The Pick problem on the polydisc and inner varieties

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If a Pick problem is not extremal, then U equals the original set of nodes. If G is a solution and $||G||_\infty < 1$, then for each polynomial p satifying $p(\lambda_i) = 0$ for each i there exists $\epsilon > 0$ such that $||G + \epsilon p||_\infty < 1$.

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For $n \geq 2$, an irreducible algebraic variety $V \subset \mathbb{C}^n$ is called **inner** if it meets $\mathbb{D}^n$ and exits $\mathbb{D}^n$ through the n-torus, i.e. $V \cap \mathbb{D}^n \neq \emptyset$ and $V \cap \partial(\mathbb{D}^n) \subset \mathbb{T}^n$.

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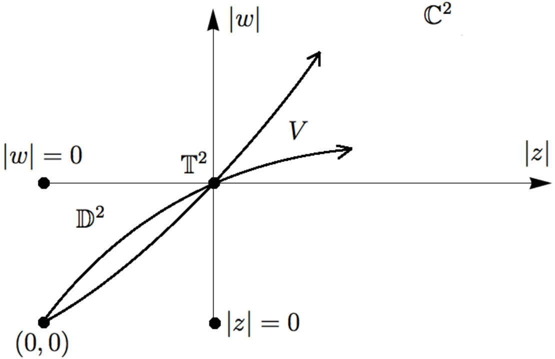

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$$(0, ..., 0), (1/2, ..., 1/2) \in \mathbb{D}^n \text{ and } 0, 1/2 \in \mathbb{D}$$

is extremal. For $V = \{(z, ..., z) : z \in \mathbb{C}\}$ all solutions agree on the set $V \cap \mathbb{D}^n$, i.e. $V \cap \mathbb{D}^n \subset U$. Furthermore, $U = V \cap \mathbb{D}^n$.

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Theorem (Agler and McCarthy, 2001): Given an extremal Pick problem on $\mathbb{D}^2$, there exists a 1-dimensional inner variety V with the property that all solutions agree on $V \cap \mathbb{D}^2$. That is, $V \cap \mathbb{D}^2 \subset U$.

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F is uniquely determined in $\mathcal{S}(\mathbb{D})$ by it's values on $\lambda_1, ..., \lambda_N$. That is, if $G \in \mathcal{S}(\mathbb{D})$ satisfies $G(\lambda_i) = F(\lambda_i)$ for each i, then $G = F$ on $\mathbb{D}$.

Generalization of Pick's theorem to $V \cap \mathbb{D}^n$.

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Theorem 1: Let $V \subset \mathbb{C}^n$ be an irreducible 1-dimensional inner variety. If F is a rational inner function on $\mathbb{D}^n$ with fewer than N zeros on $V \cap \mathbb{D}^n$ and $\lambda_1, ..., \lambda_N \in V \cap \mathbb{D}^n$ are distinct, then all solutions to the Pick problem with data $\lambda_1, ..., \lambda_N$ and $F(\lambda_1), ..., F(\lambda_N)$ agree on $V \cap \mathbb{D}^n$.

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For $a, b \neq 0$ with $|a| + |b| < 1$, the rational function defined by the following formula is inner on $\mathbb{D}^2$ and has 5 zeros on $\mathcal{N} \cap \mathbb{D}^2$.

$$F(z, w) = \frac{zw + az + bw}{1 + \bar{b}z + \bar{a}w}$$

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For any $\lambda_1, ..., \lambda_6 \in \mathcal{N} \cap \mathbb{D}^2$, Theorem 1 states that every solution to the Pick problem with data $\lambda_1, ..., \lambda_6$ and $F(\lambda_1), ..., F(\lambda_6)$ agrees with F on $\mathcal{N} \cap \mathbb{D}^2$.

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Definition

For a rational inner function F on $\mathbb{D}^n$ define $\deg(F)$, the **degree** of F, by letting $F = \frac{q}{r}$ for $q, r \in \mathbb{C}[z_1, ..., z_n]$ relatively prime and let $\deg(F) = \deg(q)$.

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Theorem 2: For $N \geq 1$ there exist points $\lambda_1, ..., \lambda_{N^n} \in \mathbb{D}^n$ with the following property. If F is a rational inner function on $\mathbb{D}^n$ of degree less than N, then the Pick problem with data $\lambda_1, ..., \lambda_{N^n}$ and $F(\lambda_1), ..., F(\lambda_{N^n})$ has a unique solution.

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Theorem 3: Let F be a rational inner function of one variable of degree less than N, $F(z, w) = f(z)$. If

$$D = \{(z, w_0) : z \in \mathbb{D}, w_0 \in \mathbb{D}^{n-1}\}$$

and $\lambda_1, ..., \lambda_N \in D$ are distinct, then the Pick problem with data $\lambda_1, ..., \lambda_N$ and $F(\lambda_1), ..., F(\lambda_N)$ has a unique solution on $\mathbb{D}^n$.

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When do there exist $F, G \in \mathcal{S}(\mathbb{D}^n)$ such that $F = G$ on $V \cap \mathbb{D}^n$ and $F \neq G$?

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For a rational inner function F on $\mathbb{D}^n$ and an algebraic variety $V \subset \mathbb{C}^n$ we write that n-deg$(V) \leq n$-deg(F) if $F = \frac{q}{r}$ on $\mathbb{D}^n$ with q, r relatively prime, $V = Z_p$ and the degree of p is less than or equal to the degree of q in each variable z_i.

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Theorem 4: Fix a rational inner function F on $\mathbb{D}^n$. If an inner variety V satisfies $n\text{-deg}(V) \leq n\text{-deg}(F)$ and F has no singular points on $V \cap \mathbb{T}^n$, then there exists a rational inner function G on $\mathbb{D}^n$ that satisfies $G = F$ on $V \cap \mathbb{D}^n$ and $G \neq F$ on $\mathbb{D}^n$.

Corollary: For each $1 \leq k \leq n$ there exists a k-dimensional inner variety $V \subset \mathbb{C}^n$ and a Pick problem such that $U = V \cap \mathbb{D}^n$.

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i. We say that V is a **Pick set** for F, if F is uniquely determined on $V \cap \mathbb{D}^n$ by its values on any N distinct points on V when N is greater than the number of zeros of F on $V \cap \mathbb{D}^n$.

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ii. We say that V is a **strong Pick set** for F, if F is uniquely determined on $\mathbb{D}^n$ by its values on V.

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ii. Any variety containing the points $\lambda_1, ..., \lambda_{N^n} \in \mathbb{D}^n$ constructed in Theorem 2 is a strong Pick set for each rational inner function F on $\mathbb{D}^n$ of degree less than N.

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Question: For a rational inner function F on $\mathbb{D}^n$ which algebraic varieties are weak/strong Pick sets for F?

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Definition

For a rational inner function F on $\mathbb{D}^n$ and a 1-dimensional inner variety $V \subset \mathbb{C}^n$, define $\deg_V(F)$, the **degree of F on V**, as the number of zeros of F on $V \cap \mathbb{D}^n$ counted with multiplicity.

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Theorem 5: If F is a rational inner function on $\mathbb{D}^n$ with n-degree $d = (d_1, ..., d_n)$ and V is a 1-dimensional inner variety with rank $m = (m_1, ..., m_n)$, then

$$\deg_V(f) \leq d \cdot m = d_1 m_1 + ... + d_n m_n.$$

Furthermore, equality holds whenever f has no singular points on $V \cap \mathbb{T}^n$.