Universality limits and de Branges spaces

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This implies $F(M) = F(UMU^*)$ for all unitary matrices U.

Consequently, $F(M)$ should depend only on the eigenvalues of M.

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$p(y) = \int_{\mathbb{R}} p(x)K_n(x, y)e^{-Q(x)}dx$

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Useful quantity is the correlation function:

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$$\lim_{n \to \infty} \frac{R_m(x + \frac{a_1}{n}, \dots, x + \frac{a_m}{n})}{n^m} = \det(\frac{\sin \pi(\lambda_i - \lambda_j)}{\pi(\lambda_i - \lambda_j)})_{1 \le i \le j \le m}.$$

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It was proved for various $Q(x)$ by P. Deift (S. Venakides, X. Zhou, J. Baik ...) using Riemann-Hilbert methods.

It was proved for more general measures by D. Lubinsky (V. Totik, B. Simon ...).

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Several natural (but vague) questions:

- Can we say anything if the measure does not have finite moments?
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We have a nest of de Branges spaces parametrized by $L > 0$.

Paley-Wiener (model) case

If μ is the Lebesgue measure we obtain the nest of Paley-Wiener spaces.

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Want similar result for measures 'sufficiently close' to the Lebesgue measure.

Main result

Theorem

Let μ be a positive Poisson summable measure that is absolutely continuous on $\mathbb{R}$ with strictly positive density. Moreover, assume that the density is continuous at a fixed point $x \in \mathbb{R}$. Then

$$\lim_{L\to\infty} \frac{K_L(x+\frac{a}{L}, x+\frac{b}{L})}{L} = \frac{\sin \pi\rho(x)(a-b)}{\pi(a-b)},$$

where $K_L(x,y)$ are the reproducing kernels from the nested family of de Branges spaces associated to μ, and $\rho(x)$ is the density of states at x.

Main ideas in the proof

Lubinski's inequality: Assume μ and μ^* are measures on $\mathbb{R}$ satisfying $\mu \leq \mu^*$. Then for all real x, y we have

$$\frac{|K_L(x,y) - K_L^*(x,y)|}{K_L(x,x)} \leq \sqrt{\frac{K_L(y,y)}{K_L(x,x)}}\sqrt{1 - \frac{K_L^*(x,x)}{K_L(x,x)}}.$$

Diagonal behavior: If μ satisfies the assumptions from the Theorem then:

$$\lim_{L\to\infty} \frac{K_L(x+\frac{a}{L}, x+\frac{a}{L})}{L} = \frac{1}{\mu'(x)}.$$

Proof: Define ν to be the same as μ on a neighborhood of x and to be $\mu'(x)$ times the Lebesgue measure outside of that neighborhood.
Define $\mu^* = \max(\mu, \nu)$ and apply the previous two results.

Thank you.