Derivational points of Banach bimodules

Z. Patrick Pan
Saginaw Valley State University

March 17-19, 2011
SouthEast Analysts' Meeting
University of Florida

Derivable maps

- A linear map δ from an algebra $\mathcal{A}$ to an $\mathcal{A}$-bimodule $\mathcal{M}$ is called a derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$.

Derivable maps

- A linear map δ from an algebra $\mathcal{A}$ to an $\mathcal{A}$-bimodule $\mathcal{M}$ is called a derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$.

- A linear map δ from an algebra $\mathcal{A}$ to an $\mathcal{A}$-bimodule $\mathcal{M}$ is called derivable at a fixed $c \in \mathcal{A}$ if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$ with $ab = c$.

Derivable maps

- A linear map δ from an algebra $\mathcal{A}$ to an $\mathcal{A}$-bimodule $\mathcal{M}$ is called a derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$.
- A linear map δ from an algebra $\mathcal{A}$ to an $\mathcal{A}$-bimodule $\mathcal{M}$ is called derivable at a fixed $c \in \mathcal{A}$ if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$ with $ab = c$.
- Clearly, a derivation is derivable at every $c \in \mathcal{A}$.

Derivable maps

- A linear map δ from an algebra $\mathcal{A}$ to an $\mathcal{A}$-bimodule $\mathcal{M}$ is called a derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$.
- A linear map δ from an algebra $\mathcal{A}$ to an $\mathcal{A}$-bimodule $\mathcal{M}$ is called derivable at a fixed $c \in \mathcal{A}$ if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$ with $ab = c$.
- Clearly, a derivation is derivable at every $c \in \mathcal{A}$.
- Question: If δ is derivable at a fixed $c \in \mathcal{A}$, what can one say about δ? (Sub-question: Why should one care?)

Why derivable maps?

The study of maps that preserve certain aspects of derivations or homomorphisms

- helps better understand the structure of these maps (derivations or homomorphisms).

Why derivable maps?

The study of maps that preserve certain aspects of derivations or homomorphisms

- helps better understand the structure of these maps (derivations or homomorphisms).
- helps better understand the structure of the underlying algebra.

Why derivable maps?

The study of maps that preserve certain aspects of derivations or homomorphisms

- helps better understand the structure of these maps (derivations or homomorphisms).
- helps better understand the structure of the underlying algebra.
- provides simpler mechanisms for construction of these maps.

Derivable maps at $c = 0$

Question:

If $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$ with $ab = 0$, what can one say about δ?

Derivable maps at $c = 0$

Question:

If $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$ with $ab = 0$, what can one say about δ?

An example:

- Let $\mathcal{A}' = \{m \in \mathcal{M} : am = ma, \ \forall a \in \mathcal{A}\}$.

Derivable maps at $c = 0$

Question:

If $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$ with $ab = 0$, what can one say about δ?

An example:

- Let $\mathcal{A}' = \{m \in \mathcal{M} : am = ma, \ \forall a \in \mathcal{A}\}$.
- For any $m \in \mathcal{A}'$, let $L_m : \mathcal{A} \mapsto \mathcal{M}$ be $L_m(a) = ma, \ \forall \ a \in \mathcal{A}$ then L_m is derivable at $c = 0$.

Derivable maps at $c = 0$

Question:

If $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$ with $ab = 0$, what can one say about δ?

An example:

- Let $\mathcal{A}' = \{m \in \mathcal{M} : am = ma, \ \forall a \in \mathcal{A}\}$.
- For any $m \in \mathcal{A}'$, let $L_m : \mathcal{A} \mapsto \mathcal{M}$ be $L_m(a) = ma, \ \forall \ a \in \mathcal{A}$ then L_m is derivable at $c = 0$.
- Check: $L_m(ab) = 0 = L_m(a)b + aL_m(b)$

Derivable maps at $c = 0$

Question:

If $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$ with $ab = 0$, what can one say about δ?

An example:

- Let $\mathcal{A}' = \{m \in \mathcal{M} : am = ma, \ \forall a \in \mathcal{A}\}$.
- For any $m \in \mathcal{A}'$, let $L_m : \mathcal{A} \mapsto \mathcal{M}$ be $L_m(a) = ma, \ \forall \ a \in \mathcal{A}$ then L_m is derivable at $c = 0$.
- Check: $L_m(ab) = 0 = L_m(a)b + aL_m(b)$

Derivable maps at $c = 0$

Question:

If $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$ with $ab = 0$, what can one say about δ?

An example:

- Let $\mathcal{A}' = \{m \in \mathcal{M} : am = ma, \ \forall a \in \mathcal{A}\}$.
- For any $m \in \mathcal{A}'$, let $L_m : \mathcal{A} \mapsto \mathcal{M}$ be $L_m(a) = ma, \ \forall \ a \in \mathcal{A}$ then L_m is derivable at $c = 0$.
- Check: $L_m(ab) = 0 = L_m(a)b + aL_m(b)$

Various results in the literature

Under certain conditions on δ, $\mathcal{A}$, and $\mathcal{M}$, if δ is derivable at 0 then there exists a derivation d from $\mathcal{A}$ to $\mathcal{M}$ such that $\delta = d + L_m$, where $m = \delta(1) \in \mathcal{A}'$.

Derivable maps at $c = 0$

Theorem 1 (Alaminos, Bresar, Extremera, and Villena, 2007)

Let δ be a bounded linear map from a unital C^*-algebra $\mathcal{A}$ to a unital Banach $\mathcal{A}$-bimodule $\mathcal{M}$. If δ is derivable at 0 then there exists a derivation d from $\mathcal{A}$ to $\mathcal{M}$ such that $\delta = d + L_m$, where $m = \delta(1) \in \mathcal{A}'$.

Derivable maps at $c = 0$

Theorem 1 (Alaminos, Bresar, Extremera, and Villena, 2007)

Let δ be a bounded linear map from a unital C^*-algebra $\mathcal{A}$ to a unital Banach $\mathcal{A}$-bimodule $\mathcal{M}$. If δ is derivable at 0 then there exists a derivation d from $\mathcal{A}$ to $\mathcal{M}$ such that $\delta = d + L_m$, where $m = \delta(1) \in \mathcal{A}'$.

Theorem 2 (J. Li and Z. Pan, 2010)

Let δ be a bounded linear map from a CSL-algebra $\mathcal{A}$ in a von Neumann algebra on a Hilbert space H. If δ is derivable at 0 then there exists a derivation d from $\mathcal{A}$ to $\mathcal{B}(H)$ such that $\delta = d + L_m$, where $m = \delta(1) \in \mathcal{A}'$.

An embedding

- Let $\delta : \mathcal{A} \mapsto \mathcal{M}$.

An embedding

- Let $\delta: \mathcal{A} \mapsto \mathcal{M}$.
- Define $\mathcal{B} = \{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a, b \in \mathcal{A}, \ m \in \mathcal{M} \}$.

An embedding

- Let $\delta : \mathcal{A} \mapsto \mathcal{M}$.
- Define $\mathcal{B} = \{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a, b \in \mathcal{A}, \ m \in \mathcal{M} \}$.
- Check: $\mathcal{B}$ is an algebra.

An interplay between derivations and homomorphisms

An embedding

- Let $\delta : \mathcal{A} \mapsto \mathcal{M}$.
- Define $\mathcal{B} = \{\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a, b \in \mathcal{A},\ m \in \mathcal{M}\}$.
- Check: $\mathcal{B}$ is an algebra.
- Define $\phi : \mathcal{A} \mapsto \mathcal{B}$ by $\phi(a) = \begin{bmatrix} a & \delta(a) \\ 0 & a \end{bmatrix}$, $\forall\ a \in \mathcal{A}$.

An interplay between derivations and homomorphisms

An embedding

- Let $\delta : \mathcal{A} \mapsto \mathcal{M}$.
- Define $\mathcal{B} = \{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a, b \in \mathcal{A}, \; m \in \mathcal{M} \}$.
- Check: $\mathcal{B}$ is an algebra.
- Define $\phi : \mathcal{A} \mapsto \mathcal{B}$ by $\phi(a) = \begin{bmatrix} a & \delta(a) \\ 0 & a \end{bmatrix}$, $\forall \; a \in \mathcal{A}$.
- $\phi(ab) = \begin{bmatrix} ab & \delta(ab) \\ 0 & ab \end{bmatrix}$.

An embedding

- Let $\delta : \mathcal{A} \mapsto \mathcal{M}$.
- Define $\mathcal{B} = \{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a, b \in \mathcal{A},\ m \in \mathcal{M} \}$.
- Check: $\mathcal{B}$ is an algebra.
- Define $\phi : \mathcal{A} \mapsto \mathcal{B}$ by $\phi(a) = \begin{bmatrix} a & \delta(a) \\ 0 & a \end{bmatrix}$, $\forall\ a \in \mathcal{A}$.
- $\phi(ab) = \begin{bmatrix} ab & \delta(ab) \\ 0 & ab \end{bmatrix}$.
- Check: $\phi(a)\phi(b) = \begin{bmatrix} ab & a\delta(b) + \delta(a)b \\ 0 & ab \end{bmatrix}$.

An interplay between derivations and homomorphisms

An embedding

- Let $\delta : \mathcal{A} \mapsto \mathcal{M}$.
- Define $\mathcal{B} = \{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a, b \in \mathcal{A},\ m \in \mathcal{M} \}$.
- Check: $\mathcal{B}$ is an algebra.
- Define $\phi : \mathcal{A} \mapsto \mathcal{B}$ by $\phi(a) = \begin{bmatrix} a & \delta(a) \\ 0 & a \end{bmatrix}$, $\forall\ a \in \mathcal{A}$.
- $\phi(ab) = \begin{bmatrix} ab & \delta(ab) \\ 0 & ab \end{bmatrix}$.
- Check: $\phi(a)\phi(b) = \begin{bmatrix} ab & a\delta(b) + \delta(a)b \\ 0 & ab \end{bmatrix}$.
- δ is a derivation iff ϕ is a homomorphism.

Derivable at $c = 0$ and zero-product preserving maps

Definition:

A linear map ϕ from an algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ is called zero-product preserving if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{A}$ with $ab = 0$.

Derivable at $c = 0$ and zero-product preserving maps

Definition:

A linear map ϕ from an algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ is called zero-product preserving if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{A}$ with $ab = 0$.

Recall:

- $\delta : \mathcal{A} \mapsto \mathcal{M}$ and $\mathcal{B} = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a, b \in \mathcal{A},\ m \in \mathcal{M} \right\}$.

Derivable at $c = 0$ and zero-product preserving maps

Definition:

A linear map ϕ from an algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ is called zero-product preserving if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{A}$ with $ab = 0$.

Recall:

- $\delta : \mathcal{A} \mapsto \mathcal{M}$ and $\mathcal{B} = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a, b \in \mathcal{A},\ m \in \mathcal{M} \right\}$.
- $\phi : \mathcal{A} \mapsto \mathcal{B}$, defined by $\phi(a) = \begin{bmatrix} a & \delta(a) \\ 0 & a \end{bmatrix}$.

Derivable at $c = 0$ and zero-product preserving maps

Definition:

A linear map ϕ from an algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ is called zero-product preserving if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{A}$ with $ab = 0$.

Recall:

- $\delta : \mathcal{A} \mapsto \mathcal{M}$ and $\mathcal{B} = \{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a, b \in \mathcal{A},\ m \in \mathcal{M}\}$.
- $\phi : \mathcal{A} \mapsto \mathcal{B}$, defined by $\phi(a) = \begin{bmatrix} a & \delta(a) \\ 0 & a \end{bmatrix}$.
- δ is derivable at $c = 0$ iff ϕ is zero-product preserving.

Zero-product preserving maps

Various results in the literature

Under certain conditions, if ϕ is a zero-product preserving map from a unital algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ then there exists a homomorphism Φ from $\mathcal{A}$ to $\mathcal{B}$ such that

$$\phi(a) = \phi(1)\Phi(a) \ \forall \ a \in \mathcal{A}.$$

Derivable maps at $c = 1$ and Jordan derivations

- A linear map δ from an algebra $\mathcal{A}$ to an $\mathcal{A}$-bimodule $\mathcal{M}$ is called a Jordan derivation if $\delta(a^2) = \delta(a)a + a\delta(a)$ for all $a \in \mathcal{A}$.

Derivable maps at $c = 1$ and Jordan derivations

- A linear map δ from an algebra $\mathcal{A}$ to an $\mathcal{A}$-bimodule $\mathcal{M}$ is called a Jordan derivation if $\delta(a^2) = \delta(a)a + a\delta(a)$ for all $a \in \mathcal{A}$.

- If δ is a Jordan derivation and $\mathcal{M}$ is a unital $\mathcal{A}$-bimodule then $\delta(1) = 0$.

Derivable maps at $c = 1$ and Jordan derivations

- A linear map δ from an algebra $\mathcal{A}$ to an $\mathcal{A}$-bimodule $\mathcal{M}$ is called a Jordan derivation if $\delta(a^2) = \delta(a)a + a\delta(a)$ for all $a \in \mathcal{A}$.
- If δ is a Jordan derivation and $\mathcal{M}$ is a unital $\mathcal{A}$-bimodule then $\delta(1) = 0$.
- If δ is a Jordan derivation then $\delta(bab) = \delta(b)ab + b\delta(a)b + ba\delta(b)$, for all $a, b \in \mathcal{A}$.

Derivable maps at $c = 1$ and Jordan derivations

- A linear map δ from an algebra $\mathcal{A}$ to an $\mathcal{A}$-bimodule $\mathcal{M}$ is called a Jordan derivation if $\delta(a^2) = \delta(a)a + a\delta(a)$ for all $a \in \mathcal{A}$.
- If δ is a Jordan derivation and $\mathcal{M}$ is a unital $\mathcal{A}$-bimodule then $\delta(1) = 0$.
- If δ is a Jordan derivation then $\delta(bab) = \delta(b)ab + b\delta(a)b + ba\delta(b)$, for all $a, b \in \mathcal{A}$.
- There are many papers in the literature studying when Jordan derivations are derivations (dating back to 1957, I.N. Herstein and I. Kaplansky)

Derivable maps at $c = 1$ and Jordan derivations

Fact

Every Jordan derivation from $\mathcal{A}$ to $\mathcal{M}$ is derivable at the unit 1.

Derivable maps at $c = 1$ and Jordan derivations

Fact

Every Jordan derivation from $\mathcal{A}$ to $\mathcal{M}$ is derivable at the unit 1.

Proof:

- Suppose δ is a Jordan derivation from $\mathcal{A}$ to $\mathcal{M}$.

Derivable maps at $c = 1$ and Jordan derivations

Fact

Every Jordan derivation from $\mathcal{A}$ to $\mathcal{M}$ is derivable at the unit 1.

Proof:

- Suppose δ is a Jordan derivation from $\mathcal{A}$ to $\mathcal{M}$.
- For any $a, b \in \mathcal{A}$ such that $ab = 1$, we need to show

$$\delta(a)b + a\delta(b) = \delta(1) = 0.$$

Derivable maps at $c = 1$ and Jordan derivations

Fact

Every Jordan derivation from $\mathcal{A}$ to $\mathcal{M}$ is derivable at the unit 1.

Proof:

- Suppose δ is a Jordan derivation from $\mathcal{A}$ to $\mathcal{M}$.
- For any $a, b \in \mathcal{A}$ such that $ab = 1$, we need to show

$$\delta(a)b + a\delta(b) = \delta(1) = 0.$$

- Note $\delta(b) = \delta(bab) = \delta(b)ab + b\delta(a)b + ba\delta(b)$.

Derivable maps at $c = 1$ and Jordan derivations

Fact

Every Jordan derivation from $\mathcal{A}$ to $\mathcal{M}$ is derivable at the unit 1.

Proof:

- Suppose δ is a Jordan derivation from $\mathcal{A}$ to $\mathcal{M}$.
- For any $a, b \in \mathcal{A}$ such that $ab = 1$, we need to show

$$\delta(a)b + a\delta(b) = \delta(1) = 0.$$

- Note $\delta(b) = \delta(bab) = \delta(b)ab + b\delta(a)b + ba\delta(b)$.
- It follows $b(\delta(a)b + a\delta(b)) = 0$. □

Derivable maps at a general element c

Definitions:

- $\mathcal{A}$ = Norm-closed unital subalgebra of $B(X)$

Derivable maps at a general element c

Definitions:

- $\mathcal{A}$ = Norm-closed unital subalgebra of $B(X)$
- For any $x \in X$ and $f^* \in X^*$, we use $x \otimes f^*$ to denote the rank-one operator satisfying $x \otimes f^*(u) = f^*(u)x, \ \forall \ u \in X$.

Derivable maps at a general element c

Definitions:

- $\mathcal{A}$ = Norm-closed unital subalgebra of $B(X)$
- For any $x \in X$ and $f^* \in X^*$, we use $x \otimes f^*$ to denote the rank-one operator satisfying $x \otimes f^*(u) = f^*(u)x, \ \forall \ u \in X$.

Derivable maps at a general element c

Definitions:

- $\mathcal{A}$ = Norm-closed unital subalgebra of $B(X)$
- For any $x \in X$ and $f^* \in X^*$, we use $x \otimes f^*$ to denote the rank-one operator satisfying $x \otimes f^*(u) = f^*(u)x, \ \forall \ u \in X$.

Theorem 3 (J. Li and Z. Pan, 2011)

Suppose $\vee\{x : \ \forall \ x \otimes f^* \in \mathcal{A}\} = X$ and δ is a linear map from $\mathcal{A}$ to $B(X)$. If δ is derivable at $C \in \mathcal{A}$ and $ran(C)$ is dense in X then δ is a derivation.

Derivable maps at a general element c

Definitions:

- $\mathcal{A}$ = Norm-closed unital subalgebra of $B(X)$
- For any $x \in X$ and $f^* \in X^*$, we use $x \otimes f^*$ to denote the rank-one operator satisfying $x \otimes f^*(u) = f^*(u)x, \ \forall \ u \in X$.

Theorem 3 (J. Li and Z. Pan, 2011)

Suppose $\vee\{x : \ \forall \ x \otimes f^* \in \mathcal{A}\} = X$ and δ is a linear map from $\mathcal{A}$ to $B(X)$. If δ is derivable at $C \in \mathcal{A}$ and $ran(C)$ is dense in X then δ is a derivation.

Corollary:

If $\vee\{x : \ \forall \ x \otimes f^* \in \mathcal{A}\} = X$ then every Jordan derivation from $\mathcal{A}$ to $B(X)$ is a derivation.

Algebras that satisfy $\vee\{x : \forall\, x \otimes f^* \in \mathcal{A}\} = X$ include

- Standard algebras

Algebras that satisfy $\vee\{x : \forall\, x \otimes f^* \in \mathcal{A}\} = X$ include

- Standard algebras
- many relexive algebras

Reflexive algebras

Definitions

- $\mathcal{L}$ = lattice of subspaces of X with operations "$\vee$" and "$\wedge$"

Definitions

- $\mathcal{L}$ = lattice of subspaces of X with operations " $\vee$ " and " $\wedge$ "
- alg$\mathcal{L} = \{T \in B(X) : TL \subseteq L, \ \forall \ L \in \mathcal{L}\}$

Definitions

- $\mathcal{L}$ = lattice of subspaces of X with operations "$\vee$" and "$\wedge$"
- alg$\mathcal{L} = \{T \in B(X) : TL \subseteq L, \ \forall\ L \in \mathcal{L}\}$
- $L_{-} = \vee\{M \in \mathcal{L} : L \not\subseteq M\}$.

  (L_{-} is called the predecessor of L)

Definitions

- $\mathcal{L}$ = lattice of subspaces of X with operations "$\vee$" and "$\wedge$"
- $\text{alg}\mathcal{L} = \{T \in B(X) : TL \subseteq L, \ \forall \ L \in \mathcal{L}\}$
- $L_- = \vee\{M \in \mathcal{L} : L \not\subseteq M\}$.

  (L_- is called the predecessor of L)
- Define $\mathcal{J}(\mathcal{L}) = \{L \in \mathcal{L} : L \neq 0 \ \text{ and } \ L_- \neq X\}$, i.e.

  $\mathcal{J}(\mathcal{L})$ is the set of L with non-trivial predecessors.

Reflexive algebras that satisfy $\vee\{x : \forall\, x \otimes f^* \in \mathcal{A}\} = X$

An equivalent condition (for reflexive algebras only)

- For $E \subseteq X$, define $E^{\perp} = \{f^* \in X^* : \; f^*|_E = 0\}$.

Reflexive algebras that satisfy $\vee\{x : \forall\, x \otimes f^* \in \mathcal{A}\} = X$

An equivalent condition (for reflexive algebras only)

- For $E \subseteq X$, define $E^{\perp} = \{f^* \in X^* : \; f^*|_E = 0\}$.
- $x \otimes f^* \in alg\mathcal{L}$ iff there exists an $E \in \mathcal{J}(\mathcal{L})$ such that $x \in E$ and $f^* \in (E_-)^{\perp}$ (W. Longstaff).

Reflexive algebras that satisfy $\vee\{x : \ \forall\ x \otimes f^* \in \mathcal{A}\} = X$

An equivalent condition (for reflexive algebras only)

- For $E \subseteq X$, define $E^{\perp} = \{f^* \in X^* : \ f^*|_E = 0\}$.
- $x \otimes f^* \in alg\mathcal{L}$ iff there exists an $E \in \mathcal{J}(\mathcal{L})$ such that $x \in E$ and $f^* \in (E_-)^{\perp}$ (W. Longstaff).
- $\vee\{x : \ \forall\ x \otimes f^* \in \mathcal{A}\} = X$ iff $\vee\{E : E \in \mathcal{J}(\mathcal{L})\} = X$.

Reflexive algebras that satisfy $\vee\{x : \forall\, x \otimes f^* \in \mathcal{A}\} = X$

An equivalent condition (for reflexive algebras only)

- For $E \subseteq X$, define $E^\perp = \{f^* \in X^* : \ f^*|_E = 0\}$.
- $x \otimes f^* \in alg\mathcal{L}$ iff there exists an $E \in \mathcal{J}(\mathcal{L})$ such that $x \in E$ and $f^* \in (E_-)^\perp$ (W. Longstaff).
- $\vee\{x : \ \forall\, x \otimes f^* \in \mathcal{A}\} = X$ iff $\vee\{E : E \in \mathcal{J}(\mathcal{L})\} = X$.

Reflexive algebras that satisfy $\vee\{x : \forall\, x \otimes f^* \in \mathcal{A}\} = X$

An equivalent condition (for reflexive algebras only)

- For $E \subseteq X$, define $E^\perp = \{f^* \in X^* : \ f^*|_E = 0\}$.
- $x \otimes f^* \in alg\mathcal{L}$ iff there exists an $E \in \mathcal{J}(\mathcal{L})$ such that $x \in E$ and $f^* \in (E_-)^\perp$ (W. Longstaff).
- $\vee\{x : \forall\, x \otimes f^* \in \mathcal{A}\} = X$ iff $\vee\{E : E \in \mathcal{J}(\mathcal{L})\} = X$.

Lattices that satisfy $\vee\{E : E \in \mathcal{J}(\mathcal{L})\} = X$ include

1. completely distributive subspace lattice $\mathcal{L}$, in particular a nest. ($\mathcal{L}$ is completely distributive iff $E = \wedge\{L_- : L \in \mathcal{L},\ L \not\subseteq E\}$.)

Reflexive algebras that satisfy $\vee\{x : \ \forall\ x \otimes f^* \in \mathcal{A}\} = X$

An equivalent condition (for reflexive algebras only)

- For $E \subseteq X$, define $E^{\perp} = \{f^* \in X^* : \ f^*|_E = 0\}$.
- $x \otimes f^* \in alg\mathcal{L}$ iff there exists an $E \in \mathcal{J}(\mathcal{L})$ such that $x \in E$ and $f^* \in (E_-)^{\perp}$ (W. Longstaff).
- $\vee\{x : \ \forall\ x \otimes f^* \in \mathcal{A}\} = X$ iff $\vee\{E : E \in \mathcal{J}(\mathcal{L})\} = X$.

Lattices that satisfy $\vee\{E : E \in \mathcal{J}(\mathcal{L})\} = X$ include

1. completely distributive subspace lattice $\mathcal{L}$, in particular a nest. ($\mathcal{L}$ is completely distributive iff $E = \wedge\{L_- : L \in \mathcal{L},\ L \not\subseteq E\}$.)
2. $\mathcal{J}$-subspace lattices.

Reflexive algebras that satisfy $\vee\{x : \forall\ x \otimes f^* \in \mathcal{A}\} = X$

An equivalent condition (for reflexive algebras only)

- For $E \subseteq X$, define $E^{\perp} = \{f^* \in X^* : \ f^*|_E = 0\}$.
- $x \otimes f^* \in alg\mathcal{L}$ iff there exists an $E \in \mathcal{J}(\mathcal{L})$ such that $x \in E$ and $f^* \in (E_-)^{\perp}$ (W. Longstaff).
- $\vee\{x : \ \forall\ x \otimes f^* \in \mathcal{A}\} = X$ iff $\vee\{E : E \in \mathcal{J}(\mathcal{L})\} = X$.

Lattices that satisfy $\vee\{E : E \in \mathcal{J}(\mathcal{L})\} = X$ include

1. completely distributive subspace lattice $\mathcal{L}$, in particular a nest. ($\mathcal{L}$ is completely distributive iff $E = \wedge\{L_- : L \in \mathcal{L},\ L \not\subseteq E\}$.)
2. $\mathcal{J}$-subspace lattices.
3. $\mathcal{L}$ that satisfies $X_- \neq X$.

Reflexive algebras that satisfy $\vee\{x : \forall\, x \otimes f^* \in \mathcal{A}\} = X$

An equivalent condition (for reflexive algebras only)

- For $E \subseteq X$, define $E^\perp = \{f^* \in X^* : \ f^*|_E = 0\}$.
- $x \otimes f^* \in alg\mathcal{L}$ iff there exists an $E \in \mathcal{J}(\mathcal{L})$ such that $x \in E$ and $f^* \in (E_-)^\perp$ (W. Longstaff).
- $\vee\{x : \ \forall\, x \otimes f^* \in \mathcal{A}\} = X$ iff $\vee\{E : E \in \mathcal{J}(\mathcal{L})\} = X$.

Lattices that satisfy $\vee\{E : E \in \mathcal{J}(\mathcal{L})\} = X$ include

1. completely distributive subspace lattice $\mathcal{L}$, in particular a nest. ($\mathcal{L}$ is completely distributive iff $E = \wedge\{L_- : L \in \mathcal{L},\ L \not\subseteq E\}$.)
2. $\mathcal{J}$-subspace lattices.
3. $\mathcal{L}$ that satisfies $X_- \neq X$.
4. $\mathcal{L}$ that satisfies $\wedge\{L_- : L \in \mathcal{J}(\mathcal{L})\} = (0)$.

Reflexive algebras that satisfy $\vee\{x : \forall\, x \otimes f^* \in \mathcal{A}\} = X$

An equivalent condition (for reflexive algebras only)

- For $E \subseteq X$, define $E^\perp = \{f^* \in X^* : \ f^*|_E = 0\}$.
- $x \otimes f^* \in alg\mathcal{L}$ iff there exists an $E \in \mathcal{J}(\mathcal{L})$ such that $x \in E$ and $f^* \in (E_-)^\perp$ (W. Longstaff).
- $\vee\{x : \forall\, x \otimes f^* \in \mathcal{A}\} = X$ iff $\vee\{E : E \in \mathcal{J}(\mathcal{L})\} = X$.

Lattices that satisfy $\vee\{E : E \in \mathcal{J}(\mathcal{L})\} = X$ include

1. completely distributive subspace lattice $\mathcal{L}$, in particular a nest. ($\mathcal{L}$ is completely distributive iff $E = \wedge\{L_- : L \in \mathcal{L},\ L \not\subseteq E\}$.)
2. $\mathcal{J}$-subspace lattices.
3. $\mathcal{L}$ that satisfies $X_- \neq X$.
4. $\mathcal{L}$ that satisfies $\wedge\{L_- : L \in \mathcal{J}(\mathcal{L})\} = (0)$.
5. $\mathcal{L}$ that satisfies $(0)_+ \neq (0)$.

Derivational points

Definition:

An element $c \in \mathcal{A}$ is called a derivational point of $L(\mathcal{A}, \mathcal{M})$ if whenever $\delta \in L(\mathcal{A}, \mathcal{M})$ is derivable at c then δ is a derivation.

Derivational points

Definition:

An element $c \in \mathcal{A}$ is called a derivational point of $L(\mathcal{A}, \mathcal{M})$ if whenever $\delta \in L(\mathcal{A}, \mathcal{M})$ is derivable at c then δ is a derivation.

Theorem 3 says:

If $\vee\{x : \ \forall\, x \otimes f^* \in \mathcal{A}\} = X$ then every $C \in \mathcal{A}$ such that $ran(C)$ is dense in X is a derivational point of $L(\mathcal{A}, B(X))$.

Theorem 4:

If $\mathcal{A}$ is a nest algebra on a Hilbert space H then every $0 \neq C \in \mathcal{A}$ is a derivational point of $L(\mathcal{A}, B(H))$.

Theorem 4:

If $\mathcal{A}$ is a nest algebra on a Hilbert space H then every $0 \neq C \in \mathcal{A}$ is a derivational point of $L(\mathcal{A}, B(H))$.

Corollary:

Suppose δ is a linear map from a nest algebra $\mathcal{A}$ on a Hilbert space H to $B(H)$ and δ is derivable at $C \in \mathcal{A}$.

(i) If $C = 0$ then there exists a derivation d from $\mathcal{A}$ to $B(H)$ and a scalar λ such that $\delta = d + \lambda I$.

(ii) If $C \neq 0$ then δ must be a derivation.

Thank you for your attention!