metric spaces.
(e) Define, on the space of polynomials $\mathcal{P}$,
$$d_1(p, q) = \int_0^1 |p - q| dt.$$
(f) If (X, d) is a metric space and $Y \subset X$, then $(Y, d|_{Y \times Y})$ is a metric space and is called a *subspace* of X.

$\triangle$

Do Problem 3.1.

Proposition 3.3. If $\| \cdot \|$ is a norm on a vector space V, then the function
$$d(x, y) = \|x - y\|,$$
is a metric on V.

Remark 3.4. In the case of $\mathbb{R}^n$ with its Euclidean norm, the resulting metric is the *Euclidean distance* which will sometimes be written as d_2. Note that $(\mathbb{R}^n, d_2)$ is, as a metric space, distinct from both $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_\infty)$.

When we speak of the metric space $\mathbb{R}^n$ we mean with the Euclidean distance, unless we have indicated otherwise. ◇

Proof. With the exception of the triangle inequality, it is evident that d satisfies the axioms of a metric.

To prove that d satisfies the triangle inequality, let $x, y, z \in V$ be given and estimate, using the triangle inequality for the norm,

$$\begin{aligned} d(x,z) =& \|x - z\| \\ =& \|(x-y) + (y-z)\| \\ \le& \|x-y\| + \|y-z\| \\ =& d(x,y) + d(y,z). \end{aligned}$$

□

Proposition 3.5. Let (X, d) be a metric space.

If $p, q, r \in X$, then

$$|d(p,r) - d(q,r)| \le d(p,q).$$

If $p_1, \dots, p_n \in X$, then

$$d(p_1, p_n) \le \sum_{j=1}^{n-1} d(p_j, p_{j+1}).$$

3.2. **Open Sets.**

Definition 3.6. Let (X, d) be a metric space. A subset $U \subset X$ is *open* if for each $x \in U$ there is an $\epsilon > 0$ such that

$$N_\epsilon(x) := \{p \in X : d(p,x) < \epsilon\} \subset U.$$

The set $N_\epsilon(x)$ is the *ϵ-neighborhood* of x. More or less synonymously, an *open ball* is a set of the form $N_r(y)$ for some $y \in X$ and $r > 0$.

Proposition 3.7. Neighborhoods are open sets; i.e., if (X, d) is a metric space, $y \in X$ and $r > 0$, then the set

$$N_r(y) = \{p \in X : d(p,y) < r\}$$

is an open set.

Proof. We must show, for each $x \in N_r(y)$ there is an ϵ (depending on x) such that $N_\epsilon(x) \subset N_r(y)$. Accordingly, let $x \in N_r(y)$ be given. Thus, $d(x,y) < r$. Choose $\epsilon = r - d(x,y) > 0$. Suppose now that $p \in N_\epsilon(x)$ so that $d(x,p) < \epsilon$. Estimate, using the triangle inequality,

$$d(y,p) \le d(y,x) + d(x,p) < d(y,x) + \epsilon = d(y,x) + (r - d(y,x)) = r.$$

Thus, $p \in N_r(y)$. We have shown $N_\epsilon(x) \subset N_r(y)$ and the proof is complete. □

Do Problem 3.2.

Example 3.8. In $\mathbb{R}^2$ with the Euclidean distance, show $E = \{(x_1, x_2) : x_j > 0\}$ is an open set. $\triangle$

Example 3.9. The set $[0, 1) \subset \mathbb{R}$ is not an open, since, for every $\epsilon > 0$, the set $N_\epsilon(0) = (-\epsilon, \epsilon)$ contains negative numbers and is thus not a subset of $[0, 1)$. $\triangle$

Proposition 3.10. Let $U \subset \mathbb{R}^n$ be given. The following are equivalent,

(i) U is open in $(\mathbb{R}^n, d_1)$;
(ii) U is open in $(\mathbb{R}^n, d_2)$;
(iii) U is open in $(\mathbb{R}^n, d_\infty)$.

Sketch of proof. Let $N^j_\epsilon(x)$ denote the $\epsilon > 0$ neighborhood of x in the $j = 1, 2, \infty$ norms respectively.

Suppose U is open in $(\mathbb{R}^n, d_1)$ and let $x \in U$ be given. There is an $\epsilon > 0$ such that $N^1_\epsilon(x) \subset U$.

By the C-S inequality,

$$\begin{aligned} d_1(x, y) =& \sum_1^n |x_j - y_j|\, 1 \\ \le& \sqrt{\sum |x_j - y_j|^2} \sqrt{\sum_1^n 1} \\ =& d_2(x, y)\sqrt{n}. \end{aligned}$$

It follows that $N^2_{\frac{\epsilon}{\sqrt{n}}}(x) \subset N^1_\epsilon(x) \subset U$ and thus U is open in $(\mathbb{R}^n, d_2)$. We have proved, if U is open in d_1, then it is open in d_2.

The proof that if U is open in d_2, then U is open in d_∞ is based on the inequality,

$$d_2(x, y) \le \sqrt{n}\, d_\infty(x, y);$$

and the proof that if U is open in d_∞, then U is open in d_1 is based on the inequality

$$d_\infty(x, y) \le d_1(x, y).$$

The details are left as an exercise. $\square$

Example 3.11. Returning to the example of the set $E = \{(x, y) : x, y > 0\} \subset \mathbb{R}^2\}$ above, it is convenient to use the d_∞ metric to prove E is open; i.e., show that E is open in $(\mathbb{R}^2, d_\infty)$ and conclude that E is open in $\mathbb{R}^2$. $\triangle$

Proposition 3.12. Let (X, d) be a metric space.

(i) $\emptyset, X \subset X$ are open;
(ii) if $\mathcal{F} \subset P(X)$ is a collection of open sets, then

$$\cup_{U \in \mathcal{F}} U$$

is open; and

(iii) if $n \in \mathbb{N}^+$ and $U_1, \dots, U_n \subset X$ are open, then

$$\cap_{j=1}^n U_j$$

is open.

Example 3.13. Let $U_j = (-\frac{1}{j+1}, 1) \subset \mathbb{R}$ for $j \in \mathbb{N}$. The sets U_j are open in $\mathbb{R}$ (they are open balls). However, the set

$$[0,1) = \cap_{j=0}^\infty U_j$$

is not open. Thus it is not possible to improve on the last item in the proposition. △

Example 3.14. The set $(-\infty, 0) = \cup_{n=0}^\infty (-2n, 0) = \cup_{n=0}^\infty N_n(-n)$ and is therefore open. We could of course easily checked this directly from the definition of open set. △

Example 3.15. The set

$$\mathbb{R}^2 \supset E = \{(x_1, x_2) : x_j > 0\} = \{x : x_1 > 0\} \cap \{x_2 > 0\}.$$

This provides yet another way to prove E is open. Namely, show that each of the sets on the right hand side above is open. △

Do Problem 3.3.

3.2.1. *Relatively open sets.*

Definition 3.16. Suppose (Z, d) is a metric space and $X \subset Z$ so that $(X, d|_{X\times X})$ is also a metric space. A subset $U \subset X$ is *open relative to* X or is *relatively open*, if U is open in the metric space X.

Example 3.17. Let $X = [0, \infty) \subset Z = \mathbb{R}$. The set $[0,1)$ is open in X, but not in Z. △

Proposition 3.18. Suppose Z is a metric space and $U \subset X \subset Z$. The set U is open in X if and only if there is an open set W in Z such that $U = W \cap X$.

Proof. First, suppose $W \subset Z$ is open (in Z) and $U = W \cap X \subset X$. Given $x \in U$, there is a $\delta > 0$ such that $\{y \in Z : d(x,y) < \delta\} \subset W$ since $x \in W$ and W is open in Z. It follows that $\{y \in X : d(x,y) < \delta\} \subset W \cap X = U$ and thus U is open in X.

Now suppose $U \subset X$ is open relative to X. For each $x \in U$ there is an $\epsilon_x > 0$ such that $V_x = \{y \in X : d(x,y) < \epsilon_x\} \subset U$. Let $W_x = \{y \in Z : d(x,y) < \epsilon_x\}$, note that $V_x = W_x \cap X$, and let

$$W = \cup_{x \in U} W_x.$$

Then W is open in Z and

$$U \subset W \cap X = \cup_{x\in U} W_x \cap X = \cup_{x \in X} V_x \subset U.$$

□

3.3. **Closed Sets.**

Definition 3.19. Let (X, d) be a metric space. A subset $C \subset X$ is *closed* if $X \setminus C$ is open.

Example 3.20. (a) In $\mathbb{R}$ the set $[0, \infty)$ is closed, since its complement, $(-\infty, 0)$ is open.
(b) The set $[0, 1) \subset \mathbb{R}$ is neither open nor closed.
(c) The set $\mathbb{Q} \subset \mathbb{R}$ is neither open nor closed.
(d) The set $F = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = 0\}$ is closed.
(e) The sets X and $\emptyset$ are both open and closed. They are *clopen*.
(f) Every subset of a discrete metric space is clopen. (See Problem 3.3.)

$\triangle$

Proposition 3.21. Let (X, d) be a metric space and let $x \in X$ and $r \geq 0$ be given. The set

$$\{p \in X : d(p, x) \leq r\}$$

is a closed.

Proof. The complement of $\{p \in X : d(p, x) \leq r\}$ is the set

$$U = \{p : d(p, x) > r\}$$

and it suffices to prove that U is open. Let $y \in U$ be given. Then $d(y, x) > r$. Let $\epsilon = d(y, x) - r > 0$. If $z \in N_\epsilon(y)$ so that $d(z, y) < \epsilon$, then,

$$\begin{aligned} d(x, z) \geq& d(x, y) - d(y, z) \\ >& d(x, y) - \epsilon \\ =& r. \end{aligned}$$

It follows that $N_\epsilon(y) \subset U$ and thus, since $y \in U$ was arbitrary, U is open. $\square$

Corollary 3.22. In a metric space, singleton sets are closed; i.e., if (X, d) is a metric space and $x \in X$, then $\{x\}$ is closed.

Proposition 3.23. Let X be a metric space.

(i) X and $\emptyset$ are closed;
(ii) if $C_1, \dots, C_n$ are closed subsets of X, then $\cup_1^n C_j$ is closed; and
(iii) if C_α, $\alpha \in J$ is a family of closed subsets of X, then

$$C = \cap_{\alpha \in J} C_\alpha$$

is closed.

Corollary 3.24. A finite set F in a metric space X is closed.

Proposition 3.25. If $C \subset \mathbb{R}$ is bounded above, nonempty, and closed, then C has a largest element.

Proof. The hypotheses imply $\alpha = \sup(C)$ exist. Certainly, $\alpha \geq x$ for all $x \in C$. Thus to prove the proposition it suffices to prove $\alpha \in C$. We argue by contradiction and accordingly assume $\alpha \in \tilde{C}$. Since C is closed, $\tilde{C}$ is open and therefore there is an $\epsilon > 0$ such that $N_\epsilon(\alpha) \subset \tilde{C}$ or equivalently $C \subset \tilde{N}_\epsilon(\alpha)$. Thus, if $c \in C$, then $c \leq \alpha - \epsilon$ (since also $c \leq \alpha$). It follows that $\alpha - \epsilon$ is an upper bound for C, contradicting the least property of α. Thus $\alpha \in C$. $\square$

Example 3.26. Let $R = \mathbb{Q} \cap [0,1]$ denote the rational numbers in the interval $[0,1]$. Since $\mathbb{Q}$ is countable, so is R. Choose an enumeration $R = \{r_1, r_2, \dots\}$ of R. Fix $1 > \epsilon > 0$ and let

$$V_j = N_{\frac{\epsilon}{2^{j+1}}}(r_j)$$

and $V = \cup V_j$. Thus V is an open set which contains R.

The set $C = [0,1] \setminus V$ is closed because it is the intersection of the closed sets $[0,1]$ and $\tilde{V}$. On the other hand, its complement contains every rational in the interval $[0,1]$, but is also the union of intervals the sum[1] of whose lengths is at most

$$\sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon < 1.$$

Thus C is a closed subset of $[0,1]$ which contains no rational number, but is large in the sense that its complement can be covered by open intervals whose lengths sum to at most ϵ.

A heuristic is that open sets are nice and closed sets can be strange, while most sets are neither open nor closed. $\triangle$

Do Problem 3.4.

3.4. The interior, closure, and boundary of a set.

Definition 3.27. Let (X,d) be a metric space and $S \subset X$. The *closure* of S is

$$\overline{S} := \cap\{C \subset X : C \supset S, \quad C \text{ is closed}\}.$$

Proposition 3.28. Let S be a subset of a metric space X.

(i) $S \subset \overline{S}$;
(ii) $\overline{S}$ is closed;
(iii) if K is any other set satisfying (i) and (ii), then $\overline{S} \subset K$.

Moreover, S is closed if and only if $S = \overline{S}$.

Definition 3.29. Let (X,d) be a metric space and $S \subset X$. The *interior* of S is the set

$$S^\circ := \cup\{U \subset X : U \subset S \text{ is open}\}.$$

Proposition 3.30. Let S be a subset of a metric space X.

(i) $S^\circ \subset S$;
(ii) S° is open;
(iii) if $V \subset S$ is an open set, then $V \subset S^\circ$.

[1] Series are introduced in Problem 4.17 in the next section and will be treated in detail later, but this particular sum should be familiar from Calculus II.

Moreover, S is open if and only if $S = S^\circ$.

Definition 3.31. A point $x \in X$ is an *interior point* of S if there is an $\epsilon > 0$ such that $N_\epsilon(x) \subset S$.

Do Problems 3.5 and 3.6.

Definition 3.32. The *boundary* of a set S in a metric space X is $\partial S = \overline{S} \cap \overline{\tilde{S}}$.

Do Problem 3.7

3.5. **Exercises.**

Exercise 3.1. Show, if $a, b, c \geq 0$ and $a + b \geq c$, then

$$\frac{a}{1+a} + \frac{b}{1+b} \geq \frac{c}{1+c}.$$

Show if (X, d) is a metric space, then

$$d_*(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is a metric on X too.

Exercise 3.2. Show that the subset $S = \{(x, y) \in \mathbb{R}^2 : x \neq y\}$ is open.

Exercise 3.3. Verify that the discrete metric is indeed a distance function.

Exercise 3.4. Let X be a nonempty set and d the discrete metric. Fix a point $z \in X$. Is the closure of the set $N_1(z)$ equal to $\{x \in X : d(x, z) \leq 1\}$?

Exercise 3.5. Show that the set

$$\{(x_1, x_2) : x_1, x_2 \geq 0\} \subset \mathbb{R}^2$$

is closed.

Show that the set

$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = 1\}$$

is closed.

Exercise 3.6. By Proposition 3.3,

$$d(f, g) = \left(\int_0^1 |f - g|^2 dt\right)^{\frac{1}{2}}$$

defines a metric on the space of polynomials $\mathcal{P}$. For $n \in \mathbb{N}$, let

$$p_n(t) = \sqrt{2n+1}\, t^n.$$

Find $d(p_n, p_m)$.

Exercise 3.7. Determine the boundary of an interval $(a, b]$ in $\mathbb{R}$.

3.6. **Problems.**

Problem 3.1. Suppose (X, d_X) and (Y, d_Y) are metric spaces. Define $d : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$d((x, y), (a, b)) = d_X(x, a) + d_Y(y, b).$$

Prove d is a metric on $X \times Y$.

Problem 3.2. Describe the neighborhoods in a discrete metric space (X, d).

Problem 3.3. Determine, with proof, the open subsets of the discrete metric space (X, d).

Problem 3.4. Given a metric space Z and $F \subset X \subset Z$ define F is *relatively closed* in X. Show, F is relatively closed in X if and only if there is a closed set $C \subset Z$ such that $F = C \cap X$.

Prove that the closure of $C \subset X$, as a subset of X, is $X \cap \overline{C}$, where $\overline{C}$ is the closure of C in Z. Conclude, if C is relatively closed, then $C = \overline{C} \cap X$.

Finally, show, if

(i) $A, B \subset Z$;
(ii) $Z = A \cup B$; and
(iii) $\overline{A} \cap B = \emptyset$,

then $B = \tilde{\overline{A}} \cap Z$ and hence is open relative to Z.

Problem 3.5. Show,

$$I(S) = \{s \in S : s \text{ is an interior point of } S\} = S^\circ.$$

Here is an outline of a solution: First show

$$I(S) = \{s \in S : s \text{ is an interior point of } S\}$$

is an open set (mostly easily done by writing it as a union of neighborhoods), from which it will then follow that $I(S) \subset S^\circ$. The inclusion $S^\circ \subset I(S)$ is straightforward.

Problem 3.6. Prove,

$$\overline{S} = \widetilde{(\tilde{S})^\circ};$$

i.e., $\overline{S}$ consists of those points $x \in X$ such that for every $\epsilon > 0$, $N_\epsilon(x) \cap S \neq \emptyset$.

Suggestion: Use the properties of closure and interior. For instance, note that $\tilde{\overline{S}}$ is open and contained in $\tilde{S}$.

Problem 3.7. Prove that $x \in \partial S$ if and only if for every $\epsilon > 0$ there exists $s \in S$, $t \in \tilde{S}$ such that $d(x, s), d(x, t) < \epsilon$.

Prove S is closed if and only if S contains its boundary; and S is open if and only if S is disjoint from its boundary.

Problem 3.8. Show, in $\mathbb{R}^2$, if $x \in \mathbb{R}^2$ and $r > 0$, then the closure of

$$N_r(x) = \{y \in \mathbb{R}^2 : d(x, y) = \|x - y\| < r\}$$

is the set

$$\{y \in \mathbb{R}^2 : d(x,y) = \|x - y\| \leq r\}.$$

Is the corresponding statement true in all metric spaces?

Problem 3.9. Let S be a non-empty subset of a metric space X. Show, x is in $\overline{S}$ if and only if

$$\inf\{d(x,s) : s \in S\} = 0.$$

Problem 3.10. Prove Proposition 3.30.

Problem 3.11. Show that the closure of $\mathbb{Q}$ in $\mathbb{R}$ is all of $\mathbb{R}$. (Suggestion: Use Problem 3.6 and Theorem 2.28 item iii). Compare with Remark 2.29.

Problem 3.12. Show that the closure of $\tilde{\mathbb{Q}}$ (the irrationals) in $\mathbb{R}$ is all of $\mathbb{R}$. Combine this problem and Problem 3.11 to determine the boundary of $\mathbb{Q}$ (in $\mathbb{R}$).

Problem 3.13. Suppose (X,d) is a metric space and $x \in X$ and $r > 0$ are given. Show that the closure of $N_r(x)$ is a subset of the set

$$\{y \in X : d(x,y) \leq r\}.$$

Give an example of a metric space X, an $x \in X$, and an $r > 0$ such that the closure of $N_r(x)$ is **not** the set

$$\{y \in X : d(x,y) \leq r\}.$$

Compare with Problem 3.8.

Problem 3.14. Let (X,d) and d_* be as in Exercise 3.1. Do the metric spaces (X,d) and (X,d_*) have the same open sets?

Problem 3.15. Suppose d and d' are metrics on the set X and there is a constant C such that, for all $x,y \in X$,

$$d(x,y) \leq C d'(x,y).$$

Prove, if U is open in (X,d), then U is open in (X,d').

Thus, if there is also a constant C' such that

$$d'(x,y) \leq C' d(x,y),$$

then the metric spaces (X,d) and (X,d') have the same open sets.

4. Sequences

4.1. Definitions and examples.

Definition 4.1. A *sequence* is a function a with domain $\mathbb{N}$. It is customary to write $a_n = a(n)$ and $(a_n)_n$ or $(a_n)_{n=0}^{\infty}$ for this function.

If the a_n lie in the set X, then (a_n) is a *sequence from* X.

If (X,d) is a metric space and $L \in X$. The sequence (a_n) (from X) *converges to* L if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $d(a_n, L) < \epsilon$,

$$\lim_{n\to\infty} a_n = L$$

and L is said to be **the** *limit* of the sequence.

The sequence (a_n) *converges* if there exists an $L \in X$ such that (a_n) converges to L. A sequence which does not converge is said to *diverge*.

It is often convenient to relax the definition of sequence, allowing the domain to be a set of the form $\{n \in \mathbb{Z} : n \geq n_0\}$ for some integer n_0. In this case, we may write $(a_n)_{n=n_0}^{\infty}$.

Remark 4.2. From the (positive definite) axioms (items (i) and (ii) of Definition 3.1) of a metric, if x, y are points in a metric space (X, d) and if $d(x, y) < \epsilon$ for every $\epsilon > 0$, then $x = y$. $\diamond$

The following proposition list some of the most basic properties of limits. The first justifies the terminology *the limit* (as opposed to *a limit*) above.

Proposition 4.3. Let $(a_n)_{n=k}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be sequences from the metric space X.

(i) If (a_n) converges, then its limit is unique;
(ii) if there is an N and an ℓ such that for $n \geq N$, $b_n = a_{n+\ell}$, then (a_n) converges if and only if (b_n) converges and moreover in this case the sequences have the same limit; and
(iii) if $(a_n)_n$ is a sequence from $\mathbb{R}$ ($X = \mathbb{R}$), $c \in \mathbb{R}$, and (a_n) converges to L, then (ca_n) converges to cL.

The items (ii) and (iii) together say that we need not be concerned with keeping close track of k.

Example 4.4. The sequence $(\frac{1}{n+1})_n$ converges to 0 in $\mathbb{R}$; however it does not converge in the metric space $((0, 1], |\cdot|)$, as can be proved using the previous proposition and the fact that the sequence converges to 0 in $\mathbb{R}$.

The sequence $(\frac{n}{n+1})_n$ converges to 1 (in $\mathbb{R}$). $\triangle$

Example 4.5. If $0 \leq a < 1$, then the sequence (a^n) converges to 0.

To prove this last statement, recall that we have already shown that $\inf(\{a^n : n \in \mathbb{N}\}) = 0$. Thus, given $\epsilon > 0$ there is an N such that $0 \leq a^N < \epsilon$. It follows that, for all $n \geq N$, $|a^n - 0| \leq a^N < \epsilon$. $\triangle$

Do Problems 4.1, 4.2, 4.3 and Exercise 4.2.

We will make repeated use of the following simple identity, valid for all real r and positive integers m,

$$1 - r^m = (1 - r)(1 + r + r^2 + \dots r^{m-1}) \tag{2}$$

Proposition 4.6. In (the metric space) $\mathbb{R}$,

(a) if $\rho > 0$, then the sequence $(\rho^{\frac{1}{n}})$ converges to 1; and
(b) the sequence $(n^{\frac{1}{n}})$ converges to 1.

Proof. To prove (a), first suppose $\rho > 1$. Using Equation (2) with $m = n$ and $r = \rho^{\frac{1}{n}}$ gives

$$\rho^{\frac{1}{n}} - 1 = \frac{\rho - 1}{\sum_{j=0}^{n-1} \rho^{\frac{j}{n}}}.$$

Thus

$$|\rho^{\frac{1}{n}} - 1| < \frac{\rho - 1}{n}.$$

Now, given $\epsilon > 0$ there is, by Theorem 2.28(i) there is an N such that if $n \geq N$, then

$$\frac{1}{n} < \frac{\epsilon}{\rho - 1}.$$

Thus, for $n \geq N$,

$$|\rho^{\frac{1}{n}} - 1| < \frac{\rho - 1}{n} < \epsilon.$$

Hence $(\rho^{\frac{1}{n}})$ converges to 1.

If $0 < \rho < 1$, then $\sigma = \frac{1}{\rho} > 1$ and $(\sigma^{\frac{1}{n}})$ converges to 1. On the other hand,

$$|1 - \rho^{\frac{1}{n}}| = \rho^{\frac{1}{n}} |\sigma^{\frac{1}{n}} - 1| \leq |\sigma^{\frac{1}{n}} - 1|,$$

from which the result follows.

To prove (b) note that the Binomial Theorem gives, for $x > 0$,

$$(1 + x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j \geq \frac{n(n-1)}{2} x^2.$$

Thus, with $x = n^{\frac{1}{n}} - 1$,

$$n \geq \frac{n(n-1)}{2} x^2.$$

Hence, for $n \geq 2$,

$$\sqrt{\frac{2}{n-1}} \geq n^{\frac{1}{n}} - 1 \geq 0,$$

from which it follows that $(n^{\frac{1}{n}})$ converges to 1. Indeed, given $\epsilon > 0$ choose $N \in \mathbb{N}^+$ such that $N \geq \frac{2}{\epsilon^2} + 1$ and observe if $n \geq N$, then $N \geq 2$ and

$$\epsilon > \sqrt{\frac{2}{N-1}} \geq \sqrt{\frac{2}{n}} \geq |n^{\frac{1}{n}} - 1|.$$

□

Remark 4.7. The limit of a sequence depends only upon the notion of open sets. See Problem 4.4. ◇

4.2. **Sequences and closed sets.**

Proposition 4.8. A subset S of a metric space X is closed if and only if every sequence (a_n) from S which converges in X actually converges in S.

Proof. Suppose S is closed and (a_n) is a sequence from S which converges to $L \in X$. Since $\tilde{S}$ is open, if $y \notin S$, then there is an $\epsilon > 0$ such that $N_\epsilon(y) \cap S = \emptyset$. In particular, $d(a_n, y) \geq \epsilon$ for all n and (a_n) does not converge to y. Hence $L \in S$.

Now suppose that S is not closed, equivalently $\tilde{S}$ is not open. In this case, there exists an $L \in \tilde{S}$ such that for every $n \in \mathbb{N}$ there is an s_n such that

$$s_n \in S \cap N_{\frac{1}{n+1}}(L).$$

It is straightforward to verify that (s_n) is a sequence from S which converges to $L \notin S$. □

Do Problems 4.5, 4.6 and 4.7.

4.3. **The monotone convergence theorem for real numbers.** For *numerical sequences*, that is sequences from $\mathbb{R}$, limits are compatible with the order structure on $\mathbb{R}$.

Proposition 4.9. Suppose (a_n) and (b_n) are sequences from $\mathbb{R}$ and $c \in \mathbb{R}$. If $a_n \le b_n + c$ for all n and if both sequences converge, then

$$\lim_n a_n \le \lim_n b_n + c.$$

Further, if (a_n), (b_n), and (c_n) are all sequences from $\mathbb{R}$, if there is an N so that for $n \ge N$,

$$a_n \le b_n \le c_n$$

and if (a_n) and (b_n) converge to the same limit L, then (b_n) also converges to L.

The second part of the Proposition is a version of the *squeeze theorem* and in Problem 4.8 you are asked to provide a proof.

Proof. Let A and B denote the limits of (a_n) and (b_n) respectively. Let $\epsilon > 0$ be given. There is an N so that for $n \ge N$ both $|a_n - A| < \epsilon$ and $|b_n - B| < \epsilon$. Hence, $A - B - c = (A - a_n) + (a_n - b_n - c) + (b_n - B) < 2\epsilon$. □

Definition 4.10. A sequence (a_n) from $\mathbb{R}$ is increasing (synonymously *non-decreasing*) if $a_n \le a_{n+1}$ for all n. The sequence is strictly increasing if $a_n < a_{n+1}$ for all n.

A sequence is *eventually increasing* if there is an N so that the sequence $(a_n)_{n=N}^{\infty}$ is increasing.

The notion of a *decreasing sequence* is defined analogously. A *monotone* sequence is a sequence which is either increasing or decreasing.

Theorem 4.11. If (a_n) is an increasing sequence from $\mathbb{R}$ which is bounded above, then (a_n) converges.

Remark 4.12. Generally, results stated for increasing sequences hold for eventually increasing sequences in view of Proposition 4.3(ii). ◇

Proof. The set $R = \{a_n : n \in \mathbb{N}\}$ (the range of the sequence) is nonempty and bounded above and therefore has a least upper bound. Let $A = \sup(R)$. Given $\epsilon > 0$ there is an $r \in R$ such that $A - \epsilon < r$. There is an N so that $r = a_N$. If $n \ge N$, then, since the sequence is increasing, $0 \le A - a_n \le A - a_N < \epsilon$. Hence (a_n) converges to A. □

Proposition 4.13. In the metric space $\mathbb{R}$, if $0 \le r < 1$, then both (r^n) and (nr^n) converge to 0.

The proof uses the easily proved special case of Proposition 4.20(i) that if (a_n) and (b_n) are sequences of real numbers which converge to A and B respectively, then $(a_n + b_n)$ converges to $A + B$.

Proof. That (r^n) converges to 0 is Example 4.5.

To prove that (nr^n) converges to 0, note that, by Example 4.4, for n sufficiently large

$$\frac{n}{n+1} > r.$$

It follows that there is an N such that for $n \ge N$ the sequence (nr^n) is decreasing. Since it also bounded below by 0 it converges to some L. Hence, using (r^n) converges to 0,

$$rL = rL + 0 = r \lim nr^n + \lim r^{n+1} = \lim (n+1) r^{n+1} = L.$$

Since $r \ne 1$, it follows that $L = 0$. □

Do Problems 4.9 and 4.10.

4.3.1. *The real numbers as infinite decimals.* Here is an informal discussion of infinite *decimal* (base ten) expansions. An *infinite decimal expansion* (base 10) is an expression of the form

$$a = a_0.a_1a_2a_3\cdots,$$

where $a_0 \in \mathbb{Z}$ and $a_j \in \{0, 1, 2, \dots, 9\}$. Let

$$s_n = a_0 + \sum_{j=1}^{n} \frac{a_j}{10^j}$$

and note that the sequence (s_n) is increasing and bounded above by $a_0 + 1$. Thus the sequence (s_n) converges to some real number s and we identify a with this real number.

Conversely, given a real number s there is a smallest integer $m > s$. Let $a_0 = m - 1$. Recursively choose a_j so that, with $s_n = a_0.a_1 \cdots a_n$, we have $0 \le s - s_n \le \frac{1}{10^n}$. In this case (s_n) converges to s and we can identify s with an infinite decimal expansion.

Note that a real number can have more than one decimal expansion. For example both $0.999\cdots$ and $1.000\cdots$ represent the real number 1.

Remark 4.14. Note too it makes sense to talk of expansions with other bases, not just base 10. Base two, called *binary,* is common. Base three is called *ternary.* For $n \in \mathbb{N}$ with $n \ge 2$, expansions base n are called *n-ary.* ⋄

Remark 4.15. Here is an informal argument that a rational number has a repeating infinite decimal expansion.

Suppose x is rational, $x = \frac{m}{n}$. Note that the Euclidean division algorithm produces a decimal representation of x. At each stage there are at most

n choices of remainder. Hence, after at most n steps of the algorithm, we must have a repeat remainder. From there the decimal repeats. $\diamond$

4.3.2. *An abundance of real numbers.*

Proposition 4.16. The set $\mathbb{R}$ is uncountable; i.e., there are uncountably many real numbers.

Proof. It suffices to show if $f : \mathbb{N} \to \mathbb{R}$, then f is not onto. For notational ease, let $x_j = f(j)$.

Choose $b_0 > a_0$ such that $x_0 \notin I_0 := [a_0, b_0]$. Next choose $a_1 < b_1$ such that $a_0 \le a_1 < b_1 \le b_0$ and $x_1 \notin I_1 = [a_1, b_1]$. Continuing in this fashion, construct, by the principle of recursion, a sequence of intervals $I_j = [a_j, b_j]$ such that

(1) $I_0 \supset I_1 \supset I_2 \supset \cdots$;
(2) $b_j - a_j > 0$; and
(3) $x_j \notin I_k$ for $j \le k$.

Observe that the recursive construction of the sequences of endpoints (a_j) and (b_j) implies that $a_0 \le a_1 \le a_2 < \cdots < b_2 \le b_1 \le b_0$; i.e., (a_j) is increasing and is bounded above by each b_m. By Theorem 4.11 (a_j) converges to

$$y = \sup\{a_j : j \in \mathbb{N}\}.$$

In particular, $a_m \le y \le b_m$ for each m. Thus $y \in I_m$ for all m. On the other hand, for each k,

$$x_k \notin I_k$$

and so $y \neq x_k$. Hence y is not in the set $\{x_k : k \in \mathbb{N}\}$ which is the range of f. $\square$

Do Problem 4.11.

4.4. **Limit theorems.**

Proposition 4.17. Let $(a(n))_n$ be a sequence from $\mathbb{R}^g$ and write $a(n) = (a_1(n), \ldots, a_g(n))$. The sequence converges to $L = (L_1, \ldots, L_g) \in \mathbb{R}^g$ if and only if

$$\lim_n a_j(n) = L_j$$

for each $1 \le j \le g$.

Definition 4.18. A sequence (a_n) from a metric space X is *bounded* if there exists an $x \in X$ and $R > 0$ such that $\{a_n : n \in \mathbb{N}\} \subset N_R(x)$.

Proposition 4.19. Convergent sequences are bounded.

Proof. Suppose (a_n) converges to L in the metric space X. Observe, with $\epsilon = 1$ there is an N such that if $n \ge N$, then $d(a_n, L) < 1$. Choosing

$$R = \max(\{d(a_j, L) : 0 \le j < N\} \cup \{1\}) + 1$$

gives $\{a_n : n \in \mathbb{N}\} \subset N_R(L)$. Hence $\{a_n : n \in \mathbb{N}\}$ is bounded. $\square$

Proposition 4.20. Let (a_n) and (b_n) be sequences from $\mathbb{R}^g$ and $c \in \mathbb{R}$. If (a_n) converges to A and (b_n) converges to B, then

(i) $(a_n + b_n)$ converges to $A + B$;
(ii) (ca_n) converges to cA;
(iii) $(a_n \cdot b_n)$ converges to $A \cdot B$; and
(iv) if $g = 1$ and $b_n \neq 0$ for each n and $B \neq 0$, then $\frac{a_n}{b_n}$ converges to $\frac{A}{B}$.

Proof. Proofs of the first two items are routine and left to the reader.

To prove the third item, let $\epsilon > 0$ be given. Since the sequence (b_n) converges, it is bounded by say M. Since (a_n) and (b_n) converge to A and B respectively, there exists N_a and N_b such that if $n \geq N_a$, then

$$\|A - a_n\| \leq \frac{\epsilon}{2(M+1)}$$

and likewise if $n \geq N_b$, then

$$\|B - b_n\| < \frac{\epsilon}{2(\|A\|+1)}.$$

Choose $N = \max\{N_a, N_b\}$. If $n \geq N$, then

$$\begin{aligned} \|A \cdot B - a_n \cdot b_n\| =& \|A \cdot (B - b_n) + (A - a_n) \cdot b_n\| \\ \leq& \|A\|\|B - b_n\| + \|A - a_n\|\|b_n\| \\ \leq& \|A\|\|B - b_n\| + \|A - a_n\| \, M \\ <& \epsilon. \end{aligned}$$

To prove the last statement, it suffices to prove it under the assumption that $a_n = 1$ for all n. Since $(|b_n|)$ converges to $|B| > 0$, with $\epsilon = \frac{|B|}{2}$ there is an M such that if $n \geq M$, then $|b_n| \geq \frac{|B|}{2}$. For such n

$$|\frac{1}{B} - \frac{1}{b_n}| = \frac{|B - b_n|}{|B| \, |b_n|} \leq |B - b_n| \frac{2}{|B|^2}.$$

The remaining details are left to the gentle reader. □

Proposition 4.21. Suppose (a_n) is a sequence of nonnegative numbers, $p, q \in \mathbb{N}^+$ and $r = \frac{p}{q}$. If (a_n) converges to L, then (a_n^r) converges to L^r.

Proof. Item (iii) of Proposition 4.20 with $g = 1$ and $b_n = a_n$ shows that (a_n^2) converges to L^2. An induction argument now shows that (a_n^p) converges to L^p.

To show $(a_n^{\frac{1}{q}})$ converges to $L^{\frac{1}{q}}$, first observe that $L \geq 0$. Suppose $L > 0$. In this case, the identity,

$$(x^q - y^q) = (x - y) \sum_{j=0}^{q-1} x^j y^{q-1-j}$$

applied to $x = a_n^{\frac{1}{q}}$ and $y = L^{\frac{1}{q}}$ gives,

$$|a_n - L| = |a_n^{\frac{1}{q}} - L^{\frac{1}{q}}| \sum_{j=0}^{q-1} a_n^{\frac{j}{q}} L^{\frac{q-1-j}{q}} \geq |a_n^{\frac{1}{q}} - L^{\frac{1}{q}}| L^{\frac{q-1}{q}}.$$

From here the remainder of the argument is easy and left to the gentle reader. □

Have another look at Problem 4.10.

4.5. **Subsequences.**

Definition 4.22. Given a sequence (a_n) and an increasing sequence $n_1 < n_2 < \dots$ of natural numbers, the sequence $(a_{n_j})_j$ is a *subsequence* of (a_n).

Alternately, a sequence (b_m) is a subsequence of (a_n) if there is a strictly increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $b_m = a_{\sigma(m)}$.

Example 4.23. The sequence $(\frac{1}{j^2})$ is a subsequence of $(\frac{1}{n})$ (choosing $n_j = j^2$ for $j \geq 1$).

The constant sequences (-1) and (1) are both subsequences of $((-1)^n)$. △

Proposition 4.24. Suppose (a_n) is sequence in a metric space X. If (a_n) converges to $L \in X$, then every subsequence of (a_n) converges to L.

This proposition is an immediate consequence of Problem 4.1.
Do Problem 4.12.

Proposition 4.25. Let (x_n) be a sequence from a metric space X and let $y \in X$ be given. If for every $\epsilon > 0$ the set

$$\{n \in \mathbb{N} : d(y, x_n) < \epsilon\}$$

is infinite, then there exists a subsequence (x_{n_k}) of (x_n) such that $(x_{n_k})_k$ converges to y.

Proof. With $\epsilon = 1$ there is an n_1 such that $d(y, x_{n_1}) < 1$. Suppose now that $n_1 < n_2 \dots < n_k$ have been constructed so that $d(y, x_{n_j}) < \frac{1}{j}$ for each $1 \leq j \leq k$. Since the set $\{n : d(y, a_n) < \frac{1}{k+1}\}$ is infinite, there exists a $n_{k+1} > n_k$ such that $d(y, a_{n_{k+1}}) < \frac{1}{k+1}$. Thus, by recursion, we have constructed a subsequence (a_{n_k}) which converges to y. □

4.6. **The limits superior and inferior.**

Proposition 4.26. Given a bounded sequence (a_n) of real numbers, let

$$\alpha_n = \sup\{a_j : j \geq n\}.$$

The sequence (α_n) is decreasing and bounded below and hence converges.

The proof of the Proposition is left as an exercise (see Problem 4.13.)

Definition 4.27. The limit of the sequence (α_n) is called the *limsup* or *limit superior* of the sequence (a_n). The *liminf* is defined analogously.

Observe that $\inf\{a_j : j \geq n\} \leq a_n \leq \sup\{a_j : j \geq n\}$ for each n. Do Problem 4.14.

Example 4.28. Here are some simple examples.

(i) The $\limsup$ and $\liminf$ of $(\sin(\frac{\pi}{2}n))$ are 1 and -1 respectively.
(ii) The $\limsup$ and $\liminf$ of the sequence $((-1)^n(1+\frac{1}{n}))$ are also 1 and -1 respectively.
(iii) The $\liminf$ of the sequence $((1-(-1)^n)n)$ is 0. It has no $\limsup$. Alternately, the $\limsup$ could be interpreted as ∞.

△

Proposition 4.29. A bounded sequence (a_n) converges if and only if

$$\limsup a_n = \liminf a_n$$

and in this case (a_n) converges to this common value.

Proof sketch. For notational purposes, let $\alpha_n = \sup\{a_j : j \geq n\}$ and let $\gamma_n = \inf\{a_j : j \geq n\}$.

Suppose (a_n) converges to a. Given $\epsilon > 0$, there is an N such that if $j \geq N$, then $|a_j - a| < \epsilon$. In particular, for $j \geq N$, we have $a_j \leq a + \epsilon$ and thus $\alpha_N \leq a + \epsilon$. Consequently, if $n \geq N$, then

$$a - \epsilon < a_n \leq \alpha_n \leq \alpha_N \leq a + \epsilon$$

and therefore $|\alpha_n - a| \leq \epsilon$. It follows that (α_n) converges to a and therefore

$$\limsup a_n = a.$$

By symmetry,

$$\liminf a_n = a.$$

Now suppose

$$\limsup a_n = \liminf a_n$$

and let A denote this common value.

Observe that $\gamma_n \leq a_n \leq \alpha_n$ for all n. Hence, by the Squeeze Theorem, Problem 4.8, (a_n) converges to A. □

Do Problem 4.15.

Proposition 4.30. Suppose (a_n) is a bounded sequence of real numbers. Given $x \in \mathbb{R}$, let $J_x = \{n : a_n > x\}$ and let

$$S = \{x \in \mathbb{R} : J_x \text{ is infinite}\}.$$

Then,

$$\limsup a_n = \sup(S).$$

Proof. For notational ease, let $\alpha_m = \sup\{a_n : n \ge m\}$ and let $\alpha = \limsup a_n$

Observe that J_x is infinite if and only if for each $n \in \mathbb{N}$ there is an $m \ge n$ such $m \in J_x$; i.e., there is an $m \ge n$ such that that $a_m > x$.

To prove α is an upper bound for S, let $x \in S$ be given. Given an integer n there is an $m \ge n$ such that $a_m > x$ Hence $\alpha_n > x$. It follows that $\alpha \ge x$.

To prove that α is the least upper bound of S, suppose $x < \alpha$. Given n, it follows that $x < \alpha_n$. Hence, x is not an upper bound for the set $\{a_j : j \ge n\}$ which means there is an $m \ge n$ such that $x < a_m \le \alpha_n$. This shows J_x is infinite. Thus $x \in S$. It follows that $(-\infty, \alpha) \supset S$ and thus if β is an upper bound for S, then $\beta \ge \alpha$. Hence α is the least upper bound of S. □

Do Problem 4.16.

4.7. **Exercises.**

Exercise 4.1. Show, arguing directly from the definitions, that the numerical sequences

$$a_n = \frac{2n-3}{n+5}, \quad n \ge 0;$$
$$b_n = \frac{n+3}{n^2-n-1} \quad n \ge 2$$

converge.

Exercise 4.2. By negating the definition of convergence of a sequence, state carefully what it means for the sequence (a_n) from the metric space X to not converge.

Show that the sequence (from $\mathbb{R}$) $(a_n = (-1)^n)$ does not converge. Suggestion, show if $L \neq 1$, then (a_n) does not converge to L; and if $L \neq -1$, then (a_n) does not converge to L.

Exercise 4.3. Consider the sequence (s_n) from $\mathbb{R}$ defined by

$$s_n = \sum_{j=1}^{n} j^{-2}.$$

Show by induction that

$$s_n \le 2 - \frac{1}{n}.$$

Prove that the sequence (s_n) converges.

Exercise 4.4. Define a sequence from $\mathbb{R}$ as follows. Fix $r > 1$. Let $a_1 = 1$ and define recursively,

$$a_{n+1} = \frac{1}{r}(a_n + r + 1).$$

Show, by induction, that (a_n) is increasing and bounded above by $\frac{r+1}{r-1}$. Does the sequence converge?

Exercise 4.5. Return to Exercise 4.1, but now verify the limits using Theorem 4.20 together with a little algebra.

Exercise 4.6. Find the limit in Exercise 4.4.

Exercise 4.7. Let $\sigma : \mathbb{N} \to \mathbb{Q} \cap [0,1]$ be a bijection. What are the subsequential limits of the sequence $(\sigma(n))$?

Exercise 4.8. Suppose (a_n) is a sequence from a metric space X and $L \in X$. Show, if there is a sequence (r_n) of real numbers which converges to 0, a real number C, and positive integer M such that, for $m \geq M$,

$$d(a_m, L) \leq Cr_m,$$

then (a_n) converges to L.

4.8. **Problems.**

Problem 4.1. Suppose (a_n), a sequence in a metric space X, converges to $L \in X$. Show, if $\sigma : \mathbb{N} \to \mathbb{N}$ is one-one, then the sequence $(b_n = a_{\sigma(n)})_n$ also converges to L.

Problem 4.2. Suppose (a_n) is a sequence from $\mathbb{R}$. Show, if (a_n) converges to L, then the sequence (of Cesaro means) (s_n) defined by

$$s_n = \frac{1}{n+1} \sum_{j=0}^{n} a_j$$

also converges to L. Is the converse true?

Problem 4.3. Suppose (a_n) and (b_n) are sequences from a metric space X. Show, if both sequences converge to $L \in X$, then (c_n), defined by $c_{2n} = a_n$ and $c_{2n+1} = b_n$, also converges to L.

Problem 4.4. Suppose d and d' are both metrics on X and that the metric spaces (X, d) and (X, d') have the same open sets. Show, the sequence (a_n) from X converges in (X, d) if and only if it converges in (X, d') and then to the same limit.

Problem 4.5. Let S be a subset of a metric space X. A point $y \in X$ is a *limit point* of S if there is a sequence (s_n) from $S \setminus \{y\}$, which converges to y.

Prove that S is closed if and only if S contains all its limit points. (Often this limit point criteria is taken as the definition of closed set.)

Problem 4.6. Let S' denote the set of limit points of a subset S of a metric space X. (See Problem 4.5.) Prove that S' is closed.

Problem 4.7. Show, if C is a subset of $\mathbb{R}$ which has a supremum, say α, then there is a sequence (c_n) from C which converges to α. Use this fact, plus Proposition 4.8, to give another proof of Proposition 3.25.

Problem 4.8. [A squeeze theorem] Suppose (a_n), (b_n), and (c_n) are sequences of real numbers. Show, if $a_n \leq b_n \leq c_n$ for all n and both (a_n) and (c_n) converge to L, then (b_n) converges to L.

Problem 4.9. Suppose (a_n) is a sequence of positive real numbers and assume

$$L = \lim \frac{a_{n+1}}{a_n}$$

exists. Show, if $L < 1$, then (a_n) converges to 0 by completing the following outline (or otherwise).

(a) Choose $L < \rho < 1$.
(b) Show there is an M so that if $m \ge M$, then $a_{m+1} \le \rho a_m$;
(c) Show $a_{M+k} \le \rho^k a_M$ for $k \in \mathbb{N}$;
(d) Show $a_n \le \rho^n \frac{a_M}{\rho^M}$ for $n \ge M$;
(e) Complete the proof.

Give an example where (a_n) converges to 0 and $L = 1$; and give an example where (a_n) does not go to 0, but $L = 1$.

Prove, if $0 \le L < 1$, and p is a positive integer, then $(n^p a_n)$ converges to 0 too.

Problem 4.10. Let $a_0 = \sqrt{2}$ and define, recursively, $a_{n+1} = \sqrt{a_n + 2}$. Prove, by induction, that the sequence (a_n) is increasing and is bounded above by 2. Does the sequence converge? If so, what should the limit be?

Problem 4.11. Use Theorem 2.28 to prove for each real number r there is a sequence (q_n) of rational numbers converging to r. Use Proposition 4.8 to conclude that the closure of $\mathbb{Q}$ (in $\mathbb{R}$) is $\mathbb{R}$. (See Remark 2.29.)

Problem 4.12. Suppose (a_n) is a sequence in a metric space X. Show, if there is an $L \in X$ such that every subsequence of (a_n) has a further subsequence which converges to L, then (a_n) converges to L.

Problem 4.13. Prove Proposition 4.26.

Problem 4.14. Suppose (a_n) is a bounded sequence of real numbers. Prove

$$\liminf a_n \le \limsup a_n.$$

Give an example which shows the inequality can be strict.

Problem 4.15. Suppose both (a_n) and (b_n) are bounded sequences of real numbers. Prove,

$$\limsup(a_n + b_n) \le \limsup a_n + \limsup b_n.$$

[Hint: Observe $\{a_j + b_j : j \ge n\} \subset \{a_j + b_k : j, k \ge n\}$ from this, and the fact that $\sup(S + T) = \sup(S) + \sup(T)$, it will follow that

$$\sup(\{a_j+b_j : j \ge n\}) \le \sup(\{a_j+b_k : j, k \ge n\}) = \sup\{a_j : j \ge n\}+\sup\{b_k : k \ge n\}.$$

Give an example which shows the inequality can be strict.

Problem 4.16. Let (a_n) be a bounded sequence of real numbers. Prove there is a subsequence $(a_{n_j})_j$ which converges to $y = \limsup a_n$. Here is one way to proceed. Show, either directly or using Proposition 4.30, that for each $\epsilon > 0$ the set $\{n : |y - a_n| < \epsilon\}$ is infinite and then apply Proposition 4.25.

Problem 4.17. Given a sequence $(a_j)_{j=0}^{\infty}$ of real numbers, let

$$s_m = \sum_{j=0}^{m} a_j.$$

The expression $\sum_{n=0}^{\infty} a_n$ is called a *series* and the sequence (s_n) is its *sequence of partial sums.* If the sequence (s_n) converges, then the series is said to *converge* and if moreover, (s_n) converges to L, then the series converges to L written

$$\sum_{n=0}^{\infty} a_n = L = \lim_{m\to\infty} s_m.$$

In particular, the expression $\sum_{n=0}^{\infty} a_n$ is used both for the sequence (s_n) and the limit of this sequence, if it exists.

Show, if $a_n \geq 0$, then the series either converges or diverges to ∞ depending on whether the partial sums form a bounded sequence or not.

Show, if $0 \leq r < 1$, then, for each m,

$$(1-r)\sum_{n=0}^{m} r^n = 1 - r^{m+1}$$

and thus,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

5. Cauchy sequences and completeness

Definition 5.1. A sequence (a_n) in a metric space (X,d) is *Cauchy* if for every $\epsilon > 0$ there is an N such that for all $n, m \geq N$, $d(a_n, a_m) < \epsilon$.

Do Problem 5.1.

Proposition 5.2. Convergent sequences are Cauchy; i.e., if (a_n) is a convergent sequence in a metric space X, then (a_n) is Cauchy.

Proposition 5.3. Cauchy sequences are bounded.

Definition 5.4. A metric space X is *complete* if every Cauchy sequence in X converges (in X).

Example 5.5. Cauchy sequences in a discrete metric space are eventually constant and hence converge. Thus, a discrete metric space is complete. △

Example 5.6. The metric space $\mathbb{Q}$ is an example of an incomplete space. Exercise 5.2 gives further examples of incomplete spaces. △

Theorem 5.7. $\mathbb{R}$ is a complete metric space.

Proof. Let (a_n) be a given Cauchy sequence from $\mathbb{R}$. By Proposition 5.3, this sequence is bounded. Hence it has a limsup; i.e., with

$$\alpha_n = \sup\{a_k : k \geq n\}$$

the sequence (α_n) is decreasing and bounded below and converges to $\alpha = \limsup a_n$.

It suffices to show that (a_n) converges to α. To this end, let $\epsilon > 0$ be given. Because (α_n) converges to α, there is an M so that if $m \ge M$, then

$$\alpha \le \alpha_m < \alpha + \epsilon. \tag{3}$$

Since (a_n) is Cauchy, there is a K such that for $n, k \ge K$,

$$|a_k - a_n| < \epsilon.$$

In particular, for $k \ge n \ge K$,

$$a_k \le a_n + \epsilon.$$

Hence $a_n + \epsilon$ is an upper bound for $\{a_j : j \ge n\}$ and therefore,

$$\alpha_n \le a_n + \epsilon. \tag{4}$$

Let $N = \max\{M, K\}$. If $n \ge N$, then, by combining Equations (3) and (4),

$$\alpha + \epsilon > \alpha_n \ge a_n \ge \alpha_n - \epsilon \ge \alpha - \epsilon.$$

Thus, if $n \ge N$, then

$$|\alpha - a_n| < \epsilon$$

and the proof is complete. □

Proposition 5.8. A closed subset of a complete metric space is complete.

Proof. Apply Proposition 4.8. □

Proposition 5.9. A complete subset of a metric space is closed.

Proof. Apply Proposition 4.8. □

Definition 5.10. A sequence (x_n) from a metric space X is *super Cauchy* if there exists a $0 \le k < 1$ such that

$$d(a_{n+1}, a_n) \le k d(a_n, a_{n-1}) \tag{5}$$

for all $n \ge 1$.

The following result is a version of the *contraction mapping principle.*

Proposition 5.11. If (a_n) is super Cauchy, then (a_n) is Cauchy. In particular, super Cauchy sequences in a complete metric space converge.

Proof. First observe, by Equation (2),

$$\sum_{j=0}^{n} k^j \le \frac{1}{k-1}.$$

Next note that, by iterating the inequality of Equation (5),

$$d(a_{m+1}, a_m) \le k^m d(a_1, a_0)$$

for all m. Thus, for $\ell \geq 0$,

$$\begin{aligned} d(a_{n+\ell}, a_n) &\leq \sum_{j=0}^{\ell-1} d(a_{n+j+1}, a_{n+j}) \\ &\leq \sum_{j=0}^{\ell-1} k^{n+j} d(a_1, a_0) \\ &= k^n d(a_1, a_0) \sum_{j=0}^{\ell-1} k^j \\ &\leq k^n \frac{d(a_1, a_0)}{1-k}. \end{aligned}$$

The remainder of the proof is a straightforward exercise based on the fact that (k^n) converges to 0. $\square$

Note that Proposition 5.11 holds under the weaker assumption that there is an N such that the inequality of Equation (5) holds just for all $n \geq N+1$; i.e., (a_n) just need be eventually super Cauchy.

Example 5.12. For $n \in \mathbb{N}^+$, let

$$s_n = \sum_{j=2}^{n} \frac{1}{j}.$$

Note that

$$s_{2^n} = \sum_{k=0}^{n-1} \sum_{j=2^k+1}^{2^{k+1}} \frac{1}{j} \geq \frac{n}{2}$$

and thus (s_n) is not a bounded sequence and is therefore not Cauchy.

On the other hand,

$$|s_{n+2} - s_{n+1}| = \frac{1}{n+2} < \frac{1}{n+1} = |s_{n+1} - s_n|.$$

$\triangle$

5.1. **Exercises.**

Exercise 5.1. Define a sequence of real numbers recursively as follows. Let $a_1 = 1$ and

$$a_{n+1} = 1 + \frac{1}{1+a_n}.$$

Show (a_n) is not monotonic (that is neither increasing or decreasing). Show that $a_n \geq 1$ for all n and then use Proposition 5.11 to show that (a_n) is Cauchy. Conclude that the sequences converges and find its limit.

Exercise 5.2. Suppose y is a limit point (see Problem 4.5) of the metric space X. Show $Y = X \setminus \{y\}$ is not complete.

Exercise 5.3. Show directly that the sequence $((-1)^n)$ is not Cauchy and conclude that it doesn't converge. Compare with Exercise 4.2.

5.2. **Problems.**

Problem 5.1. Suppose (x_n) is a Cauchy sequence in a metric space X. Show, if (x_n) has a subsequence (x_{n_k}) which converges to some $y \in X$, then (x_n) converges to y.

Problem 5.2. Fix $A > 0$ and define a sequence from $\mathbb{R}$ as follows. Let $a_0 = 1$. For $n \geq 1$, recursively define

$$a_{n+1} = A + \frac{1}{a_n}.$$

Show, for all $n \geq 1$, $a_n \geq A$ and $a_n a_{n+1} \geq 1 + A^2$. Use Proposition 5.11 to prove that (a_n) converges. What is the limit?

Problem 5.3. The *diameter* of a set S in a metric space X is

$$\operatorname{diam}(S) = \sup\{d(s,t) : s,t \in S\}.$$

(In the case that the set of values $d(s,t)$ is not bounded above this supremum is interpreted as plus infinity.)

Prove, if X is a complete metric space, $S_1 \supset S_2 \supset \dots$ is a nested decreasing sequence of nonempty closed subsets of X, and the sequence $(\operatorname{diam}(S_n))_n$ converges to 0, then

$$\cap S_n$$

contains exactly one point.

Show that this result fails if any of the hypotheses - completeness, closedness of the S_n, or that the diameters tend to 0 - are omitted.

Problem 5.4. Suppose $U_1, U_2, \dots$ is a sequence of open sets in a nonempty complete metric space X. Show, if, for each j, the closure of U_j is all of X, then

$$\cap_1^\infty U_j \neq \emptyset.$$

This is a version of the Baire Category Theorem.

Here is an outline of a proof. Observe that for each $x \in X$, $r > 0$, and j, that $N_r(x) \cap U_j \neq \emptyset$ and let $B_r(x) = \{y \in X : d(x,y) \leq r\}$ (the closed ball of radius r with center x).

Pick a point $x_1 \in U_1$. There is an $r_1 \leq 1$ such that $B_{r_1}(x_1) \subset U_1$. There is a point $x_2 \in N_{r_1}(x) \cap U_2$. There is an $0 < r_2 < \frac{r_1}{2}$ such that $B_{r_2}(x_2) \subset U_2$. Continuing in this fashion constructs a sequence of sets $B_{r_j}(x_j)$. Apply an earlier problem to complete the proof.

Problem 5.5. Complete the following outline that $\mathbb{R}$ is complete. Let (a_n) be a given Cauchy sequence from $\mathbb{R}$. Explain why

$$\alpha = \limsup a_n$$

exists. There is a subsequence (a_{n_j}) of (a_n) which converges to α (by Problem 4.16); and thus the sequence itself converges to α (by Problem 5.1).

Problem 5.6. Given metric spaces (X, d_X) and (Y, d_Y) let Z denote the metric space built from X and Y as in Problem 3.1. Show, if X and Y are complete, then so is $X \times Y$.

Problem 5.7. Show that the sequence (a_n) from Exercise 5.1 is not eventually monotone. As a suggestion, first show that, for each n, $a_{n+1} \neq a_n$ as otherwise a_n would be irrational.

6. Compact Sets

6.1. Definitions and Examples.

Definition 6.1. An *open cover* $\mathcal{U}$ of a subset S of a metric space X is a subset of $P(X)$ such that each $U \in \mathcal{U}$ is open and

$$S \subset \cup\{U : U \in \mathcal{U}\} = \cup_{U \in \mathcal{U}} U.$$

A *subcover* of the open cover $\mathcal{U}$ is a subset $\mathcal{V} \subset \mathcal{U}$ which is also an open cover of S.

A subset K of a metric space X is *compact* provided every open cover of K has a finite subcover.

Remark 6.2. Often it is convenient to view covers as an indexed family of sets, rather than a subset of $\mathcal{P}(X)$. In this case an open cover of S consists of an index set $\mathcal{J}$ and a collection of open sets $\mathcal{U} = \{U_j : j \in \mathcal{J}\}$ whose union contains S. A subcover is then a collection $\mathcal{V} = \{U_k : k \in K\}$, for some subset K of J. A set K is compact if for each collection $\{U_j : j \in J\}$ such that

$$K \subset \cup_{j \in J} U_j,$$

there is a finite subset $K \subset J$ such that

$$K \subset \cup_{k \in K} U_k.$$

◇

Example 6.3. Consider the set $K = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$ as a subset of the metric space $\mathbb{R}$.

Let $\mathcal{U}$ be a given open cover of K. There is then a $U_0 \in \mathcal{U}$ such that $0 \in U_0$. Since U_0 is open, there is an $\epsilon > 0$ such that $N_\epsilon(0) \subset U_0$. Since $\frac{1}{n}$ converges to 0, there is an N such that if $n \geq N$, then $\frac{1}{n} \in N_\epsilon(0)$. For each $j = 1, 2, \dots, N-1$ there is a $U_j \in \mathcal{U}$ such that $\frac{1}{j} \in U_j$. It follows that $\mathcal{V} = \{U_0, \dots, U_{N-1}\} \subset \mathcal{U}$ is a finite subcover (of K). Thus K is compact. △

Do Problems 6.1 and 6.2.

Example 6.4. Let $S = (0, 1] \subset \mathbb{R}$ and consider the indexed family of sets $U_j = (\frac{1}{j}, 2)$ for $j \in \mathbb{N}^+$. It is readily checked that

$$S \subset \cup_{j=1}^{\infty} U_j$$

and of course each U_j is open. Thus $\mathcal{U} = \{U_j : j \in \mathbb{N}^+\}$ is an open cover of S.

Let $\mathcal{V}$ be a given finite subset of $\mathcal{U}$. In particular, there is an N such that $\mathcal{V} \subset \{U_j : 1 \le j \le N\}$ and therefore,

$$\cup_{V\in\mathcal{V}} V \subset \cup_{j=1}^{N} U_j = (\frac{1}{N}, 2).$$

Thus $\mathcal{V}$ is not a cover of S and hence $\mathcal{U}$ contains no finite subset which covers S. Thus S is not compact. △

Theorem 6.5. Closed bounded intervals in $\mathbb{R}$ are compact.

Proof. Let $[a, b]$ be a given closed bounded interval and let $\mathcal{U}$ be a given open cover of $[a, b]$.

Let

$$S = \{x \in [a, b] : [a, x] \text{ has a finite subcover from } \mathcal{U}\}.$$

There is a $U \in \mathcal{U}$ such that $a \in U$ and hence $[a, a] \subset U$. It follows that $a \in S$ and thus S is nonempty. It is also bounded above by b. It follows that $\sup(S)$ exists and is at most b.

To prove that $b \in S$, observe that there is a $U_0 \in \mathcal{U}$ such that $\sup(S) \in U_0$ since $\sup(S) \in [a, b]$ and $\mathcal{U}$ is an open cover of $[a, b]$. Because U_0 is open, there is an $\epsilon > 0$ such that $N_\epsilon(\sup(S)) \subset U_0$. There is an $s \in S$ such that $\sup(S) - \epsilon < s \le \sup(S)$. Since $s \in S$, there is a finite subcover $\mathcal{V} \subset \mathcal{U}$ of $[a, s]$; i.e., $\mathcal{V}$ is finite and

$$[a, s] \subset \cup\{U : U \in \mathcal{V}\}.$$

It follows that

$$[a, \sup(S) + \frac{\epsilon}{2}] \subset [a, s] \cup [\sup(S) - \frac{\epsilon}{2}, \sup(S) + \frac{\epsilon}{2}] \subset \cup\{U : U \in \mathcal{V}\} \cup \{U_0\}.$$

Thus, for each $t \in [a, b] \cap [\sup(S), \sup(S) + \frac{\epsilon}{2}]$, the collection $\mathcal{W} = \mathcal{V} \cup \{U_0\}$ is a finite subset of $\mathcal{U}$ which covers $[a, t]$. Thus, each such t is in S. In particular, $\sup(S) \in S$. On the other hand, if $\sup(S) < b$, then there is a $t \in s \in [a, b] \cap (\sup(S), \sup(S) + \frac{\epsilon}{2}]$ in violation of the least property of $\sup(S)$. Thus, $\sup(S) = b$ and moreover

$$[a, b] \subset \{U : U \in \mathcal{V}\} \cup \{U_0\}.$$

Thus $[a, b]$ is compact. □

Do Problem 6.3 which says that a subset K of a discrete metric space X is compact if and only if K is finite. In particular, if the set K in Example 6.3 is considered with the discrete metric, then it is not Compact.

Theorem 6.6. If Y is a metric space and $K \subset X \subset Y$, then K is compact in X if and only if K is compact in Y.

Remark 6.7. The proposition says that compactness is intrinsic and thus, unlike for open and closed sets, we we can speak of compact sets without reference to a larger ambient metric space. ⋄

Proof. First suppose K is compact in X. To prove K is compact in Y, let $\mathcal{U} \subset P(Y)$ an open (in Y) cover of K be given. Let $\mathcal{W} = \{U \cap X : U \in \mathcal{U}\}$. Then $\mathcal{W} \subset P(X)$ is an open (in X) cover of K. Hence there is a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\{U \cap X : U \in \mathcal{V}\}$ covers K. It follows that $\mathcal{V}$ is a finite subset of $\mathcal{U}$ which covers K and hence K is compact as a subset of Y.

Conversely, suppose K is compact in Y. To prove that K is compact in X, let Let $\mathcal{U} \subset P(X)$ be a given open (in X) cover of K. For each $U \in \mathcal{U}$ there exists an open in Y set W_U such that $U = X \cap W_U$. The collection $\mathcal{W} = \{W_U : U \in \mathcal{U}\} \subset P(Y)$ is an open cover of X. Hence there is a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\{W_U : U \in \mathcal{V}\}$ covers K. It follows that $\mathcal{V}$ is a finite subset of $\mathcal{U}$ which covers K. Hence K is compact in X. □

Do Problems 6.4 and 6.5.

6.2. **Compactness and closed sets.**

Definition 6.8. A subset B of a metric space X is *bounded* if there exists $x \in X$ and $R > 0$ such that $B \subset N_R(x)$.

Equivalently, B is bounded if for every $y \in X$ there is a $C > 0$ such that $B \subset N_C(y)$.

Proposition 6.9. Compact sets are closed and bounded.

Proof. Suppose K is a compact subset of a metric space X. If $\tilde{K}$ is empty, then it is open and K is closed. Suppose now that $\tilde{K}$ is not empty. Let $y \notin K$ be given. Let $V_n = \{x \in X : d(x,y) > \frac{1}{n}\}$. The sets V_n are open and

$$\cup_{n=1}^{\infty} V_n \supset X \setminus \{y\} \supset K.$$

Since K is compact, there is an N so that

$$V_N = \cup_{n=1}^{N} V_n \supset K.$$

It follows that, for each $x \in K$, $d(x,y) > \frac{1}{N}$. Hence $N_{\frac{1}{N}}(y) \subset \tilde{K}$ and so $\tilde{K}$ is open and K is closed.

To prove that K is bounded, fix $x_0 \in X$ and let $W_n = \{x \in X : d(x_0, x) < n\}$. Then

$$K \subset X = \cup W_n.$$

By compactness of K, there is an N so that $K \subset W_N$ and thus K is bounded. □

Proposition 6.10. A closed subset of a compact set is compact.

Proof. Suppose X is a metric space, $C \subset K \subset X$, K is compact, and C is closed.

To prove C is compact, let $\mathcal{U}$ be a given open cover of C. Then $\mathcal{W} = \mathcal{U} \cup \{\tilde{C}\}$ is an open cover of K. Hence some finite subset of $\mathcal{W}$ covers K; but then a finite subset of $\mathcal{U}$ covers C. □

Corollary 6.11. Closed bounded subsets of $\mathbb{R}$ are compact. Thus a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Proof. Suppose $K \subset \mathbb{R}$ is both closed and bounded. Since K is bounded, there is a positive real M such that $K \subset [-M, M]$. Now K is a closed subset of the compact set $[-M, M]$ and is hence itself compact. □

It turns out that this corollary is true with $\mathbb{R}$ replaced by $\mathbb{R}^g$, a result which is called the Heine-Borel Theorem. A proof, based upon the Lebesgue number Lemma, and the concomitant fact that compactness and sequential compactness are the same for a metric space, is in Subsection 6.4 below.

Remark 6.12. If X is an infinite set with the discrete metric, then X is closed and bounded, but not compact. Hence, in general, closed and bounded does not imply compact. While this example may seem a bit contrived, we will encounter other more natural metric spaces for which closed and bounded is not the same as compact. (See for instance Problem 6.7.) ◇

6.3. **Sequential Compactness.**

Definition 6.13. A subset K of a metric space X is *sequentially compact* if every sequence in K has a subsequence which converges in K; i.e., if (a_n) is a sequence from K, then there exists $p \in K$ and a subsequence $(a_{n_j})_j$ of (a_n) which converges to p.

Remark 6.14. The notion of sequentially compact does not actually depend upon the larger metric space X, just the metric space K. ◇

Proposition 6.15. If X is sequentially compact, then X is complete.

Problem 6.8 asks you to provide a proof of this Proposition.

Proposition 6.16. Let X be a metric space. If X is compact, then X is sequentially compact.

Proof. Let (s_n) be a given sequence from X. If there is an $s \in X$ such that for every $\epsilon > 0$ the set $J_\epsilon(s) = \{n : s_n \in N_\epsilon(s)\}$ is infinite, then, by Proposition 4.25, (s_n) has a convergent subsequence (namely one that converges to s).

Arguing by contradiction, suppose for each $s \in X$ there is an $\epsilon_s > 0$ such that $J(s) = \{n : s_n \in N_{\epsilon_s}(s)\}$ is a finite set. The collection $\{N_{\epsilon_s}(s) : s \in X\}$ is an open cover of X. Since X is compact there is a finite subset $F \subset X$ such that $\mathcal{V} = \{N_{\epsilon_t}(t) : t \in F\}$ is a cover of X; i.e.,

$$X \subset \cup\{N_{\epsilon_t}(t) : t \in F\}.$$

For each n there is a $t \in F$ such that $s_n \in N_{\epsilon_t}(t)$ and thus $\mathbb{N} = \cup_{t \in F} J_{\epsilon_t}(t)$. But then, for some $u \in F$, the set $J_{\epsilon_u}(u)$ is infinite, a contradiction. □

Do Problem 6.9.

Proposition 6.17. If X is compact, then X is complete.

Corollary 6.18. The metric space $\mathbb{R}$ is complete.

Proof. Suppose (a_n) is a Cauchy sequence from $\mathbb{R}$. It follows that (a_n) is bounded and hence there is a number $R > 0$ such that each a_n is in the interval $I = [-R, R]$. Since I is compact, it is complete. Hence (a_n) converges in I and thus in $\mathbb{R}$. □

The remainder of this section is devoted to proving the converse of Proposition 6.16.

Lemma 6.19. [Lebesgue number lemma] If K is a sequentially compact metric space and if $\mathcal{U}$ is an open cover of K, then there is a $\delta > 0$ such that for each $x \in K$ there is a $U \in \mathcal{U}$ such that $N_\delta(x) \subset U$.

Proof. We argue by contradiction. Accordingly, suppose for every $n \in \mathbb{N}^+$ there is an $x_n \in K$ such that, for each $U \in \mathcal{U}$, $N_{\frac{1}{n}}(x_n)$ is not a subset of U. The sequence (x_n) has a subsequence $(x_{n_k})_k$ which converges to some $w \in K$ because K is sequentially compact. There is a $W \in \mathcal{U}$ such that $w \in W$. Hence there is an $\epsilon > 0$ such that $N_\epsilon(w) \subset W$. Choose k so that $\frac{1}{n_k} < \frac{\epsilon}{2}$ and also so that $d(x_{n_k}, w) < \frac{\epsilon}{2}$. Then $N_{\frac{1}{n_k}}(x_{n_k}) \subset N_\epsilon(w) \subset W$, a contradiction. □

Definition 6.20. A metric space X is *totally bounded* if, for each $\epsilon > 0$, there exists a finite set $F \subset X$ such that

$$X = \cup_{x \in F} N_\epsilon(x).$$

Proposition 6.21. If X is sequentially compact, then X is totally bounded.

Proof. We prove the contrapositive. Accordingly, suppose X is not totally bounded. Then there exists an $\epsilon > 0$ such that for every finite subset F of X,

$$X \neq \cup_{x \in F} N_\epsilon(x).$$

Choose $x_1 \in X$. Choose $x_2 \notin N_\epsilon(x_1)$. Recursively choose,

$$x_{n+1} \notin \cup_1^n N_\epsilon(x_j).$$

The sequence (x_n) has no convergent subsequence since, for $j \neq k$, $d(x_k, x_j) \geq \epsilon$. Thus X is not sequentially compact. □

Proposition 6.22. If X is sequentially compact, then X is compact.

Proof. Let $\mathcal{U}$ be a given open cover of X. From the Lebesgue Number Lemma, there is a $\delta > 0$ such that for each $x \in X$ there is a $U \in \mathcal{U}$ such that $N_\delta(x) \subset U$.

Since X is totally bounded, there exists a finite set $F \subset X$ so that

$$X = \cup_{x \in F} N_\delta(x).$$

For each $x \in F$, there is a $U_x \in \mathcal{U}$ such that $N_\delta(x) \subset U_x$. Hence,

$$X = \cup_{x \in F} U_x;$$

i.e., $\{U_x : x \in F\} \subset \mathcal{U}$ is an open cover of X. Hence X is compact. □

6.4. **The Heine-Borel theorem.**

Lemma 6.23. Cubes in $\mathbb{R}^g$ are compact.

Proof for the case $g = 2$. Either an induction argument or an argument similar to the proof below for $g = 2$ handles the case of general d.

Consider the cube $C = [a, b] \times [c, d]$. It suffices to prove that every sequence (z_n) from C has a subsequence which converges in C; i.e., that C is sequentially compact. To this end, let $(z_n) = (x_n, y_n)$ be a given sequence from C. Since $[a, b]$ is compact, there is a subsequence $(x_{n_k})_k$ of (x_n) which converges to some $x \in [a, b]$. Similarly, since $[c, d]$ is compact the sequence $(y_{n_k})_k$ has a subsequence $(y_{n_{k_j}})_j$ which converges to a $y \in [c, d]$. It follows that $(z_{n_{k_j}})_j$ converges to $z = (x, y) \in C$. □

Theorem 6.24. [Heine-Borel] A subset K of $\mathbb{R}^g$ is compact if and only if it is closed and bounded.

Proof. We have already seen that compact implies closed and bounded in any metric space.

Suppose now that K is closed and bounded. There is a cube C such that $K \subset C \subset \mathbb{R}^g$. The cube C is compact and K is a closed subset of C and is therefore compact. □

Do Problem 6.12.

Corollary 6.25. $\mathbb{R}^g$ is complete.

The proof is similar to that of Corollary 6.18. The details are left as an exercise for the gentle reader.

6.5. **Exercises.**

Exercise 6.1. Let X be a metric space. Show, if there is an $r > 0$ and sequence (x_n) from X such that $d(x_n, x_m) \geq r$ for $n \neq m$, then X is not compact.

Exercise 6.2. Suppose X has the property that each closed bounded subset of X is compact. Show X is complete.

Exercise 6.3. Show, if X is totally bounded, then X is bounded. Give an example of a bounded metric space X which is not totally bounded.

6.6. **Problems.**

Problem 6.1. Prove, if X is a metric space and $(a_n)_{n=1}^{\infty}$ is a sequence in X which converges to A, then $\{A, a_1, a_2, \dots\}$ is compact.

Problem 6.2. Prove a finite subset of a metric space X is compact.

More generally, prove a finite union of compact sets is compact.

Problem 6.3. Show, a subset K of a discrete metric space X is compact if and only if it is finite. In particular, if X is infinite, then X is closed and bounded, but not compact.

Problem 6.4. [*The finite intersection property (fip)*] Suppose X is a compact metric space and $\mathcal{F} \subset P(X)$. Show, if each $C \in \mathcal{F}$ is closed and for each finite subset $F \subset \mathcal{F}$ the set

$$\cap_{C \in F} C \neq \emptyset,$$

then in fact

$$\cap_{C \in \mathcal{F}} C \neq \emptyset.$$

As a corollary, show if $C_1 \supset C_2 \supset$ is a nested decreasing sequence of non-empty compact sets in a metric space X, then $\cap C_j$ is non-empty too.

Show the result fails if X is not assumed compact. On the other hand, even if X is not compact, the result is true if it assumed that there is a $D \in \mathcal{F}$ which is compact. Compare with Problem 5.3.

Problem 6.5. Prove that any open cover of $\mathbb{R}$ has an *at most countable subcover*.

More generally, prove, if there exists a sequence $K_1, K_2, \dots$ of compact subsets of a metric space X such that $X = \cup K_j$, then every open cover of X has an at most countable subcover.

Problem 6.6. Let ℓ^∞ denote the set of bounded sequences $a = (a(n))$ of real numbers. The function $d : \ell^\infty \times \ell^\infty \to \mathbb{R}$ defined by

$$d(a, b) = \sup\{|a(n) - b(n)| : n \in \mathbb{N}\}$$

is a metric on ℓ^∞.

Let e_j denote the sequence from ℓ^∞ (so a sequence of sequences) with $e_j(j) = 1$ and $e_j(k) = 0$ if $k \neq j$. Find, $d(e_j, e_\ell)$.

Let 0 denote the zero sequence in ℓ^∞. Is

$$B = \{a \in \ell^\infty : d(a, 0) \leq 1\}$$

closed? Is it bounded? Is it compact?

As a challenge, show ℓ^∞ is complete.

Problem 6.7. This problem assumes Problem 4.17. Let ℓ^2 denote the set of sequences $(a(n))$ of real numbers such that

$$\sum_0^\infty |a(n)|^2$$

converges (to a finite number). Use the Cauchy Schwartz inequality to show, if $a, b \in \ell^2$, then

$$\langle a, b \rangle := \sum_0^\infty a(j) b(j)$$

converges and that $\langle a, b \rangle$ is an inner product on ℓ^2. Let

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}$$

denote the resulting metric.

Let e_j denote the sequence with $e_j(j) = 1$ and $e_j(k) = 0$ if $j \neq k$. What is $d(e_j, e_k)$? Does the sequence (of sequences) (e_j) have a convergent subsequence? Let 0 denote the zero sequence. Is the set

$$B = \{x \in \ell^2 : d(x, 0) \leq 1\}$$

closed? Is it bounded? Is it compact?

As a challenge, prove that ℓ^2 is complete.

Problem 6.8. Prove Proposition 6.15. (See Problem 5.1.)

Problem 6.9. Suppose K is a nonempty compact subset of a metric space X and $x \in X$. Show, there is a point $p \in K$ such that, for all other $q \in K$,

$$d(p, x) \leq d(q, x).$$

[Suggestion: Let $S = \{d(x, y) : y \in K\}$ and show there is a sequence (q_n) from K such that $(d(x, q_n))$ converges to $\inf(S)$.]

Give an example where this conclusion fails if the hypothesis that K is compact is replaced by K is closed and bounded.

Problem 6.10. Suppose B is a compact subset of a metric space X and $a \notin B$. Show there exists disjoint open sets U and V such that $a \in U$ and $B \subset V$. Suggestion, first use Problem 6.9 to show, for each $b \in B$ there is an $\epsilon_b > 0$ such that $N_{\epsilon_b}(b) \cap N_{\epsilon_b}(a) = \emptyset$.

Problem 6.11. Show if A and B are disjoint compact sets in a metric space X, then there exists disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Suggestion, by the previous problem, for each $a \in A$ there exists disjoint open sets U_a and V_a such that $a \in U_a$ and $B \subset V_b$.

Problem 6.12. Show that K compact can be replaced by K closed in Problem 6.9 in the case that $X = \mathbb{R}^g$.

Problem 6.13. Given metric spaces (X, d_X) and (Y, d_Y) let Z denote the metric space built from X and Y as in Problem 3.1. Show, if X and Y are compact, then so is $X \times Y$.

7. Connected sets

Definition 7.1. A metric space X is *disconnected* if there exists sets $U, V \subset X$ such that

(i) U and V are open;
(ii) $U \cap V = \emptyset$;
(iii) $X = U \cup V$; and
(iv) $U \neq \emptyset \neq V$;

The metric space X is *connected* if it is not disconnected.

A *subset S of X is connected* if the metric space (subspace) S is connected.

Do Problem 7.1.

Remark 7.2. A metric space X is connected if and only if the only subsets of X which are both open and closed are X and $\emptyset$.

By Proposition 3.18, subsets U_0 and V_0 of S are open relative to S if and only if there exists subsets U, V of X which are open (in X) such that $U_0 = U \cap S$ and $V_0 = V \cap S$. Thus, a subset S of a metric space X is connected if and only if given subsets U and V of X such that

(i) U and V are open;
(ii) $U \cap S \cap V = \emptyset$; and
(iii) $S \subset U \cup V$

it follows that either $U \cap S$ or $V \cap S$ is empty.

Note, if U, V satisfy (ii) and (iii), then $\tilde{V} \cap S = U \cap S$. $\diamond$

Problem 7.2 gives an alternate condition for a subset S of a metric space X to be connect in terms of subsets of X. Do also Problem 7.3.

Proposition 7.3. A nonempty subset I of $\mathbb{R}$ is connected if and only if $x, y \in I$ and $x < z < y$ implies $z \in I$.

In particular, intervals in $\mathbb{R}$ are connected.

Proof. Suppose I has the property that $x, y \in I$ and $x < z < y$ implies $z \in I$. To prove that I is connected, it suffices to show, if $U, V \subset \mathbb{R}$ satisfy condition (i), (ii), and (iii) in Remark 7.2, then either $U \cap I$ or $V \cap I$ is empty. Arguing by contradiction, suppose $U \cap I$ and $V \cap I$ are both nonempty and choose $u \in U \cap I$ and $v \in V \cap I$. Without loss of generality, $u < v$. By hypothesis $[u, v] \subset I$. Consider $A = U \cap [u, v]$ and $B = V \cap [u, v]$ and observe that $A \cup B = [u, v]$ and $A \cap B = \emptyset$. Hence $\tilde{B} \cap [u, v] = A$ and therefore, as $\tilde{B} = \tilde{V} \cup \widetilde{[u, v]}$, $A = \tilde{V} \cap [u, v]$. In particular, A is closed and bounded. It follows that A has a largest element $a \in A$. Since $v \in B$, we find $a < v$. Since U is open, there is an ϵ such that $v - a > \epsilon > 0$ and $N_\epsilon(a) \subset U$. In particular, $(a, a+\epsilon) \subset U \cap [u, v] = A$. But then say $a + \frac{\epsilon}{2} \in A$, a contradiction.

To prove the converse, suppose there exists $x, y \in I$ and $z \notin I$ such that $x < z < y$. In this case, let $U = (-\infty, z)$ and $V = (z, \infty)$. Then $U \cap V = \emptyset$, U and V are open, $U \cap I$ and $V \cap I$ are nonempty, and $I \subset U \cup V$, thus I is not connected. $\square$

Do Problem 7.4.

Proposition 7.4. If $\mathcal{C}$ is a nonempty collection of connected subsets of a metric space X and if

$$\cap\{C : C \in \mathcal{C}\} \neq \emptyset,$$

then $\Gamma = \cup\{C : C \in \mathcal{C}\}$ is connected.

Proof. Suppose $U, V \subset X$ are open, $U \cap \Gamma \cap V = \emptyset$, and $\Gamma \subset U \cup V$. It suffices to show that either $\Gamma \cap U = \emptyset$ or $\Gamma \cap V = \emptyset$. Arguing by contradiction, suppose both are not empty. Then there exists $C_U, C_V \in \mathcal{C}$ such that $C_U \cap U \neq \emptyset$ and $C_V \cap V \neq \emptyset$. Now U, V are open; $C_U \subset U \cup V$; and $U \cap C_U \cap V \subset U \cap \Gamma \cap V = \emptyset$.

Thus, since C_U is connected, either $C_U \cap U = \emptyset$ or $C_U \cap V = \emptyset$. It follows that $C_U \cap V = \emptyset$ and hence $C_U \subset U$. By symmetry, $C_V \subset V$ and thus,

$$C_U \cap C_V \subset U \cap \Gamma \cap V = \emptyset,$$

contradicting the assumption that the intersection of the sets C in $\mathcal{C}$ is nonempty. □

Do Problems 7.5 and 7.6.

Corollary 7.5. Given a point x in a subset S of a metric space X there is a largest connected set C_x containing x and contained in S; i.e.,

(i) $x \in C_x \subset S$,
(ii) $C_x \subset X$ is connected; and
(iii) if $x \in D \subset S$ and $D \subset X$ is connected, then $D \subset C_x$.

The set C_x of the Corollary is called the *connected component* containing x.

Proof. Note that $\{x\}$ is connected. Let $\mathcal{C}$ denote the collection of connected sets containing x and contained in S and apply the previous proposition to conclude that $\Gamma = \cup\{C : C \in \mathcal{C}\}$ is connected. By construction, if D is connected and $x \in D$, then $D \subset \Gamma$. □

Do Problems 7.7, 7.8 and 7.9.

7.1. Exercises.

Exercise 7.1. Determine the connected subsets of a discrete metric space.

Exercise 7.2. Let $I = [0,1] \subset \mathbb{R}$. If $0 < x < 1$, is $I \setminus \{x\}$ connected?

Let $S \subset \mathbb{R}^2$ denote the unit circle, $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. If $x \in S$, is $S \setminus \{x\}$ connected? If $x \neq y$ are both in S, is $S \setminus \{x,y\}$ connected?

Let $R \subset \mathbb{R}^2$ denote the unit square $R = [0,1] \times [0,1]$. If $F \subset R$ is finite, is $R \setminus F$ connected?

Exercise 7.3. Let $S = \{\frac{1}{n} : n \in \mathbb{N}^+\} \subset \mathbb{R}$ and let

$$C = (K \times [0,1]) \cup ([0,1] \times \{0\}) \subset \mathbb{R}^2.$$

Draw a picture of C. Is it connected?

Let $D = C \cup \{(0,1)\}$. Is D connected? Can you draw a path from $(0,0)$ to $(0,1)$ without leaving D?

Exercise 7.4. Show if A, B, C are connected subsets of X and $A \cap B \neq \emptyset$ and $A \cap C \neq \emptyset$, then $A \cup B \cup C$ is connected. A more general statement, requiring a more elaborate proof, can be found in Problem 7.5.

7.2. **Problems.**

Problem 7.1. Show singleton sets are connected, but finite sets with more than one element are not.

Problem 7.2. Prove, $S \subset X$ is disconnected if and only if there exists subsets $A, B \subset X$ such that

(i) both A and B are nonemtpy;
(ii) $A \cup B = S$;
(iii) $\overline{A} \cap B = \emptyset$; and
(iv) $A \cap \overline{B} = \emptyset$.

(Here the closures are taken with respect to X.) You may wish to use Problem 3.4.

Problem 7.3. Show, if S is a connected subset of a metric space X, then $\overline{S}$ is also connected. In fact, each $S \subset T \subset \overline{S}$ is connected.

Problem 7.4. Suppose $I \subset \mathbb{R}$ is open. Prove that I is also connected if and only if either

(i) I is an open interval;
(ii) there is an $a \in \mathbb{R}$ such that $I = (a, \infty)$;
(iii) there is a $b \in \mathbb{R}$ such that $I = (-\infty, b)$; or
(iv) I is empty or all of $\mathbb{R}$.

The term *open interval* is expanded to refer to a set of any of the above forms.

Problem 7.5. Prove the following stronger variant of Proposition 7.4. Suppose $\mathcal{C}$ is a nonempty collection of connected subsets of a metric space X and $B \in \mathcal{C}$. and if, for each $A \in \mathcal{C}$, $A \cap B \neq \emptyset$, then $\Gamma = \cup\{C : C \in \mathcal{C}\}$ is connected.

Problem 7.6. Must the intersection of two connected sets be connected?

Problem 7.7. Let X be a metric space. For each $x \in X$, let C_x denote the connected component containing x. Prove that the collection $\{C_x : x \in X\}$ is a partition of X; i.e., if $x, y \in X$ then either $C_x = C_y$ or $C_x \cap C_y = \emptyset$ and $X = \cup_{x \in X} C_x$.

Problem 7.8. Prove, if $O \subset \mathbb{R}$ is open, then each connected component of O is open; i.e., if $U \subset O$ is connected in $\mathbb{R}$ and if $U \subset V \subset O$ is connected implies $U = V$, then U is open.

Problem 7.9. Prove that every open subset O of $\mathbb{R}$ is a disjoint union of open intervals (in the sense of Problem 7.4). Further show that this union is at most countable by noting that each component must contain a rational.

8. Continuous Functions

8.1. **Definitions and Examples.**

Definition 8.1. Suppose X, Y are metric spaces, $a \in X$ and $f : X \to Y$. The function f is *continuous at* a if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $d_X(a, x) < \delta$, then $d_Y(f(a), f(x)) < \epsilon$.

If f is continuous at every point $a \in X$, then f is said to be *continuous.*

Example 8.2. (a) Constant functions are continuous.

(b) For a metric space X, the identity function $id : X \to X$ given by $id(x) = x$ is continuous.

(c) If $f : X \to Y$ is continuous and $Z \subset X$, then $f|_Z : Z \to Y$ is continuous.

(d) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$ is nowhere continuous.

To prove this last statement, given $x \in \mathbb{R}$, choose $\epsilon_0 = \frac{1}{2}$.

(e) The function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 0$ if $x \notin \mathbb{Q}$ and $f(x) = \frac{1}{q}$, where $x = \frac{p}{q}$, $p \in \mathbb{N}$, $q \in \mathbb{N}^+$, and $\gcd(p, q) = 1$, is continuous precisely at the irrational points.

Lets prove that f is continuous at irrational points, leaving the fact that it is not continuous at each rational point as an easy exercise.

Suppose $x \notin \mathbb{Q}$ ($x \in [0, 1]$) and let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}^+$ so that $\frac{1}{N} < \epsilon$. Let

$$\delta = \min\{|x - \frac{m}{n}| : m, n \leq N,\ m, n \in \mathbb{N}^+\}.$$

This minimum exists and is positive since it is a minimum over a finite set and 0 is not an element of the set (since $x \notin \mathbb{Q}$). If $|x - y| < \delta$ and $y \in [0, 1]$, then either $y \notin \mathbb{Q}$ in which case $|f(x) - f(y)| = |0 - 0| = 0$; or $y \in \mathbb{Q}$ and $y = \frac{p}{q}$ (in reduced form) where $q > N$ in which case $|f(x) - f(y)| = \frac{1}{q} < \epsilon$.

(f) If X is a metric space and $a \in X$, then the function $f : X \to \mathbb{R}$ given by $f(x) = d(a, x)$ is continuous.

Fix x and let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. If $d(x, y) < \delta$, then

$$|f(x) - f(y)| = |d(x, a) - d(a, y)| \leq d(x, y) < \delta = \epsilon.$$

(g) Given $\gamma \in \mathbb{R}^g$, the function $p_\gamma : \mathbb{R}^g \to \mathbb{R}$ defined by

$$p_\gamma(x) = \langle x, \gamma \rangle$$

is continuous.

△

Do Problems 8.1 and 8.2.

Proposition 8.3. A function $f : X \to Y$ is continuous if and only if $f^{-1}(U) \subset X$ is open for every open set $U \subset Y$.

Note that the result doesn't change if Y is replaced by any Z with $f(X) \subset Z \subset Y$.

Proof. Suppose f is continuous and $U \subset Y$ is open. To prove $f^{-1}(U)$ is open, let $x \in f^{-1}(U)$ be given. Since U is open and $f(x) \in U$, there is an

$\epsilon > 0$ such that $N_\epsilon(f(x)) \subset U$. Since f is continuous at x, there is a $\delta > 0$ such that if $d_X(x,z) < \delta$, then $d_Y(f(x), f(z)) < \epsilon$. Thus, if $z \in N_\delta(x)$, then $f(z) \in N_\epsilon(f(x)) \subset U$ and thus $z \in f^{-1}(U)$. Hence $N_\delta(x) \subset f^{-1}(U)$. We have proved that $f^{-1}(U)$ is open.

Conversely, suppose that $f^{-1}(U)$ is open in X whenever U is open in Y. Let $x \in X$ and $\epsilon > 0$ be given. The set $U = N_\epsilon(f(x))$ is open and thus $f^{-1}(U)$ is also open. Since $x \in f^{-1}(U)$, there is a $\delta > 0$ such that $N_\delta(x) \subset f^{-1}(U)$; i.e., if $d_X(x,z) < \delta$, then $f(z) \in U$ which means $d_Y(f(x), f(z)) < \epsilon$. Hence f is continuous at x; and thus f is continuous. □

Corollary 8.4. A function $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed (in X) for every closed set C (in Y).

Do Problems 8.3 and 8.4. See also Problem 3.9.

Proposition 8.5. Suppose X, Y, Z are metric spaces, $f : X \to Y$ and $g : Y \to Z$. If both f and g are continuous, then so is $h = g \circ f : X \to Z$.

Proof. Let V an open subset of Z be given. Since g is continuous, $U = g^{-1}(V)$ is open in Y. Since f is continuous, $f^{-1}(U)$ is open in X. Thus, $h^{-1}(V) = f^{-1}(U)$ is open and hence h is continuous. □

There are local versions of Propositions 8.5 and 8.3 (See Problems 8.6 and 8.5). Here is a sample whose proof is left to the reader.

Proposition 8.6. Suppose X, Y, Z are metric spaces, $f : X \to Y$ and $g : Y \to Z$. If f is continuous at a and g is continuous at $b = f(a)$, then $h = g \circ f$ is continuous at a.

8.2. **Continuity and Limits.**

Definition 8.7. Let S be a subset of a metric space X. A point $p \in X$ is a *limit point* of S if, for every $\delta > 0$, the set $S \cap N_\delta(p)$ is infinite.

A point $p \in S$ is an *isolated point* of S if p is not a limit point of S.

Do Exercise 8.1 and compare with Problem 4.5.

Example 8.8. (a) If $S \neq \emptyset$ is an open set in $\mathbb{R}^g$, then every point of S is a limit point of S. In fact, as an exercise, show in this case the set of limit points of S is the closure of S.

(b) The set $\mathbb{Z}$ in $\mathbb{R}$ has no limit points.

(c) The only limit point of the set $\{\frac{1}{n} : n \in \mathbb{N}^+\}$ is 0.

△

Definition 8.9. Let X and Y be metric spaces and let $a \in X$ and $b \in Y$. Suppose a is a limit point of X and either $f : X \to Y$ or $f : X \setminus \{a\} \to Y$. Then f has *limit b as x approaches a*, written

$$\lim_{x \to a} f(x) = b,$$

if for every $\epsilon > 0$ there is a δ such that if $0 < d_X(a,x) < \delta$, then $d_Y(b, f(x)) < \epsilon$.

Remark 8.10. The limit b, if it exists, is unique. ⋄

Proposition 8.11. Suppose $f : X \to Y$ and $a \in X$ is a limit point of X. The function f is continuous at a if and only if $\lim_{x\to a} f(x)$ exists and equals $f(a)$.

If $f : X \setminus \{a\} \to Y$ and $\lim_{x\to a} f(x)$ exists and equals b, then the function $g : X \to Y$ defined by $g(x) = f(x)$ for $x \neq a$ and $g(a) = b$ is continuous at a.

If a is not a limit point of X and $h : X \to Y$, then h is continuous at a.

Proposition 8.12. Suppose $a \in X$ and $f : W \to Y$, where $W = X$ or $W = X \setminus \{a\}$. If $\lim_{x\to a} f(x) = b$ and if $g : Y \to Z$ is continuous at b, then $\lim_{x\to a} g \circ f(x) = g(b)$. In particular, if f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

Proof. The function $h : X \to Y$ defined by $h(x) = f(x)$ if $x \neq a$ and $h(a) = b$ is continuous at a by Proposition 8.11. Hence $g \circ h$ is continuous at a by Proposition 8.5. It follows that

$$\lim_{x\to a} g \circ f(x) = \lim_{x\to a} g \circ h(x) = g(h(a)) = g(b).$$

□

For a variation on this composition law for limits, see Problem 8.7.

The following Proposition gives a sequential formulation of limit.

Proposition 8.13. Suppose X is a metric space, a is a limit point of X, and $f : Z \setminus \{a\} \to Y$ where Z is either X or $X \setminus \{a\}$. The limit $\lim_{x\to a} f(x)$ exists and equals $b \in Y$ if and only if for every sequence (a_n) from Z which converges to a, $(f(a_n))$ converges to b.

If $f : X \to Y$, then f is continuous at a if and only if for every sequence (a_n) from $X \setminus \{a\}$ converging to a, $(f(a_n))$ converges to $f(a)$.

Proof. To prove the the first part of the lemma in the case $Z = X \setminus \{a\}$, first suppose $\lim_{x\to a} f(x) = b$ and (a_n) converges to a. To see that $(f(a_n))$ converges to b, let $\epsilon > 0$ be given. There is a $\delta > 0$ such that if $0 < d_X(a, x) < \delta$, then $d_Y(b, f(x)) < \epsilon$. There is an N so that if $n \geq N$, then $0 < d_X(a, a_n) < \delta$. Hence, if $n \geq N$, then $d_Y(b, f(a_n)) < \epsilon$ and thus $(f(a_n))$ converges to b.

Conversely, suppose $\lim_{x\to a} f(x) \neq b$. Then there is an $\epsilon_0 > 0$ such that for each n there exists a_n such that $d_X(a, a_n) < \frac{1}{n}$, but $d_Y(b, f(a_n)) \geq \epsilon_0$. The sequence (a_n) converges to a, but $(f(a_n))$ does not converge to b.

The second part of the proposition follows readily from the first part. □

8.3. **Continuity of Rational Operations.**

Proposition 8.14. Let X be a metric space and $a \in X$ be a limit point of X. Suppose $f : Y \to \mathbb{R}^k$ where Y is either X or $X \setminus \{a\}$. Write $f = (f_1, \dots, f_k)$ with $f_j : X \to \mathbb{R}$.

The limit $\lim_{x\to a} f(x)$ exists and equals $A = (A_1, \ldots, A_k) \in \mathbb{R}^k$ if and only if, for each j, the limit $\lim_{x\to a} f_j(x)$ exists and equals A_j. In particular, if $f : X \to \mathbb{R}^k$, then f is continuous at a if and only if each f_j is continuous at a.

Proof. Let (a_n) be a given sequence from $X \setminus \{a\}$ which converges to a. By Proposition 4.17, the sequence $A_n = f(a_n)$ converges to A if and only if $(f_j(a_n))_n$ converges to A_j for each j. An application of Proposition 8.13 thus completes the proof. □

Proposition 8.15. Suppose $a \in X$ is a limit point of the metric space X, W is either X or $X\setminus\{a\}$ and $f, g : W \to \mathbb{R}^k$. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and equal A and B respectively, then

(i) $\lim_{x\to a} f(x) \cdot g(x) = A \cdot B$;
(ii) $\lim_{x\to a}(f + g)(x) = A + B$;
(iii) if $k = 1$, g is never 0 and $B \neq 0$, then $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{B}$.

Proof. To prove item (i), suppose (a_n) is a sequence in $X \setminus \{a\}$ which converges to a. From Proposition 8.13, $(f(a_n))$ and $(g(a_n))$ converge to A and B respectively. Hence $(f(a_n)\cdot g(a_n))$ converges to $A\cdot B$, by Proposition 4.20. Finally, another application of Proposition 8.13 completes the proof.

The proofs of the other items are similar. □

Corollary 8.16. If $f, g : X \to \mathbb{R}^k$ are continuous at a, then so are $f \cdot g$ and $f + g$. If $k = 1$ and g is never 0, then $\frac{1}{g}$ is continuous at a.

Example 8.17. For each j, the function $\pi_j : \mathbb{R}^d \to \mathbb{R}$ given by $\pi_j(x) = x_j$ is continuous since it can be expressed as

$$\pi_j(x) = \langle x, e_j\rangle = x \cdot e_j,$$

where e_j is the j-th standard basis vector of $\mathbb{R}^d$; i.e., e_j has a 1 in the j-th entry and 0 elsewhere.

If $p(x_1, \ldots, x_d)$ and $q(x_1, \ldots, x_d)$ are polynomials, then the rational function

$$r(x) = \frac{p(x)}{q(x)}$$

is continuous (wherever it is defined). △

Do Problems 8.8 and 8.9.

8.4. **Continuity and Compactness.**

Proposition 8.18. If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact; i.e., the continuous image of a compact set is compact.

Proof. Let $\mathcal{W}$ be a given open cover of $f(X)$. Then,

$$\mathcal{U} = \{f^{-1}(U) : U \in \mathcal{W}\}$$

is an open cover of X. Hence there is a finite subset $\mathcal{F} \subset \mathcal{W}$ such that $\{f^{-1}(U) : U \in \mathcal{F}\}$ is a cover of X.

Using the fact that $f(f^{-1}(B)) \subset B$, it follows that

$$\cup\{U : U \in \mathcal{F}\} \supset \cup\{f(f^{-1}(U)) : U \in \mathcal{F}\} = f(\cup\{f^{-1}(U) : U \in \mathcal{F}\}) \supset f(X).$$

Thus, $\{U : U \in \mathcal{F}\}$ is a finite subcover of $f(X)$. □

Do Problem 8.10.

Corollary 8.19 (Extreme Value Theorem)**.** If $f : X \to \mathbb{R}$ is continuous and X is non-empty and compact, then there exists $x_0 \in X$ such that $f(x_0) \geq f(x)$ for all $x \in X$; i.e., f has a maximum on X.

Proof. By the previous proposition, the set $f(X)$ is a compact subset of $\mathbb{R}$. It is also non-empty. In view of Proposition 3.25, non-empty compact subsets of $\mathbb{R}$ have a largest element; i.e., there is an $M \in f(X)$ such that $M \geq f(x)$ for all $x \in X$. Since $M \in f(X)$, there is an $x_0 \in X$ such that $M = f(x_0)$. □

Return to Problem 6.9.

Corollary 8.20. If X is compact, and if $f : X \to Y$ is one-one, onto and continuous, then f^{-1} is continuous.

Proof. Let $C \subset X$, a closed set, be given. Since X is compact, so is C. Hence $f(C)$ is compact and thus closed in Y. Thus $(f^{-1})^{-1}(C) = f(C)$ is closed. It follows, from Corollary 8.4 that f^{-1} is continuous. □

Example 8.21. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and define $f : [0, 2\pi) \to \mathbb{T}$ by $f(t) = \exp(it) = (\cos(t), \sin(t))$. Then f is continuous and invertible, but f^{-1} is not continuous at 1.

In fact, if $g : \mathbb{T} \to [0, 2\pi)$ is continuous, then it is not onto since its image $g(\mathbb{T})$ will then be a compact, and hence proper, subset of $[0, 2\pi)$. △

8.5. **Uniform Continuity and Compactness.**

Definition 8.22. A function $f : X \to Y$ is *uniformly continuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \in X$ and $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \epsilon$.

Given $S \subset X$, f is *uniformly continuous on* S if $f|_S : S \to Y$ is uniformly continuous.

Proposition 8.23. If $f : X \to Y$ is continuous on X and if X is compact, then f is uniformly continuous on X.

Proof. Let $\epsilon > 0$ be given. For each $x \in X$ there is a $r_x > 0$ such that if $d_X(x, y) < r_x$, then $d(f(x), f(y)) < \frac{\epsilon}{2}$.

The collection $\mathcal{U} = \{N_{\frac{r_x}{2}}(x) : x \in X\}$ is an open cover of X. Since X is compact, there is a finite subset $F \subset X$ such that $\mathcal{V} = \{N_{\frac{r_x}{2}}(s) : s \in F\}$ is a cover of X.

Let $\delta = \frac{1}{2}\min\{r_x : x \in F\}$. Suppose $y, z \in X$ and $d_X(y, z) < \delta$. There is an $x \in F$ such that $y \in N_{\frac{r_x}{2}}(x)$; i.e., $d_X(x, y) < \frac{r_x}{2}$. Hence

$$d_X(x, z) \leq d_X(x, y) + d_X(y, z) < \frac{r_x}{2} + \delta \leq r_x.$$

Consequently,

$$d_Y(f(y), f(z)) \le d_Y(f(y), f(x)) + d_Y(f(x), f(z)) < \epsilon.$$

□

Example 8.24. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not uniformly continuous.

Choose $\epsilon_0 = 1$. Given $\delta > 0$, let $x = \frac{2}{\delta}$ and $y = \frac{2}{\delta} + \frac{\delta}{2}$. Then $|x - y| < \delta$, but,

$$|f(y) - f(x)| = 2 + \frac{\delta^2}{4} \ge \epsilon_0 = 1.$$

On the other hand, the function from Problem 8.1 is uniformly continuous.
△

Do Problems 8.12 and 8.11.

8.6. **Continuity and Connectedness.**

Proposition 8.25. If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.

Proof. Suppose U and V are open subsets of $f(X)$ such that $f(X) = U \cup V$ and $U \cap V = \emptyset$.

The sets $A = f^{-1}(U)$ and $B = f^{-1}(V)$ are open, $X = A \cup B$ and $A \cap B = \emptyset$ (since if $x \in A \cap B$, then $f(x) \in U \cap V$). Hence, without loss of generality, $A = X$. Hence, $f(A) = f(X) = f(f^{-1}(U)) \subset U$ and $V = \emptyset$. It follows that $f(X)$ is connected. □

Example 8.26. Returning to Example 8.21, there does not exist a one-one onto continuous mapping $f : [0, 2\pi] \to \mathbb{T}$. If there were, then $g = f^{-1}$ would be a continuous one-one mapping of $\mathbb{T}$ onto $[0, 2\pi]$. Let $z = f(\pi)$ and $Z = \mathbb{T} \setminus \{z\}$. Now Z is connected and $g|_Z : Z \to [0, \pi) \cup (\pi, 2\pi]$ is one-one and onto. But then $g|_Z(Z) = [0, \pi) \cup (\pi, 2\pi]$ is connected which is a contradiction. △

Do Problems 8.13, 8.14, and 8.15.

Corollary 8.27. [Intermediate Value Theorem] If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$, then there is a point $a < c < b$, such that $f(c) = 0$.

Definition 8.28. Let S denote a subset of $\mathbb{R}$. A function $f : S \to \mathbb{R}$ is increasing (synonymously *non-decreasing*) if $x, y \in S$ and $x \le y$ implies $f(x) \le f(y)$. The function is strictly increasing if $x, y \in S$ and $x < y$ implies $f(x) < f(y)$.

Corollary 8.29. If $f : [a, b] \to \mathbb{R}$ is continuous and increasing, then $f([a, b]) = [f(a), f(b)]$.

Example 8.30. Returning to the discussion in Subsection 2.4, fix a positive integer n and let $f : [0,\infty) \to [0,\infty)$ denote the function with rule $f(x) = x^n$. To show that f is onto, let $y \in [0,\infty)$ be given. With b the larger of 1 and y, consider $g = f|_{[0,b]} : [0,b] \to \mathbb{R}$. Since $f(b) \geq y$, it follows that y is in the interval $[0, g(b)]$. By Corollary 8.29, y is in the range of g and hence in the range of f. The conclusion is then that positive numbers have n-th roots. △

8.6.1. *More on connectedness - optional.* The following property of a metric space X is sometimes expressed by saying that X is *completely normal*. It is evidently stronger than the statement that disjoint closed sets can be separated by disjoint open sets, a property known as *normality*. Compare with Problem 6.11.

Proposition 8.31. If A, B are subset of a metric space such that $\overline{A} \cap B \neq \emptyset$ and $A \cap \overline{B} \neq \emptyset$, then there exists $U, V \subset X$ such that

(i) U and V are open;
(ii) $A \subset U$, $B \subset V$; and
(iii) $U \cap V = \emptyset$.

Proof. If either A or B is empty, then the result is immediate. Accordingly, suppose that $A \neq \emptyset$ and $B \neq \emptyset$ and of course that $\overline{A} \cap B = \emptyset$ and $\overline{B} \cap A = \emptyset$¿ By Problem 8.1, the function $f : X \to \mathbb{R}$ given by

$$f(x) = d(x; B) - d(x; A)$$

is continuous. Observe, if $x \in A$, then $x \notin \overline{B}$ and hence $d(x; A) = 0$, but $d(x; B) > 0$ by Problem 3.9. Thus, $f(x) > 0$ for $x \in A$. Similarly, $f(x) < 0$ for $x \in B$. Let $U = f^{-1}(0,\infty)$ and $V = f^{-1}(-\infty, 0)$. It follows that U and V are open, $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$. Thus U and V satisfy conditions (i)–(iv). □

Remark 8.32. Proposition 8.31 gives another characterization of connected subsets S of a metric space X. Namely, S is not connected if and only if there exist nonempty, open, disjoint subsets U, V of X such that $S \subset U \cup V$. ⋄

8.7. **Exercises.**

Exercise 8.1. Let S be a subset of the metric space X and suppose $p \in X$. Explain why the following conditions are equivalent.

(i) p is a limit point of S;
(ii) For every $\delta > 0$ the set $(S \setminus \{p\}) \cap N_\delta(p) \neq \emptyset$; and
(iii) There is a sequence (s_n) from $S \setminus \{p\}$ which converges to p.

Explain why $p \in S$ is an isolated point of S if and only if the set $\{p\}$ is an open set in S; i.e., open relative to S.

Exercise 8.2. Show that $f : (0,1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous, but not uniformly so.

Exercise 8.3. Use the Intermediate Value Theorem 8.27 along with Corollary 8.20 to argue that the function $\sqrt[n]{}$ is continuous.

Exercise 8.4. Use Exercise 8.3 to show if the sequence (a_n) of nonnegative real numbers converges to A, and $r = \frac{m}{n}$ $(m, n \in \mathbb{N}^+)$ is a rational number, then (a_n^r) converges to A^r.

Exercise 8.5. Give an alternate proof of the statement of Problem 6.9 using Example 8.2(f) and Corollary 8.19.

8.8. **Problems.**

Problem 8.1. Let A be a nonempty subset of a metric space X. Define $f : X \to [0, \infty)$ by $f(x) = \inf\{d(x, a) : a \in A\}$. Prove that f is continuous.

Problem 8.2. Let X be a metric space and Y a discrete metric space.

(i) Determine all continuous functions $f : Y \to X$.
(ii) Determine all continuous functions $g : \mathbb{R} \to Y$;

Problem 8.3. Prove Corollary 8.4.

Problem 8.4. Show, if $f : X \to \mathbb{R}$ is continuous, then the zero set of f,

$$Z(f) = \{x \in X : f(x) = 0\}$$

is a closed set.

Show that the set

$$\{(x, y) : xy = 1\} \subset \mathbb{R}^2$$

is a closed set.

Problem 8.5. Prove the following local version of Proposition 8.3.

Suppose $f : X \to Y$ and $a \in X$. The function f is continuous at a if and only if for every open set U containing $b = f(a)$, there is an open set V containing a so that $V \subset f^{-1}(U)$.

Problem 8.6. Prove Proposition 8.6.

Problem 8.7. Suppose X is a metric space, $a \in X$ is a limit point of X and $f : X \setminus \{a\} \to Y$. Show, if

(a) $\lim_{x \to a} f(x)$ exists and equals b;
(b) $g : Z \to X$ is continuous at c;
(c) $g(c) = a$; and
(d) $g(z) \neq a$ for $z \neq c$,

then

$$\lim_{z \to c} f \circ g(z) = b.$$

Problem 8.8. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $f(x) = \sin(\frac{1}{x})$. Show

(i) f does not have a limit at 0;
(ii) does $g(x) = xf(x)$ have a limit at 0;

(iii) more generally, show if $h : \mathbb{R} \to \mathbb{R}$ is continuous at 0 and $h(0) = 0$, then hf has a limit at 0.

Problem 8.9. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Is f continuous at $0 = (0,0)$?

Define $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Is g continuous at $0 = (0,0)$?

Problem 8.10. Suppose X is compact and $f : X \to Y$. Let Z denote the metric space $Z = X \times Y$ with distance function

$$d((x_1,y_1),(x_2,y_2)) = \max\{d_X(x_1,x_2), d_Y(y_1,y_2)\}.$$

Prove, if $f : X \to Y$ is continuous, then $F : X \to Z$ defined by $F(x) = (x, f(x))$ is also continuous.

Prove, if f is continuous, then the graph of f,

$$\text{graph}(f) = \{(x, f(x)) \in Z : x \in X\} \subset Z$$

is compact.

As a challenge, show, if the graph of f is compact, then f is continuous. As a suggestion, consider the function $H : \text{graph}(f) \to X$ defined by $H(x, f(x)) = x$.

Problem 8.11. Prove if $f : X \to Y$ is uniformly continuous and (a_n) is a Cauchy sequence from X, then $(f(a_n))$ is Cauchy in Y.

Problem 8.12. Given a metric space Y, a point $L \in Y$, and $f : [0,\infty) \to Y$, *f has limit $L \in Y$ at infinity,* written,

$$\lim_{x\to\infty} f(x) = L,$$

if for every $\epsilon > 0$ there is a $C > 0$ such that if $x > C$, then $d_Y(f(x), L) < \epsilon$.

Prove, if $f : [0,\infty) \to Y$ is continuous and has a limit at infinity, then f is uniformly continuous.

Problem 8.13. A function $f : X \to Y$ is a *homeomorphism* if it is one-one and onto and both f and f^{-1} are continuous.

Suppose $f : X \to Y$ is a homeomorphism. Show, if $Z \subset X$, then $f|_Z : Z \to f(Z)$ is also a homeomorphism. In particular, if Z is connected, then so is $f(Z)$.

Problem 8.14. Does there exist a continuous onto function $f : [0,1] \to \mathbb{R}$?

Does there exist a continuous onto function $f : (0,1) \to (-1,0) \cup (0,1)$?

Problem 8.15. Suppose $f : [0,1] \times [0,1] \to \mathbb{R}$. Prove, if f is continuous, then f is not one-one.

Problem 8.16. Let $I = (c,d)$ be an interval and suppose $a \in I$. Let E denote either I or $I \setminus \{a\}$ and suppose $f : E \to \mathbb{R}$. We say f *has a limit as x approaches a from the right (above)* if the function $f|_{(a,d)} : (a,d) \to \mathbb{R}$ has a limit at a. The limit, if it exists, is denoted,

$$\lim_{x \to a^+} f(x) = \lim_{a < x \to a} f(x).$$

The limit from the left (below) is defined similarly.

Show f has a limit at a if and only if both the limits from the right and left at a exist and are equal.

Problem 8.17. Suppose $f : (c,d) \to \mathbb{R}$ is monotone increasing and $c < a < d$. Show, f has a limit from the left at a and this limit is

$$\sup\{f(t) : c < t < a\}.$$

Problem 8.18. Suppose $f : [a,b] \to [c,d]$ is one-one and onto and (strictly) monotone increasing. Prove f is continuous.

Problem 8.19. A function $f : X \to X$ is a *contraction mapping* if there is an $0 \le r < 1$ such that

$$d(f(x), f(y)) \le r d(x,y)$$

for all $x, y \in X$.

A point p is a *fixed point* of f if $f(p) = p$.

Prove that a contraction mapping can have at most one fixed point.

Prove, if f is a contraction mapping and X is complete, then f has a (unique) fixed point. In fact, show, for any point $x \in X$, the sequence (x_n) defined recursively by $x_0 = x$ and $x_{n+1} = f(x_n)$ converges to this fixed point. (See Proposition 5.11.)

Problem 8.20. Suppose K is compact and $f : K \to K$. Show, if f is continuous, then the function $g : K \to [0, \infty)$

$$g(x) = d(f(x), x)$$

attains its infimum (achieves a minimum). Show further that if $g(z)$ is the minimum value, then

$$d(f(f(z)), f(z)) \ge d(f(z), z).$$

Show that x is a fixed point of f if and only if $g(x) = 0$.

Suppose now that f satisfies

$$d(f(x), f(y)) < d(x,y)$$

for all $x \ne y$ in K.

Prove f has a unique fixed point.

Show by example, that the hypothesis that K is compact can not be dropped.

Problem 8.21. Suppose $f : X \to Y$ maps convergent sequences to convergent sequences; i.e., if (a_n) converges in X, then $(f(a_n))$ converges in Y.

Show, if (a_n) converges to a, and (b_n) is the sequence defined by $b_{2n} = a_n$ and $b_{2n+1} = a$, then (b_n) converges to a. Now prove that $f(b_n)$ converges to $f(a)$.

Prove f is continuous.

Problem 8.22 (Pasting Lemma)**.** Suppose $f : X \to Y$ and $X = S \cup T$, where S and T are closed. Show, if the restriction of f to both S and T is continuous, then f is continuous. The same is true if both S and T are open.

Problem 8.23. Show, if $f : X \to X$ is continuous, X is compact, and f does not have a fixed point, then there is an $\epsilon > 0$ such that $d(x, f(x)) \geq \epsilon$ for all $x \in X$.

Index