B(H)-SUBIDEALS OF OPERATORS

ABSTRACT. Techniques developed in the last decade generalize to arbitrary ideals the 1983 Fong-Radjavi determination of which principal ideals of compact operators are also B(H) ideals. This involves generalizing a notion of soft ideals introduced by Kaftal-Weiss. Knowing this leads to a characterization of all principal ideals inside an arbitrary B(H) ideal. Modest generalizations of these are presented as evidence of a completely general conjecture. Joint with Sasmita Patnaik.

1983 Fong-Radjavi-On ideals and Lie ideals in the ideal of compact operators.

In this talk, we focus on FR ideal results (instead of FR Lie ideal results) which dominate FR and which we consider to be the central issue.

- K(H)-principal ideal $(A)_{K(H)}$ is a B(H)-ideal if and only if $s_{nk}(A) = o(s_n(A))$ for some $k \in \mathbb{N}$.

Recall distance formula from A to the rank n-1 operators: $s_n(A) := inf\{||A - F|| \mid rank\ F < n\}$

How does this fit the ideal theory developments/perspectives of the last decade?
Will comment as we encounter them.

First-In the language of Kaftal-W (2004) this o-condition is softness of the B(H)-principal ideal: $(A) = K(H)(A)$.

A small surprise-existence of nonlinear subideals:

What is a ring?
Commutative group under +, associative multiplication, left and right distributive laws.

What is an ideal in a ring?
A subring closed under left and right multiplication from the ring.

F-R (Intro): "By an ideal in $K(H)$ or in $B(H)$ we shall mean a two-sided ideal."
• **Observe "*linear*" is absent, though recently Radjavi confirmed linear was meant.**
"A Lie ideal $\mathfrak{L}$ in $B(H)$ is a **linear** subspace of $B(H)$ with $A \in \mathfrak{L}, B \in B(H) \Rightarrow AB - BA \in \mathfrak{L}$.
An ideal J in $K(H)$ is not necessarily an ideal in $B(H)$.
At least in one interesting special case, J is an ideal in $B(H)$ if and only if it is a Lie ideal.
In this case, i.e., the case where J is generated by a single positive operator P,
we also give other conditions equivalent to J being an ideal in $B(H)$,
including one in terms of the singular number sequence $s(P) := < s_n(P) >$
(that $s_{kn}(A) = o(s_n(A))$ for some $k \in \mathbb{N}$)."

• Interesting ideal lattices: $\mathcal{I}(B(H)) \subset \mathcal{I}(K(H)) \subset \mathcal{I}(J)$
for arbitrary proper $B(H)$-ideals J since $B(H) \supset K(H) \supset J$.

• Explored in Dykema-Figiel-W-Wodzicki: $\mathcal{I}(B(H))$ is a commutative semiring (ring without additive inverses-Wikipedia), that is, a commutative semigroup (hence associative) under both ideal addition and multiplication $I + J, IJ$ (with distributive law).
Useful tidbit: $IJ = JI$ turns some seemingly noncommutative operator problems into commutative ones.

The zero ideal is clearly the $\mathcal{I}(B(H))$ unique additive identity and it is easy to see that $B(H)$ is its unique multiplicative identity.
Note then $\mathcal{I}(B(H))$ we call linear in that it is closed under linear combinations, but it is not a ring since its elements lack additive inverses, its a semiring.

• General principal subideals-start with a $B(H)$-ideal J.
Observe: all such J must be linear because $cI \in B(H)$ implies $(cI)A \in J$ for all $A \in J$.
We shall see that this often fails for $B(H)$-subideals.

• J-principal ideals look like $(A)_J \ = \ \mathbb{Z}A + JA + AJ + \sum_{finite} JAJ$

• J-linear principal ideals look like $< A >_J = \ \mathbb{C}A + JA + AJ + \sum_{finite} JAJ$

Proofs: $\subset$: LHS is the intersection of all ideals containing A and the RHS is a J-ideal (although not necessarily linear-**the small surprise**)
$\supset$: RHS single products are clearly contained in any ideal containing A.

Remark: These forms reflect FR intermediate equivalences: $A = \sum_{finite} J_i A K_i$ with various constraints on the coefficients.

• $(A)_J \ = < A >_J \Leftrightarrow \ \frac{1}{2}A \in \mathbb{Z}A + JA + AJ + \sum_{finite} JAJ \ \Leftrightarrow \ A \in JA + AJ + \sum_{finite} JAJ.$

Since $JA + AJ + \sum_{finite} JAJ$ is a J-ideal, hence closed under scalar multiplicaton.

Interesting also $\sum_{finite} JAJ$ is a B(H)-ideal, indeed, $\sum_{finite} JAJ \subset J(A)J$ and with a little work one gets equality.

Also using Lemma 6.3 (Dykema, Figiel, W, Wodzicki), $J(A)J$ traditionally defined as the linear span of triple products becomes a single triple product.

• Proving, simplifying, generalizing and unifying Fong-Radjavi
(without positive case and other preliminaries-absent a couple of Lie ideal results):

$(A)_J$ (equivalently, $< A >_J$) is a $B(H)$-ideal if and only if
$A \in JA + AJ + \sum_{finite} JAJ$ if and only if
(A) is J-soft $((A) = J(A))$ if and only if
for some k, $diag\ < s_{nk}(A)/s_n(A) > \in J$.

• Proof: Let Φ be any surgective isometry $H \to H \oplus H$ and assume $< A >_J$ is a $B(H)$-ideal.

Then $\Phi A \Phi^{-1}$ preserves singular numbers and practically everything else,
in particular, s-number sequences and hence via Calkin, ideals.

So $\Phi^{-1}(A \oplus 0)\Phi, \Phi^{-1}(A \oplus 0)\Phi$ lie in $\mathbb{C}A + JA + AJ + \sum_{finite} JAJ$,

that is, $\Phi^{-1}(A \oplus 0)\Phi = \alpha A + KA + AL + \sum_{finite} JAJ$,
$\Phi^{-1}(0 \oplus A)\Phi = \beta A + K'A + AL' + \sum_{finite} JAJ$.

Playing with a few cases: $\Phi^{-1}(\alpha' A \oplus \beta' A)\Phi \in JA + AJ + \sum_{finite} JAJ$ for α', β' not both 0.
Multiplying by a cannonical projection and suitable unitary operators:
$\Phi^{-1}(A \oplus 0)\Phi \in JA + AJ + \sum_{finite} JAJ \subset J(A)J = (A)J^2 \subset (A)J$,
hence also $A \in (A)J$ and so $(A) \subset (A)J$ with the reverse inclusion automatic, yielding J-softness.
The converse depends on the polar decomposition of A and diagonalizing the positive part, and is an easy construction.

• Finitely generated $B(H)$-ideals are really principal ideals. Not necessarily so for J-ideals.
But still, our generalized FR results and characterization of finitely generated J-ideals holds.

Sasmita is currently working on the countably generated and general cases, and the Lie ideal generalizations.