Essential normality of the cyclic submodule generated by any polynomial

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Outline

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Hilbert module

Definition [Douglas,Paulsen]

Let A be a function algebra, and let H be a Hilbert space. The space H is called a Hilbert module over A provided that H is equipped with a mapping $A \times H \to H$, which we denote $(f, h) \to f \cdot h$, satisfying:
(1) $1 \cdot h = h$,
(2) $(fg) \cdot h = f \cdot (g \cdot h)$,
(3) $(f + g) \cdot h = f \cdot h + g \cdot h$,
(4) $f \cdot (\alpha h + \beta k) = \alpha(f \cdot h) + \beta(f \cdot h)$,
(5) $\exists K_f, \|f \cdot h\| \leq K_f \|h\|$,
for every f, $g \in A$, h, $k \in H$, $\alpha, \beta \in \mathbb{C}$.

Hilbert module

Example

Let $\mathbf{T} = (T_1, \cdots, T_n)$ be a tuple of commuting operators acting on a Hilbert space H. Then, one naturally makes H into a Hilbert module over the polynomial ring $C[z_1, \cdots, z_n]$. The $C[z_1, \cdots, z_n]$ module structure is defined by

$$p \cdot \xi = p(T_1, \cdots, T_n)\xi, \quad p \in C[z_1, \cdots, z_n], \ \xi \in H.$$

The tuples $\mathbf{T}, \mathbf{T}'$ are unitarily equivalent if and only if the corresponding Hilbert modules are unitarily equivalent as $\mathbb{C}[z_1, \cdots, z_n]$ module.

Hilbert module

Example

Let M be a subspace of $\mathbb{C}[z_1, \cdots, z_n]$ module H. We call M submodule if for any polynomial $p \in \mathbb{C}[z_1, \cdots, z_n]$, $h \in M$,

$$p \cdot h \in M.$$

The subspace $H \ominus M$ has the natural quotient structure which is defined by

$$p \cdot h = P_{H \ominus M}(p \cdot h), \quad \forall h \in H \ominus M$$

Essentially normal Hilbert module

Definition

A $\mathbb{C}[z_1, \cdots, z_n]$ Hilbert module H is said to be essentially normal if the commutators

$$[T_k^*, T_j] = T_k^* T_j - T_j T_k^* \in \mathcal{K},$$

where T_i denotes the action of z_i on H for any $1 \leq i \leq n$, $\mathcal{K}$ is the compact operator ideal.

By the spectrum theory, the essentially normal Hilbert module yields an extension:

$$0 \to \mathcal{K} \hookrightarrow C^*(H) \xrightarrow{\pi} C(\sigma_e(H)) \to 0.$$

p-essential normal

Definition

A $\mathbb{C}[z_1, \cdots, z_n]$ Hilbert module is said to be p-essentially normal if the commutators

$$[T_k^*, T_j] = T_k^* T_j - T_j T_k^* \in \mathcal{L}^p,$$

where T_i denotes the action of z_i on H for any $1 \leq i \leq n$, $0 < p \leq \infty$ and $\mathcal{L}^p$ is the Schatten-p class.

Example: Arveson-Drury module and Bergman module over the unit ball $\mathbb{B}_n$ are p-essentially normal for $p > n$.

BDF Theory

- Given a compact metric X, we call (H, E, π) is an extension of $C(X)$ by the compact operator ideal $\mathcal{K}$ if

$$0 \to \mathcal{K} \to E \xrightarrow{\pi} C(X) \to 0,$$

is exact.

Equivalence $\sim$

We say the extension (H_1, E_1, π_1) and (H_2, E_2, π_2) are equivalent if there exists a unitary operator $U : H_1 \to H_2$ making the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{K} & \to & E_1 & \to & C(X) & \to & 0 \\ & & \downarrow U & & \downarrow U & & \| id & & \\ 0 & \to & \mathcal{K} & \to & E_2 & \to & C(X) & \to & 0 \end{array}$$

BDF Theory

Let $Ext(X)$ be the equivalent classes of extensions on X, that is

$$Ext(X) = \{(H, E, \pi) : 0 \to \mathcal{K} \to E \xrightarrow{\pi} C(X) \to 0\}/\sim,$$

Brown, Douglas and Fillmore showed that

- $Ext(X)$ is a group and it is a homotopic invariant.
- $Ext(X) = K_1(X)$.

Example

Example[Arveson]

Let M be the submodule generated by the polynomial $z_1^d + z_2^d - z_3^d$ and $\Gamma = \{z \in \mathbb{C}^3 : \ z_1^d + z_2^d = z_3^d\} \cap \partial\mathbb{B}_3$ be the corresponding zero variety. If M is essentially normal, then we have

$$0 \to \mathcal{K} \hookrightarrow C^*(M^{\perp}) \to C(\Gamma) \to 0,$$

which yields a K-homology element of zero variety Γ.

Question

Conjecture[Arveson]

If M is a graded submodule of $H_n^2 \otimes \mathbb{C}^r$, then M is p-essential normal for $p > n$.

Conjecture[Douglas]

If M is a submodule of $L_a^2(\mathbb{B}_n) \otimes \mathbb{C}^r$ generated by vector homogeneous polynomials, then M is p-essentially normal for $p > n$.

Affirmative results

- Arveson established the p-essential normality in the case of monomials.
- Douglas generalized it to cases in which H_n^2 is replaced by more general weighted shifts.
- Arveson reduced the problem of establishing essential normality of submodules to cases of linearized submodules.

Affirmative results

- Guo showed that the grade submodule of H_n^2 is essentially normal in the dimension $n = 2$.
- Guo and Wang proved each graded principal submodule of H_n^2 is p-essentially normal for $p > n$.
- Xia showed that, for any radical set H and $V = H \cap \partial\mathbb{B}_n$, there exists a quotient module of $L_a^2(\partial\mathbb{B}_n)$ is 1-essentially normal Hilbert module with the essential spectrum V.

cyclic submodule generated by any polynomial

Theorem

If $\mathcal{M} = [p]$ is the submodule of the Bergman space $L_a^2(\mathbb{B}_n)$ generated by a polynomial p, then $\mathcal{M}$ is t-essentially normal for $t > n$.

the weighted Bergman space $L_a^2(|p|^2dm)$

Corollary

Let p be a polynomial with degree m. Then

- The embedding mapping $L_a^2(|p|^2dm) \to L_a^2((1-z)^{2m}dm)$ is bounded.
- $L_a^2(|p|^2dm)$ is isomorphism to $[p]$, and then it's essentially normality.
- For $f \in L^2(dm)$, $f \in [p]$ iff $f = ph$ for some analytic function h on $\mathbb{B}_d$.

Outline of Proof

Notation

- Define the number operator N by $N(z^\alpha) = |\alpha| z^\alpha$ for any non-negative multi-index α.
- Let $\partial_i = \partial_{z_i}, \bar{\partial}_i = \partial_{\bar{z}_i}$ be the partial derivatives with respect to $z_i, \overline{z_i}$.
- Let $R(f) = \sum_{i=1}^n z_i \partial_i(f)$ be the radial derivative.
- Denote $L_{j,i} p = \bar{z}_i \partial_j p - \bar{z}_j \partial_i p$ the complex tangential derivative.

Outline of Proof

Proposition

For polynomials $f, p \in \mathbb{C}[z_1, \cdots, z_n]$ and $1 \le j \le n$, the equation

$$M_{z_j}^* M_p f - M_p M_{z_j}^* f = \sum_{k=0}^{\infty} \frac{1}{(N+1+n)^{k+1}} [(M_{\partial_j R^k p} - M_{z_j}^* M_{R^{k+1} p}) f]$$

holds on the Bergman space $L_a^2(\mathbb{B}_n)$.

Outline of Proof

Notation

Define $D_j : p\mathbb{C}[z_1, \cdots, z_n] \subset L^2_a(\mathbb{B}_n) \to L^2_a(\mathbb{B}_n)$ by

$$D_j(pf) = \sum_{k=0}^{\infty} \frac{1}{(N+1+n)^{k+1/2}}[M_{\partial_j R^k p} - M^*_{z_j} M_{R^{k+1}p}](f).$$

By the above proposition, we have for any polynomial f that

$$[M^*_{z_j}, P_M](pf) = P_{\mathcal{M}^\perp} M^*_{z_j}(pf) = P_{\mathcal{M}^\perp} \frac{1}{(N+1+n)^{1/2}} D_j(pf).$$

Outline of Proof

By the identity

$$\partial_j g - \overline{z_j} R g = (1 - |z|^2)\partial_j g + \sum_{i=1, i \neq j}^{n} z_i L_{j,i}(g),$$

and some algebraic reduction we divide $\|D_j(pf)\|$ to the combination of $\|L_{j,i} R^k(p) f\|_{2k+1}$ and $\|(1 - |z|^2)\partial_j R^k(p) f\|_{2k+1}$.

Outline of Proof

Proposition

For positive integers n and m, there is a positive constant $C(n,m) > 1$ such that for every polynomial $p \in \mathbb{C}[z_1, \cdots, z_n]$ with degree m, the following inequalities hold:

$$(1) \quad \|(R^l p) f\|_{2k}^2 \leq \frac{c_{2k} C(n,m)^{k+1}}{c_{2k-2l}} \|pf\|_{2k-2l}^2,$$

$$(2) \quad \|(L_{j,i} p) f\|_{2k+1}^2 \leq \frac{c_{2k+1} C(n,m)^{k+1}}{c_{2k}} \|pf\|_{2k}^2,$$

$$(3) \quad \|(\partial_j p) f\|_{2k+2}^2 \leq \frac{c_{2k+2} C(n,m)^{k+1}}{c_{2k}} \|pf\|_{2k}^2,$$

for any polynomial $f \in \mathbb{C}[z_1, \cdots, z_n]$ and nonnegative integer k, where $c_t = \frac{(n+t)!}{n!t!}$ for $t \in \mathbb{N}$.

Outline of Proof

Using the above estimate, we show that D_j is bounded and

$$P_{M^\perp} M_{z_j}^* P_M = P_{M^\perp} \frac{1}{(N+1+n)^{1/2}} D_j$$

is Schatten 2p-class.

Thank you