Tridiagonal Kernels and Subnormality

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The Bilateral Bergman Shift
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Main Question:

• If H is a Hilbert space of analytic functions on the unit disk $\mathbb{D}$ with o.n. basis $\{f_n(z) = (a_n + b_n z)z^n : n \geq 0\}$, when is $S = M_z$ a bounded subnormal operator on H?

• Recall $S \in B(X)$ is a *subnormal operator* if there is a normal operator $N \in B(Y)$ where $X \subset Y$ such that $S = N|_X$.

• In the above setting $H = H(K)$ is an analytic reproducing kernel Hilbert space with kernel $K(z, w) = \sum_{i,j=0}^{\infty} f_n(z)\overline{f_n(w)}$.

$Dom(K) = \mathcal{D}(K) \cup \{z_0\}$ where $\mathcal{D}(K)$ is the disc

$$\mathcal{D}(K) = \{z : \sum_n [|z|^n(|a_n| + |b_n|)]^2 < \infty\}.$$

• When $f_n(z) = \sqrt{a_n} z^n$, $H(K)$ is a diagonal space and
• M_z is a weighted shift with weight sequence $\{\alpha_n\} = \{\sqrt{\frac{a_n}{a_{n+1}}}\}$.
• Given $a_0 = 1$ and $\alpha_n \nearrow 1$, subnormality means there is a Berger measure μ on $[0, 1]$ such that

$$(\alpha_0 \alpha_1 \cdots \alpha_{n-1})^2 = \frac{1}{a_n} = \int_0^1 t^{2n} d\mu(t)$$

• Classical and truncated moment problem: Hausdorff, Stieltjes, Hamburger, Bernstein, Widder, Akheizer, Curto, Fialkow, Putinar, Stochel, Szafraniec, Atzmon, Ando, . . . (a hundred year history). The Hankel matrix below must be positive for all n

$$H_{\{a_n\}} = \begin{pmatrix} \frac{1}{a_0} & \frac{1}{a_1} & \cdots & \frac{1}{a_n} \\ \\ \frac{1}{a_1} & \frac{1}{a_2} & \cdots & \frac{1}{a_{n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{a_n} & \frac{1}{a_{n+1}} & \cdots & \frac{1}{a_{2n}} \end{pmatrix}$$

- Hardy space H^2, $a_n = 1$, $d\mu(te^{i\theta}) = \delta_1(t)\ \frac{dtd\theta}{2\pi}$,

$$K(z,w) = \frac{1}{1-\bar{w}z} = \sum_{n=0}^{\infty} (\bar{w}z)^n\ .$$

- Bergman space L_a^2, $a_n = \sqrt{n+1}$, $d\mu(te^{i\theta}) = 2t\ \frac{dtd\theta}{2\pi}$,

$$K(z,w) = \frac{1}{(1-\bar{w}z)^2} = \sum_{n=0}^{\infty} (n+1)(\bar{w}z)^n\ .$$

- "Bergman" s-space L_s^2, $a_n = \sqrt{\frac{(s+n-1)!}{n!}}$,

$d\mu(te^{i\theta}) = \frac{2}{(s-2)!} t(1-t^2)^{s-2}\ \frac{dtd\theta}{2\pi}$,

$$K(z,w) = \frac{1}{(1-\bar{w}z)^s} = \sum_{n=0}^{\infty} \frac{(s+n-1)!}{n!}(\bar{w}z)^n\ .$$

• (Aronszajn, 1950) If K_1 and K_2 are reproducing kernels on a common domain Ω, then $K = K_1 + K_2$ is a reproducing kernel and

$$H(K) = H(K_1)\widehat{\oplus}H(K_2) = \{f_1 + f_2 : f_i \in H(K_i)\}$$

with

$$||f||^2 = inf\{||f_1||^2 + ||f_2||^2 : f = f_1 + f_2\}.$$

• Question: How do changes in K affect subnormality and measures?

• $K(z,w) = \frac{1+\alpha\bar{w}z}{1-\bar{w}z}$ adds a point mass at the origin and

$$H(K) = H^2 \ \widehat{\oplus} \ \sqrt{\alpha}zH^2.$$

- $K(z,w) = \frac{1+\alpha\bar{w}z}{(1-\bar{w}z)^2}$ has measure

$$d\mu(te^{i\theta}) \;=\; \frac{2t}{1+\alpha}\; t^{\frac{1-\alpha}{1+\alpha}}\; \frac{dt\;\,d\theta}{2\pi}.$$

and

$$H(K) = L^2_a \;\widehat{\oplus}\; \sqrt{\alpha} z L^2_a.$$

- $K(z,w) = \frac{1}{2}\left(\frac{1}{(1-\bar{w}z)^2} + \frac{1}{(1-\bar{w}z)^3}\right)$ has measure

$$d\mu(te^{i\theta}) \;=\; \frac{4}{3}t\;(1-t^6))\;\frac{dt\;\,d\theta}{2\pi}.$$

and

$$H(K) = L^2_a \;\widehat{\oplus}\; L^2_s.$$

- M_z on $K(z,w) = \frac{1+(\bar{w}z)^2}{1-\bar{w}z}$ is not a subnormal operator and

$$H(K) = H^2 \mathbin{\widehat{\oplus}} z^2 H^2.$$

- M_z on $K(z,w) = \frac{1}{1-\bar{w}z} + \frac{1}{(1-\bar{w}z)^3}$ is not subnormal and

$$H(K) = H^2 \mathbin{\widehat{\oplus}} L^2_s.$$

With $v(z) = (1, z, z^2, \cdots)$, a tridiagonal kernel has the form

$$K(z,w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)} = v(z)LL^*v(w)^*$$

where

$$L = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ b_0 & a_1 & 0 & \ddots \\ 0 & b_1 & a_2 & \ddots \\ \vdots & \vdots & b_2 & \ddots \end{pmatrix}.$$

Here $A = LL^*$ is the coefficient matrix of the Taylor series expansion of K about $(0,0)$ and $H(K)$ is unitarily equivalent to the range space of L in l^2_+ which is $\{L\vec{x} : \vec{x} \in l^2_+\}$ with $||L\vec{x}|| = ||\vec{x}||$.

• When $H(K)$ contains the polynomials, the Grammian matrix

$$G = \left[< z^i, z^j > \right] = A^{-1} = (LL^*)^{-1} = L^{-1*}L^{-1}$$

• $H(LL^*)$ contains the polynomials if and only if the sequence

$$\{1, \frac{b_n}{a_{n+1}}, \frac{b_n b_{n+1}}{a_{n+1}a_{n+2}}, \frac{b_n b_{n+1} b_{n+2}}{a_{n+1}a_{n+2}a_{n+3}}, \ldots\}$$

is square summable for each n.

• Note that if $K_2(z,w) = f(z)\overline{f(w)}K_1(z,w)$, then $H(K_2) = f(z)H(K_1)$ and $||fg||_2 = ||g||_1$.

Trivial case: If M_z is subnormal on a diagonal space $H(K_1)$ and $f(z) = a + bz$, then M_z on $H(K_2) = f(z)H(K_1)$ is subnormal where $K_2(z,w) = f(z)\overline{f(w)}K_1(z,w)$ is tridiagonal.

$$\widehat{M_z} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \frac{a_0}{a_1} & 0 & 0 & 0 & \ddots \\ \mathbf{c_0} & \frac{a_1}{a_2} & 0 & 0 & \ddots \\ \frac{-c_0 b_2}{a_3} & \mathbf{c_1} & \frac{a_2}{a_3} & 0 & \ddots \\ \frac{c_0 b_2 b_3}{a_3 a_4} & \frac{-c_1 b_3}{a_4} & \mathbf{c_2} & \frac{a_3}{a_4} & \ddots \\ \frac{-c_0 b_2 b_3 b_4}{a_3 a_4 a_5} & \frac{c_1 b_3 b_4}{a_4 a_5} & \frac{-c_2 b_4}{a_5} & \mathbf{c_3} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \text{ where } c_n = \frac{b_n}{a_{n+2}} - \frac{a_n}{a_{n+1}} \frac{b_{n+1}}{a_{n+2}}.$$

Formally, $\widehat{M_z} = S_\psi + S^2 (I + S_\omega)^{-1} D$

Note $c_n = 0$ means $H(K) = (1 + \alpha z) H(K_{diag})$ and $\widehat{M_z} = S_\psi$.

- Consider

$$K(z,w) = \frac{1+(1-z)(1-\bar{w})}{1-\bar{w}z} = v(z)Av(w)^*$$

$$= (1, z, z^2, \cdots) \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 3 & -1 & 0 & \ddots \\ 0 & -1 & 3 & -1 & \ddots \\ 0 & 0 & -1 & 3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ \bar{w} \\ \bar{w}^2 \\ \vdots \end{pmatrix}.$$

Here

$$H(K) = H^2 \ \widehat{\oplus} \ (1-z) \ H^2.$$

Factoring $A = LL^*$ where

$$L = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ b_0 & a_1 & 0 & \cdots \\ 0 & b_1 & a_2 & \ddots \\ 0 & 0 & b_2 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

we arrive at the recursion $a_0 = \sqrt{2}$, $b_n = \frac{-1}{a_n}$, and $a_n^2 + \frac{1}{a_{n-1}^2} = 3$. This is easily solved to produce $a_n = \frac{\alpha_{2(n+1)}}{\alpha_{2n}}$ where α_n is the n-th Fibonacci number.

In this case the zeros of $f_n(z)$ are 0 and $\frac{-a_n}{b_n} = a_n^2$. The latter

converges to ϕ^2 where $\phi = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio. Let

$\lambda = \frac{1}{\phi^2} = \frac{(3-\sqrt{5})}{2}$ and $\zeta = S(z) = \frac{z-\lambda}{1-\lambda z}$. With

$z = S^{-1}(\zeta) = \frac{z+\lambda}{1+\lambda z}$ and $w = S^{-1}(\omega) = \frac{\omega+\lambda}{1+\lambda\omega}$,

$$K(z,\bar{w}) = K(S^{-1}(\zeta), S^{-1}(\bar{\omega})) = \frac{\frac{\sqrt{5}-1}{2} + \frac{\sqrt{5}+1}{2}\,\bar{\omega}\zeta}{1-\bar{\omega}\zeta}.$$

So $K(z,w)$ is the composition of a diagonal kernel and a Mobius map and the resulting measure turns out to be $P_\lambda(\theta)d\theta + c\ \delta_\lambda$, where $P_\lambda(\theta)$ is the Poisson kernel and δ_λ is a point mass at λ.

- Now consider

$$K(z,w) = \frac{1 + (1-z)(1-\bar{w})}{(1-\bar{w}z)^2} = v(z)Av(w)^*$$

$$= (1, z, z^2, \cdots) \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 5 & -2 & 0 & \ddots \\ 0 & -2 & 8 & -3 & \ddots \\ 0 & 0 & -3 & 11 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ \bar{w} \\ \bar{w}^2 \\ \vdots \end{pmatrix}.$$

Here

$$H(K) = L_a^2 \ \widehat{\oplus} \ (1-z) \ L_a^2.$$

Factoring $A = LL^*$ where

$$L = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ b_0 & a_1 & 0 & \cdots \\ 0 & b_1 & a_2 & \ddots \\ 0 & 0 & b_2 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

we arrive at the recursion $a_0 = \sqrt{2}$, $b_n = \frac{-(n+1)}{a_n}$, and

$$a_n^2 + \frac{n^2}{a_{n-1}^2} = 2 + 3n$$

or if $a_n = \sqrt{\frac{t_{n+1}}{t_n}}$,

$$t_{n+1} = (2+3n)t_n - n^2 t_{n-1}.$$

This type of recursion is *not* easy to solve.

However, the same transformation as before yields $K(z, \bar{w})$

$$= K(S^{-1}(\zeta), S^{-1}(\bar{\omega})) = Q(\zeta)\overline{Q(\omega)} \left[\frac{1 \; + \; \frac{\lambda^2 \; + \; (1-\lambda)^2}{1+(1-\lambda)^2} \; \bar{\omega}\zeta}{1 - \bar{\omega}\zeta} \right],$$

where

$$Q(\zeta) = \frac{\sqrt{1 + (1-\lambda)^2}}{1 - \lambda^2)} \; (1 + \lambda\zeta).$$

So S^{-1} transforms K to a new kernel which is a linear function times a diagonal space which has Berger measure $g(t)dt$ with

$$g(t) = (1 + \frac{1}{\sqrt{5}}) \; t^{\frac{1}{\sqrt{5}}}.$$

The resulting transformed measure on $H(K)$ is given by

$$||h||^2_{H(K)} = \frac{\lambda}{\pi}\int_0^{2\pi}\int_0^1 |h(re^{i\theta}|^2 \left|\frac{re^{i\theta}-\lambda}{1-\lambda re^{i\theta}}\right|^{\frac{1}{\sqrt{5}}} \frac{r\ dr\ d\theta}{|re^{i\theta}-\lambda|\ |1-\lambda re^{i\theta}|}$$

Lagniappe for John:

The recursion $a_n^2 + \frac{n^2}{a_{n-1}^2} = 2 + 3n$ with $a_0 = \sqrt{2}$ has solution

$$a_n = \sqrt{\frac{(n+1)\int_0^{2\pi}\int_0^1 r^{n+1}e^{i(n+1)\theta}(\sqrt{2}-\frac{1}{\sqrt{2}}re^{-i\theta})\left|\frac{re^{i\theta}-\frac{1}{\phi^2}}{1-\frac{1}{\phi^2}re^{i\theta}}\right|^{\frac{1}{\sqrt{5}}}\frac{r\ dr\ d\theta}{|re^{i\theta}-\frac{1}{\phi^2}|\ |1-\frac{1}{\phi^2}re^{i\theta}|}}{\int_0^{2\pi}\int_0^1 r^n e^{in\theta}(\sqrt{2}-\frac{1}{\sqrt{2}}re^{-i\theta})\left|\frac{re^{i\theta}-\frac{1}{\phi^2}}{1-\frac{1}{\phi^2}re^{i\theta}}\right|^{\frac{1}{\sqrt{5}}}\frac{r\ dr\ d\theta}{|re^{i\theta}-\frac{1}{\phi^2}|\ |1-\frac{1}{\phi^2}re^{i\theta}|}}}$$

Proposition 1. K is a tridiagonal kernel and $H(K)$ contains the polynomials if, and only if, for $m \geq n+1$,

$$\kappa_n = \frac{< z^n, z^m >}{< z^{n+1}, z^m >}$$

is independent of m.

In the previous two examples κ_n converges to $\lambda = \frac{1}{\phi^2}$. In the next family of examples the measure is constructed to satisfy Proposition 1.

A class of subnormal operators on tridiagonal spaces. Let $a \geq 0$, $p \geq 0$, and $b \in \mathbb{R}$ be such that *if* $a > 0$, *then* $a > \frac{b}{p+2}$ and *if* $a = 0$, *then* $b < 0$. For $|\lambda| > 1$, define

$$u(r,\theta) = \sum_{j=-\infty}^{\infty} \left[\frac{a - (1-r)(b + 2|j|a)}{\lambda^{|j|}}\right] r^{|j|+p} e^{ij\theta}$$

Let $d\mu(re^{i\theta}) = u(r,\theta)\ drd\theta$. A calculation shows

$$u(r,\theta) = [\,a - (1-r)b\,]r^p + 2r^p\,\mathcal{R}e\left[\frac{re^{i\theta}}{\lambda}\left(\frac{a-(1-r)b}{1-\frac{1}{\lambda}re^{i\theta}} - \frac{2(1-r)a}{(1-\frac{1}{\lambda}re^{i\theta})^2}\right)\right].$$

Also, for $m \geq n$,

$$< z^n, z^m >_\mu = \frac{2\pi(2n+p+2)a-b}{\lambda^{m-n}(2m+p+1)(2m+p+2)}$$

and for $m \geq n+1$,

$$\kappa_n = \frac{(2n+p+2)a-b}{\lambda(2n+p+4)a-b} \to \frac{1}{\lambda}.$$

The functions

$$\{g_n(z) = (1 - \kappa_n z)z^n : n = 0,\ 1, \ldots\}$$

are orthogonal and can be normalized to the o.n. basis

$$\{f_n(z) = \frac{g_n(z)}{||g_n||} : n = 0,\ 1, \ldots\}.$$

Since the nontrivial zero of f_n has limit λ, it is tempting to think the transformation $S(x) = \frac{x - \frac{1}{\lambda}}{1 - \frac{1}{\bar{\lambda}} x}$ may be a simplifying one for some choices of a, p, b, and λ.

The norm of g_n is given by

$$\sqrt{2\pi}\sqrt{\frac{a(2n+p+2)-b}{(2n+p+2)^2} - \kappa_n \left[\frac{(2n+p+2)a-b}{(2n+p+3)(2n+p+4)}\right]\left(\frac{1}{\lambda}+\frac{1}{\bar{\lambda}}\right) + \kappa_n^2 \left[\frac{(2n+p+4)a-b}{(2n+p+3)(2n+p+4)}\right]}$$

• What if the nontrivial zero of $f_n(z) = (a_n + b_n z)z^n$ diverges to infinity?

In particular suppose $f_n(z) = (n + 1 + z)z^n$? In this case

$$K(z, w) = \frac{1 + \bar{w}z}{(1 - \bar{w}z)^3} + \frac{\bar{w} + z}{(1 - \bar{w}z)^2} + \frac{\bar{w}z}{(1 - \bar{w}z)},$$

while for $m \geq n$,

$$< z^n, z^m > = \; n! \sum_{j=m}^{\infty} \frac{m!(-1)^m}{(j!)^2},$$

and

$$\kappa_n = \frac{< z^n, z^m >}{< z^{n+1}, z^m >} \; = \; \frac{-1}{n+1}.$$

Is M_z subnormal?

The following table consists of examples of such kernels for which M_z passes several numerical tests for subnormality. The tests consisted of checking positivity of commutators of the operator as well as polynomials of the operator.

It suggests that large classes of tridiagonal kernels exist for which M_z is a subnormal operator and which are not of the types described above.

a_n	b_n	range of k	conjecture
$(n+1)^2$	$n+k$	$-3 \leq k \leq 3$	subnormal?
$(n+2)^2$	$n+k$	$-7 \leq k \leq 8$	subnormal?
$(n+3)^2$	$n+k$	$-12 \leq k \leq 15$	subnormal?
$(n+4)^2$	$n+k$	$-18 \leq k \leq 22$	subnormal?
$(n+5)^2$	$n+k$	$-26 \leq k \leq 31$	subnormal?
$(n+6)^2$	$n+k$	$-35 \leq k \leq 41$	subnormal?
$(n+7)^2$	$n+k$	$-45 \leq k \leq 52$	subnormal?
$(n+8)^2$	$n+k$	$-57 \leq k \leq 64$	subnormal?
$(n+9)^2$	$n+k$	$-69 \leq k \leq 78$	subnormal?
$(n+10)^2$	$n+k$	$-83 \leq k \leq 93$	subnormal?
$(n+11)^2$	$n+k$	$-99 \leq k \leq 109$	subnormal?
$(n+1)^4$	$(n+k)^2$	$-3 \leq k \leq 3$	subnormal?
$(n+2)^4$	$(n+k)^2$	$-7 \leq k \leq 8$	subnormal?
$(n+1)^5$	$(n+k)^2$	$-6 \leq k \leq 5$	subnormal?
$(n+1)^6$	$(n+k)^2$	$-9 \leq k \leq 8$	subnormal?
$(n+1)^6$	$n+k$	$-84 \leq k \leq 83$	subnormal?
$\sqrt{n+1}$	$\sqrt[4]{n+1}$	none	subnormal?
$1+\sqrt{n+1}$	$\sqrt[4]{n+1}$	none	subnormal?

Assume $K_1(z,w) = \sum a_n(\bar{w}z)^n$ and $K_2(z,w) = \sum b_n(\bar{w}z)^n$ satisfy

1. $\lim \frac{a_n}{a_{n+1}} = \lim \frac{b_n}{b_{n+1}} = 1$

2. $\lim a_n = \lim b_n = +\infty$

3. $\dfrac{1}{a_n} = \displaystyle\int_0^1 t^{2n}\, d\nu_1(t)$ and $\dfrac{1}{b_n} = \displaystyle\int_0^1 t^{2n}\, d\nu_2(t)$ for all n.

Note $\nu_1\{1\} = \nu_2\{1\} = 0$, $H(K_i) = P^2(\mu_i)$ where $d\mu_i(re^{i\theta}) = \nu_i(r)\frac{dr\ d\theta}{2\pi}$, and M_z is a bounded subnormal operator. So condition (1) ensures $\mathbb{D}$ is a common domain and condition (2) ensures the measures live only on $\mathbb{D}$ and not on the boundary.

Conjecture: Multiplication by z on $H(K_1+K_2) = H(K_1) \mathbin{\widehat{\oplus}} H(K_2)$ is a subnormal operator. This is equivalent to either (1) or (2).

1. The parallel sum of $\frac{1}{a_n}$ and $\frac{1}{b_n}$ defined as

$$\left(\left(\frac{1}{a_n}\right)^{-1} + \left(\frac{1}{b_n}\right)^{-1}\right)^{-1} = \frac{1}{a_n + b_n}$$

   is a moment sequence.

2. The harmonic mean of $\frac{1}{a_n}$ and $\frac{1}{b_n}$ defined as

$$2\left(\left(\frac{1}{a_n}\right)^{-1} + \left(\frac{1}{b_n}\right)^{-1}\right)^{-1} = \frac{2}{a_n + b_n}$$

   is a moment sequence.

3. The Hankel matrix below must be positive for all n

$$H_{\{\frac{1}{a_n+b_n}\}} = \begin{pmatrix} \frac{1}{a_0+b_0} & \frac{1}{a_1+b_1} & \cdots & \frac{1}{a_n+b_n} \\ \\ \frac{1}{a_1+b_1} & \frac{1}{a_2+b_2} & \cdots & \frac{1}{a_{n+1}+b_{n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{a_n+b_n} & \frac{1}{a_{n+1}+b_{n+1}} & \cdots & \frac{1}{a_{2n}+b_{2n}} \end{pmatrix}$$

4. For each $k \geq 0$,

$$(-1)^k \Delta^k \left(\frac{1}{a_n + b_n} \right) \geq 0$$

where Δ^k denotes the k-th finite difference.

General Question: If K_1 and K_2 are measure spaces on a common open domain Ω with the measures supported only on Ω, under what conditions is the space $H(K_1) \mathbin{\widehat{\oplus}} H(K_2)$ with kernel $K_1 + K_2$ a measure space?

Thank you John B. Conway and

best wishes for the future!