Hankel Operators and the $\overline{\partial}$-Neumann Problem

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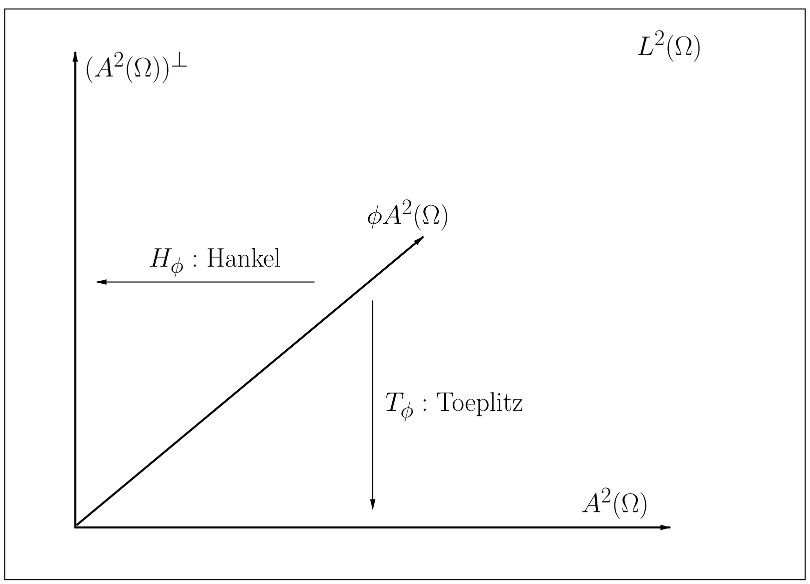
Hankel Operators on Bergman Spaces

Let $\Omega \subset \mathbb{C}^n$ be a domain.
$A^2(\Omega)$: the Bergman space on Ω $(H(\Omega) \cap L^2(\Omega))$.
$P : L^2(\Omega) \to A^2(\Omega)$ Bergman (orthogonal) projection.
The Hankel operator with symbol $\phi \in L^\infty(\Omega)$ is defined as

$$H_\phi(f) = (Id - P)(\phi f) = \phi f - P(\phi f) \text{ for } f \in A^2(\Omega).$$



Some Classical Results

[Axler '86] Let $\phi \in H(\mathbb{D})$ where $\mathbb{D}$ is the open unit disc in $\mathbb{C}$. Then $H_{\overline{\phi}}$ is compact $\Leftrightarrow (1-|z|^2)|\phi'(z)| \to 0$ as $|z| \to 1^-$.

Peloso in 1994 extended Axler's result to Bergman spaces on smooth bounded strongly pseudoconvex domains in $\mathbb{C}^n$.

Li in 1994 characterized bounded and compact Hankel operators with symbols in L^2 on smooth bounded strongly pseudoconvex in $\mathbb{C}^n$.

The method: integral representation

$$H_\phi(f)(z) = \int_\Omega K(z,w)(\phi(z)-\phi(w))f(w)dV(w) \text{ for } f \in A^2(\Omega).$$

Some Observations on Previous Results

1. H_ϕ is compact $\Leftrightarrow$ ϕ .... in Ω.
2. The method is integral representation.
3. Symbols are "general".
4. Domains are "simple":
   Unit ball or more generally strongly pseudoconvex domains.

Some New Results

From now on Ω will be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, unless otherwise is stated, and $\phi \in C^\infty(\overline{\Omega})$

Theorem (Čučković-Ş. '09)

Let $\Omega \subset \mathbb{C}^2$ and H_ϕ be compact. Then $\phi \circ f$ is holomorphic for all holomorphic $f : \mathbb{D} \to \partial\Omega$.

Theorem (Čučković-Ş. '09)

Let Ω be convex in $\mathbb{C}^2$. Then H_ϕ is compact iff $\phi \circ f$ is holomorphic for all holomorphic $f : \mathbb{D} \to \partial\Omega$.

Our Method: The $\overline{\partial}$-Neumann problem

$\overline{\partial}$-Neumann Problem

$(0,1)$-form: $\sum f_j d\overline{z}_j$

$(0,2)$-form: $\sum f_{jk} d\overline{z}_j \wedge d\overline{z}_k$

$(0,3)$-form: $\sum f_{jkl} d\overline{z}_j \wedge d\overline{z}_k \wedge d\overline{z}_l$

$\overline{\partial}$-complex:
$$L^2(\Omega) \overset{\overline{\partial}}{\to} L^2_{(0,1)}(\Omega) \overset{\overline{\partial}}{\to} L^2_{(0,2)}(\Omega) \overset{\overline{\partial}}{\to} \cdots$$
$$L^2(\Omega) \underset{\overline{\partial}^*}{\leftarrow} L^2_{(0,1)}(\Omega) \underset{\overline{\partial}^*}{\leftarrow} L^2_{(0,2)}(\Omega) \underset{\overline{\partial}^*}{\leftarrow} \cdots$$

$$\square = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : L^2_{(0,1)}(\Omega) \to L^2_{(0,1)}(\Omega).$$

[Kohn and Hörmander '60's] $\square$ has a bounded inverse N if Ω is bounded pseudoconvex.

$\overline{\partial}$-Neumann operator: $N = \square^{-1}$

$\overline{\partial}$-Neumann problem: solving $\square f = g$

Kohn's Formula: $P = I - \overline{\partial}^* N \overline{\partial}$

$H_\phi(f) = \overline{\partial}^* N \overline{\partial}(\phi f) = \overline{\partial}^* N(f \overline{\partial}\phi)$ for $f \in A^2(\Omega)$ and $\phi \in C^1(\overline{\Omega})$.

N is compact ($\Rightarrow \overline{\partial}^* N$ is compact) $\Rightarrow H_\phi$ is compact.

N is compact on:

- the unit ball,
- strongly pseudoconvex domains,
- convex domains with no analytic discs in the boundary.

Fact: There exists a convex domain $\Omega \subset \mathbb{C}^n$ and $\phi \in C^\infty(\overline{\Omega})$ such that H_ϕ is not compact.

The Converse

Fact: N is compact $\Rightarrow H_\phi$ is compact.

The converse is still open in general. However, on non-pseudoconvex domains the converse is false.

Result 1 (Çelik and Ş.)

There exists a smooth bounded non-pseudoconvex domain $\Omega \subset \mathbb{C}^3$ such that H_ϕ is compact for all $\phi \in C(\overline{\Omega})$ and the $\overline{\partial}$-Neumann problem on Ω is bounded but not compact.

Comparison of Methods

Integral Rep.:
$H_\phi(f)(z) = \int_\Omega K(z,w)(\phi(z) - \phi(w))f(w)dV(w)$

- The symbols are "general"
- The domains are restrictive.
- Relates compactness of H_ϕ to the behavior of ϕ in the domain.

$\overline{\partial}$-Neumann Problem: $H_\phi(f) = \overline{\partial}^* N(f\overline{\partial}\phi)$

- The symbols are restrictive (as least continuous up to the boundary).
- The domains are "general" (pseudoconvex).
- Relates compactness of H_ϕ to the behavior of ϕ on the boundary.

References

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