Some new developments in the theory of Toeplitz operators with matrix almost periodic symbols

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Based on joint work with Cristina Câmara and Yuri Karlovich.

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and (time permitting, the last couple of sentences)

with Alex Brudnyi and Leiba Rodman.

An almost periodic (AP for short) polynomial by definition is a linear combination of the exponential functions $e_\lambda(x) =: e^{i\lambda x}$ with real parameters λ over the complex field $\mathbb{C}$; the set of all AP polynomials is denoted APP. In other words, $f \in APP$ if and only if it is a finite sum of the form

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for some $\lambda_j \in \mathbb{R}$, $c_j \in \mathbb{C}$. The Wiener norm of f given by (1) is

$$\|f\|_W = \sum_j |c_j| \, ; \tag{2}$$

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the closure of APP with respect to this norm is the algebra APW of Wiener AP functions. It consists of all series (1), finite or not, for which the right hand side of (2) is finite. On the other hand, the closure of APP with respect to the usual uniform norm is the algebra AP of Bohr AP functions.

According to Bohr mean value theorem, for every $f \in AP$ there exists

$$\mathbf{M}(f) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\, dx,$$

called the mean value of f. Consequently, for all $\lambda \in \mathbb{R}$ there exist the mean values of $e_{-\lambda} f$, denoted $\widehat{f}(\lambda)$ and called the Bohr-Fourier coefficients of f. As it happens, at most countably many of them are different from zero. The respective values of λ form the Bohr-Fourier spectrum of f:

$$\Omega(f) = \{\lambda \in \mathbb{R} \colon \widehat{f}(\lambda) \neq 0\}.$$

Of course, for an APP or APW function f given by (1), $c_j = \widehat{f}(\lambda_j)$.

If $f \in AP$ is bounded away from zero, then it is actually invertible in AP. Moreover, then there exists (obviously, unique) real κ such that a continuous branch of $\log(e_{-\kappa} f)$ also belongs to AP. This κ is called the mean motion of f, and in what follows will be denoted $\kappa(f)$.

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We denote by AP^+ (APW^+, APP^+) the subalgebra of AP (resp., APW, APP) consisting of the functions f with $\Omega(f) \subset \mathbb{R}_+$; the classes AP^-, APW^-, APP^- are defined in a similar way. Finally, relations $F \in X$ for the above mentioned functional classes X in the case of a vector or a matrix function F are understood entry-wise, and $\Omega(F)$ will denote the union of the Bohr-Fourier spectra of its entries.

A (right) *AP* factorization of an $n \times n$ matrix function G is its representation in the form

$$G = G_- \Lambda G_+^{-1}, \tag{3}$$

where $G_+^{\pm 1} \in AP^+$, $G_-^{\pm 1} \in AP^-$ and the middle factor Λ is diagonal, with the diagonal entries of the form e_{λ_j} (cf. [KarlSpit89,BKS]; see [BKS1] also for the motivations behind the notion of *AP* factorization and its various applications).

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We say that (3) is an APW factorization of G if in fact $G_+^{\pm 1} \in APW^+$, $G_-^{\pm 1} \in APW^-$. Obviously, the matrix function G must lie in AP (APW) and be invertible there in order to admit an AP (resp, APW) factorization (3). Moreover, the sum of its partial AP indices is nothing but $\kappa(\det G)$. Since the algebra APW is inverse closed, an AP factorization (3) of $G \in APW$ is actually its APW factorization as soon as at least one of the factors $G_\pm, G_\pm^{-1}$ belongs to APW. Furthermore, a canonical AP factorization of an APW matrix function, if it exists, is automatically its APW factorization[Spit89], Theorem 1 (see also Section 9.4 in [BKS]).

The factorization (3) is closely related to the Riemann-Hilbert problem

$$G\psi^+ = \psi^-, \tag{4}$$

in which the unknown vector-functions $\psi^\pm$ are analytic in the upper/lower half plane $\mathbb{C}^\pm$, respectively. On the one hand, the description of all solutions to (4) can be given in terms of (3). On the other hand, the existence criterion and formulas for the factors in (3) can be given in terms of special solutions to (4).

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Recall that the set of functions $\phi = (\phi_1, \dots, \phi_n)$ satisfies the corona condition (notation: $\phi \in CP(D)$) if ϕ_j are analytic and bounded on D, and

$$\inf_{z \in D} \max_{j=1,\dots,n} |\phi_j(z)| > 0. \tag{5}$$

In what follows, $CP(\mathbb{C}^\pm)$ is abbreviated to $CP^\pm$.

Theorem

Let G be a 2×2 invertible APW matrix function with $\det G$ having zero mean motion. Then G admits an APW factorization if and only if problem (4) has an APW solution with $\psi^- \in CP^-$ and $\psi^+ \in e_{-\delta}CP^+$ for some $\delta \geq 0$. If this is the case, then the partial AP indices of G equal $\pm\delta$ and the factors $G_\pm$ can be chosen in such a way that $e_{-\delta}\psi^+$ and ψ^- form the first column of G_+ and G_-, respectively.

Because of Theorem 1, the pairs of functions in $APW \cap CP^{\pm}$ are of special interest to us. In particular, the following simple observation is useful.

Lemma

Let $\phi_1, \phi_2 \in APW^{\pm}$. Then for $(\phi_1, \phi_2) \in CP^{\pm}$ it is necessary (and if one of these functions is a monomial, also sufficient) that $0 \in \Omega(\phi_1) \cup \Omega(\phi_2)$. Moreover, in the latter case actually $(\phi_1, \phi_2) \in CP(D)$ for any set $D \subset \mathbb{C}$ the projection of which onto the y-axis is bounded from below/above.

Suppose the Bohr-Fourier spectrum of $f \in AP$ is bounded below. Then f admits (a unique) analytic extension into the upper half plane; with a slight abuse of notation, we denote the extended function by the same symbol f. If f is not identically zero, then for all but countably many values of $y \in \mathbb{R}_+$ the mean motion $\kappa(f_y)$ of the function $f_y(x) =: f(x + iy)$ will exist, and moreover (see [Levin], Chapter II.6)

$$\lim_{y \to +\infty} \kappa(f_y) = -\inf \Omega(f). \tag{6}$$

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Since the mean motion of the product equals the sum of the mean motions, (6) implies that [1]

$$\inf \Omega(fg) = \inf \Omega(f) + \inf \Omega(g) \qquad (7)$$

whenever $\inf \Omega(f), \inf \Omega(g) > -\infty$.

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Note that (7) is obvious when both $\inf \Omega(f)$ and $\inf \Omega(g)$ are attained, and fails if either of them is allowed to equal $-\infty$. An easy counterexample is delivered by any f invertible in AP^- and $g = f^{-1}$. In particular, $f = 1 + ce_{-1}$ with $|c| < 1$ will do.

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Passing to complex conjugates, it is easy to derive from (7) that

$$\sup \Omega(fg) = \sup \Omega(f) + \sup \Omega(g) \quad \text{provided that } \sup \Omega(f), \sup \Omega(g) < +\infty. \tag{8}$$

Theorem

Let G be a 2×2 invertible APW matrix function with $\det G$ having zero mean motion and $\Omega(G)$ bounded below. Suppose that problem (4) has an APW solution (ψ^+, ψ^-) in which at least one component of ψ^- is a monomial and the set $\Omega(\psi^-)$ contains its supremum δ_-. Denote $\delta_+ = \inf \Omega(\psi^+)$.

(i) *If $\delta_+ \in \Omega(\psi^+)$, then G admits an APW factorization, and its partial AP indices equal $\pm(\delta_+ - \delta_-)$.*

(ii) *If $\delta_+ \notin \Omega(\psi^+)$, then G is not AP factorable.*

Multiplying both ψ^+ and ψ^- by $e_{-\delta_-}$, we may without loss of generality suppose that $\delta_- = 0$, and therefore $\delta_+ - \delta_- = \delta_+$. According to Lemma 2 (and in the notation of its proof), $\psi^- \in CP(\mathbb{C}^- \cup S_a)$ for any $a > 0$.

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(i) Observe that all the entries of G and $\psi^\pm$ are entire functions. So, (4) holds not only on $\mathbb{R}$ but actually everywhere on $\mathbb{C}$. Since G is bounded on S_a, from $\psi^- \in CP(\mathbb{C}^- \cup S_a)$ it follows that $\psi^+ \in CP(S_a)$. Equivalently, $e_{-\delta_+}\psi^+ \in CP(S_a)$.

On the other hand, at least one of the functions $e_{-\delta_+}\psi_j^+ \in APW^+$ $(j = 1, 2)$ has a non-zero mean value. Therefore, this function is bounded away from zero on $\mathbb{C}^+ \setminus S_a$ for a large enough. This proves that $e_{-\delta_+}\psi^+ \in CP^+$. By Theorem 1. G is APW factorable with partial AP indices equal $\pm\delta_+$.

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On the other hand, at least one of the functions $e_{-\delta_+}\psi_j^+ \in APW^+$ $(j = 1, 2)$ has a non-zero mean value. Therefore, this function is bounded away from zero on $\mathbb{C}^+ \setminus S_a$ for a large enough. This proves that $e_{-\delta_+}\psi^+ \in CP^+$. By Theorem 1. G is APW factorable with partial AP indices equal $\pm\delta_+$.
(ii) If G admits a canonical AP factorization, then ψ^+ can be used as a respective column of the factor G_+ for some of them. Consequently, $\psi^+ \in CP^+$, and $0 \in \Omega(\psi_1^+) \cup \Omega(\psi_2^+)$ by Lemma 2.

On the other hand, if an AP factorization of G is non-canonical, its partial AP indices are $\delta(>0)$ and $-\delta$, and according to [CDKS]

$$\psi^{+} = fg_1^{+}, \quad \psi^{-} = e_{-\delta} f g_1^{-}. \tag{9}$$

Here $g_1^{\pm}$ are matching columns of $G_{\pm}$ from the factorization (3), while f is an AP function with $\Omega(f) \subset [0, \delta]$. Note that $\Omega(G_{-})$ is bounded from below along with $\Omega(G)$, and obviously bounded from above by 0. Consequently, formulas (7) and (8) are applicable to the equalities

$$\psi_j^{-} = e_{-\delta} f g_{1j}^{-},$$

obtained by entry-wise rewriting the second part of (9).

Choosing the value $j = 1, 2$ for which ψ_j^- is a monomial, we conclude that

$$\sup \Omega(f) + \sup \Omega(g_{1j}^-) = \inf \Omega(f) + \inf \Omega(g_{1j}^-).$$

Therefore, f is a monomial as well. Since $g_1^- \in CP^-$ and $\delta_- = 0$, the second equation in (9) implies that $f = e_\delta$. But then $e_{-\delta}\psi^+ = g_1^+$ according to the first equation in (9), and therefore lies in CP^+. In particular (Lemma 2 again), $\delta = \min \Omega(\psi^+)$.
Note that the proof of sufficiency is based on ideas similar to those of [Camara-Diogo'08].

Let

$$G = \begin{bmatrix} e_{-\lambda} & 0 \\ g & e_{\lambda} \end{bmatrix}, \tag{10}$$

where

$$g = c_{-1}e_{-\sigma} + T_+ e_\mu, \quad 0 < \sigma, \mu < \lambda, \ T_+ \in APW \text{ with } \inf \Omega(T_+) = 0. \tag{11}$$

Using the notation $\mu + \sigma = \kappa$ and $n = \left\lceil \frac{\lambda}{\kappa} \right\rceil$, suppose that

$$n\tau \leq \min\{\mu, \lambda - n\kappa\} \tag{12}$$

and

$$\tau \in \Omega(T_+) \text{ if } n\tau > \lambda - n\kappa - \sigma, \tag{13}$$

where $\tau = \sup \Omega(T_+)$.

Theorem

Under the conditions listed, the matrix function G given by (10) *is* APW *factorable if* $0 \in \Omega(T_+)$ *and not* AP *factorable otherwise. In the former case, the partial* AP *indices of* G *equal* $\pm\delta$ *with* δ *determined by*

$$\delta = \begin{cases} \min\{\sigma, (n+1)\kappa - \lambda\} & \text{if } n\tau \leq \lambda - n\kappa - \sigma, \\ \min\{\lambda - n\kappa, \mu\} - n\tau & \text{if } n\tau > \lambda - n\kappa - \sigma. \end{cases} \tag{14}$$

Let now g be given by

$$g = b + e_\alpha T_+ \tag{15}$$

with b being a binomial $c_{-1}e_{-\sigma} + c_1 e_\mu$, and

$$\mu < \alpha < \lambda \leq \alpha + \sigma, \ \inf \Omega(T_+) = 0. \tag{16}$$

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Theorem

Let (15)–(16) *hold. Denote* $\xi := \inf(\Omega(T_+) \setminus \{0\})$. *The matrix function* (10) *is not AP factorable if either*

$$\alpha < \mu + \kappa n, \quad \mathbf{M}(T_+) = 0, \tag{17}$$

or

$$\mu + \kappa n = \alpha, \quad \mathbf{M}(T_+) = -c_1 c^n \tag{18}$$

while $\lambda - \alpha > \xi \notin \Omega(T_+)$.

Otherwise, G is APW factorable, with partial AP indices $\pm\delta$ computed according to the rule:

$$\delta = \begin{cases} \min\{\mu, \alpha - \kappa n\} & \text{if } \lambda - \kappa n \leq \sigma, \\ \min\{\kappa(n+1) - \lambda, \alpha + \sigma - \lambda\} & \text{if } \lambda - \kappa n \geq \sigma \end{cases} \quad (19)$$

when either $\mu + \kappa n < \alpha$, or

$$\mu + \kappa n = \alpha \text{ and } \mathbf{M}(T_+) \neq -c_1 c^n, \quad (20)$$

or

$$\mu + \kappa n > \alpha \text{ and } \mathbf{M}(T_+) \neq 0; \quad (21)$$

$$\delta = \min\{\lambda - \kappa n, \sigma\} \tag{22}$$

when (18) holds while $\xi \geq \lambda - \alpha$, and

$$\delta = \begin{cases} \alpha + \xi - \kappa n & \text{if } \lambda - \kappa n \leq \sigma, \\ \alpha + \sigma + \xi - \lambda & \text{if } \lambda - \kappa n \geq \sigma \end{cases} \tag{23}$$

when (18) holds with $\Omega(T_+) \ni \xi < \lambda - \alpha$.

Finally, let in (10) g be a trinomial

$$g = c_{-1}e_{-\sigma} + c_1 e_\mu + c_2 e_\alpha \tag{24}$$

with non-zero coefficients c_j and $\sigma, \mu, \alpha \in (0, \lambda), \mu < \alpha$. This may be thought of as a particular case of (15), with T_+ being constant. Our additional conditions on the exponents will be different however. Since the case $\alpha + \sigma \geq \lambda$ in (24) has been studied earlier ([QRS], see also [BKS], Section 15 for the systematic exposition and a recent [KarlSpit10] for more constructive treatment), it is natural to concentrate on the case

$$\alpha + \sigma \leq \lambda. \tag{25}$$

We may also suppose that

$$\nu := 2\mu + \sigma - \alpha \neq 0,$$

because otherwise $\alpha - \mu = \mu + \sigma$, and the distances between the points of $\Omega(g)$ are commensurable — a well known situation (see e.g., Section 14.4 in [BKS]). We will, however, impose more restrictive conditions

$$2\kappa \geq \lambda, \quad \nu > 0, \tag{26}$$

along with

$$k\nu < \alpha - \mu, \tag{27}$$

where

$$k = \left\lceil \frac{\sigma}{\nu} \right\rceil - 1. \tag{28}$$

Note that we could allow the equality in (27) but this would again yield the commensurability of the distances between the points of $\Omega(g)$ and therefore is not interesting.

Theorem

Let G be of the form (10) *with g given by* (24) *under conditions* (25)–(27)*. Then G admits an APW factorization, and its partial AP indices equal $\pm\delta$ with δ given by*

$$\delta = \min\{2\kappa - \lambda, \sigma - k\nu, \lambda - (\alpha + \sigma), k\nu + 2\mu - \alpha\}. \qquad (29)$$

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