Distance formulas in Nevanlinna-Pick interpolation.

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joint work with
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Real title

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Tangential Nevanlinna-Pick theorems for subalgebras of multiplier algebras of finite-rank, irreducible, complete Nevanlinna-Pick kernels and property $\mathbb{A}_1(1)$.

RKHS

- A set X
- A kernel on $X \times X$, i.e., a function $K : X \times X \to \mathbb{C}$ such that $[K(x_i, x_j)]_{i,j=1}^n \geq 0$ for $\{x_1, \ldots, x_n\} \subseteq X$.
- A Hilbert space H_K that is associated to K. The evaluations $f \mapsto f(x)$ are bounded. $f(x) = \langle f, k_x \rangle$.

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- A Hilbert space H_K that is associated to K. The evaluations $f \mapsto f(x)$ are bounded. $f(x) = \langle f, k_x \rangle$.
- An algebra $M(H_K)$. The pointwise multipliers of H_K.

Multiplier algebras

The **multiplier algebra** of H_K is

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- are bounded, by the closed-graph theorem;
- have adjoints with lots of eigenvectors $M_f^* k_x = \overline{f(x)} k_x$;
- are abelian **dual** operator algebras.

Nevanlinna-Pick Interpolation

Given $x_1, \ldots, x_n \in X$, and $w_1, \ldots, w_n \in \mathbb{C}$. We have an associated extremal problem

$$\inf\{\|f\|_{M(H_K)} \,:\, f \in M(H_K), f(x_j) = w_j\}$$

Distance problems

- All solutions $f_0 + g$, where
  - $f_0(x_j) = w_j$ is a **particular solution** and
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- $\mathcal{I} = \{g \in M(H_K) \,:\, g(x_j) = 0\}$ is an ideal.

$$\begin{aligned}\inf\{\|f\|_{M(H_K)} \,:\, f \in M(H_K), f(x_j) = w_j\} &= \inf\{\|f_0 + g\| \,:\, g \in \mathcal{I}\} \\ &= \|f_0 + \mathcal{I}\|_{M(H_K)/\mathcal{I}}\end{aligned}$$

$M(H_K)/\mathcal{I}$ is an n-dimensional operator algebra, compute its norm.

Representations

A natural representation of $M(H_K)/\mathcal{I}$. L be another RKHS with kernel K_L

- Suppose $M(H_K) \subseteq M(L)$; $\rho : M(H_K) \to B(L)$.

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The representation theory of $M(H_K)/\mathcal{I}$ is a measure of the complexity of the interpolation problem.

What is a distance formula?

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Banach space duality is one way to get them.

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The particular solution $f_0(x_j) = w_j$

$$\left[(1 - f(x_i)\overline{f(x_j)})K(x_i, x_j)\right] = [(1 - w_i\overline{w_j})K(x_i, x_j)] \geq 0.$$

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norm on the span of kernel functions $\Leftrightarrow$ matrix positivity condition

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- Davidson-Paulsen-R-Singh (2008). $f'(0) = 0$. Later extended to arbitrary weak*-closed subalgebras of H^∞.

Subalgebras

We want to generalize well-known theorems (see history) to the case where $\mathcal{A}$ is a unital weak*-closed subalgebra of $M(H_K)$.

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Why?

Motivation. Unification

- A generalization of Abrahamse's theorem if we view $H^\infty(R)$ as a subalgebra of $H^\infty(\mathbb{D})$. Fixed-point algebras of H^∞. (R-2009).

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- A Nevanlinna-Pick theorem on the Neil parabola $\{z^2 = w^3\} \subseteq \mathbb{D}^2$. (DPRS-2009).

Motivation. Töplitz corona problems.

Theorem (Töplitz Corona. Arveson, Schubert)

If $f_1, \ldots, f_n \in H^\infty$, with $T_{f_1} T_{f_1}^* + \cdots + T_{f_1} T_{f_n}^* \geq \delta^2 I$ in $B(H^2)$, then there exists $g_1, \ldots, g_n \in H^\infty$ such that $f_1 g_1 + \cdots + f_n g_n = 1$ and $\sup_{z\in\mathbb{D}} \sum_{j=1}^n |g_j(z)|^2 \leq \delta^{-2}$.

Theorem (R.-Wick, 2010)

Corresponding theorem for Riemann surfaces and fixed-point algebras. Generalizes Ball (1981)

Motivation. Interpolating sequences 1.

$R : M(H_K) \to \ell^\infty$, $R(f) = (f(z_j))_{j=1}^\infty$, R surjective.

Theorem (Interpolating sequences. Carleson)

A sequence $z_j \in \mathbb{D}$ is interpolating for H^∞ if and only if it is strongly separated, i.e.,

$$\inf_{j \geq 1} \prod_{i \neq j} \left| \frac{z_i - z_j}{1 - \overline{z_j} z_i} \right| > 0.$$

Motivation. Interpolating sequences 2.

Theorem (R.-Wick. 2010)

Let $\mathcal{A} \subseteq H^\infty$ *be a fixed-point algebra. Let* $E = \{z_j\}$ *be a sequence of points in the unit disk. Let* $E_j = E \setminus \{z_j\}$*. The following are equivalent:*

1. E *is an interpolating sequence for* $\mathcal{A}$*.*

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3. $\inf_{j \geq 1} d(z_j, E_j) > 0$, *i.e., strong separation.*

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4. $\sum_{j=1}^{\infty} |f(z_j)|^2 K(z_j, z_j)^{-1} \leq C \|f\|_2^2$, *i.e., Carleson measure, and (z_j) is weakly separated.*

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$$\mathcal{A}/\mathcal{I} \mapsto B(\bigoplus_h \mathcal{K}_h) \text{ is isometric.}$$

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In the presence of a group action you can do better, h can be chosen character automorphic, and there is a bounded (norm-equivalent to image) representation into $B(\mathcal{K}_1)$, because the modules $\overline{\mathcal{A}h}$ and $\overline{\mathcal{A}} = H_{K_1}$ are similar.

Complete Nevanlinna-Pick kernels

Definition (CNP Kernel)

K on $X \times X$ is a complete Nevanlinna-Pick kernel (cnp) iff for all $x_1, \dots, x_n$, all matrices $W_1, \dots, W_n \in M_d(\mathbb{C})$

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$$F(x_j) = W_j \text{ with } \|F\| \leq 1 \Leftrightarrow [(1 - W_i W_j^*) K(x_i, x_j)] \geq 0.$$

Examples: Hardy space (Pick), Dirichlet space (Agler), Sobolev space (Agler).

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The natural representation of

$$M(H_K)/\mathcal{I} \mapsto B(\text{span}\{k_{x_1}, \ldots, k_{x_n}\})$$

is a complete isometry.

The Drury-Arveson space

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Theorem (McCullough, Quiggin, Agler-McCarthy)

Every irreducible CNP space is of the form

$$H^2_{d,S} = \mathrm{span}\{k_x \,:\, x \in S \subseteq \mathbb{B}_d\}$$

for some set S and some d.

Tangential interpolation

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Tangential version $x_1, \ldots, x_n \in X$, $w_1, \ldots, w_n \in \mathbb{C}$ **and** vectors $v_1, \ldots, v_n \in \ell^2$.

$$\inf\{\|F\| : \langle F(x_j), v_j \rangle = w_j, F \in C(M(H_K))\} = \|F_0 + \mathcal{J}\|$$

$\mathcal{J} = \{F \in C(M(H_K)) : \langle F(x_j), v_j \rangle = 0\}$

Tangential Pick families

$\mathcal{A} \subseteq M(H_K)$. $\mathcal{L} \subset \mathrm{Lat}(\mathcal{A})$ a *tangential Nevanlinna-Pick family* if for every choice of points $x_1, \ldots, x_n \in X$, $w_1, \ldots, w_n \in \mathbb{C}$ and $v_1, \ldots, v_n \in \ell^2$, we have

$$d(F, \mathcal{J}) = \sup_{L \in \mathcal{L}} \| P_L M_F^* |_{\mathcal{K}_L} \| .$$

Where $\mathcal{J} = \{\langle F(x_j), v_j \rangle = 0\}$, $\mathcal{K}_L = \mathrm{span}\{k_{x_1}^L \otimes v_1, \ldots, k_{x_n}^L \otimes v_n\}$.

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Theorem (Hamilton-R. 2011)

If $\mathcal{A} \subset M(H_K)$, then the **cyclic** *invariant subspaces in $Lat(\mathcal{A})$ form a tangential NP-family.*

Extends results of Davidson-Hamilton (2010).

Free semigroup algebra

Definition (The noncommutative analytic Töplitz algebra)

$\mathcal{L}_d$ is the WOT-closed algebra generated by an d-tuple $(S_1, \ldots, S_d)$ of shifts acting on the space $\ell^2(\mathbb{F}_d^+)$ where $\mathbb{F}_d^+$ is the free semigroup.

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Proposition

Any multiplier algebra $M(H_K)$, where K is a CNP kernel, is a (complete) quotient of $\mathcal{L}_d$.

Property $A_1(1)$

Definition (Bercovici-Foias-Pearcy (1985))

$\mathcal{A} \subseteq B(H)$, unital, weak*-closed has $\mathbb{A}_1(1)$ iff every weak*-continuous linear functional ϕ, $\|\phi\| < 1$, on $\mathcal{A}$ is $\phi(A) = \langle Ax, y \rangle$ for some $x, y \in H$, with $\|x\| \, \|y\| < 1$.

- $\mathcal{A} \subseteq B(H)$ has $\mathbb{A}_1(1)$ when viewed as
$$\mathcal{A} \otimes I \subseteq B(H \otimes \ell^2).$$
- Starting point for the Arveson (1975), McCullough (1996) and R-Wick (2010).

Proof

Theorem (Bercovici (1998))

An algebra of operators has the property $A_1(1)$ if its commutant contains two isometries with pairwise orthogonal ranges.

So $\mathcal{L}_d$, and $\mathcal{L}_d \otimes B(\ell^2)$, have $\mathbb{A}_1(1)$, in fact has more.

Using the fact that $\mathcal{A}$ is a subalgebra of a quotient we get that $C(\mathcal{A}) \subseteq B(H_K \oplus (H_K \otimes \ell^2))$ has $\mathbb{A}_1(1)$.

The rest of the proof is calculations and unwinding.