C^*-algebras Generated by Linear-fractionally-induced Composition Operators

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Notation

- Let $\mathbb{D}$ denote the open unit disk and $\mathbb{T}$ the unit circle.
- $H^2(\mathbb{D})$ is the space of all functions $f = \sum_{n=0}^{\infty} a_n z^n$ that are analytic on $\mathbb{D}$ and satisfy

$$||f||_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

- For $\varphi : \mathbb{D} \to \mathbb{D}$ analytic, $C_\varphi f = f \circ \varphi$ for all $f \in H^2(\mathbb{D})$.
- $U_\varphi =$ partial isometry in the polar decomposition of C_φ
- $T_z =$ the forward shift on $H^2(\mathbb{D})$.
- $\mathcal{K} =$ the ideal of compact operators on $H^2(\mathbb{D})$

More Notation

- We're interested in composition operators induced by linear-fractional maps $\varphi(z) = \dfrac{az+b}{cz+d}$ that
  - map $\mathbb{D}$ into but not onto $\mathbb{D}$
  - and fix a point on $\mathbb{T}$.
- Without loss of generality, take the fixed point to be 1.
- Let $\mathcal{F} := \{C_\varphi : \varphi : \mathbb{D} \to \mathbb{D} \text{ non-auto, LFT}, \varphi(1) = 1\}$.

Main Questions for Today's Talk

Let $\mathcal{F} := \{C_\varphi \,|\, \varphi : \mathbb{D} \to \mathbb{D}$ non-auto, LFT$, \varphi(1) = 1\}$.

1. What is the structure of $C^*(\mathcal{F})$, modulo the ideal of compact operators?
2. If $C_\varphi \in \mathcal{F}$, what is the structure of $C^*(C_\varphi, \mathcal{K})$, modulo the ideal of compact operators?
3. What spectral information about algebraic combinations of composition operators and their adjoints can we obtain from these structure results?

Composition Operators Induced by Parabolic Non-automorphisms

- A linear-fractional, non-automorphism self-map of $\mathbb{D}$ that fixes the point 1 is parabolic if $\varphi'(1) = 1$.
- In this case, $\varphi = \rho_a = \frac{(2-a)z+a}{-az+(2+a)}$, where $\operatorname{Re} a > 0$.
- Let $\mathbb{P} = \{C_{\rho_a} : \operatorname{Re} a > 0\}$.

Theorem (Kriete, MacCluer, and Moorhouse)

If $C_\varphi \in \mathbb{P}$, then $C^*(C_\varphi, \mathcal{K}) = C^*(C_\varphi) = C^*(\mathbb{P})$ and there exists a unique $*$-isomorphism

$$\Gamma : C([0,1]) \to C^*(\mathbb{P})/\mathcal{K}$$

such that, for all $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$, $\Gamma(x^a) = [C_{\rho_a}]$.

Rewriting Operators in $\mathcal{F}$

For $t > 0$, define the map $\Psi_t(z) = \dfrac{(t+1)z + (1-t)}{(1-t)z + (1+t)}$, which is an automorphism of $\mathbb{D}$.

If $C_\varphi \in \mathcal{F}$, set $t = \varphi'(1)$ and $a = \frac{\varphi''(1) - t^2 + t}{t}$. Then

$$C_\varphi = C_{\rho_a} C_{\Psi_t}.$$

Applying results of Bourdon and MacCluer, Jury, and Kriete, MacCluer, and Moorhouse, we can then rewrite C_φ as

$$C_\varphi = \frac{1}{\sqrt{t}} C_{\rho_a} U_{\Psi_t} + K,$$

where $K \in \mathcal{K}$ and U_{Ψ_t} is the unitary operator appearing in the polar decomposition of C_{Ψ_t}.

Composition Operators Induced by Automorphisms

Let G be a collection of automorphisms of $\mathbb{D}$ that form an abelian group under composition.

Theorem (Jury, 2007)

$$\begin{aligned} C^*(T_z, \{C_\gamma : \gamma \in G\})/\mathcal{K} &= C^*(T_z, \{U_\gamma : \gamma \in G\})/\mathcal{K} \\ &\cong C(\mathbb{T}) \rtimes_\alpha G_d \end{aligned}$$

- $\alpha_\gamma(f) = f \circ \gamma$ *for all* $f \in C(\mathbb{T})$ *and* $\gamma \in G$
- G_d *denotes the locally compact group obtained from* G *by applying the discrete topology.*

Crossed Product C*-algebras (Discrete Version)

- Let G be a discrete group, and let $\mathcal{A}$ be a C*-algebra.
- An action α of G on $\mathcal{A}$ is a homomorphism $\alpha : G \to Aut(\mathcal{A})$, $g \mapsto \alpha_g$.
- The crossed product $\mathcal{A} \rtimes_\alpha G$ is the completion of

$$C_c(G, \mathcal{A}) = \left\{ \sum_{s \in G} A_s \chi_s : \begin{array}{c} A_s \in \mathcal{A}, A_s = 0 \text{ for all} \\ \text{but finitely many } s \end{array} \right\}$$

in a norm that is built from a set of representations of $C_c(G, \mathcal{A})$ that come from covariant representations of $(\mathcal{A}, G, \alpha)$.

1. What is the structure of $C^*(\mathcal{F})/\mathcal{K}$?

- $C^*(\mathcal{F})/\mathcal{K} \subset C^*(\mathbb{P}, \{U_{\Psi_t} : t \in \mathbb{R}^+\})/\mathcal{K}$
- $C^*(\mathbb{P})/\mathcal{K} \cong C([0,1])$ Kriete, MacCluer, and Moorhouse
- $C^*(T_z, \{U_{\Psi_t} : t \in \mathbb{R}^+\})/\mathcal{K} \cong C(\mathbb{T}) \rtimes \mathbb{R}^+_d$ Jury
- We want to show that $C^*(\mathbb{P}, \{U_{\Psi_t} : t \in \mathbb{R}^+\})/\mathcal{K}$ is also a crossed product.

1. What is the structure of $C^*(\mathcal{F})/\mathcal{K}$?

$$C^*(\mathbb{P}, \{U_{\Psi_t} : t \in \mathbb{R}^+\})/\mathcal{K} = C^*\left(C^*(\mathbb{P})/\mathcal{K}, \{[U_{\Psi_t}] : t \in \mathbb{R}^+\}\right)$$

- $\{[U_{\Psi_t}] : t \in \mathbb{R}^+\}$ is an abelian group of cosets of unitary operators. (Jury 2007)
- $[U_{\Psi_t}]\Gamma(g)[U^*_{\Psi_t}] = \Gamma(\beta_t(g))$

  for all $g \in C([0,1])$ and $t \in \mathbb{R}^+$, where $\beta_t(g)(x) = g(x^t)$ and Γ is the $*$-isomorphism from $C([0,1])$ onto $C^*(\mathbb{P})/\mathcal{K}$

  (Obtained by applying results of Bourdon, MacCluer 2007; Jury 2007: Kriete, MacCluer, Moorhouse 2007, 2009)

We can show that the action β is topologically free and then apply the machinery of Karlovich or Lebedev.

Theorem (Q)

$$C^*(\mathbb{P}, \{U_{\Psi_t} : t \in \mathbb{R}^+\})/\mathcal{K} \cong C([0,1]) \rtimes_\beta \mathbb{R}_d^+.$$

The $*$*-isomorphism*
$F : C([0,1]) \rtimes_\beta \mathbb{R}_d^+ \to C^*(\mathbb{P}, \{U_{\Psi_t} : t \in \mathbb{R}^+\})/\mathcal{K}$ *satisfies*

$$F(\sum_{finite} g_t \chi_t) = \sum_{finite} \Gamma(g_t)[U_{\Psi_t}]$$

for all $\sum_{finite} g_t \chi_t \in C_c(\mathbb{R}_d^+, C([0,1]))$.

Corollary

$C^*(\mathcal{F})/\mathcal{K}$ *is isomorphic to a subalgebra of* $C([0,1]) \rtimes_\beta \mathbb{R}_d^+$.

Identifying the Full Fixed Point Algebra

- All cosets of words in $\mathcal{F} \cup \mathcal{F}^*$ look like $[bC_{\rho_a} U_{\Psi_t}]$.
- Let $C_0([0,1]) := \{g \in C([0,1]) : g(0) = 0\}$ and set

$$N = \left\{ \sum_{\text{finite}} \Gamma(g_t)[U_{\Psi_t}] : g_t \in C_0([0,1]) \right\}.$$

  Then $\mathbb{C}[I] + N$ is dense in $C^*(\mathcal{F})/\mathcal{K}$.

- Under the iso, N maps onto $C_c(\mathbb{R}_d^+, C_0([0,1]))$.

Theorem (Q)

Define $\beta : \mathbb{R}_d^+ \to Aut(C_0([0,1]))$ by $\beta_t(g)(x) := g\left(x^t\right)$ for all $t \in \mathbb{R}_d^+$, $g \in C_0([0,1])$, and $x \in [0,1]$.

Then $C^*(\mathcal{F})/\mathcal{K}$ is isometrically $*$-isomorphic to the unitization of $C_0([0,1]) \rtimes_\beta \mathbb{R}_d^+$.

2. If $C_\varphi \in \mathcal{F}$ and $\varphi'(1) \neq 1$, what is the structure of $C^*(C_\varphi, \mathcal{K})/\mathcal{K}$?

Theorem (Q)

Let φ be a linear-fractional, non-automorphism self-map of $\mathbb{D}$ with $\varphi(1) = 1$ and $\varphi'(1) = t \neq 1$.

Define $\beta^t : \mathbb{Z} \to Aut(C_0([0,1]))$ by $\beta^t_n g(x) := g(x^{t^n})$ for all $n \in \mathbb{Z}$, $g \in C_0([0,1])$, and $x \in [0,1]$.

Then $C^*(C_\varphi, \mathcal{K})/\mathcal{K}$ is isometrically $*$-isomorphic to the unitization of $C_0([0,1]) \rtimes_{\beta^t} \mathbb{Z}$.

Essential Spectra

Consider cosets in $C^*(C_\varphi, \mathcal{K})/\mathcal{K}$ of the form

$$[A] = \sum_{n=-N}^{N} \Gamma(g_n)\left[U^*_{\Psi_{t^n}}\right]$$

for $t = \varphi'(1)$, $N \in \mathbb{N}$, $g_n \in C([0,1])$, and $g_n(0) = 0$ for $n \neq 0$.

For $x \in [0,1]$, define $\pi_x([A]) \in \mathcal{B}(\ell^2(\mathbb{Z}))$ by $\pi_x([A]) =$

$$\begin{bmatrix} \ddots & \ddots & \ddots & & & & & & 0 \\ & g_1(x^{t^{-2}}) & g_0(x^{t^{-2}}) & g_{-1}(x^{t^{-2}}) & & & & & \\ & & g_1(x^{t^{-1}}) & g_0(x^{t^{-1}}) & g_{-1}(x^{t^{-1}}) & & & & \\ & & & g_1(x) & g_0(x) & g_{-1}(x) & & & \\ & & & & g_1(x^{t}) & g_0(x^{t}) & g_{-1}(x^{t}) & & \\ & & & & & g_1(x^{t^2}) & g_0(x^{t^2}) & g_{-1}(x^{t^2}) & \\ 0 & & & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

Trajectorial Approach: $[A]$ is invertible $\Leftrightarrow$ the discrete operator $\pi_x([A])$ is invertible for all $x \in [0,1]$.

When is $\pi_x([A])$ invertible?

If $x \in (0,1)$ and $t > 1$, then $\pi_x([A]) =$

$$\left[\begin{array}{cccc|cccc}
\ddots & \ddots & \ddots & & & & & 0 \\
 & g_1(1) & g_0(1) & g_{-1}(1) & & & & \\
 & & g_1(1) & g_0(1) & 0 & & & \\
\hline
 & & & 0 & g_0(0) & g_{-1}(0) & & \\
 & & & & g_1(0) & g_0(0) & g_{-1}(0) & \\
 & & & & & g_1(0) & g_0(0) & g_{-1}(0) \\
0 & & & & & & \ddots & \ddots \quad \ddots
\end{array}\right] + \mathrm{K}$$

By results of Gohberg and Fel′dman, $\pi_x([A])$ is Fredholm with index zero if and only if

$$p_{A,0}(z) := \sum_{n=-N}^{N} g_n(0) z^n \quad \text{and} \quad p_{A,1}(z) := \sum_{n=-N}^{N} g_n(1) z^n$$

do not vanish on $\mathbb{T}$ and have the same winding number.

When is $\pi_x([A])$ invertible?

If $x \in (0,1)$ and $t > 1$, then $\pi_x([A]) =$

$$\left[\begin{array}{cccc|ccccc}
\ddots & \ddots & \ddots & & & & & & 0 \\
 & g_1(1) & g_0(1) & g_{-1}(1) & & & & & \\
 & & g_1(1) & g_0(1) & 0 & & & & \\
\hline
 & & & 0 & g_0(0) & 0 & & & \\
 & & & & 0 & g_0(0) & 0 & & \\
 & & & & & 0 & g_0(0) & 0 & \\
0 & & & & & & \ddots & \ddots & \ddots
\end{array}\right] + K$$

By results of Gohberg and Fel′dman, $\pi_x([A])$ is Fredholm with index zero if and only if

$$p_{A,0}(z) := g_0(0) \quad \text{and} \quad p_{A,1}(z) := \sum_{n=-N}^{N} g_n(1) z^n$$

do not vanish on $\mathbb{T}$, and $p_{A,1}(z)$ has winding number 0.

When is $\pi_x([A]) = \left[g_{i-j}\left(x^{t^j}\right)\right]_{i,j=-\infty}^{\infty}$ invertible?

If $g_0(0) \neq 0$, $p_{A,1}$ does not vanish on $\mathbb{T}$, and $p_{A,1}$ has winding number 0, define

$$\pi_x([A])^\nu = [g_{i-j}(x^{t^j})]_{i,j=-\nu+1}^{\nu-1}$$

$$\pi_x([A])_\mu^\nu = [g_{i-j}(x^{t^j})]_{i,j\in\{-\nu+1,\ldots,-\mu,\mu,\ldots,\nu-1\}}$$

Theorem (Karlovich and Kravchenko, 1984)

$\pi_x([A])$ is invertible on $\ell^2(\mathbb{Z})$ if and only if the conditions above hold and there exists $\mu_0 > 0$ such that for all $\mu \geq \mu_0$,

$$\lim_{\nu\to\infty} \frac{\det \pi_x([A])^\nu}{\det \pi_x([A])_\mu^\nu} \neq 0. \qquad (1)$$

If $\pi_x([A])$ is lower-triangular, then (1) is equivalent to the condition that $g_0(x^{t^j}) \neq 0$ for all j.

Theorem

Let $t = \varphi'(1) \neq 1$, and suppose $A \in C^*(C_\varphi, \mathcal{K})$ satisfies

$$[A] = \sum_{n=0}^{N} \Gamma(g_n) \left[U^*_{\Psi_{t^n}} \right]$$

for some $N \in \mathbb{N}$, $g_0 \in C([0,1])$, and $g_1, \ldots, g_n \in C_0([0,1])$.

Then $\sigma_e(A) = g_0([0,1]) \cup p_{A,1}(\overline{\mathbb{D}})$.

Example

Let $b_1, \dots, b_n \in \mathbb{C}$ and suppose that $\varphi_1, \dots, \varphi_n$ are linear fractional, non-automorphism self-maps of $\mathbb{D}$ that fix the point 1 and satisfy $\varphi_1'(1) = \dots = \varphi_n'(1) = s \neq 1$.

If $A = \sum_{j=1}^{n} b_j C_{\varphi_j}$, then

$$p_{A,1}(z) = \left(\frac{1}{\sqrt{s}} \sum_{j=1}^{n} b_j \right) z \quad \text{and} \quad g_0 \equiv 0,$$

so

$$\sigma_e \left(\sum_{j=1}^{n} b_j C_{\varphi_j} \right) = \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{\sqrt{s}} \left| \sum_{j=1}^{n} b_j \right| \right\}.$$

K-theory

We can apply the Pimsner-Voiculescu exact sequence for crossed products by $\mathbb{Z}$ and the six-term exact sequence to determine the K-theory of $C^*(C_\varphi, \mathcal{K})$.

Theorem (Q, 2009)

If φ is a linear-fractional, non-automorphism self-map of $\mathbb{D}$ with $\varphi(1) = 1$ and $\varphi'(1) \neq 1$, then

$$K_0(C^*(C_\varphi, \mathcal{K})) \cong \mathbb{Z} \oplus \mathbb{Z} \quad \textit{and} \quad K_1(C^*(C_\varphi, \mathcal{K})) \cong 0.$$

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Slides will be posted at
http://www.math.jmu.edu/~querteks/Research.html

Example of an Essential Spectrum Calculation

Let $a, c_1, c_2 \in \mathbb{C}$ with $\mathrm{Re}\, a > 0$. Suppose that φ is a linear-fractional, non-automorphism self-map of $\mathbb{D}$ with $\varphi(1) = 1$ and $\varphi'(1) = s \neq 1$.

Then, there exists $b \in \mathbb{C}$ with $\mathrm{Re}\, b > 0$ such that

$$[c_1 C_{\rho_a} + c_2 C_\varphi] = \Gamma(c_1 x^a) \left[U^*_{\Psi_{(1/s)^0}} \right] + \Gamma\left(\frac{c_2}{\sqrt{s}} x^b \right) \left[U^*_{\Psi_{(1/s)^1}} \right].$$

Thus, $g_0(x) = c_1 x^a$, and $p_{(c_1 C_{\rho_a} + c_2 C_\varphi),1}(z) = c_1 + \frac{c_2}{\sqrt{s}} z$.

Hence

$$\sigma_e(c_1 C_{\rho_a} + c_2 C_\varphi) = \{c_1 x^a : x \in [0,1]\} \cup \left\{ \lambda \in \mathbb{C} : |\lambda - c_1| \leq \frac{c_2}{\sqrt{s}} \right\}.$$