Spectral properties of operators associated with the Cesàro operator on weighted Bergman spaces

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Definition

- *Let* m *denote planar Lebesgue measure and let* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
- *For* $\alpha > -1$, *define the measure* m_α *on* $\mathbb{D}$ *given by* $dm_\alpha(z) = (1 - |z|^2)^\alpha \, dm(z)$, *and for* $p \geq 1$, *let* $L^p_a(\mathbb{D}, m_\alpha)$ *denote the weighted Bergman space*

$$L^p_a(\mathbb{D}, m_\alpha) = L^p(\mathbb{D}, m_\alpha) \cap \mathcal{H}(\mathbb{D}).$$

- *For* $\operatorname{Re} \nu > 0$, *define*

$$C_\nu f(z) = \frac{1}{z^\nu} \int_0^z \frac{f(\omega)\, \omega^{\nu-1}}{\omega - 1} \, d\omega \quad (f \in \mathcal{H}(\mathbb{D}),\ z \in \mathbb{D}).$$

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C_1 is the classical Cesàro operator and if $f(z) = \sum_{n=0}^\infty a_n z^n$, then

$$C_\nu f(z) = \sum_{n=0}^\infty \frac{1}{n+\nu} \left(\sum_{k=0}^n a_k \right) z^n.$$

1918 Hardy. C_1 bounded on H^2.

1965 Brown, Halmos and Shields. C_1 is hyponormal on H^2; $\sigma(C_1, H^2) = \overline{B(1,1)}$.

1971 Kriete and Trutt. C_1 on H^2 is unitarily equivalent to M_z on $P^2(\mu)$ and thus subnormal.

1984 Cowen. Subnormality of C_1 on H^2 via semigroups. Subnormality of the semigroup through the Paley-Wiener Theorem.

1987 Siskakis. Used Cowen's semigroups to establish $\sigma(C_1, H^p) = \overline{B(\frac{p}{2}, \frac{p}{2})}$.

1998 M.M. and Smith. Used Siskakis's semigroup analysis to determine spectral picture of C_1 on H^p, $p > 1$, and an analytic $L : \mathbb{C} \setminus \sigma_e(C_1, H^p) \to \mathcal{L}(H^p)$ with $L(\lambda)(\lambda - C_1) = I$ and $\log \|L(\lambda)\|$ integrable on circles intersecting $\sigma_e(C_1)$. Consequently, C_1 has a decomposable extension.

2001 M.M. Extend preceding to Bergman spaces $L^p_a(\mathbb{D}, m_0)$, $p \geq 4$.

2002 M.M. Extend Paley-Wiener to $L^2_a(\mathbb{P}, m_0)$ and use Cowen's ideas to show C_2 is subnormal on $L^2_a(\mathbb{D}, m_0)$; C_1 has a generalized scalar extension and C_1 is subdecomposable for $2 \leq p$.

2003 Dalhner. Uses non-semigroup methods to obtain spectral picture and subdecomposabilty on $L^p_a(\mathbb{D}, m_\alpha)$, $1 < p < \infty$; an explicit decomposable extension is given.

2008 Persson. Extends Dahlner's results to more general spaces including nonrelexive spaces H^1 and $L^1_a(\mathbb{D}, m_\alpha)$ for $\alpha \geq 0$.

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- In every case, the subdecomposabilty of C_1 and decomposibilty of Dahlner's extension follow from growth estimates of an analytic left resolvent. There is no analogous condition sufficient for an operator to be subscalar.

Theorem

Let $\alpha > -1$. *Then*

1. *If* $\operatorname{Re}\nu > 0$ *then* $C_\nu : L^2_a(\mathbb{D}, m_\alpha) \to L^2_a(\mathbb{D}, m_\alpha)$ *is bounded with* $\sigma_p(C_\nu) = \{\frac{1}{\nu+n} : 0 \leq n < \frac{\alpha+2}{2} - \operatorname{Re}\nu\}$, $\sigma(C_\nu) = \sigma_p(C_\nu) \cup \overline{B}$ *where* $B = B(\frac{1}{\alpha+2}, \frac{1}{\alpha+2})$, *and* $\sigma_e(C_\nu) = \partial B$.

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2. $C_{\alpha+2} : L^2_a(\mathbb{D}, m_\alpha) \to L^2_a(\mathbb{D}, m_\alpha)$ *is subnormal.*

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2. $C_{\alpha+2} : L^2_a(\mathbb{D}, m_\alpha) \to L^2_a(\mathbb{D}, m_\alpha)$ *is subnormal.*
3. *If* $n \in \mathbb{Z}$, $n + \alpha > 0$, *then* $C_{\alpha+n}$ *is subscalar.*
4. *If* $(\alpha+2)/2 \le \alpha + n < \alpha + 2$ *then* $C_{\alpha+n}$ *is not hyponormal.*

Proof. We first follow Cowen's argument to show $C_{\alpha+2}$ is subnormal. Define $k_t(z) = (e^{-t} - 1)z + 1$ and $\psi_t(z) = \frac{e^{-t}z}{k_t(z)}$. Then $1 \le e^t|k_t(z)|$, $\operatorname{Re} k_t(z) \le 2e^t - 1$ for $z \in \mathbb{D}$ and $\psi_t(\mathbb{D}) \subset \mathbb{D}$. Then for $t \ge 0$,

$$\Gamma_\nu(t)f(z) := \left(\frac{\psi_t(z)}{z}\right)^\nu f(\psi_t(z),$$

is a strongly continuous semigroup of weighted composition operators on $L^2_a(\mathbb{D}, m_\alpha)$ with generator $\Lambda_\nu f(z) = -(1-z)(zf'(z) + \nu f(z))$. $\operatorname{dom}(\Lambda_\nu) = \{f : (1-z)zf' \in L^2_a(\mathbb{D}, m_\alpha)\}$.

Define a measure on the upper half plane $\mathbb{P}$ by $d\mu_\alpha(\omega) = (\operatorname{Im}\omega)^\alpha\, dm(\omega)$ and denote the corresponding Bergman space to be $L^2_a(\mathbb{P}, m_\alpha) = L^2(\mathbb{P}, m_\alpha) \cap \mathcal{H}(\mathbb{P})$.

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$\varphi(\omega) = \frac{\omega - i}{\omega + i}$ maps $\mathbb{P}$ onto $\mathbb{D}$ and $W : L^2_a(\mathbb{D}, m_\alpha) \to L^2_a(\mathbb{P}, m_\alpha)$ given by

$$Wf(\omega) := \frac{2^{\alpha+1} i^{(\alpha+2)/2}}{(\omega + i)^{\alpha+2}} f(\varphi(\omega)),$$

is an isometry with inverse

$$W^{-1}g(z) = \frac{2 i^{(\alpha+2)/2}}{(1 - z)^{\alpha+2}} g(\varphi^{-1}(z)).$$

For $a > 0$ and $\operatorname{Im} b \geq 0$, define a composition operator $\widetilde{C}_{a,b}$ on $L^2_a(\mathbb{P}, m_\alpha)$ by $\widetilde{C}_{a,b} f(\omega) = f(a\omega + b)$.

For $a > 0$ and $\operatorname{Im} b \geq 0$, define a composition operator $\widetilde{C}_{a,b}$ on $L_a^2(\mathbb{P}, m_\alpha)$ by $\widetilde{C}_{a,b} f(\omega) = f(a\omega + b)$.
Then

$$\Gamma_{\alpha+2}(t) = e^{-t(\alpha+2)} W^{-1} \widetilde{C}_{e^{-t},(1-e^{-t})i} W.$$

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Cowen derived properties of $\widetilde{C}_{a,b}$ using classical Payley-Wiener theorem. [MM'02] used an extension of the Fourier transform to the unweighted Bergman space $L_a^2(\mu_0)$ to obtain analogous results. The following plays the same role for weighted Bergman spaces.

Theorem (Duren, Gallardo-Gutiérrez, Montes-Rodríguez)

For $\beta > 0$ define τ_β on $(0, \infty)$ by $d\tau_\beta(t) = 2^{\beta-1}\pi\Gamma(\beta)\, t^{-\beta}\, dt$. Then the inverse Fourier transform

$$Uf(\omega) = \int_0^\infty f(t)e^{i\omega t}\, dt$$

is an isometry from $L^2(\mathbb{R}^+, \tau_{\alpha+1})$ onto $L^2_a(\mathbb{P}, m_\alpha)$.

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If $s > 0$ and $\operatorname{Re} t \le 0$, define $T_{s,t}$ on $L^2(\mathbb{R}^+, \tau_\beta)$ by $T_{s,t}f(x) = e^{tx}f(sx)$. Since Duren's U^{-1} is the same extension of the Fourier transform to the weighted Bergman spaces the following relation holds in this setting, too.

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Corollary (Cowen, MM'02)

If $a > 0$ and $\operatorname{Im} b \geq 0$, then $\widetilde{C}_{a,b} = \frac{1}{a} U T_{\frac{1}{a}, \frac{ib}{a}} U^*$.

Theorem (Cowen, MM'02)

Let $\beta > 0$ and assume $s > 0$ and $\operatorname{Re} t \leq 0$.

1. $\|T_{s,t}\| = s^{(\beta-1)/2}$ *and if* $s \neq 1$ *or* $\operatorname{Re} t < 0$ *then* $\sigma(T_{s,t}) = \overline{B(0, \|T_{s,t}\|)}$.
2. *If* $s > 1$ *then* $T_{s,t}$ *is unitarily equivalent to an subnormal operator-weighted bilateral shift;*

It follows that for all $t \geq 0$, $\Gamma_{\alpha+2}(t)$ is subnormal on $L^2_a(\mathbb{D}, m_\alpha)$ with $\|\Gamma_{\alpha+2}(t)\| = e^{-(\alpha+2)t/2}$.
$\lim_{t\to\infty} \frac{\log\|\Gamma_{\alpha+2}(t)\|}{t} = -\frac{\alpha+2}{2}$ implies that $C_{\alpha+2} = R(0, \Lambda_{\alpha+2})$ is bounded with $\sigma(C_{\alpha+2}) \subseteq \overline{B}$, where $B = B(\frac{1}{\alpha+2}, \frac{1}{\alpha+2})$.

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By a theorem if Ito, the semigroup $(\Gamma_{\alpha+2}(t))_{t\geq 0}$ has a normal extension; in particular, $C_{\alpha+2}$ is subnormal.
Similarly, if $\operatorname{Re} \nu \geq \alpha + 2$, then $(\Gamma_\nu(t))_{t\geq 0}$ has type $\leq -\frac{\alpha+2}{2}$ and it follows that $C_\nu = R(0, \Lambda_\nu)$ is bounded with $\sigma(C_\nu) \subset \overline{B}$.

Let $S, Q : L^2_a(\mathbb{D}, m_\alpha) \to L^2_a(\mathbb{D}, m_\alpha)$ be defined by $Sf(z) = zf(z)$ and $Qf(z) = \frac{f(z)-f(0)}{z}$. Then $QS = I$ and $SQ = I - e_0 \otimes e_0$, where $e_n(z) = z^n$, $n \geq 0$ is the orthogonal basis for $L^2_a(\mathbb{D}, m_\alpha)$. $\|e_n\|^2 \sim n^{-(\alpha+1)}$.

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If $\operatorname{Re} \nu > 0$, choose $n \geq 0$ so that $\operatorname{Re} \nu + n \geq \alpha + 2$. Then $C_{\nu+n}$ is bounded and

$$C_\nu|_{\operatorname{ran} S^n} = S^n C_{\nu+n} Q^n|_{\operatorname{ran} S^n}. \qquad (*)$$

Thus $C_\nu|_{\operatorname{ran} S^n}$ is bounded and, since $C_\nu e_\ell = \sum_{k=\ell}^\infty \frac{1}{k+\nu} e_k \in L^2_a(\mathbb{D}, m_\alpha)$ for $0 \leq \ell < n$. $C_\nu : L^2_a(\mathbb{D}, m_\alpha) \to L^2_a(\mathbb{D}, m_\alpha)$ is bounded.

To determine the spectrum, we solve the equation

$$(\lambda - C_\nu)f = h \tag{1}$$

for $f, h \in \mathcal{H}(\mathbb{D})$, $\lambda \neq 0$. (1) implies that

$$(\mu(z)f(z))' = \frac{1}{\lambda}\left(\frac{z}{1-z}\right)^{-1/\lambda}(z^\nu h(z))' \quad (z \notin (-1,0]) \tag{2}$$

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In particular, $(\lambda - C_\nu)f = 0$ implies that $f(z) = cz^{1/\lambda-\nu}(1-z)^{-1/\lambda}$, and since $(1-z)^{-1/\lambda} \in L^2_a(\mathbb{D}, m_\alpha) \Leftrightarrow \mathrm{Re}(\frac{1}{\lambda}) < \frac{\alpha+2}{2}$,

$$\sigma_p(C_\nu) = \left\{\frac{1}{n+\nu} : 0 \leq n < \frac{\alpha+2}{2} - \mathrm{Re}\,\nu\right\}.$$

Assume that $\operatorname{Re}\nu \geq \alpha + 2$ and $\operatorname{Re}\frac{1}{\lambda} < \frac{\alpha+2}{2}$. Then, as in Persson, for $h \in L_a^2(\mathbb{D}, m_\alpha)$, (1) has unique solution

$$\begin{aligned} f(z) &= \frac{1}{\lambda}h(z) + \frac{1}{\lambda^2\mu(z)} \int_0^z \omega^{\nu-1/\lambda-1}(1-\omega)^{1/\lambda-1}h(\omega)\,d\omega \\ &= \frac{1}{\lambda}h(z) + \frac{1}{\lambda^2} \int_0^\infty \Gamma_\nu(t)h(z)e^{t/\lambda}\,dt, \end{aligned}$$

norm convergent since $\|\Gamma_\nu(t)\| \leq e^{|\operatorname{Im}\nu|\pi/2}\, e^{-(\alpha+2)t/2}$.

The essential spectrum is identified through the following.

Theorem (Dahlner, Persson ($\nu = 1$ general $p \geq 1$))

Let $\operatorname{Re}\nu > 0$ *and define*

$$L_{\lambda,\nu}(f) = \int_0^1 f(t) t^{\nu - 1/\lambda - 1}(1-t)^{1/\lambda - 1}\, dt.$$

If $\frac{\alpha+2}{2} < \operatorname{Re}\frac{1}{\lambda} < \operatorname{Re}\nu + m$ *then*

1. $L_{\lambda,\nu} : \operatorname{ran} S^m \to \mathbb{C}$ *is continuous, and*
2. $(\lambda - C_\nu)\operatorname{ran} S^m = \ker L_{\lambda,\nu} \cap \operatorname{ran} S^m$.

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Corollary

If $\operatorname{Re}\nu \geq \alpha + 2$, *then* $B = B(\frac{1}{\alpha+2}, \frac{1}{\alpha+2}) = \rho_e(C_\nu) \cap \sigma(C_\nu)$.

Proof: $\lambda \in B \Leftrightarrow \frac{\alpha+2}{2} < \operatorname{Re}\frac{1}{\lambda}$. Choose m so that $\operatorname{Re}\frac{1}{\lambda} < \operatorname{Re}\nu + m$.

Recall $(*)$: If $\operatorname{Re}\nu > 0$ and $n \geq 0$ is such that $\operatorname{Re}\nu + n \geq \alpha + 2$, then

$$C_\nu|_{\operatorname{ran} S^n} = S^n C_{\nu+n} Q^n|_{\operatorname{ran} S^n}.$$

Since $\operatorname{ran} S^n$ has finite codimension, it follows that for $\sigma_\bullet = \sigma_{\ell e}, \sigma_{re}$ and σ_e,

$$\sigma_\bullet(C_\nu) = \sigma_\bullet(C_\nu, \operatorname{ran} S^n) = \sigma_\bullet(S^n C_{\nu+n} Q^n, \operatorname{ran} S^n) = \sigma_\bullet(C_{\nu+n}).$$

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If $T \in \mathcal{L}(X)$, then $\partial\sigma(T) \setminus \sigma_{\ell e}(T)$ consists of isolated eigenvalues. Thus we obtain $\sigma(C_\nu) \setminus \sigma_p(C_\nu) = \overline{B}$.

Finally, we establish that $C_{n+\alpha}$ is subscalar for all $n \in \mathbb{Z}$, $n > -\alpha$. We show that if C_ν is subscalar, then $C_{\nu+1}$ is subscalar, and, if $\operatorname{Re}\nu > 1$, that $C_{\nu-1}$ is subscalar.

An intrinsic characterization of subscalar operators is due to Eschmeier and Putinar: Let X be a Banach space and let $\mathcal{E}(\mathbb{C}, X)$ denote the Frècet space of infinitely differentiable X-valued functions. If $T \in \mathcal{L}(X)$, then T is subscalar $\Leftrightarrow$ the mapping $T_{\mathbb{C}}f(\lambda) := (\lambda - T)f(\lambda)$ is injective with closed range in $\mathcal{E}(\mathbb{C}, X)$.

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Suppose that $(f_n)_n \subset \mathcal{E}(\mathbb{C}, L^2_a(\mathbb{D}, m_\alpha))$ is such that $(C_{\nu+1})_{\mathbb{C}} f_n \to 0$in $\mathcal{E}(\mathbb{C}, L^2_a(\mathbb{D}, m_\alpha))$. Then, by $(*)$, $QC_\nu S = C_{\nu+1}$ and so $Q(C_\nu)_{\mathbb{C}} S = (C_{\nu+1})_{\mathbb{C}}$. Thus $(C_\nu)_{\mathbb{C}} Sf_n = SQ(C_\nu)_{\mathbb{C}} Sf_n \to 0$, and therefore $Sf_n \to 0$ in $\mathcal{E}(\mathbb{C}, L^2_a(\mathbb{D}, m_\alpha))$. Thus $f_n = QSf_n \to 0$ and it follows that $C_{\nu+1}$ is subscalar.

Now suppose that $\operatorname{Re}\nu > 1$ and $(C_{\nu-1})_{\mathbb{C}} f_n \to 0$ in $\mathcal{E}(\mathbb{C}, L_a^2(\mathbb{D}, m_\alpha))$. Write $f_n(\lambda) = \sum_{k=0}^{\infty} a_{n,k}(\lambda) e_k$, where each $a_{n,k} \in \mathcal{E}(\mathbb{C})$. Then

$$(\lambda - \tfrac{1}{\nu-1}) a_{n,0}(\lambda) e_0 = (I - SQ)(\lambda - C_{\nu-1}) f_n(\lambda) \to 0 \text{ in } \mathcal{E}(\mathbb{C}, L_a^2(\mathbb{D}, m_\alpha));$$

thus $a_{n,0} \to 0$ in $\mathcal{E}(\mathbb{C})$ by a theorem of L. Schwartz. Since $f_n = a_{n,0} e_0 + SQf_n$, it follows from $(*)$ that

$$(C_\nu)_{\mathbb{C}} Q f_n = S (C_\nu)_{\mathbb{C}} Q (SQf_n) = (C_{\nu-1})_{\mathbb{C}} (SQf_n) \to 0 \text{ in } \mathcal{E}(\mathbb{C}, L_a^2(\mathbb{D}, m_\alpha)).$$

Since C_ν is subscalar, it follows that $f_n \to 0$ as required.

Finally, if $\frac{\alpha+2}{2} \leq n+\alpha < \alpha+2$, then $\sigma_p(C_{\alpha+n}) = \emptyset$ and $C_{\alpha+n}$ has spectral radius $r(C_{\alpha+n}) = \frac{2}{\alpha+2}$.

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THANKS!