Boundary values of functions in the range of a truncated Toeplitz operator

A. Hartmann and W. Ross

Bordeaux/Richmond

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Model spaces

- Θ inner function
- $\mathcal{K}_\Theta = H^2 \ominus \Theta H^2$ model space
- $P_\Theta : L^2 \to \mathcal{K}_\Theta$
- $k_\lambda = \dfrac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \overline{\lambda} z}$

Pseudocontinuations

Theorem (Douglas-Shapiro-Shields-1970)

For $f \in H^2$ the following are equivalent:

1. $f \in \mathcal{K}_\Theta$.
2. f/Θ *has a pseudocontinuation to an* $F \in H^2(\mathbb{D}_e)$ *with* $F(\infty) = 0$.
3. $f \in H^2 \cap \Theta\overline{H_0^2}$.

Analytic continuation

- $\sigma(\Theta) = \{\lambda \in \mathbb{D}^- : \liminf_{z \to \lambda} |\Theta(z)| = 0\}$
- $\sigma(\Theta) = Z(b_\Theta)^- \cup \operatorname{supt}(\mu_\Theta)$

Theorem (Livsic, Moeller)

1. *Every $f \in \mathcal{K}_\Theta$ has an analytic continuation across $\partial\mathbb{D} \setminus \sigma(\Theta)$*
2. *If $A_z f = P_\Theta(zf)$, then $\sigma(A_z) = \sigma(\Theta)$.*

What happens on $\sigma(\Theta)$?

Theorem (Ahern-Clark - 1970)

For Θ inner and $\zeta \in \partial\mathbb{D}$, TFAE:

1. *Every $f \in \mathcal{K}_\Theta$ is a finite nt-limit at ζ.*
2. *Θ has a finite angular derivative at ζ.*
3. $$k_\zeta(z) = \frac{1 - \overline{\Theta(\zeta)}\Theta(z)}{1 - \overline{\zeta} z} \in H^2$$
4. $$\sum_{\lambda \in Z(b_\Theta)} \frac{1 - |\lambda|^2}{|\zeta - \lambda|^2} + \int \frac{d\mu_\Theta(\xi)}{|\xi - \zeta|^2} < \infty.$$

What's behind Ahern-Clark

- $A_z f = P_\Theta(zf)$, compressed shift
- $f(\lambda) = \langle f, (I - \overline{\lambda} A_z)^{-1} P_\Theta 1 \rangle$ for $f \in \mathcal{K}_\Theta$.
- $k_\lambda = (I - \overline{\lambda} A_z)^{-1} P_\Theta 1 \to k_\zeta$ weakly as $\lambda \to \zeta$ n.t.

Our starting point

Look at

$$f(\lambda) = \langle f, (I - \overline{\lambda} A_z)^{-1} P_\Theta 1 \rangle.$$

What happens when we change this to

$$\langle f, (I - \overline{\lambda} A_z)^{-1} P_\Theta h \rangle, \quad h \in H^\infty?$$

Proposition

$$\langle f, (I - \overline{\lambda} A_z)^{-1} P_\Theta h \rangle = [A_{\overline{h}} f](\lambda),$$

$$A_{\overline{h}} f = P_\Theta(\overline{h} f).$$

$$k_\lambda^h = (I - \overline{\lambda} A_z)^{-1} P_\Theta h$$

Towards the main result

Suppose Θ is a Blaschke product with zeros $\{\lambda_n\}_{n \geq 1}$.

$$\gamma_n(z) = \frac{\sqrt{1 - |\lambda_n|^2}}{1 - \overline{\lambda_n} z} \prod_{k=1}^{n-1} b_{\lambda_k}(z).$$

$$b_\lambda(z) = \frac{z - \lambda}{1 - \overline{\lambda} z}$$

Theorem (Takanaka-Walsh)

$\{\gamma_n : n \geq 1\}$ *is an o.n. basis for* $\mathcal{K}_\Theta$.

Towards the main result

- $A_{\overline{h}} f = P_\Theta(\overline{h} f)$
- $\langle f, (I - \overline{\lambda} A_z)^{-1} P_\Theta h \rangle = [A_{\overline{h}} f](\lambda)$
- $k_\lambda^h = (I - \overline{\lambda} A_z)^{-1} P_\Theta h$
- When does $k_\lambda^h \to k_\zeta^h$ weakly as $\lambda \to \zeta$ n.t.?

The main result

Theorem (Hartmann-R)

Suppose Θ is a Blaschke product with zeros $\{\lambda_n\}_{n \geq 1}$ and $h \in H^\infty$. Then every function in $Rng A_{\overline{h}}$ has a finite non-tangential limit at $\zeta \in \partial\mathbb{D}$ if and only if

$$\sum_{n=1}^{\infty} |(A_{\overline{h}} \gamma_n)(\zeta)|^2 < \infty.$$

Note then when $h = 1$, we get

$$\sum_{n=1}^{\infty} \left| \frac{\sqrt{1 - |\lambda_n|^2}}{1 - \overline{\lambda_n}\zeta} \prod_{k=1}^{n-1} b_{\lambda_k}(\zeta) \right|^2 = \sum_{n=1}^{\infty} \frac{1 - |\lambda_n|^2}{|\zeta - \lambda_n|^2}.$$

The main result

Theorem (Hartmann-R)

Suppose Θ is a Blaschke product with zeros $\{\lambda_n\}_{n\geq 1}$ and $h \in H^\infty$. Then every function in $Rng A_{\overline{h}}$ has a finite non-tangential limit at $\zeta \in \partial\mathbb{D}$ if and only if

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A check

Proposition

let Λ be a Blaschke sequence and $h \in H^\infty$. Then, writing

$$\prod_{l=1}^{n}(z-\lambda_l) = \prod_{l=1}^{r}(z-\mu_l)^{k_l},$$

we have, for any $\zeta \in \partial\mathbb{D}$,

$$(A_{\overline{h}}\gamma_n)(\zeta) = \sqrt{1-|\lambda_n|^2}\sum_{l=1}^{r}\frac{1}{(k_l-1)!}\overline{\frac{d^{k_l-1}}{d\mu_l^{k_l-1}}\left[\frac{h(\mu_l)\prod_{m=1}^{n-1}(1-\overline{\lambda_m}\mu_l)}{(1-\overline{\zeta}\mu_l)\prod_{j=1,j\neq l}^{r}(\mu_l-\mu_j)^{k_j}}\right]}.$$

A check

Note that

$$A_{\overline{h}}\mathcal{K}_{\Theta} \subset \mathcal{K}_{\Theta}$$

It is difficult to see that

$$\sum_{n\geq 1} |(A_{\overline{h}}\gamma_n)(\zeta)|^2 < \infty$$

is better than Ahern-Clark

$$\sum_{n=1}^{\infty} \frac{1-|\lambda_n|^2}{|\zeta-\lambda_n|^2} < \infty.$$

Theorem (Hartmann-R)

Suppose Θ is an interpolating[a] *Blaschke product with zeros $\{\lambda_n\}_{n\geq 1}$ and $h \in H^\infty$. Then every function in Rng$A_{\overline{h}}$ has a finite non-tangential limit at $\zeta \in \partial\mathbb{D}$ if and only if*

$$\sum_{n=1}^{\infty}(1-|\lambda_n|^2)\left|\frac{h(\lambda_n)}{\zeta-\lambda_n}\right|^2 < \infty.$$

[a] Can fail if Θ is not interpolating

General questions

1. What happens when Θ is not a Blaschke product?
2. When happens when $A_{\overline{h}}$, $h \in H^\infty$, is replaced by A_h, $h \in L^\infty$?

A final comment

Recall Ahern-Clark: Every $f \in \mathcal{K}_\Theta$, Θ a BP, has a finite non-tangential limit at 1 IFF

$$\sum_{n=1}^{\infty} \frac{1-|\lambda_n|^2}{|1-\lambda_n|^2} < \infty.$$

Let

$$\lambda_n = r_n e^{i\theta_n}, \quad \theta_n = \frac{1}{2^n}, \quad 1 - r_n = x_n \theta_n^2, \quad x_n \to 0.$$

$$\sum_{n=1}^{\infty} \frac{1 - |\lambda_n|^2}{|1 - \lambda_n|^2} \simeq \sum_{n=1}^{\infty} x_n.$$

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Example

1. If $x_n = \frac{1}{n}$, then every $f \in \mathcal{K}_\Theta$ satisfies

$$|f(r)| \lesssim \|f\| \log \log \frac{1}{1 - r}.$$

2. If $x_n = \frac{1}{n \log n}$, then every $f \in \mathcal{K}_\Theta$ satisfies

$$|f(r)| \lesssim \|f\| \log \log \log \frac{1}{1 - r}.$$