Conditional expectations onto maximal abelian *-subalgebras

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Introduction

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This is related to a famous 1959 paper of Kadison and Singer, and in fact we recover one of their main results by different methods.

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If you don't care or know about vNas, you can still enjoy the talk by focusing on MASAs of $B(\ell^2)$, which are easy to describe. Up to isomorphism, they are (by the spectral theorem)

- ℓ^∞ ("discrete");
- $L^\infty = L^\infty([0, 1], m)$ ("continuous");
- $L^\infty \oplus \ell^\infty_n$ for some $n \in \{1, 2, \ldots, \infty\}$.

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For a discrete (diagonal) MASA in $B(H)$, Kadison and Singer showed that there is a unique CE, implemented by zeroing out the off-diagonal terms, e.g., $\left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right)$.

A way to build a CE onto a MASA

Let W be the collection of finite sets of projections in $\mathcal{A}$ with sum 1. These "partitions" are partially ordered by refinement; i.e., $F \geq G$ if every element of F is dominated by an element of G.

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For every $F \in V$, we have a "paving" operator on $\mathcal{M}$:

$$x \mapsto x_F = \sum_{p \in F} pxp, \qquad \text{e.g.,} \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \mapsto \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

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Question: Are there any improper CEs onto MASAs?

Facts: old, new, and both

Theorem (Arveson 1967, generalizing Kadison-Singer for $B(\ell^2)$)

If a CE onto a MASA is weak continuous, it is the unique proper CE.*

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The following limited converse is a key step in our work:

Theorem

If $\mathcal{A}$ has a sequential full subset, and there is only one proper CE, then it is weak continuous.*

Whenever $\mathcal{A}$ acts on a separable Hilbert space, or (weaker) is singly-generated, it has a sequential full subset.

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Whenever $\mathcal{A}$ acts on a separable Hilbert space, or (weaker) is singly-generated, it has a sequential full subset.

Kadison and Singer remarked that there is no weak* continuous CE onto a continuous MASA of $B(\ell^2)$, relying on a 1952 result of Kaplansky. From this we get the immediate

Corollary

There is more than one (proper) CE onto a continuous MASA of $B(\ell^2)$.

What Kadison and Singer did

Kadison and Singer wanted to know if a pure state[1] of a MASA of $B(\ell^2)$ has a unique state extension to all of $B(\ell^2)$. This was suggested by Dirac's text on quantum mechanics.

[1] A state is a positive linear functional of norm one, and a pure state is an extreme point of the convex, weak* compact state space.

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Whenever there are multiple CEs onto a MASA, some pure states of the MASA have nonunique extensions.

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Kadison and Singer showed that there is more than one CE onto a continuous MASA of $B(\ell^2)$ via some detailed calculations with Fourier series (=the Corollary of the previous slide).

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The converse of the observation is not known to hold. So although the CE from $B(\ell^2)$ onto a discrete MASA is unique, the uniqueness of pure state extensions in this case is still OPEN.

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Theorem

If $\mathcal{A}$ is a singly-generated MASA in a semifinite vNa $\mathcal{M}$, then there is a unique CE if and only if $\mathcal{A}$ is generated by projections that are abelian[a] in $\mathcal{M}$. In particular, this can only happen if $\mathcal{M}$ is type I.

[a] A projection $p \in \mathcal{M}$ is *abelian* if $p\mathcal{M}p$ is an abelian algebra.

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Reason: it is a standard fact that there is a weak* continuous CE, so by Arveson's theorem this is the only proper CE. By the theorem above there are others.