Boundaries of Holomorphic Chains within Vector Bundles over Complex Projective Space

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March 19, 2011
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A holomorphic $p$-chain is a (locally finite) linear combination of $p$-dimensional analytic varieties with integer multiplicities.

e.g., $2 \cdot \mathbb{V}(w - e^z) - 3 \cdot \mathbb{V}(w^2 - z^3)$
Let $M$ be a closed rectifiable current of dimension $2p - 1$ with support satisfying condition $A_{2p-1}$ in a complex manifold $Z$.

Definition: We say that $M$ bounds a holomorphic $p$-chain within $Z$ if there exists a holomorphic $p$-chain $T$ in $Z \setminus \text{spt } M$ with a simple extension as a current to $Z$ such that

- $dT = M$ (in the sense of currents, namely $\int_T d\omega = \int_M \omega$ for compactly supported $(2p - 1)$-forms $\omega$)
- $\text{spt } T \subseteq Z$
Some Known Characterizations

\( \mathbb{C}^n, p = 1 \) - vanishing moment conditions (Wermer 1958, Harvey, Lawson 1975, Dinh 1998)
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\( \mathcal{O}_{\mathbb{C}P^1}(d), p = 1 \) - relations on Wermer moments, but their nature depends on \( d \) (W. 2010)
Let $E$ be a vector bundle over a connected complex manifold $X$ of complex dimension $p$ (e.g., $\mathbb{CP}^p$) with projection map $\pi : E \to X$. Suppose that $T$ is a holomorphic $p$-chain bounded by $M$ within $E$. Furthermore suppose such that $\pi|_{spt \, T}$ is a proper map.
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Moreover $\overline{\partial} \pi_*(w^\alpha T)(\zeta) = \left[\pi_*(w^\alpha M)\right]_{0,1}$.

So for $M$ to bound in $E$ it is necessary that:

For all $\alpha \geq 0$, $\left[\pi_*(w^\alpha M)\right]_{0,1}$ is $\overline{\partial}$-exact, i.e., $\left[\pi_*(w^\alpha M)\right]_{0,1}$ corresponds to zero in $H^1_{\text{cpt}}(X, S|_\alpha|(E))$.

$p_\alpha(\zeta) = \pi_*(w^\alpha T)(\zeta) = \sum_j n_j w_j^\alpha$, where $T \cap \{z = \zeta\} = \sum_j n_j w_j$. 

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$$\bar{\partial}_\pi (w^\alpha T) = [\pi_*(w^\alpha M)]^{0,1}.$$
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![Diagram of vector bundle and manifold](image)

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The essential issue can be seen on level of fibers. So let us consider the 0-chain $S = \sum_j n_j \cdot (w_{j,1}, w_{j,2}, \ldots, w_{j,r})$ in $\mathbb{C}^r$. Let $p_\alpha$ denote its power sum

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Define the generating function $P[\lambda] = \sum_{\alpha \geq 0, \alpha \neq 0} p_{\alpha} \lambda^\alpha$. By standard geometric series techniques, it holds that

$$P[\lambda] = \sum_j \frac{w_{j} \cdot \lambda}{1 - w_{j} \cdot \lambda} = \sum_j \frac{w_{j,1} \lambda_1 + w_{j,2} \lambda_2 + \cdots + w_{j,r} \lambda_r}{1 - (w_{j,1} \lambda_1 + w_{j,2} \lambda_2 + \cdots + w_{j,r} \lambda_r)}.$$
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\]

Define $E[\lambda] = 1 + \sum_{\alpha \geq 0, \alpha \neq 0} e_\alpha \lambda^\alpha = \prod_j (1 + w_j \cdot \lambda_j)^{n_j}$, as the generating function of the extended elementary multisymmetric functions $e_\alpha$ of $S$. (If the 0-chain is has non-negative multiplicities, then $E[\lambda]$ becomes a standard finite generating function of the elementary multisymmetric functions.)
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\( E[\lambda] \) can be readily constructed from \( P[\lambda] \) by means of the following generalization of the Newton formulae

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E[\lambda] = \exp \left( \int_0^1 -P[-t\lambda] \, dt \right).
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\[ E[\lambda] = \exp \left( \int_0^1 -P[-t\lambda] \, dt \right). \]

Furthermore, \( E[\lambda] \) corresponds to a finite 0-chain if and only if \( E[\lambda] \) is a rational function that completely splits into linear factors in terms of \( \lambda \). (Furthermore, this correspondence is unique with the only exception of the multiplicity of the point \( w = 0 \).)
Consider the multivariate power series $E[\lambda] = 1 + \sum_{\alpha \geq 0, \alpha \neq 0} e_{\alpha} \lambda^{\alpha}$. Given prescribed bounds $M$ and $N$, $E[\lambda]$ is a rational function with numerator having degree bounded by $M$ and denominator having degree bounded by $N$ if and only if certain determinantal relations (in essence due to criteria by Kroenecker) on $e_{\alpha}$ are satisfied.
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Moreover, a (multivariate) polynomial of degree \( d \) completely splits into linear factors is equivalent to the coefficients of the polynomial residing in the Chow variety of 0-dimensional varieties of degree \( d \) in \( \mathbb{C}^r \).
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This latter condition is vacuous in the case $r = 1$ (or when $d = 0, 1$, but not when $r \geq 2$ and $d \geq 2$.

For example $1 + e_{0,1} \lambda_2 + e_{1,0} \lambda_1 + e_{0,2} \lambda_2^2 + e_{1,1} \lambda_1 \lambda_2 + e_{2,0} \lambda_1^2$ splits into a product of two linear factors if and only if

$$e_{1,1}^2 - e_{1,0} e_{0,1} e_{1,1} + e_{0,1}^2 e_{2,0} + e_{1,0}^2 e_{0,2} - 4 e_{2,0} e_{0,2} = 0$$
Note, if $E[\lambda t]$ is rational with respect to $t$ for all $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, this implies that $E[\lambda]$ is rational with respect to $\lambda$, but it does not imply that $E[\lambda]$ completely splits into linear factors.
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Example: $1 + \lambda_1 + \lambda_2 + k\lambda_1\lambda_2$, $k \neq 0, 1$. 

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Theorem

Let \( M \) be a rectifiable \( 2p - 1 \) chain with support satisfying condition \( A_{2p-1} \) such that \( \pi(sptM) \) satisfies condition \( A_{2p-1} \) and \( \pi|_{sptM\setminus S} \) is injective for some \( H^{2p-1} \)-measure zero subset \( S \) of \( sptM \). \( M \) bounds a holomorphic 1-chain within the vector bundle \( E \) if and only if the following hold

1. \( M \) is maximally complex,
2. there exist \( p_\alpha \in H^0_{cpt}(X, S|\alpha|(E)) \) such that \( \bar{\partial}p_\alpha = \pi_*(w^\alpha \gamma)^{0,1} \) (in other words \( \pi_*(w^\alpha \gamma)^{0,1} \) is \( \bar{\partial} \)-exact in \( H^1_{cpt}(\mathbb{CP}^1, S|\alpha|(E)) \)), and
3. in a neighborhood of some \( \zeta^* \in X \setminus \pi(sptM) \) there exist \( r_\alpha \) in \( Z^{0,0}(X, S|\alpha|(E)) \) such that
   \[
   \exp \left( -\int_0^1 \sum_{\alpha \geq 0, \alpha \neq 0} (p_\alpha + r_\alpha)(-t\lambda)\alpha dt \right)
   \]
   is a rational function that completely splits into linear factors.