Boundaries of Holomorphic Chains within Vector Bundles over Complex Projective Space

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A holomorphic p-chain is a (locally finite) linear combination of p-dimensional analytic varieties with integer multiplicities.

e.g., $2 \cdot \mathbb{V}(w - e^z) - 3 \cdot \mathbb{V}(w^2 - z^3)$

Preliminaries and Background

Let M be a closed rectifiable current of dimension $2p-1$ with support satisfying condition A_{2p-1} in a complex manifold Z.

Definition: We say that M *bounds a holomorphic p-chain within Z* if there exists a holomorphic p-chain T in $Z \backslash \mathrm{spt}\ M$ with a simple extension as a current to Z such that

- $dT = M$ (in the sense of currents, namely $\int_T d\omega = \int_M \omega$ for compactly supported $(2p-1)$-forms ω)
- $\mathrm{spt}\ T \Subset Z$

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$\mathcal{O}_{\mathbb{CP}^1}(d)$, $p = 1$ - relations on Wermer moments, but their nature depends on d (W. 2010)

Let E be a vector bundle over a connected complex manifold X of complex dimension p (e.g., $\mathbb{CP}^p$) with projection map $\pi : E \to X$. Suppose that T is a holomorphic p-chain bounded by M within E. Furthermore suppose such that $\pi|_{\operatorname{spt} T}$ is a proper map.

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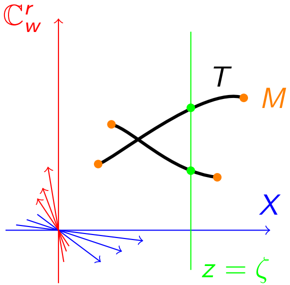

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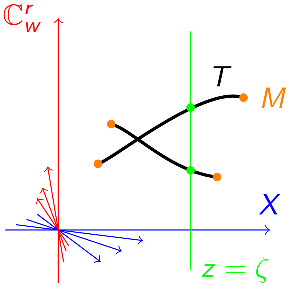

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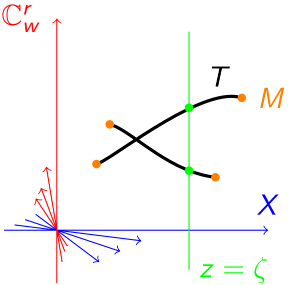

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So for M to bound in E it is necessary that:

For all $\alpha \geq 0$, $[\pi_*(w^\alpha M)]^{0,1}$ is $\bar\partial$-exact, i.e., $[\pi_*(w^\alpha M)]^{0,1}$ corresponds to zero in $H^1_{\mathrm{cpt}}(X, S^{|\alpha|}(E))$

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The essential issue can be seen on level of fibers. So let us consider the 0-chain $S = \sum_j n_j \cdot (w_{j,1}, w_{j,2}, \ldots, w_{j,r})$ in $\mathbb{C}^r$. Let p_α denote its power sum

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Define the generating function $P[\lambda] = \sum_{\alpha \geq 0, \alpha \neq 0} p_\alpha \lambda^\alpha$. By standard geometric series techniques, it holds that $P[\lambda] = \sum_j \frac{w_j \cdot \lambda}{1 - w_j \cdot \lambda} = \sum_j \frac{w_{j,1}\lambda_1 + w_{j,2}\lambda_2 + \cdots + w_{j,r}\lambda_r}{1 - (w_{j,1}\lambda_1 + w_{j,2}\lambda_2 + \cdots + w_{j,r}\lambda_r)}$.

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Define $E[\lambda] = 1 + \sum_{\alpha \geq 0, \alpha \neq 0} e_\alpha \lambda^\alpha = \prod_j (1 + w_j \cdot \lambda_j)^{n_j}$, as the generating function of the *extended elementary multisymmetric functions* e_α of S. (If the 0-chain is has non-negative multiplicities, then $E[\lambda]$ becomes a standard finite generating function of the elementary multisymmetric functions.)

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$$E[\lambda] = \exp\left(\int_0^1 -P[-t\lambda]\, dt \right).$$

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Furthermore, $E[\lambda]$ corresponds to a finite 0-chain if and only if $E[\lambda]$ is a rational function that completely splits into linear factors in terms of λ. (Furthermore, this correspondence is unique with the only exception of the multiplicity of the point $w = 0$.)

Consider the multivariate power series $E[\lambda] = 1 + \sum_{\alpha \geq 0, \alpha \neq 0} e_\alpha \lambda^\alpha$. Given prescribed bounds M and N, $E[\lambda]$ is a rational function with numerator having degree bounded by M and denominator having degree bounded by N if and only if certain determinantal relations (in essence due to criteria by Kroenecker) on e_α are satisfied.

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For example $1 + e_{0,1}\lambda_2 + e_{1,0}\lambda_1 + e_{0,2}\lambda_2^2 + e_{1,1}\lambda_1\lambda_2 + e_{2,0}\lambda_1^2$ splits into a product of two linear factors if and only if

$$e_{1,1}^2 - e_{1,0}e_{0,1}e_{1,1} + e_{0,1}^2 e_{2,0} + e_{1,0}^2 e_{0,2} - 4e_{2,0}e_{0,2} = 0$$

Note, if $E[\lambda t]$ is rational with respect to t for all $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, this implies that $E[\lambda]$ is rational with respect to λ, but it does not imply that $E[\lambda]$ completely splits into linear factors.

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Example: $1 + \lambda_1 + \lambda_2 + k\lambda_1\lambda_2$, $k \neq 0, 1$.

Theorem

Let M be a rectifiable $2p-1$ chain with support satisfying condition A_{2p-1} such that $\pi(\mathrm{spt} M)$ satisifes condition A_{2p-1} and $\pi|_{\mathrm{spt} M \setminus S}$ is injective for some H^{2p-1}-measure zero subset S of $\mathrm{spt} M$. M bounds a holomorphic 1-chain within the vector bundle E if and only if the following hold

1. *M is maximally complex,*
2. *there exist $p_\alpha \in H^0_{cpt}(X, S^{|\alpha|}(E))$ such that $\bar{\partial} p_\alpha = \pi_*(w^\alpha \gamma)^{0,1}$ (in other words $\pi_*(w^\alpha \gamma)^{0,1}$ is $\bar{\partial}$-exact in $H^1_{cpt}(\mathbb{CP}^1, S^{|\alpha|}(E))$), and*
3. *in a neighborhood of some $\zeta^* \in X \setminus \pi(\mathrm{spt} M)$ there exist r_α in $Z^{0,0}(X, S^{|\alpha|}(E))$ such that $\exp\left(-\int_0^1 \sum_{\alpha \geq 0, \alpha \neq 0} (p_\alpha + r_\alpha)(-t\lambda)^\alpha \, dt\right)$ is a rational function that completely splits into linear factors.*