MAA 6617 Course Notes Spring 2025

1. Normed vector spaces

In this section $\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{X}$ be a vector space over $\mathbb{F}$.

1.1. **Definitions and preliminary results.**

Definition 1.1. A normed vector space $\mathcal{X} = (\mathcal{X}, \|\cdot\|)$ consists of a vector space $\mathcal{X}$ over $\mathbb{F}$ together with a norm $\|\cdot\| : \mathcal{X} \to [0,\infty)$ (see definition **??** – which does not change with $\mathbb{R}$ replaced by $\mathbb{C}$). We often denote the normed vector space as $\mathcal{X}$, with the norm $\|\cdot\|$ implicit.

As we noted before, using the properties of a norm, it is straightforward to check that $d : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ defined by

$$d(x,y) := \|x-y\|$$

is a metric on $\mathcal{X}$. The resulting topology is the *norm topology* and it is the *default* topology on $\mathcal{X}$.

Definition 1.2. A normed vector space $\mathcal{X}$ is a *Banach space* if it is complete (with its norm topology). □

Definition 1.3. Two norms $\|\cdot\|_1, \|\cdot\|_2$ on $\mathcal{X}$ are *equivalent* if there exist constants $C, c > 0$ such that

$$c\|x\|_1 \le \|x\|_2 \le C\|x\|_1,$$

for all $x \in \mathcal{X}$. □

Remark 1.4. Equivalent norms determine the same topology on $\mathcal{X}$ and the same Cauchy sequences (Problem 1.4). In particular, it follows that if $\mathcal{X}$ is equipped with two equivalent norms $\|\cdot\|_1, \|\cdot\|_2$ then it is complete (a Banach space) in one norm if and only if it is complete in the other.

Equivalence of norms is an equivalence relation on the set of norms on $\mathcal{X}$. □

The next proposition is simple but fundamental; it says that the norm and the vector space operations are continuous in the norm topology.

Proposition 1.5 (Continuity of vector space operations)**.** *Let $\mathcal{X}$ be a normed vector space over $\mathbb{F}$.*

a) *If (x_n) converges to x in $\mathcal{X}$, then $(\|x_n\|)$ converges to $\|x\|$ in $\mathbb{R}$.*
b) *If (k_n) converges to k in $\mathbb{F}$ and (x_n) converges to x in $\mathcal{X}$, then $(k_n x_n)$ converges to kx in $\mathcal{X}$.*
c) *If (x_n) converges to x and (y_n) converges to y in $\mathcal{X}$, then $(x_n + y_n)$ converges to $x + y$ in $\mathcal{X}$.*

Proof. The proofs follow readily from the properties of the norm, and are left as exercises. □

The following proposition gives a convenient criterion for a normed vector space to be complete.

Definition 1.6. Given a sequence (x_n) from a normed vector space $\mathcal{X}$, the expression $\sum_{n=1\infty} x_n$ denotes the sequence $(s_N = \sum_{n=1}^N x_n)$, called the sequence of *partial sums* of the series. The *series converges* if the sequence of partial sums converges in the norm topology. In this case we use $\sum_{n=1\infty} x_n$ to also denote the limit of this sequence and call it the sum.

Explicitly, the series $\sum_{n=1}^\infty x_n$ converges means there is an $x \in \mathcal{X}$ such that for each $\epsilon > 0$ there is an N such that $\|s_n - x\| < \epsilon$ for all $n \geq N$.

The series $\sum_{n=1}^\infty x_n$ *converges absolutely* if the series $\sum_{n=1}^\infty \|x_n\|$ converges (in the normed vector space $(\mathbb{R}, |\cdot|)$). □

Proposition 1.7. *A normed space $(\mathcal{X}, \|\cdot\|)$ is complete if and only if every absolutely convergent series in $\mathcal{X}$ is convergent.*

Before proving the Proposition we collect two lemmas. A definition is needed for the first.

Definition 1.8. *A sequence (y_k) from a normed vector space $\mathcal{X}$ is* super-cauhcy *if the series $\sum_{k=1}^\infty (y_{k+1} - y_k)$ converges absolutely.*

Lemma 1.9. *If (x_n) is a Cauchy sequence from a normed vector space $\mathcal{X}$, then there is a subsequence (y_k) of (x_n) that is super-cauchy.*

Proof. With $\epsilon = \frac{1}{2}$, there exists an N_1 such that $\|x_n - x_m\| < \frac{1}{2}$ for all $m, n \geq N_1$ since (x_n) is Cauchy. Assuming $N_1 < N_2 < \cdots < N_k$ have been chosen so that $\|x_n - x_m\| < \frac{1}{2^j}$ for $1 \leq j \leq k$, there is an $N_{k+1} < N_k$ such that $\|x_n - x_m\| < \frac{1}{2^{k+1}}$ since (x_n) is Cauchy. Hence by recursion we have constructed a (strictly)increasing sequence of integers N_k such that $\|x_n - x_m\| < \frac{1}{2^k}$ for all $m, n \geq N_k$. Set $y_k = x_{N_k}$ and note that $\|y_{k+1} - y_k\| < \frac{1}{2^k}$, from which it follows that (y_k) is a super-cauchy subsequence of (x_n). □

The proof will also use the following standard lemma from advanced calculus.

Lemma 1.10. *If (x_n) is a Cauchy sequence from a metric space (X, d) and if (x_n) has a subsequence (y_k) that converges to some x, then (x_n) converges to x.*

Proof of Proposition 1.7. First suppose $\mathcal{X}$ is complete and $\sum_{n=1}^\infty x_n$ is absolutely convergent. Write $s_N = \sum_{n=1}^N x_n$ for the N^{th} partial sum and let $\epsilon > 0$ be given. Since $\sum_{n=1}^\infty \|x_n\|$ is convergent, there exists an L such that $\sum_{n=L}^\infty \|x_n\| < \epsilon$. If $N > M \geq L$, then

$$\|s_N - s_M\| = \left\|\sum_{n=M+1}^{N} x_n\right\| \le \sum_{n=M+1}^{N} \|x_n\| < \epsilon.$$

Thus the sequence (s_N) is Cauchy in $\mathcal{X}$, hence convergent by the completeness hypothesis.

Conversely, suppose every absolutely convergent series in $\mathcal{X}$ is convergent and that (x_n) is given Cauchy sequence from X. By Lemma 1.9 there is a super-cauchy subsequence (y_k) of (x_n). Since (y_k) is super-cauchy, the series $\sum_{k=1}^{\infty}(y_{k+1} - y_k)$ is absolutely convergent and hence, by hypothesis, convergent in $\mathcal{X}$. Thus there is an $z \in \mathcal{X}$ such that the sequence of partial sums

$$\sum_{k=1}^{n}(y_{k+1} - y_k) = y_{n+1} - y_1$$

converges to z. Rearranging, $(x_{N_{n+1}} = y_{n+1})$ converges to $x = z+y_1$. Hence (x_n) is Cauchy and has a convergent subsequence. Thus (x_n) converges (to x) by Lemma 1.10. $\square$

1.2. **Examples.**

1.2.1. *Euclidean space.* Observe that the Euclidean norm on the complex vector space $\mathbb{C}^n$ agrees with the Euclidean norm on the real vector space $\mathbb{R}^{2n}$ (via that natural real linear isomomorphism $\mathbb{R}^2 \to \mathbb{C}$ sending (x, y) to $x + iy$). Thus, $\mathbb{F}^n$ with the usual Euclidean norm $\|(x_1, \dots x_n)\| = \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2}$ is a Banach space.

The vector space $\mathbb{F}^n$ can also be equipped with the *ℓ^p-norms*

$$\|(x_1, \dots x_n)\|_p := \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}$$

for $1 \le p < \infty$, and the *ℓ^∞-norm*

$$\|(x_1, \dots x_n)\|_\infty := \max(|x_1|, \dots |x_n|).$$

For $1 \le p < \infty$ and $p \ne 2$, it is not immediately obvious that $\|\cdot\|_p$ defines a norm. We will prove this assertion later. It is not too hard to show that all of the ℓ^p norms $(1 \le p \le \infty)$ are equivalent on $\mathbb{F}^n$ (though the constants c, C depend on the dimension n). For instance, for $n \in \mathbb{N}$,

$$\|x\|_\infty \le \|x\|_2 \le \|x\|_1 \le n\|x\|_\infty.$$

The first and third inequalities are evident. For the middle inequality, observe

$$(\|x\|_1)^2 = \sum_{j,k=1}^{n} |x_j|\,|x_k| \ge \sum_{j=1}^{n} |x_j|^1 = \|x\|_2^2.$$

Given a normed vector space $\mathcal{X} = (\mathcal{X}, \|\cdot\|)$, denote its closed *unit ball* by

$$\mathcal{X}_1 = \{x \in \mathcal{X} : \|x\| \le 1\}.$$

It is instructive to sketch the closed unit ball in $\mathbb{R}^2$ with the three norms above.

It turns out that *any* two norms on a finite-dimensional vector space are equivalent. As a corollary, every finite-dimensional normed space is a Banach space. See Problem 1.5.

Lemma 1.11. *If $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on $\mathcal{X}$ and there is a constant $C > 0$ such that $\|x\|_1 \leq C\|x\|_2$ for all $x \in \mathcal{X}$, then the mapping $\iota : (\mathcal{X}, \|\cdot\|_2) \to (\mathcal{X}, \|\cdot\|_1)$ is (uniformly) continuous.*

Proof. For $x, y \in \mathcal{X}$, we have $\|\iota(x) - \iota(y)\|_1 = \|\iota(x-y)\|_1 = \|x-y\|_1 \leq C\|x-y\|_2$. $\square$

Proposition 1.12. *If $\|x\|$ is a norm on $\mathbb{R}^n$, then $\|x\|$ is equivalent to the Euclidean norm $\|\cdot\|_2$.*

Sketch of proof. Let $\{e_1, \ldots, e_n\}$ denote the usual basis for $\mathbb{R}^n$. Given $x = \sum a_j e_j \in \mathbb{R}^n$,

$$\|x\| \leq \sum |a_j|\,\|e_j\| = \sum |a_j|\|e_j\| \leq M\,\|x\|_1 \leq n\,M\,\|x\|_2,$$

where $M = \max\{\|e_1\|, \ldots, \|e_n\|\}$. It now follows that the map $\iota : (\mathbb{R}^n, \|\cdot\|_2) \to (\mathbb{R}^n, \|\cdot\|)$ is continuous and therefore so is the map $f : (\mathbb{R}^n, \|\cdot\|_2) \to [0, \infty)$ defined by $f(x) = \|x\|$. Since

$$S^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$$

(the unit sphere) is compact in $\mathbb{R}^n$, by the Extreme Value Theorem, f attains its infimum; that is, there is a point $p \in S^{n-1}$ such that $f(p) \leq f(x)$ for all $x \in S^{n-1}$. But $f(p) = \|p\| > 0$ since $p \neq 0$. Let $c = f(p) = \|p\|$. We conclude that if $\|x\|_2 = 1$ then $\|x\| \geq c\|x\|_2$, from which it follows by homogeneity that $\|x\| \geq c\|x\|_2$ for all $x \in \mathbb{R}^n$. $\square$

Corollary 1.13. *All norms on a finite dimensional vector space are equivalent. Further, if V is a finite dimensional normed vector space, then V_1 is compact and V is a Banach space.*

Proof. Suppose V is a normed vector space of dimension n and let $\{v_1, \ldots, v_n\}$ denote a basis for V. The function $\|\cdot\|' : V \to [0, \infty)$ defined by

$$\|v\|' = \|\sum a_j v_j\|' = \sum |a_j|$$

is easily seen to be a norm.

Now let $\|\cdot\|$ be a given norm on V. This norm induces a norm $\|\cdot\|_*$ on $\mathbb{R}^n$ given by

$$\|\sum a_j e_j\|_* = \|\sum a_j v_j\|.$$

Since all norms in $\mathbb{R}^n$ are equivalent, the norm $\|\cdot\|_*$ is equivalent to the norm $\|\cdot\|_1$. Hence there exist constants $0 < c < C$ such that

$$c\|v\|' = c\sum |a_j| = c\|\sum a_j e_j\|_1 \leq \|\sum a_j e_j\|_* \leq C\|\sum a_j e_j\|_1 = C\sum |a_j| = C\|v\|'.$$

Thus, as $\|\sum a_j e_j\|_* = \|\sum a_j v_j\|$,

$$c\|v\|' \leq \|v\| \leq C\|v\|'$$

for all $v \in V$. Thus all norms on V are equivalent.

Further, by definition, $f : (V, \|\cdot\|) \to (\mathbb{R}^n, \|\cdot\|_*)$ is bijective and isometric. Thus, f^{-1} is continuous, $f^{-1}(S)$ where S is the unit ball in $(\mathbb{R}^n, \|\cdot\|_*$, is the unit ball in $(V, \|\cdot\|)$ and is compact as its the continuous image of a compact set. It is now routine to pass from compactness of the unit ball in $(V, \|\cdot\|)$ to completeness of $(V, \|\cdot\|)$. □

1.2.2. *The Banach space of bounded functions.* If V is a vector space over $\mathbb{F}$ and $\varnothing \neq T$ is a set, then $F(T, V)$, the set of functions $f : T \to V$ is a vector space over $\mathbb{F}$ under pointwise operations; e.g., if $f, g \in F(T, V)$ then $f + g : T \to V$, is the function defined by $(f + g)(t) = f(t) + g(t)$.

Definition 1.14. A subset R of a normed vector space $\mathcal{X}$ is bounded if there is a C such that $\|x\| \leq C$ for all $x \in R$; that is, $R \subseteq C\mathcal{X}_1$.

A function $f : T \to \mathcal{X}$ is bounded if $f(T) \subseteq \mathcal{X}$ is bounded.

Let $F_b(T, \mathcal{X})$ denote the vector space (subspace of $F(T, \mathcal{X})$) of bounded functions $f : T \to \mathcal{X}$.

Remark 1.15. The function $\|\cdot\|_\infty : F_b(T, \mathcal{X}) \to [0, \infty)$ defined by

$$\|f\|_\infty = \sup\{|f(t)| : t \in T\}$$

is a norm on $F_b(T, \mathcal{X})$ as you should verify. Let d_∞ denote the resulting metric: $d_\infty(f, g) = \|f - g\|_\infty$.

Note that convergence of a sequence in the metric space $(F_b(T, \mathcal{X}), d_\infty)$ is uniform convergence; in particular, a sequence is Cauchy in $F_b(T, \mathcal{X})$ if and only if it is uniformly Cauchy. (Exercise.)

Proposition 1.16. *If $\mathcal{X}$ is a Banach space, then $F_b(T, \mathcal{X})$ is also Banach space.*

Proof. We are to show $F_b(T, \mathcal{X})$ is complete, assuming $\mathcal{X}$ is complete. Accordingly, suppose (f_n) is a Cauchy sequence from $F_b(T, \mathcal{X})$ and $\mathcal{X}$ is complete. In particular, given $\epsilon > 0$ there is an N such that $d_\infty(f_n, f_m) = \sup\{\|f_n(t) - f_m(t)\| : t \in T\} < \epsilon$. It follows that, for each $s \in T$, the sequence $(f_n(s))$ is a Cauchy in $\mathcal{X}$ and hence converges to some $x \in \mathcal{X}$. Define $f : T \to \mathcal{X}$ by $f(s) = x$. It remains to see that f is bounded and (f_n) converges to f.

Since Cauchy sequences are bounded and (f_n) is Cauchy in the metric space $F_b(T, \mathcal{X})$, there is a C such that

$$\sup\{\|f_n(t)\| : t \in T\} = d_\infty(f_n, 0) \leq C$$

for all n. It follows from Proposition 1.5 that $(\|f_n(t)\|)_n$ converges to $|f(t)$ and hence $\|f(t)\| \leq C$ for all $t \in T$. Thus f is bounded; that is $f \in F_b(T, \mathcal{X})$.

It only remains to show that (f_n) converges to f in $F_b(T, \mathcal{X})$. To do so let $\epsilon > 0$ be given. There is an N such that if $m, n \geq N$, then $\|f_n(t) - f_m(t)\| < \epsilon$ for all $t \in T$.

Given $s \in T$, there is an $M \geq N$ such that $\|f_m(s) - f(s)\| < \epsilon$ for all $m \geq N$. Since, $(f_m(s) - f_n(s))_m$ converges (with m) to $(f(s) - f_n(s))$ in $\mathcal{X}$, another application of Proposition 1.5 gives $(\|f_m(s) - f_n(s)\|)_m$ converges to $\|f(s) - f_n(s)\|$. Thus

$$\|f(s) - f_n(s)\| \leq \epsilon,$$

for all $s \in T$. Hence $d_\infty(f, f_n) = \|f - f_n\| \leq \epsilon$ and the proof is complete. □

There are important Banach spaces of continuous functions. Before going further, we remind the reader of the following result from advanced calculus.

Theorem 1.17. *Suppose X, Y are metric spaces, (f_n) is a sequence $f_n : X \to Y$ and $x \in X$. If each f_n is continuous at x and if (f_n) converges uniformly to f, then f is continuous at x. Hence if each f_n is continuous, then so is f.*

Proof. Let x and $\epsilon > 0$ be given. Choose N such that if $n \geq N$ and $y \in X$, then $d_Y(f_n(y), f(y)) < \epsilon$. Since f_N is continuous at x, there is a $\delta > 0$ such that if $d_X(x, y) < \delta$, then $d_Y(f_N(x), f_N(y)) < \epsilon$. Thus, if $d_X(x, y) < \delta$, then

$$\begin{aligned} d_Y(f(x), f(y)) &\leq d_Y(f(x), f_N(x)) + d_Y(f_N(x), f_N(y)) + d_Y(f_N(y), f(y)) \\ &< 3\epsilon, \end{aligned}$$

proving the theorem. □

Given a normed vector space $\mathcal{Y}$, let $C_b(X, \mathcal{Y})$ denote the subspace of $F_b(X, \mathcal{Y})$ consisting of continuous functions. Since uniform convergence is the same as convergence in the normed vector space $(F_b(X, \mathcal{Y}), d_\infty)$, by Theorem 1.17, $C_b(X, \mathcal{Y})$ is a closed subspace of $F_b(X, \mathcal{Y})$. In particular, in the case $\mathcal{Y}$ is a Banach space, so is $C_b(X, \mathcal{Y})$.

When X be a compact metric space, let $C(X) = C(X, \mathbb{F})$ denote the set of continuous functions $f : X \to \mathbb{F}$. Thus $C(X)$ is a subspace of $F_b(X, \mathbb{F})$ and we endow $C(X)$ with the norm it inherits from $F_b(X, \mathbb{F})$. Since $\mathbb{F}$ is complete, $C(X)$ is a Banach space. Of course here we could replace $\mathbb{F}$ by a Banach space $\mathcal{X}$ and obtain the analogous conclusion for the space $C(X, \mathcal{X})$.

Now let X be a locally compact metric space. In this case, a function $f : X \to \mathbb{F}$ *vanishes at infinity* if for every $\epsilon > 0$, there exists a compact set $K \subseteq X$ such that $\sup_{x \notin K} |f(x)| < \epsilon$. Let $C_0(X)$ denote the subspace of $F_b(X, \mathbb{F})$ consisting of continuous functions $f : X \to \mathbb{F}$ that vanish at infinity. Then $C_0(X)$ is a normed vector space with the norm it inherits from $C(X)$ (equivalently $F_b(X, \mathbb{F})$. It is routine to check that $C_0(X)$ is complete.

1.2.3. *L^1 spaces over $\mathbb{R}$.* Let $(X, \mathscr{M}, \mu)$ be a measure space and let $L^1(\mu)$ denote the (real) vector space of (real-valued) absolutely integrable functions on X from Theorem **??**. We saw that

$$\|f\|_1 := \int_X |f| \, dm$$

defines a norm on $L^1(\mu)$, provided we agree to identify f and g when $f = g$ a.e. (Indeed the chief motivation for making this identification is that it makes $\|\cdot\|_1$ into a norm.

Proposition 1.18. *The real vector space $L^1(\mu)$ is a Banach space.*

We will construct a complex vector space analog of $L^1(\mu)$ a bit later.

Proof. It suffices to verify the hypotheses of Proposition 1.7. Accordingly suppose $\sum_{n=1}^{\infty} f_n$ is absolutely convergent (so that $\sum_{n=1}^{\infty} \|f_n\|_1 < \infty$). By Tonelli's summation theorem, Theorem **??**,

$$\int \sum_{n=1}^{\infty} |f_n| \, dm = \sum_{n=1}^{\infty} \int |f_n| \, dm = \sum_{n=1}^{\infty} \|f_n\|_1 < \infty.$$

Thus the function $g := \sum_{n=1}^{\infty} |f_n|$ belongs to L^1 and is thus finite m-a.e. In particular the sequence of partial sums $s_N = \sum_{n=1}^{N} f_n$ is a sequence of measurable functions with $|s_N| \leq g$ that converges pointwise a.e. to a measurable function f. Hence by the DCT and its corollary, $f \in L^1$ and the partial sums $(s_N)_N$ converge to f in L^1. □

1.2.4. *Complex $L^1(\mu)$ spaces.* In this subsection we describe the extension of $L^1(\mu)$ to a complex vector space of complex valued functions (equivalence classes of functions).

Again we work on a fixed measure space $(X, \mathscr{M}, \mu)$. As a topological space, $\mathbb{C}$ and $\mathbb{R}^2$, are the same. A function $f : X \to \mathbb{C} = \mathbb{R}^2$ is *measurable* if and only if it is $\mathscr{M} - \mathscr{B}_2$ measurable. Measurability of f can also be described in terms of the real and imaginary parts of f.

Proposition 1.19. *Suppose $(X, \mathscr{M})$ is a measurable space and $f : X \to \mathbb{C}$. Writing $f : X \to \mathbb{C}$ as $f = u + iv$, where $u, v : X \to \mathbb{R}$, the function f is measurable if and only if both u and v are.*

Moreover, if f is measurable, then so is $|f| : X \to [0, \infty)$.

We begin with the following elementary lemma whose proof is left to the reader.

Lemma 1.20. *Suppose $(X, \mathscr{M})$ is a measure space and Y and Z are topological spaces. If $f : X \to Y$ is $\mathscr{M} - \mathscr{B}_Y$ measurable and $g : Y \to Z$ is $\mathscr{B}_Y - \mathscr{B}_Z$ measurable, then $g \circ f$ is $\mathscr{M} - \mathscr{B}_Z$ measurable. In particulr, the result holds if g is continuous.*

Sketch of proof of Proposition 1.19. The Borel σ-algebra $\mathscr{B}_2$ is generated by open rectangles; that is, a set $U \subseteq \mathbb{C}$ is open if and only if it is a countable union of open rectangles (with rational vertices even). For an open rectagle $I = J \times K = (a, b) \times (c, d)$ observe that

$$f^{-1}(J) = u^{-1}(J) \cap v^{-1}(K).$$

Thus, if u and v are measurable, then $f^{-1}(J) \in \mathscr{M}$. Consequently, by Propition **??**, f is measurable. Hence if u, v are both measurable, then so is f.

Now suppose f is measurable. In this case

$$\mathscr{M} \ni f^{-1}((t,\infty)\times\mathbb{R}) = \{u > t\}.$$

Since the sets $\{(t,\infty) : t\}$ generate $\mathscr{B}_1$, Proposition **??** implies u is measurable. By symmetry v is measurable.

To prove the second statement, since f is measurable and $g:\mathbb{C}\to[0,\infty)$ defined by $g(z)=|z|$ is continuous, the function $g\circ f = |f|$ is measurable by Lemma 1.20. □

Definition 1.21. A measurable $f : X \to \mathbb{C}$ is *integrable* (or *absolutely integrable*) if $|f|$ is integrable.

Remark 1.22. From the inequalities

$$|\mathrm{Re} f|,\ |\mathrm{Im} f| \le |f| \le |\mathrm{Re} f| + |\mathrm{Im} f|$$

it follows that $f : X \to \mathbb{C}$ is (absolutely) integrable if and only if $\mathrm{Re}f$ and $\mathrm{Im}f$ are.

Definition 1.23. If f is complex-valued and absolutely integrable (that is, f is measurable and $|f|$ is integrable), we define the integal of f by

$$\int f = \int \mathrm{Re} f + i \int \mathrm{Im} f.$$

We also write $\|f\|_1 := \int_X |f|\, d\mu$ in the complex case. Finally, we write $L^1 = L^1(\mu)$ to denote the set of absolutely integrable complex-valued functions on X.

Generally, we leave it to context to indicate if we are considering the real or complex version of L^1; but for the following theorem we temporarily adopt the notation $L^1_{\mathbb{R}}$ and $L^1_{\mathbb{C}}$ to distinguish between the real and complex vector space versions of $L^1(\mu)$.

Theorem 1.24 (L^1 as a $\mathbb{C}$ normed vector space)**.** *The set $L^1_{\mathbb{C}}$ of is a vector space over $\mathbb{C}$ (with the usual addition and scalar multiplication of functions). Morever, if $f, g \in L^1_{\mathbb{C}}$ and $c \in \mathbb{C}$, then*

(a) the mapping $\Lambda : L^1 \to \mathbb{C}$ defined by $\Lambda(f) = \int f$ is linear;
(b) $\left|\int f\right| \le \int |f|$.
(c) $\|cf\|_1 = |c|\|f\|_1$.
(d) $\|f+g\|_1 \le \|f\|_1 + \|g\|_1$.

Sketch of proof. Write $f = u + iv$ and $g = x + iy$. In particular, u, v, x, y are all $L^1_{\mathbb{R}}$. Given $c = a + ib$, the functions au, bv, av, bu are all $L^1_{\mathbb{R}}$ and so are $au - bv$ and $av + bu$ since $L^1_{\mathbb{R}}$ is a real vector space. Therefore, $cf = (au-bv)+i(av+bu)$ is in $L^1_{\mathbb{C}}$. A similar, but easier, argument shows $f+g$ is in $L^1_{\mathbb{C}}$. Hence $L^1_{\mathbb{C}}$ is a vector space over $\mathbb{C}$. Moreover, since the integral is real linear on $L^1_{\mathbb{R}}$,

$$\begin{aligned}\Lambda(cf) &= \Lambda((au+bv)+i(av+bu)) = \Lambda((au+bv)) + i\Lambda((av+bu))\\ &= a\Lambda(u) + b\Lambda(v) + i[a\Lambda(v) + b\Lambda(u)]\\ &= (a+ib)[\Lambda(u) + i\Lambda(v)] = c\Lambda(f).\end{aligned}$$

Likewise $\Lambda(f+g) = \Lambda(f) + \Lambda(g)$. Thus Λ is $\mathbb{C}$-linear on $L^1_{\mathbb{C}}$ and item (a) is proved.

If $\int f = 0$, then $f = 0$ almost everywhere and the last three items hold. Otherwise, write $\int f = re^{it}$ in polar coordinates and observe

$$e^{-it} \int f \in \mathbb{R}_+.$$

Thus, from the definition and linearity of the integral

$$\mathbb{R}_+ \ni e^{-it} \int f = \int e^{-it} f = \int \operatorname{real} e^{-it} f \, + \, i \int \operatorname{image} e^{-it} f.$$

Thus $\int \operatorname{image} e^{-it} f = 0$ and using results for $L^1_{\mathbb{R}}$,

$$\left| \int f \right| = \left| e^{-it} \int f \right| = \int e^{-it} f = \int \operatorname{real} e^{-it} f \le \int |\operatorname{real} e^{-it} f| \le \int |f|,$$

proving item (b).

Next,

$$\int |cf| = \int |c| \, |f| = |c| \int |f|.$$

Hence item (c) holds. Similarly,the triangle inequality, item (d), follows from $|f+g| \le |f| + |g|$ (pointwise). $\square$

Remark 1.25. Proposition 1.24 says $\| \cdot \|_1$ is a semi-norm on L^1. As usual, we identify functions that differ by a null vector; that is, $f \sim g$ if $\|f-g\|_1 = 0$; equivalently, identifying functions that are equal a.e., we obtain a normed complex vector space of L^1 *functions* (which of course are not actually functions).

1.2.5. *Sequence spaces.* Define

$$\begin{aligned} c_0 &:= \{f : \mathbb{N} \to \mathbb{F} | \lim_{m \to \infty} |f(m)| = 0\} \\ \ell^\infty &:= \{f : \mathbb{N} \to \mathbb{F} | \sup_{m \in \mathbb{N}} |f(m)| < \infty\} \\ \ell^1 &:= \{f : \mathbb{N} \to \mathbb{F} | \sum_{m=0}^{\infty} |f(m)| < \infty\}. \end{aligned}$$

Note that $\ell^\infty = F_b(\mathbb{N}, \mathbb{F})$ and is a Banach space with the norm

$$\|f\|_\infty = \sup_m |f(m)|.$$

Further, $c_0 \subseteq \ell^\infty$ is the subspace $C_0(\mathbb{N})$ of ℓ^∞ again with the norm $\| \cdot \|_\infty$. In particular, c_0 is a Banach space.

Observe that ℓ^1 is the space $(\mathbb{N}, P(\mathbb{N}), c)$, where c is counting measure on $\mathbb{N}$ and $\| \cdot \|_1$ is the corresponding ℓ^1 norm. Since only set of measure zero in this measure space is the emptyset, two functions in $\ell^1 = L^1(c)$ are equivalent if and only if they are equal.

Along with these spaces it is also helpful to consider the vector space

$$c_{00} := \{f : \mathbb{N} \to \mathbb{F} | f(n) = 0 \text{ for all but finitely many } n\}$$

Notice that c_{00} is a vector subspace of each of c_0, ℓ^1 and ℓ^∞. Thus it can be equipped with either the $\|\cdot\|_\infty$ or $\|\cdot\|_1$ norms. It is not complete in either of these norms, however. What is true is that c_{00} is *dense* in c_0 and ℓ^1 (but not in ℓ^∞). (See Problem 1.11).

1.2.6. *L^p spaces.* Again let $(X, \mathscr{M}, m)$ be a measure space. For $1 \leq p < \infty$ let $L^p(m)$ denote the set of measurable functions f for which

$$\|f\|_p := \left(\int_X |f|^p \, dm \right)^{1/p} < \infty$$

(again we identify f and g when $f = g$ a.e.). It turns out that this quantity is a norm on $L^p(m)$, and $L^p(m)$ is complete, though we will not prove this yet (it is not immediately obvious that the triangle inequality holds when $p > 1$).

Choosing $(X, \mathscr{M}, \mu) = (\mathbb{N}, P(\mathbb{N}), c)$, counting measure on $\mathbb{N}$, obtains the sequence spaces ℓ^p; that is, the $\mathbb{F}$-vector space of functions $f : \mathbb{N} \to \mathbb{F}$ such that

$$\|f\|_p := \left(\sum_{n=1}^{\infty} |f(n)|^p \right)^{1/p} < \infty$$

and this quantity is a norm making ℓ^p into a Banach space.

When $p = \infty$, we define $L^\infty(\mu)$ to be the set of all functions $f : X \to \mathbb{K}$ with the following property: there exists $M > 0$ such that

$$|f(x)| \leq M \quad \text{for } \mu - \text{a.e. } x \in X; \tag{1}$$

as for the other L^p spaces we identify f and g when there are equal a.e. When $f \in L^\infty$, let $\|f\|_\infty$ be the smallest M for which (1) holds. Then $\|\cdot\|_\infty$ is a norm making $L^\infty(\mu)$ into a Banach space.

1.2.7. *Subspaces and direct sums.* If $(\mathcal{X}, \|\cdot\|)$ is a normed vector space and $\mathcal{Y} \subseteq \mathcal{X}$ is a vector subspace, then the restriction of $\|\cdot\|$ to $\mathcal{Y}$ is clearly a norm on $\mathcal{Y}$. If $\mathcal{X}$ is a Banach space, then $(\mathcal{Y}, \|\cdot\|)$ is a Banach space if and only if $\mathcal{Y}$ is *closed* in the norm topology of $\mathcal{X}$. (This is just a standard fact about metric spaces—a subspace of a complete metric space is complete in the restricted metric if and only if it is closed.)

If $\mathcal{X}, \mathcal{Y}$ are vector spaces then the *algebraic direct sum* is the vector space of ordered pairs

$$\mathcal{X} \oplus \mathcal{Y} := \{(x, y) : x \in \mathcal{X}, y \in \mathcal{Y}\}$$

with entrywise operations. If $\mathcal{X}$, $\mathcal{Y}$ are equipped with norms $\|\cdot\|_{\mathcal{X}}, \|\cdot\|_{\mathcal{Y}}$, then each of the quantities

$$\|(x,y)\|_\infty := \max(\|x\|_{\mathcal{X}}, \|y\|_{\mathcal{Y}}),$$
$$\|(x,y)\|_1 := \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$$
$$\|(x,y)\|_2 := \sqrt{\|x\|_{\mathcal{X}}^2 + \|y\|_{\mathcal{Y}}^2}$$

is a norm on $\mathcal{X} \oplus \mathcal{Y}$. These three norms are equivalent; indeed it follows from the definitions that

$$\|(x,y)\|_\infty \leq \|(x,y)\|_2 \leq \|(x,y)\|_1 \leq 2\|(x,y)\|_\infty.$$

If $\mathcal{X}$ and $\mathcal{Y}$ are both complete, then $\mathcal{X} \oplus \mathcal{Y}$ is complete in each of these norms. The resulting Banach spaces are denoted $\mathcal{X} \oplus_\infty \mathcal{Y}$, $\mathcal{X} \oplus_1 \mathcal{Y}$ and $\mathcal{X} \oplus_2 \mathcal{Y}$.

1.2.8. *Qoutient spaces.* If $\mathcal{X}$ is a normed vector space and $\mathcal{M}$ is a proper subspace, then one can form the *algebraic quotient* $\mathcal{X}/\mathcal{M}$, defined as the collection of distinct cosets $\{x + \mathcal{M} : x \in \mathcal{X}\}$. From linear algebra, $\mathcal{X}/\mathcal{M}$ is a vector space under the standard operations. Let $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$ denote the quotient map.

Proposition 1.26. *If $\mathcal{M}$ is a* closed *subspace of a normed vector space $\mathcal{X}$, then the quantity*

$$\|\pi(x)\| = \|x + \mathcal{M}\| := \inf_{y \in \mathcal{M}} \|x - y\|$$

is a norm on $\mathcal{X}/\mathcal{M}$. Moreover, if $\mathcal{X}$ is a Banach space, then so is $\mathcal{X}/\mathcal{M}$.

The norm in Proposition 1.26 is called the *quotient norm*. Geometrically, $\|x + \mathcal{M}\|$ is the *distance* in $\mathcal{X}$ from x to the closed set $\mathcal{M}$. The assumption that $\mathcal{M}$ is closed in needed to ensure that the quotient norm is indeed a norm. For instance $M = C([0,1])$ is dense subspace of $L^1([0,1])$ (with Lebesgue measure) and hence for any

$$\inf_{g \in \mathcal{M}} \|f - g\| = 0$$

for all $f \in L^1([0,1])$.

Proof. We will verify a couple of the axioms of a norm for the quotient norm, leaving the remainder of the proof as an exercise. First suppose $x \in \mathcal{X}$ and $\|\pi(x)\| = 0$. It follows that there is a sequence (m_n) form $\mathcal{M}$ such that $(\|x - m_n\|)$ converges to 0; that is, (m_n) converges to x. Since $\mathcal{M}$ is closed, $x \in \mathcal{M}$ and hence $\pi(x) = 0$.

Now let $x, y \in \mathcal{X}$ and $\epsilon > 0$ be given. There exists $m, n \in \mathcal{M}$ such that

$$\|x - m\| \leq \|\pi(x)\| + \epsilon, \quad \|y - n\| \leq \|\pi(y)\| + \epsilon.$$

Hence

$$\|\pi(x) + \pi(y)\| = \|\pi(x+y)\| \leq \|x + y - (m+n)\| \leq \|x - m\| + \|y - n\| \leq \|\pi(x)\| + \|\pi(y)|\| + 2\epsilon,$$

from which it follows that the triangle inequality holds and we have proved the quotient norm is indeed a norm.

To prove $\mathcal{X}/\mathcal{M}$ is complete (with the quotient norm) under the assumption that $\mathcal{X}$ is a Banach space (complete), suppose (y_n) is a sequence from $\mathcal{X}/\mathcal{M}$ and $\sum y_n$ is absolutely convergent. For each n there exists $x_n \in \mathcal{X}$ such that $\|x_n\| \le \|y_n\| + \frac{1}{n^2}$ and $\pi(x_n) = y_n$. It follows that $\sum x_n$ is absolutely convergent. Since $\mathcal{X}$ is a Banach space the sequence of partial sums $s_N = \sum_{n=1}^N x_n$ converges to some $x \in \mathcal{X}$. In partiulcar,

$$\|s_N - x\| \ge \|\pi(s_N - x)\| = \|\pi(s_N) - \pi(x)\| = \|\sum_{n=1}^{N} y_n - \pi(x)\|.$$

Since $(\|s_N - x\|)$ converges to 0, it follows that $\sum y_n$ converges to $\pi(x)$. Hence $\mathcal{X}/\mathcal{M}$ is complete by Proposition 1.7. □

More examples are given in the exercises and further examples will appear after the development of some theory.

1.3. **Linear transformations between normed spaces.**

Definition 1.27. Let $\mathcal{X}, \mathcal{Y}$ be normed vector spaces. A linear transformation $T : \mathcal{X} \to \mathcal{Y}$ is *bounded* if there exists a constant $C \ge 0$ such that $\|Tx\|_{\mathcal{Y}} \le C\|x\|_{\mathcal{X}}$ for all $x \in \mathcal{X}$.

Remark 1.28. Note that in Definition 1.27 it suffices to require that $\|Tx\|_{\mathcal{Y}} \le C\|x\|_{\mathcal{X}}$ just for all $x \ne 0$, or for all x with $\|x\|_{\mathcal{X}} = 1$ (why?). □

The importance of boundedness and the following simple proposition is hard to overstate. Recall, a mapping $f : X \to Y$ between metric spaces is *Lipschitz continuous* if there is a constant $C > 0$ such that $d(f(x), f(y)) \le Cd(x, y)$ for all $x, y \in X$. A simple exercise shows Lipschitz continuity implies (uniform) continuity.

Proposition 1.29. *If $T : \mathcal{X} \to \mathcal{Y}$ is a linear transformation between normed spaces, then the following are equivalent:*

(i) T is bounded.
(ii) T is Lipschitz continuous.
(iii) T is uniformly continuous.
(iv) T is continuous.
(v) T is continuous at 0.

Moreover, in this case,

$$\begin{aligned}\|T\| &:= \sup\{\|Tx\| : \|x\| = 1\}\\ &= \sup\{\frac{\|Tx\|}{\|x\|} : x \ne 0\}\\ &= \inf\{C : \|Tx\| \le C\|x\| \text{ for all } x \in \mathcal{X}\}\end{aligned}$$

and $\|T\|$ is the smallest number (the infimum is attained [1]*) such that*

$$\|Tx\| \le \|T\| \, \|x\| \tag{2}$$

for all $x \in \mathcal{X}$.

Proof. Suppose T is bounded; that is, there exists a $C \ge 0$ such that $\|Tx\| \le C\|x\|$ for all $x \in \mathcal{X}$. Thus, if $x, y \in \mathcal{X}$, then, $\|Tx - Ty\| = \|T(x-y)\| \le C\|x-y\|$ by linearity of T. Hence (i) implies (ii). The implications (ii) implies (iii) implies (iv) implies (v) are evident.

The proof of (v) implies (i) exploits the homogeneity of the norm and the linearity of T and not nearly the full strength of the continuity assumption. By hypothesis, with $\epsilon = 1$ there exists $\delta > 0$ such that if $\|x\| = \|x - 0\| < 2\delta$, then $\|Tx\| = \|Tx - T0)\| < 1$. Given a nonzero vector $x \in \mathcal{X}$, the vector $\delta x/\|x\|$ has norm less than δ, so

$$1 > \left\| T\left(\frac{\delta x}{\|x\|}\right)\right\| = \delta \frac{\|Tx\|}{\|x\|}.$$

Rearranging this we find $\|Tx\| \le (1/\delta)\|x\|$ for all $x \ne 0$.

Assuming T is bounded, it is immediate that $\sup\{\|Tx\| : \|x\| = 1\}$ exists (and is a real number). From homogeneity of the norm, it is also clear that

$$\sup\{\|Tx\| : \|x\| = 1\} = \sup\{\frac{\|Tx\|}{\|x\|} : x \ne 0\}.$$

Likewise assuming T is bounded the set $S = \{C : \|Tx\| \le C\|x\| \text{ for all } x \in \mathcal{X}\} \subseteq [0, \infty)$ is not empty and bounded below (by 0) and hence the infimum exists. From the definition of $\|T\|$ we see that $\|T\| \in S$. Hence the infimum is at most $\|T\|$. On the other hand, if $C' < \inf S$, then there is an $x \in \mathcal{X}$ such that $\|Tx\| > C'\|x\|$ so that $\frac{\|Tx\|}{\|x\|} > C'$. Thus $C' < \|T\|$. ∎

The set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted $B(\mathcal{X}, \mathcal{Y})$. It is a vector space under the operations of pointwise addition and scalar multiplication. The quantity $\|T\|$ is easily seen to be a norm. It is called the *operator norm* of T.

Problem 1.1. Prove the $\|\cdot\|_1$ and $\|\cdot\|_\infty$ norms on c_{00} are not equivalent. Conclude from your proof that the identity map on c_{00} is bounded from the $\|\cdot\|_1$ norm to the $\|\cdot\|_\infty$ norm, but not the other way around.

Problem 1.2. Consider c_0 and c_{00} equipped with the $\|\cdot\|_\infty$ norm. Prove there is no bounded operator $T : c_0 \to c_{00}$ such that $T|_{c_{00}}$ is the identity map. (Thus the conclusion of Proposition 1.31 can fail if $\mathcal{Y}$ is not complete.)

Proposition 1.30. *For normed vector spaces $\mathcal{X}$ and $\mathcal{Y}$, the operator norm makes $B(\mathcal{X}, \mathcal{Y})$ into a normed vector space that is complete if $\mathcal{Y}$ is complete.*

[1]The suprema need not be attained.

Proof. That $B(\mathcal{X},\mathcal{Y})$ is a normed vector space follows readily from the definitions and is left as an exercise.

Suppose now $\mathcal{Y}$ is complete, and let T_n be a cauchy sequence in $B(\mathcal{X},\mathcal{Y})$. Let $E = \mathcal{X}_1$ denote the closed unit ball in $\mathcal{X}$. For $x \in E$,

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \le \|T_n - T_m\|\|x\| \le \|T_n - T_m\|. \tag{3}$$

Hence the sequence $(T_n|_E)$ is a cauchy sequence (so uniformly cauchy) from $C_b(E,\mathcal{Y})$, the space of bounded continuous functions from B to $\mathcal{Y}$. Since $\mathcal{Y}$ is complete, there is an $F \in C_b(E,\mathcal{Y})$ such that $(T_n|_E)$ converges to F in $C_b(E,\mathcal{Y})$ and moreover $\|F(x)\| \le C := \sup\{\|T_n\| : n\} < \infty$. See Subsection 1.2.2. An exercise shows, given $x, y \in B$ and $c \in \mathbb{F}$ if $x + y \in B$ and $c\,x \in B$, then $F(x + y) = F(x) + F(y)$ and $F(cx) = cF(x)$. Hence F extends, by homogeneity, to a linear map $T : \mathcal{X} \to \mathcal{Y}$ such that $\|T\| \le C$ and, by equation (3), (T_n) converges to T in $B(X,Y)$. □

If $T \in B(\mathcal{X},\mathcal{Y})$ and $S \in B(\mathcal{Y},\mathcal{Z})$, then two applications of the inequality (2) give, for $x \in \mathcal{X}$,

$$\|STx\| \le \|S\|\|Tx\| \le \|S\|\|T\|\|x\|$$

and it follows that $ST \in B(\mathcal{X},\mathcal{Z})$ and $\|ST\| \le \|S\|\|T\|$. In the special case that $\mathcal{Y} = \mathcal{X}$ is complete, $B(\mathcal{X}) := B(\mathcal{X},\mathcal{X})$ is an example of a *Banach algebra*.

The following proposition is very useful in constructing bounded operators—at least when the codomain is complete. Namely, it suffices to define the operator (and show that it is bounded) on a dense subspace.

Proposition 1.31 (Extending bounded operators)**.** *Let $\mathcal{X}$, $\mathcal{Y}$ be normed vector spaces with $\mathcal{Y}$ complete, and $\mathcal{E} \subseteq \mathcal{X}$ a dense linear subspace. If $T : \mathcal{E} \to \mathcal{Y}$ is a bounded linear operator, then there exists a unique bounded linear operator $\widetilde{T} : \mathcal{X} \to \mathcal{Y}$ extending T (so $\widetilde{T}|_{\mathcal{E}} = T$). Further $\|\widetilde{T}\| = \|T\|$.*

Sketch of proof. Recall, if X, Y are metric spaces, Y is complete, $D \subseteq X$ is dense and $f : D \to Y$ is uniformly continuous, then f has a unique continuous extension $\tilde{f} : X \to Y$. Moreover, this extension can be defined as follows. Given $x \in X$, choose a sequence (x_n) from D converging to x and let $\tilde{f}(x) = \lim f(x_n)$ (that the sequence $f(x_n)$ is Cauchy follows from uniform continuity; that it converges from the assumption that $\mathcal{Y}$ is complete and finally it is an exercise to show $\tilde{f}(x)$ is well defined independent of the choice of (x_n)). Thus, it only remains to verify that the extension $\tilde{T}$ of T is in fact linear and $\|T\| = \|\tilde{T}\|$. Both are routine exercises. □

Example 1.32. Equip c_0 and c_{00} with the sup norm, $\|\cdot\|_\infty$ and consider the identity map $\iota : c_{00} \to c_{00}$. If T is an extension of ι to the completion c_0 of c_{00} (in the sup norm), then, letting $s_n \in c_{00}$ denote the sequence $s_n(m) = \frac{1}{m}$ for $m \le n$ and $s_n(m) = 0$ for $m > n$, the sequence (s_n) is converge in c_0 to the sequence s with $s(m) = \frac{1}{m}$ for all m. Hence $(T(s_n) = s_n)$ converges to some $t \in c_{00}$. But now there is a K such that $t(k) = 0$

for all $k \geq K$ so that $\|s_n - t\| \geq \frac{1}{K}$ for all $n \geq K$, a contradiction. This example shows completeness of $\mathcal{Y}$ is essential in Proposition 1.31. □

Definition 1.33. A bounded linear transformation $T \in B(\mathcal{X}, \mathcal{Y})$ is said to be *invertible* if it is bijective (being bijective, automatically T^{-1} exists and is a linear transformation) and T^{-1} is bounded from $\mathcal{Y}$ to $\mathcal{X}$. Two normed spaces $\mathcal{X}, \mathcal{Y}$ are said to be *(boundedly) isomorphic* if there exists an invertible linear transformation $T : \mathcal{X} \to \mathcal{Y}$.

Example 1.34. As an example, given equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space $\mathcal{X}$, the identity mapping $\iota : (\mathcal{X}, \|\cdot\|_1) \to (\mathcal{X}, \|\cdot\|_2)$ is (boundedly) invertible and witnesses the fact that these two normed vector spaces are boundedly isomorphic.

Definition 1.35. An operator $T : \mathcal{X} \to \mathcal{Y}$ such that $\|Tx\| = \|x\|$ for all $x \in \mathcal{X}$ is an *isometry*. Note that an isometry is automatically injective and if it is also surjective then it is automatically invertible and T^{-1} is also an isometry. The normed vector spaces are *isometrically isomorphic* if there is an invertible isometry $T : \mathcal{X} \to \mathcal{Y}$.

Example 1.36. If $\mathcal{X}$ is a finite dimensional vector space and $T : \mathcal{X} \to \mathcal{X}$ is an isometry, then T is onto. However, when $\mathcal{X}$ is not finite dimensional, an isometry need not be surjective. As examples, let $\ell^p = \ell^p(\mathbb{N})$ denote the sequence spaces from Subsection 1.2.5. The linear map $S : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ defined by $Sf(n) = 0$ if $n = 0$ and $f(n-1)$ if $n > 0$ (for $f = (f(n))_n \in \ell^p$) is the *shift operator*. It is straightforward to verify that S is an isometry but not onto.

Example 1.37. Following up on the previous example, a linear map $T : \mathcal{X} \to \mathcal{X}$ can be one-one and have dense range without being (boundedly) invertible. Let $e_n \in \ell^2(\mathbb{N})$ denote the function $e_n(m) = 1$ if $n = m$ and 0 otherwise for non-negative integers $0 \leq m, n$. The set of $c_{00} = \{\sum_{n=0}^{N} a_n e_n : N \in \mathbb{N}, c_n \in \mathbb{F}\}$ is dense in $\ell^2(\mathbb{N})$ and the mapping $D : c_{00} \to \ell^2(\mathbb{N})$ defined by

$$D(\sum_{n=0}^{N} a_n e_n) = \sum_{n=0}^{N} \frac{a_n}{n+1} e_n$$

is easily seen to be bounded with $\|D\| = 1$. It is also injective. Hence D extends to an injective bounded operator, still denoted D, from $\ell^2 \to \ell^2$, with $\|D\| = 1$. The range of D contains $\{e_n : n \in \mathbb{N}\}$ and is thus dense in $\ell^2(\mathbb{N})$.

Since $\sum_{n=0}^{\infty} |\frac{1}{n+1}|^2 < \infty$, the vector $f = \sum_{n=1}^{\infty} \frac{1}{n+1} e_n$ is in $\ell^2(\mathbb{N})$. On the other hand, if $g \in \ell^2(\mathbb{N})$ and $Dg = f$, then

$$\frac{1}{n+1} g(n) = (Dg)(n) = f(n) = \frac{1}{n+1}$$

and thus $g(n) = 1$ for all n; however, since $\sum |g(n)|^2 = \infty$, we obtain a contradiction. Hence f is not in the range of D.

1.4. **Examples.**

(a) If $\mathcal{X}$ is a finite-dimensional normed space and $\mathcal{Y}$ is any normed space, then every linear transformation $T : \mathcal{X} \to \mathcal{Y}$ is bounded. See Problem 1.16.

(b) Let $\mathcal{X}$ denote c_{00} equipped with the $\|\cdot\|_1$ norm, and $\mathcal{Y}$ denote c_{00} equipped with the $\|\cdot\|_\infty$ norm. Then the identity map $id_{\mathcal{X},\mathcal{Y}} : \mathcal{X} \to \mathcal{Y}$ is bounded (in fact its norm is equal to 1), but its inverse, the identity map $\iota_{\mathcal{Y},\mathcal{X}} : \mathcal{Y} \to \mathcal{X}$, is unbounded. To verify this claim, For positive integers n, let f_n denote the element of c_{00} defined by

$$f_n(m) = \begin{cases} 1 & \text{if } m \le n \\ 0 & \text{if } m > n. \end{cases}$$

Now $\|\iota_{\mathcal{Y},\mathcal{X}}(f_n)\|_1 = n$, but $\|f_n\|_\infty = 1$.

(c) Consider c_{00} with the $\|\cdot\|_\infty$ norm. Let $a : \mathbb{N} \to \mathbb{F}$ be any function and define a linear transformation $T_a : c_{00} \to c_{00}$ by

$$T_a f(n) = a(n)f(n). \tag{4}$$

The mapping T_a is bounded if and only if $M = \sup_{n\in\mathbb{N}} |a(n)| < \infty$, in which case $\|T_a\| = M$. In this case, T_a extends uniquely to a bounded operator from c_0 to c_0 by Proposition 1.31, and one may check that the formula (4) defines the extension. All of these claims remain true if we use the $\|\cdot\|_1$ norm instead of the $\|\cdot\|_\infty$ norm. In this case, we get a bounded operator from ℓ^1 to itself.

(d) Define $S : \ell^1 \to \ell^1$ as follows: given the sequence $(f(n))_n$ from ℓ^1 let $Sf(1) = 0$ and $Sf(n) = f(n-1)$ for $n > 1$. (Viewing f as a sequence, S shifts the sequence one place to the right and fills in a 0 in the first position). This S is an isometry, but is not surjective. In contrast, if $\mathcal{X}$ is finite-dimensional, then the rank-nullity theorem from linear algebra guarantees that every injective linear map $T : \mathcal{X} \to \mathcal{X}$ is also surjective.

(e) Let $C^\infty([0,1])$ denote the vector space of functions on $[0,1]$ with continuous derivatives of all orders. The differentiation map $D : C^\infty([0,1]) \to C^\infty([0,1])$ defined by $Df = \frac{df}{dx}$ is a linear transformation. Since, for $t \in \mathbb{R}$, we have $De^{tx} = te^{tx}$, it follows that there is *no* norm on $C^\infty([0,1])$ such that $\frac{d}{dx}$ is bounded.

1.5. **Problems.**

Problem 1.3. Prove Proposition 1.5.

Problem 1.4. Prove equivalent norms define the same topology and the same Cauchy sequences.

Problem 1.5. (a) Prove all norms on a finite dimensional vector space $\mathcal{X}$ are equivalent. Suggestion: Fix a basis $e_1, \dots e_n$ for $\mathcal{X}$ and define $\|\sum a_k e_k\|_1 := \sum |a_k|$. It is routine to check that $\|\cdot\|_1$ is a norm on $\mathcal{X}$. Now complete the following outline.

(i) Let $\|\cdot\|$ be the given norm on $\mathcal{X}$. Show there is an M such that $\|x\| \leq M\|x\|_1$. Conclude that the mapping $\iota : (\mathcal{X}, \|\cdot\|_1) \to (\mathcal{X}, \|\cdot\|)$ defined by $\iota(x) = x$ is continuous;

(ii) Show that the *unit sphere* $S = \{x \in \mathcal{X} : \|x\|_1 = 1\}$ in $(\mathcal{X}, \|\cdot\|_1)$ is compact in the $\|\cdot\|_1$ topology;

(iii) Show that the mapping $f : S \to (\mathcal{X}, \|\cdot\|)$ given by $f(x) = \|x\|$ is continuous and hence attains its infimum. Show this infimum is not 0 and finish the proof.

(b) Combine the result of part (a) with the result of Problem 1.4 to conclude that every finite-dimensional normed vector space is complete.

(c) Let $\mathcal{X}$ be a normed vector space and $\mathcal{M} \subseteq \mathcal{X}$ a finite-dimensional subspace. Prove $\mathcal{M}$ is closed in $\mathcal{X}$.

Problem 1.6. Finish the proofs from the examples subsections.

Problem 1.7. A function $f : [0,1] \to \mathbb{F}$ is called *Lipschitz continuous* if there exists a constant C such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all $x, y \in [0,1]$. Define $\|f\|_{Lip}$ to be the best possible constant in this inequality. That is,

$$\|f\|_{Lip} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}$$

Let $\mathrm{Lip}[0,1]$ denote the set of all Lipschitz continuous functions on $[0,1]$. Prove $\|f\| := |f(0)| + \|f\|_{Lip}$ is a norm on $\mathrm{Lip}[0,1]$, and that $\mathrm{Lip}[0,1]$ is complete in this norm.

Problem 1.8. Let $C^1([0,1])$ denote the space of all functions $f : [0,1] \to \mathbb{R}$ such that f is differentiable in $(0,1)$ and f' extends continuously to $[0,1]$. Prove

$$\|f\| := \|f\|_\infty + \|f'\|_\infty$$

is a norm on $C^1([0,1])$ and that C^1 is complete in this norm. Do the same for the norm $\|f\| := |f(0)| + \|f'\|_\infty$. (Is $\|f'\|_\infty$ a norm on C^1?)

Problem 1.9. Let $(X, \mathscr{M})$ be a measurable space. Let $M(X)$ denote the (real) vector space of all signed measures on $(X, \mathscr{M})$. Prove the *total variation norm* $\|\mu\| := |\mu|(X)$ is a norm on $M(X)$, and $M(X)$ is complete in this norm.

Problem 1.10. Prove, if $\mathcal{X}, \mathcal{Y}$ are normed spaces, then the operator norm is a norm on $B(\mathcal{X}, \mathcal{Y})$.

Problem 1.11. Prove c_{00} is dense in c_0 and ℓ^1. (That is, given $f \in c_0$ there is a sequence f_n in c_{00} such that $\|f_n - f\|_\infty \to 0$, and the analogous statement for ℓ^1.) Using these facts, or otherwise, prove that c_{00} is *not* dense in ℓ^∞. (In fact there exists $f \in \ell^\infty$ with $\|f\|_\infty = 1$ such that $\|f - g\|_\infty \geq 1$ for all $g \in c_{00}$.)

Problem 1.12. Prove c_{00} is not complete in the $\|\cdot\|_1$ or $\|\cdot\|_\infty$ norms. (After we have studied the Baire Category theorem, you will be asked to prove that there is *no* norm on c_{00} making it complete.)

Problem 1.13. Consider c_0 and c_{00} equipped with the $\|\cdot\|_\infty$ norm. Prove there is no bounded operator $T : c_0 \to c_{00}$ such that $T|_{c_{00}}$ is the identity map. (Thus the conclusion of Proposition 1.31 can fail if $\mathcal{Y}$ is not complete.)

Problem 1.14. Prove the $\|\cdot\|_1$ and $\|\cdot\|_\infty$ norms on c_{00} are not equivalent. Conclude from your proof that the identity map on c_{00} is bounded from the $\|\cdot\|_1$ norm to the $\|\cdot\|_\infty$ norm, but not the other way around.

Problem 1.15. a) Prove $f \in C_0(\mathbb{R}^n)$ if and only if f is continuous and $\lim_{|x|\to\infty} |f(x)| = 0$. b) Let $C_c(\mathbb{R}^n)$ denote the set of continuous, compactly supported functions on $\mathbb{R}^n$. Prove $C_c(\mathbb{R}^n)$ is dense in $C_0(\mathbb{R}^n)$ (where $C_0(\mathbb{R}^n)$ is equipped with sup norm).

Problem 1.16. Prove, if $\mathcal{X}, \mathcal{Y}$ are normed spaces and $\mathcal{X}$ is finite dimensional, then every linear transformation $T : \mathcal{X} \to \mathcal{Y}$ is bounded. Suggestion: Let d denote the dimension of X and let $\{e_1, \dots, e_d\}$ denote a basis. The function $\|\cdot\|_1$ on $\mathcal{X}$ defined by $\|\sum x_j e_j\|_1 = \sum |x_j|$ is a norm. Apply Problem 1.5.

Problem 1.17. Prove the claims in Example 1.4(c).

Problem 1.18. Let $g : \mathbb{R} \to \mathbb{K}$ be a (Lebesgue) measurable function. The map $Mg : f \to gf$ is a linear transformation on the space of measurable functions. Prove, if $g \notin L^\infty(\mathbb{R})$, then there is an $f \in L^1(\mathbb{R})$ such that $gf \notin L^1(\mathbb{R})$. Conversely, show if $g \in L^\infty(\mathbb{R})$, then M_g is bounded from $L^1(\mathbb{R})$ to itself and $\|M_g\| = \|g\|_\infty$.

Problem 1.19. Prove the claims about direct sums.

Problem 1.20. Let $\mathcal{X}$ be a normed vector space and $\mathcal{M}$ a proper *closed* subspace. Prove for every $\epsilon > 0$, there exists $x \in \mathcal{X}$ such that $\|x\| = 1$ and $\inf_{y\in\mathcal{M}} \|x - y\| > 1 - \epsilon$. (Hint: take any $u \in \mathcal{X} \setminus \mathcal{M}$ and let $a = \inf_{y\in\mathcal{M}} \|u - y\|$. Choose $\delta > 0$ small enough so that $\frac{a}{a+\delta} > 1 - \epsilon$, and then choose $v \in \mathcal{M}$ so that $\|u - v\| < a + \delta$. Finally let $x = \frac{u-v}{\|u-v\|}$.)

Note that the distance to a (closed) subspace need not be attained. Here is an example. Consider the Banach space $C([0,1])$ (with the sup norm of course and either real or complex valued functions) and the closed subspace

$$T = \{f \in C([0,1]) : f(0) = 0 = \int_0^1 f\, dt\}.$$

Using machinery in the next section it will be evident that T is a closed subspace of $C([0,1])$. For now, it can be easily verified directly. Let g denote the function $g(t)=t$. Verify that, for $f\in T$, that

$$\frac{1}{2}=\int g\,dt=\int (g-f)\,dt\le \|g-f\|_\infty.$$

In particular, the distance from g to T is at least $\frac{1}{2}$.

Note that the function $h=x-\frac{1}{2}$, while not in T, satisfies $\|g-h\|_\infty=\frac{1}{2}$.

On the other hand, for any $\epsilon>0$ there is an $f\in T$ so that $\|g-f\|_\infty\le\frac{1}{2}+\epsilon$ (simply modify h appropriately). Thus, the distance from g to T is $\frac{1}{2}$. Now verify, using the inequality above, that h is the only element of $C([0,1])$ such that $\int h\,dt=0$ and $\|g-h\|_\infty=\frac{1}{2}$.

Problem 1.21. Prove, if $\mathcal{X}$ is an infinite-dimensional normed space, then the unit ball $ball(\mathcal{X}):=\{x\in\mathcal{X}:\|x\|\le 1\}$ is not compact in the norm topology. (Hint: use the result of Problem 1.20 to construct inductively a sequence of vectors $x_n\in\mathcal{X}$ such that $\|x_n\|=1$ for all n and $\|x_n-x_m\|\ge\frac{1}{2}$ for all $m<n$.)

Problem 1.22. (The quotient norm) Let $\mathcal{X}$ be a normed space and $\mathcal{M}$ a proper closed subspace.

a) Prove the quotient norm is a norm.
b) Show that the quotient map $x\to x+\mathcal{M}$ has norm 1. (Use Problem 1.20.)
c) Prove, if $\mathcal{X}$ is complete, so is $\mathcal{X}/\mathcal{M}$.

Problem 1.23. A normed vector space $\mathcal{X}$ is called *separable* if it is separable as a metric space (that is, there is a countable subset of $\mathcal{X}$ which is dense in the norm topology). Prove c_0 and ℓ^1 are separable, but ℓ^∞ is not. (Hint: for ℓ^∞, show that there is an uncountable collection of elements $\{f_\alpha\}$ such that $\|f_\alpha-f_\beta\|=1$ for $\alpha\ne\beta$.)

END FALL TERM

2. Linear functionals and the Hahn-Banach theorem

If there is a *fundamental theorem of functional analysis,* it is the Hahn-Banach theorem. The theorem is somewhat abstract-looking at first, but its importance will be clear after studying some of its corollaries.

Definition 2.1. Let $\mathcal{X}$ be a normed vector space over the field $\mathbb{F}$. A *linear functional* on $\mathcal{X}$ is a linear map $L:\mathcal{X}\to\mathbb{F}$. The *dual space* of $\mathcal{X}$, denoted $\mathcal{X}^*$ is the space $B(\mathcal{X},\mathbb{F})$ of bounded linear functionals on $\mathcal{X}$.

Remark 2.2. Since $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ is complete, the vector space of bounded linear functionals is itself a Banach space (complete normed vector space) and is known as the . It is not yet obvious that $\mathcal{X}^*$ need be non-trivial (that is, that there are any bounded linear functionals on $\mathcal{X}$ besides 0). One corollary of the Hahn-Banach theorem is there exist enough bounded linear functionals on $\mathcal{X}$ to separate points.

2.1. **Examples.** This subsection contains some examples of bounded linear functionals and dual spaces.

Example 2.3. For each of the sequence spaces c_0, ℓ^1, ℓ^∞, for each n the map $f \to f(n)$ is a bounded linear functional. That is, $\lambda_n : \mathcal{X} \to \mathbb{F}$ defined by $\lambda_n(f) = f(n)$ for $f : \mathbb{N} \to \mathbb{F}$ in $\mathcal{X}$, where $\mathcal{X}$ is any one of c_0, ℓ^1, ℓ^∞, is continuous since in each case it is immediate that

$$|\lambda_n(f)| = |f(n)| \leq \|f\|_{\mathcal{X}}.$$

Example 2.4. *Given $g \in \ell^1$, if $f \in c_0$, then*

$$\sum_{n=0}^{\infty} |f(n)g(n)| \leq \|f\|_\infty \sum_{n=0}^{\infty} |g(n)| = \|g\|_1 \|f\|_\infty. \tag{5}$$

Thus $\sum_{n=0}^{\infty} f(n)g(n)$ converges and we obtain a functional $L_g : c_0 \to \mathbb{F}$ defined by

$$L_g(f) := \sum_{n=0}^{\infty} f(n)g(n). \tag{6}$$

The inequality of equation (5) says L_g is bounded (continuous) and $\|L_g\| \leq \|g\|_1$. Moreover, it is immediate that $\Phi : \ell^1 \to c_0^*$ defined by $\Phi(g) = L_g$ is bounded and linear and $\|\Phi\| \leq 1$. In fact, Φ is onto so that every bounded linear functional on c_0 is of the form L_g for some $g \in \ell^1$.

Proposition 2.5. *The map $\Phi : \ell^1 \to c_0^*$ defined by $\Phi(g) = L_g$ is an isometric isomorphism from ℓ^1 onto the dual space c_0^*.*

Proof. We have already seen that each $g \in \ell^1$ gives rise to a bounded linear functional $L_g \in c_0^*$ via

$$L_g(f) := \sum_{n=0}^{\infty} g(n)f(n),$$

that $\|L_g\| \leq \|g\|_1$ and the the mapping Φ is bounded and linear. We will prove simultaneously that this map is onto and that $\|L_g\| \geq \|g\|_1$.

Let $L \in c_0^*$. We will first show that there is unique $g \in \ell^1$ so that $L = L_g$. Let $e_n \in c_0$ be the indicator function of n, that is

$$e_n(m) = \delta_{nm}.$$

Define a function $g : \mathbb{N} \to \mathbb{F}$ by

$$g(n) = L(e_n).$$

We claim that $g \in \ell^1$ and $L = L_g$. To see this, fix an integer N and define $h = h_N : \mathbb{N} \to \mathbb{F}$ by

$$h(n) = \begin{cases} \overline{g(n)}/|g(n)| & \text{if } n \le N \text{ and } g(n) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus $h = \sum_{n=0}^{N} h(n)e_n$. Further, by $h \in c_{00} \subseteq c_0$ and $\|h\|_\infty \le 1$. Now

$$\sum_{n=0}^{N} |g(n)| = \sum_{n=0}^{N} h(n)g(n) = L(h) = |L(h)| \le \|L\|\|h\| \le \|L\|.$$

It follows that $g \in \ell^1$ and $\|g\|_1 \le \|L\|$. By construction $L = L_g$ when restricted to c_{00}, so by the uniqueness of extensions of bounded operators, Proposition 1.31, $L = L_g$. Thus the map $g \to L_g$ is onto and

$$\|g\|_1 \le \|L\| = \|L_g\| \le \|g\|_1.$$

□

Example 2.6. Given $g \in \ell^\infty$, if $f \in \ell^1$, then equation (5) shows $|L_g(f)| \le \|g\|_\infty \|f\|_1$, where L_g is defined as in equation (6). Thus $\|L_g\| \le \|g\|_\infty$ and we obtain a bounded linear map $\Psi : \ell^\infty \to (\ell^1)^*$

Proposition 2.7. *The map* Ψ *is an isometric isomorphism from* $(\ell^1)^*$ *onto* ℓ^∞.

Proof. The proof follows the same lines as the proof of the previous proposition; the details are left as an exercise. □

Remark 2.8. The same mapping $g \to L_g$ also shows that every $g \in \ell^1$ gives a bounded linear functional on ℓ^∞, but it turns out these do not exhaust $(\ell^\infty)^*$ (see Problem **??**).

Regarding ℓ^1 and ℓ^∞ as L^1 and L^∞ for counting measure on $\mathbb{N}$, it is not surprising that, given a measure space $(X, \mathscr{M}, \mu)$, a function $g \in L^\infty(\mu)$ (see Subsection 1.2.6 for the definition of $L^\infty(\mu)$) defines a linear functional $L_g : L^1(\mu) \to \mathbb{F}$ by

$$L_g(f) := \int_X f g \, dm$$

for $f \in L^1(\mu)$ is a bounded linear functional of norm at most M. We will prove in Section **??** that the norm of L_g is in fact $\|g\|_\infty$, and every bounded linear functional on $L^1(m)$ is of this type (at least when m is σ-finite). □

Example 2.9. A regular Borel measure μ on a locally compact set X such that $\mu(K) < \infty$ for compact subsets of X determines a linear functional $\lambda : C_c(X) \to \mathbb{F}$ by

$$\lambda(f) = \lambda_\mu(f) = \int_X f \, d\mu.$$

An $f \in C_c(X)$ is a *positive function* (really non-negative), written $f \ge 0$, if $f(x) \ge 0$ for all $x \in X$. The linear functional λ_μ is a *positive linear functional* in the sense that if $f \in C_c(X)$ is positive, then $\lambda_\mu(f) \ge 0$.

As a second example, let $X = [0,1]$ and note that the mapping $I : C([0,1]) \to \mathbb{C}$ defined by

$$I(f) = \int_0^1 f\,dx,$$

where the integral is in the Riemann sense, is a positive linear functional on $C([0,1])$.

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Theorem 2.10 (Riesz-Markov Representation Theorem: positive version)**.** *Let $X = (X,\tau)$ be a locally compact Hausdorff space. If $\lambda : C_c(X) \to \mathbb{C}$ is a positive linear functional, then there exists a unique Borel measure μ on the Borel σ-algebra $\mathscr{B}_X$, such that*

$$\lambda(f) = \int f\,d\mu$$

for $f \in C_c(X)$. Moreover, μ is regular *in the sense that*

(i) if $K \subseteq X$ is compact, then $\mu(K) < \infty$;
(ii) if $E \in \mathscr{B}_X$, then $\mu(E) = \inf\{\mu(U) : E \subseteq U,\, U \text{ open}\}$; and
(iii) if $E \in \mathscr{B}_X$ and $\mu(E) < \infty$, then $\mu(E) = \sup\{\mu(K) : K \subseteq E,\, K \text{ compact}\}$.

Remark 2.11. In general elements of $C_c(X)^*$ correspond to *signed measures* that will appear later in these notes. □

We close this subsection with the following result that should be compared with item (a) from Subsection 1.4.

Proposition 2.12. *If V is an infinite dimensional normed vector space, then there exists a linear map $f : V \to \mathbb{F}$ that is not continuous.*

For a Banach space $\mathcal{X}$, there are notions of a basis that reference the norm. For instance, a *Schauder basis* for $\mathcal{X}$ is a sequence $(e_n)_{n=1}^\infty$ such that for each $x \in \mathcal{X}$ there exists a unique choice of scalars $x_n \in \mathbb{F}$ such that the series $\sum_{n=1}^\infty x_n e_n$ converges to x. Forgetting the norm structure, a *Hamel basis* $B \subseteq \mathcal{X}$ for $\mathcal{X}$ is a basis in the sense of linear algebra. Explicitly, letting $F_{00}(B)$ denote the functions $a : B \to \mathbb{F}$ such that $a_b = a(b)$ is zero for all but finitely many $b \in B$, the set B is a Hamel basis for $\mathcal{X}$ if for each $v \in \mathcal{X}$ there exist is a unique function $a \in F_{00}(B)$ such that

$$v = \sum_{b\in B} a_b\, b = \sum_{b\in B}^{\text{finite}} a_b\, b.$$

In this case any choice of $c : B \to \mathbb{F}$ determines uniquely a linear functional $\lambda : \mathcal{X} \to \mathbb{F}$ via the rule

$$\lambda(v) = \sum_{b\in B} c_b\, a_b,$$

where $c_b = c(b)$. Often this process is described informally as: *let $\lambda(b) = c(b)$ and extend by linearity.* Finally, an argument using Zorn's Lemma, which we will soon encounter in the proof of the Hahn-Banach Theorem, shows that every vector space has a basis. While it is true that every basis for a vector space V has the same cardinality, all that we need to make sense of the statement V *is an infinite dimensional vector space* is the fact that V has a basis that is infinite, then all bases for V are infinite, which is an immediate consequence of the fact that all bases for a finite dimensional vector space have the same cardinality. Thus, we can take the statement $\mathcal{X}$ *is infinite dimensional* to mean that $\mathcal{X}$ has a Hamel basis B that contains a countable set B_0.

Proof of Proposition 2.12. Let B denote a Hamel basis for V. By assumption, B has a countable subset B_0. Write $B_0 = \{b_1, b_2, \dots\}$ (so choose a bijection $\psi : \mathbb{N} \to B_0$) and assume, without loss of generality that $\|b_j\| = 1$. Let $\lambda : V \to \mathbb{F}$ denote the linear functional determined by $\lambda(b_j) = j$ for $b_j \in B_0$ and $\lambda(b) = 0$ for $b \in B \setminus B_0$ and observe that λ is not bounded. □

2.2. **The Hahn-Banach Extension Theorem.** To state and prove the Hahn-Banach Extension Theorem, we first work in the setting $\mathbb{F} = \mathbb{R}$, then extend the results to the complex case.

Definition 2.13. Let $\mathcal{X}$ be a real vector space. A *Minkowski functional* is a function $p : \mathcal{X} \to \mathbb{R}$ such that $p(x + y) \le p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for all $x, y \in \mathcal{X}$ and nonnegative $\lambda \in \mathbb{R}$.

For examples, if $L : \mathcal{X} \to \mathbb{R}$ is any linear functional, then the function $p : \mathcal{X} \to \mathbb{R}$ defined by $p(x) := |L(x)|$ is a Minkowski functional; and if $\|\cdot\|$ is a seminorm on $\mathcal{X}$, then $p : \mathcal{X} \to \mathbb{R}$ defined by $p(x) = \|x\|$ is a Minkowski functional.

Theorem 2.14 (The Hahn-Banach Extension[2] Theorem, real version)**.** *Let $\mathcal{X}$ denote a real vector space, p a Minkowski functional on $\mathcal{X}$, and $\mathcal{M}$ a subspace of $\mathcal{X}$. If L is a linear functional on $\mathcal{M}$ such that $L(x) \le p(x)$ for all $x \in \mathcal{M}$, then there exists a linear functional L' on $\mathcal{X}$ such that*

(i) $L'|_{\mathcal{M}} = L$ (L' extends L)
(ii) $L'(x) \le p(x)$ for all $x \in \mathcal{X}$ (L' is dominated by p).

Remark 2.15. In the statement of Theorem 2.14, $\mathcal{X}$ is a vector space, not a normed vector space and correspondingly $\mathcal{M}$ is a subspace in the sense of linear algebra (sometimes referred to as a linear manifold). □

The proof will invoke *Zorn's Lemma*, a result that is equivalent to the axiom of choice (as well as the well-ordering principle and the Hausdorff maximality principle). A *partial order* $\preceq$ on a set S is a relation that is reflexive, symmetric and transitive; that is, for all $x, y, z \in S$

[2]There is also the Hahn-Banach Separation Theorem. Both theorems are often simple called *the* (sic) Hahn-Banach Theorem.

(i) $x \preceq x$,
(ii) if $x \preceq y$ and $y \preceq x$, then $x = y$, and
(iii) if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

We call $\mathcal{S}$, or more precisely $(S, \preceq)$, a *partially ordered set* or *poset*. A subset T of S is *totally ordered*, if for each $x, y \in T$ either $x \preceq y$ or $y \preceq x$. A totally ordered subset T is often called a *chain*. An upper bound z for a chain T is an element $z \in S$ such that $t \preceq z$ for all $t \in T$. A maximal element for S is a $w \in S$ that has no successor; that is there does not exist an $s \in S$ such that $s \neq w$ and $w \preceq s$. An *upper bound* for a subset A of S is an element $s \in S$ such that $a \preceq s$ for all $a \in A$.

Theorem 2.16 (Zorn's Lemma)**.** *Suppose S is a partially ordered set. If every chain in S has an upper bound, then S has a maximal element.*

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The following Lemma is at the heart of the proof of Theorem 2.14.

Lemma 2.17. *With the hypotheses of Theorem 2.14, if $0 \neq x \in \mathcal{X} \setminus \mathcal{M}$, then the conclusion of Theorem 2.14 holds with the subspace $\mathcal{M} + \mathbb{R}x$ in place of $\mathcal{X}$.*

Proof. For any $m_1, m_2 \in \mathcal{M}$, by hypothesis,

$$L(m_1) + L(m_2) = L(m_1 + m_2) \leq p(m_1 + m_2) \leq p(m_1 - x) + p(m_2 + x).$$

Rearranging gives, for $m_1, m_2 \in \mathcal{M}$,

$$L(m_1) - p(m_1 - x) \leq p(m_2 + x) - L(m_2)$$

and thus

$$\sup_{m \in \mathcal{M}} \{L(m) - p(m - x)\} \leq \inf_{m \in \mathcal{M}} \{p(m + x) - L(m)\}.$$

Now choose any real number λ satisfying

$$\sup_{m \in \mathcal{M}} \{L(m) - p(m - x)\} \leq \lambda \leq \inf_{m \in \mathcal{M}} \{p(m + x) - L(m)\}.$$

In particular, for $m \in \mathcal{M}$,

$$\begin{aligned} L(m) - \lambda &\leq p(m - x) \\ L(m) + \lambda &\leq p(m + x). \end{aligned} \tag{7}$$

Let $\mathcal{N} = \mathcal{M} + \mathbb{R}x$ and define $L' : \mathcal{N} \to \mathbb{R}$ by $L'(m + tx) = L(m) + t\lambda$ for $m \in \mathcal{M}$ and $t \in \mathbb{R}$. Thus L' is linear and agrees with L on $\mathcal{M}$ by definition. Moreover, by construction and equation 7,

$$\begin{aligned} L(m - x) &\leq p(m - x) \\ L(m + x) &\leq p(m + x). \end{aligned} \tag{8}$$

We now check that $L'(y) \le p(y)$ for all $y \in \mathcal{M} + \mathbb{R}x$. Accordingly, suppose $y \in \mathcal{N}$ so that there exists $m \in \mathcal{M}$ and $t \in \mathbb{R}$ such that $y = m + tx$. If $t = 0$ there is nothing to prove. If $t > 0$, then, in view of the second inequality of equation (8),

$$L'(y) = L'(m+tx) = t\left(L(\frac{m}{t}) + \lambda\right) \le t\, p(\frac{m}{t} + x) = p(m+tx) = p(y)$$

and a similar estimate, using the first inequality of equation (8), shows that

$$L'(m+tx) \le p(m+tx)$$

for $t < 0$. We have thus successfully extended L to a linear map $L' : \mathcal{N} \to \mathbb{R}$ satisfying $L'(n) \le p(n)$ for all $n \in \mathbb{N}$ and the proof is complete. □

We make one further observation before turning to the proof of the Hahn-Banach Theorem. If T is a totally ordered set and $(\mathcal{N}_\alpha)_{\alpha\in T}$ are subspaces of a vector space $\mathcal{X}$ that are nested increasing in the sense that $\mathcal{N}_\alpha \subseteq \mathcal{N}_\beta$ for $\alpha \preceq \beta$, then $\mathcal{N} = \cup_{\alpha\in T}\mathcal{N}_\alpha$ is again a subspace of $\mathcal{X}$. By contrast, if $\mathcal{X}$ is a normed vector space and $\mathcal{N}_\alpha$ are (closed) subspaces of $\mathcal{X}$, then $\mathcal{N}$ will not necessarily be a (closed) subspace of $\mathcal{X}$.

Proof of Theorem 2.14. Let $\mathcal{L}$ denote the set of pairs $(L', \mathcal{N})$ where $\mathcal{N}$ is a subspace of $\mathcal{X}$ containing $\mathcal{M}$, and L' is an extension of L to $\mathcal{N}$ obeying $L'(y) \le p(y)$ on $\mathcal{N}$. Declare $(L_1', \mathcal{N}_1) \preceq (L_2', \mathcal{N}_2)$ if $\mathcal{N}_1 \subseteq \mathcal{N}_2$ and $L_2'|_{\mathcal{N}_1} = L_1'$. This relation $\preceq$ is a partial order on $\mathcal{L}$; that is $(\mathcal{L}, \preceq)$ is a partially ordered set. Further, Lemma 2.17 says if $(L', \mathcal{N})$ is maximal element, then $\mathcal{N} = \mathcal{X}$.

An exercise shows, given any increasing chain $(L_\alpha', \mathcal{N}_\alpha)$ in $\mathcal{L}$ has as an upper bound $(L', \mathcal{N})$ in $\mathcal{L}$, where $\mathcal{N} := \bigcup_\alpha \mathcal{N}_\alpha$ and $L'(n_\alpha) := L_\alpha'(n_\alpha)$ for $n_\alpha \in \mathcal{N}_\alpha$. By Zorn's Lemma the collection $\mathcal{L}$ has a maximal element $(L', \mathcal{N})$ with respect to the order $\preceq$ and the proof is complete. □

The use of Zorn's Lemma in the proof of Theorem 2.14 is a typical - one knows how to carry out a construction one step a time, but there is no clear way to do it all at once. As an exercise, use Zorn's Lemma to prove that if V Is a vectors space and $S \subseteq V$ is a linearly independent set, then there is a basis B for V such that $B \supseteq S$.

In the special case that p is a seminorm, since $L(-x) = -L(x)$ and $p(-x) = p(x)$ the inequality $L \le p$ is equivalent to $|L| \le p$.

Corollary 2.18. *Suppose $\mathcal{X}$ is a real normed vector space, $\mathcal{M}$ is a subspace of $\mathcal{X}$, and L is a bounded linear functional on $\mathcal{M}$. If $C \ge 0$ and $|L(x)| \le C\|x\|$ for all $x \in \mathcal{M}$, then there exists a bounded linear functional L' on $\mathcal{X}$ extending L such that $\|L'\| \le C$.*

Proof. Apply the Hahn-Banach theorem with the Minkowski functional $p(x) = C\|x\|$. □

Before obtaining further corollaries, we extend Thoerem 2.14 to complex normed spaces. First, if $\mathcal{X}$ is a vector space over $\mathbb{C}$, then trivially it is also a vector space over $\mathbb{R}$, and there is a simple relationship between the $\mathbb{R}$- and $\mathbb{C}$-linear functionals.

Lemma 2.19. *Let $\mathcal{X}$ be a vector space over $\mathbb{C}$. If $L : \mathcal{X} \to \mathbb{C}$ is a $\mathbb{C}$-linear functional, then $u(x) = \operatorname{real} L(x)$ defines an $\mathbb{R}$-linear functional on $\mathcal{X}$ and $L(x) = u(x) - iu(ix)$. Conversely, if $u : \mathcal{X} \to \mathbb{R}$ is $\mathbb{R}$-linear then $L(x) := u(x) - iu(ix)$ is $\mathbb{C}$-linear. If, in addition, $p : \mathcal{X} \to \mathbb{R}$ is a seminorm, then $|u(x)| \leq p(x)$ for all $x \in \mathcal{X}$ if and only if $|L(x)| \leq p(x)$ for all $x \in \mathcal{X}$.*

Proof. Problem **??**.

To prove the last statement, it is immediate that $|u(x)| \leq |L(x)|$ for all $x \in \mathcal{X}$.

Conversely, given x there is a unimodular α such that $\alpha L(x) = |L(x)|$. Hence,

$$|L(x)| = L(\alpha x) = |u(\alpha x)| \leq p(\alpha x) = |\alpha|\, p(x) = p(x). \qquad \square$$

Remark 2.20. Note that in passing from the real to the complex case, we must give up the generality of a Minkowski functional and instead content ourselves with seminorms p.

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