Carleson Measures for Besov-Sobolev Spaces and Non-Homogeneous Harmonic Analysis

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Talk Outline

- Motivation of the Problem
  - Besov-Sobolev Spaces of analytic functions on $\mathbb{B}_n$
  - Carleson Measures for Besov-Sobolev Spaces
  - Connections to Non-Homogeneous Harmonic Analysis
- Main Results and Sketch of Proof
  - T(1)-Theorem for Bergman-type operators
  - Characterization of Carleson measures for Besov-Sobolev Spaces

Besov-Sobolev Spaces

- The space $B_2^\sigma(\mathbb{B}_n)$ is the collection of holomorphic functions f on the unit ball $\mathbb{B}_n := \{z \in \mathbb{C}^n : |z| < 1\}$ such that

$$\left\{\sum_{k=0}^{m-1}\left|f^{(k)}(0)\right|^2 + \int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m+\sigma} f^{(m)}(z)\right|^2 d\lambda_n(z)\right\}^{\frac{1}{2}} < \infty,$$

where $d\lambda_n(z) = \left(1-|z|^2\right)^{-n-1} dV(z)$ is the invariant measure on $\mathbb{B}_n$ and $m+\sigma > \frac{n}{2}$.

- Various choices of σ give important examples of classical function spaces:
  - $\sigma = 0$: Corresponds to the Dirichlet Space;
  - $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space;
  - $\sigma = \frac{n}{2}$: Classical Hardy Space;
  - $\sigma > \frac{n}{2}$: Bergman Spaces.

Besov-Sobolev Spaces

- The spaces $B_2^\sigma(\mathbb{B}_n)$ are examples of reproducing kernel Hilbert spaces.
- Namely, for each point $\lambda \in \mathbb{B}_n$ there exists a function $k_\lambda^\sigma \in B_2^\sigma(\mathbb{B}_n)$ such that

$$f(\lambda) = \langle f, k_\lambda^\sigma \rangle_{B_2^\sigma(\mathbb{B}_n)}$$

- A computation shows that the kernel function $k_\lambda^\sigma(z)$ is given by:

$$k_\lambda^\sigma(z) = \frac{1}{\left(1 - \overline{\lambda} z\right)^{2\sigma}}$$

  - $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space; $k_\lambda^{\frac{1}{2}}(z) = \frac{1}{1-\overline{\lambda}z}$
  - $\sigma = \frac{n}{2}$: Classical Hardy Space; $k_\lambda^{\frac{n}{2}}(z) = \frac{1}{(1-\overline{\lambda}z)^n}$
  - $\sigma = \frac{n+1}{2}$: Bergman Space; $k_\lambda^{\frac{n+1}{2}}(z) = \frac{1}{(1-\overline{\lambda}z)^{n+1}}$

Carleson Measures for Besov-Sobolev Spaces

Definition (Carleson Measures for $B_2^\sigma(\mathbb{B}_n)$)

A non-negative Borel measure μ is a $B_2^\sigma(\mathbb{B}_n)$-Carleson measure if

$$\int_{\mathbb{B}_n} |f(z)|^2 d\mu(z) \leq C(\mu)^2 \|f\|^2_{B_2^\sigma(\mathbb{B}_n)} \quad \forall f \in B_2^\sigma(\mathbb{B}_n).$$

Carleson measures play an important role in both the function theory of the space and more generally play a predominant role in harmonic analysis:

- Interpolating Sequences;
- Characterization of Multipliers for spaces of function;
- Corona Theorems;
- Paraproducts, Commutators, Hankel Operators;

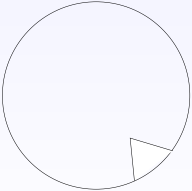
Carleson Measures for Besov-Sobolev Spaces

Question

Give a 'geometric' characterization of Carleson measures for the Besov-Sobolev spaces $B_\sigma^2(\mathbb{B}_n)$.

Testing on the reproducing kernel k_λ^σ we always have a necessary geometric condition for the measure μ to be Carleson:

$$\mu\left(T(B_r(\xi))\right) \lesssim r^{2\sigma} \quad \forall \xi \in \partial\mathbb{B}_n,\ r > 0$$



When $\sigma \geq \frac{n}{2}$ then this necessary condition is also sufficient.

Carleson Measures for Besov-Sobolev Spaces

When $0 \leq \sigma \leq \frac{1}{2}$ then a geometric characterization of Carleson measures is known:

- If $n = 1$, the results can be expressed in terms of capacity conditions. More precisely,

  $$\mu\left(T(\Omega)\right) \lesssim \text{Cap}_{\sigma}\left(\Omega\right) \quad \forall \text{open}\, \Omega \subset \mathbb{T}.$$

  See for example Stegenga, Maz'ya, Verbitsky, Carleson.
- If $n > 1$ there are two different characterizations of Carleson measures for $B_2^{\sigma}(\mathbb{B}_n)$:
  - One method via integration operators on trees (dyadic structures on the ball $\mathbb{B}_n$) by Arcozzi, Rochberg and Sawyer.
  - One method via "T(1)" conditions by E. Tchoundja.

Question (Main Problem: Characterization in the Difficult Range)

Characterize the Carleson measures when $\frac{1}{2} < \sigma < \frac{n}{2}$.

Operator Theoretic Characterization of Carleson Measures

The following observations hold in an arbitrary Hilbert space with a reproducing kernel.

- Let $\mathcal{J}$ be a Hilbert space of functions on a domain X with reproducing kernel function j_x, i.e.,

$$f(x) = \langle f, j_x \rangle_{\mathcal{J}} \quad \forall f \in \mathcal{J}.$$

- A measure μ is Carleson exactly if the inclusion map ι from $\mathcal{J}$ to $L^2(X;\mu)$ is bounded, or

$$\int_X |f(x)|^2 d\mu(z) \leq C(\mu) \|f\|_{\mathcal{J}}^2.$$

We can give a characterization of Carleson measures for the space $\mathcal{J}$ in terms of information about the boundedness of a certain linear operator related to the reproducing kernel j_x.

Operator Theoretic Characterization of Carleson Measures

Proposition (Arcozzi, Rochberg, Sawyer)

A measure μ is a $\mathcal{J}$-Carleson measure if and only if the linear map

$$f(z) \to T_{\mu,\mathcal{J}}(f)(z) = \int_X \operatorname{Re} j_x(z) f(x) d\mu(x)$$

is bounded on $L^2(X;\mu)$.

When we apply this proposition to the spaces $B_2^\sigma(\mathbb{B}_n)$ this suggests that we study the operator $T_{\mu,2\sigma} : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$ given by

$$T_{\mu,2\sigma}(f)(z) = \int_{\mathbb{B}_n} \operatorname{Re}\left(\frac{1}{(1-\overline{w}z)^{2\sigma}}\right) f(w) d\mu(w)$$

and find some conditions that will let us determine when it is bounded.

Calderón-Zygmund Operators

A function $k(x,y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ will be called a Calderón-Zygmund kernel if it satisfies the following estimates:

$$|k(x,y)| \lesssim \frac{1}{|x-y|^d} \quad \forall x,y \in \mathbb{R}^d;$$

If $|x-x'| \leq \frac{1}{2}|x-y|$ then

$$|k(y,x)-k(y,x')| + |k(x,y)-k(x',y)| \lesssim \frac{|x-x'|^\tau}{|x-y|^{d+\tau}}$$

provided that, with some (fixed) $0 < \tau \leq 1$. Given the kernel, we define the operator by

$$T(f)(x) = \int_{\mathbb{R}^n} k(x,y)f(y)dy$$

Then for $1 < p < \infty$ $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$.

Connections to T(1)-Theorem

Guessing the Characterization?

Theorem (David and Journé)

If T is a Calderón-Zygmund operator then $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ if and only if $T(1), T^*(1) \in BMO(\mathbb{R}^n)$ and T is weak bounded.

The condition $T(1) \in BMO(\mathbb{R}^n)$ can be phrased in a more geometric way:

Lemma

Suppose that T is a Calderón-Zygmund operator. The following are equivalent:

(i) $T(1) \in BMO(\mathbb{R}^n)$;

(ii) For all $Q \subset \mathbb{R}^n$

$$\int_Q |T\chi_Q(x)|^2 dx \lesssim |Q|.$$

Calderón–Zygmund Estimates for $T_{\mu,2\sigma}$

If we define

$$\Delta(z,w) := \begin{cases} \left||z|-|w|\right| + \left|1-\frac{z\overline{w}}{|z||w|}\right| & : \quad z,w\in\mathbb{B}_n\setminus\{0\} \\ |z|+|w| & : \quad \text{otherwise.} \end{cases}$$

Then Δ is a pseudo-metric and makes the ball into a space of homogeneous type.

A computation demonstrates that the kernel of $T_{\mu,2\sigma}$ satisfies the following estimates:

$$|K_{2\sigma}(z,w)| \lesssim \frac{1}{\Delta(z,w)^{2\sigma}} \quad \forall z,w\in\mathbb{B}_n;$$

If $\Delta(\zeta,w) < \frac{1}{2}\Delta(z,w)$ then

$$|K_{2\sigma}(\zeta,w) - K_{2\sigma}(z,w)| \lesssim \frac{\Delta(\zeta,w)^{1/2}}{\Delta(z,w)^{2\sigma+1/2}}.$$

Calderón-Zygmund Estimates for $T_{\mu,2\sigma}$

- These estimates on $K_{2\sigma}(z,w)$ say that it is a Calderón-Zygmund kernel of order 2σ with respect to the metric Δ.
  - Unfortunately, we can't apply the standard T(1) technology (adapted to a space of homogeneous type) to study the operators $T_{\mu,2\sigma}$. We would need the estimates of order n instead of 2σ.
- However, the measures we want to study (the Carleson measures for the space) satisfy the growth estimate

$$\mu\left(T(B_r)\right) \lesssim r^{2\sigma}$$

  and this is exactly the issue that will save us!
- This places us in the setting of non-homogeneous harmonic analysis as developed by Nazarov, Treil and Volberg. We have an operator with a Calderón-Zygmund kernel satisfying estimates of order 2σ, a measure μ of order 2σ, and are interested in $L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$ bounds.

Euclidean Variant of the Question

There is a natural extension of these questions/ideas to the Euclidean setting $\mathbb{R}^d$.

More precisely, for $m \leq d$ we are interested in Calderón-Zygmund kernels that satisfy the following estimates:

$$|k(x,y)| \lesssim \frac{1}{|x-y|^m},$$

and

$$|k(y,x)-k(y,x')|+|k(x,y)-k(x',y)| \lesssim \frac{|x-x'|^\tau}{|x-y|^{m+\tau}}$$

provided that $|x-x'| \leq \frac{1}{2}|x-y|$, with some (fixed) $0<\tau\leq 1$.

Euclidean Variant of the Question

Additionally the kernels will have the following property

$$|k(x,y)| \leq \frac{1}{\max(d(x)^m, d(y)^m)},$$

where $d(x) := \operatorname{dist}(x, \mathbb{R}^d \setminus H)$ and H being an open set in $\mathbb{R}^d$.

Key Example: Let $H = \mathbb{B}_d$, the unit ball in $\mathbb{R}^d$ and

$$k(x,y) = \frac{1}{(1 - x \cdot y)^m}.$$

A Calderón-Zygmund kernel k on a closed $X \subset \mathbb{R}^d$ if $k(x,y)$ is defined only on $X \times X$ and the previous properties of k are satisfied whenever $x, x', y \in X$.

We say that a $L^2(\mathbb{R}^d;\mu)$ bounded operator is a Calderón-Zygmund operator with kernel k if,

$$T_{\mu,m} f(x) = \int_{\mathbb{R}^d} k(x,y) f(y) d\mu(y) \quad \forall x \notin \operatorname{supp} f.$$

Main Results

T(1)-Theorem for Bergman-Type Operators

Theorem (T(1)-Theorem for Bergman-Type Operators, Volberg, BDW (Amer. J. Math., to appear))

Let $k(x,y)$ be a Calderón-Zygmund kernel of order m on $X \subset \mathbb{R}^d$, $m \leq d$ with Calderón-Zygmund constants C_{CZ} and τ. Let μ be a probability measure with compact support in X and all balls such that $\mu(B_r(x)) > r^m$ lie in an open set H. Let also

$$|k(x,y)| \leq \frac{1}{\max(d(x)^m, d(y)^m)},$$

where $d(x) := \operatorname{dist}(x, \mathbb{R}^d \setminus H)$. *Finally, suppose also that:*

$$\|T_{\mu,m}\chi_Q\|^2_{L^2(\mathbb{R}^d;\mu)} \leq A\,\mu(Q)\,,\ \|T^*_{\mu,m}\chi_Q\|^2_{L^2(\mathbb{R}^d;\mu)} \leq A\,\mu(Q)\,.$$

Then $\|T_{\mu,m}\|_{L^2(\mathbb{R}^d;\mu)\to L^2(\mathbb{R}^d;\mu)} \leq C(A,m,d,\tau)$.

Remarks about T(1)-Theorem for Bergman-Type Operators

This theorem gives an extension of the non-homogeneous harmonic analysis of Nazarov, Treil and Volberg to "Bergman-type" operators.

- The balls for which we have $\mu(B(x,r)) > r^m$ are called "non-Ahlfors balls".
  - Non-Ahlfors balls are enemies, their presence make the estimate of Calderón-Zygmund operator basically impossible.
  - The key hypothesis is that we can capture all the non-Ahlfors balls in some open set H.
  - This is just a restatement of the Carleson measure condition in this context.
- To handle this difficulty we suppose that our Calderón-Zygmund kernels have an additional estimate in terms of the behavior of the distance to the complement of H (namely that they are Bergman-type kernels).

Main Results

Characterization of Carleson Measures for $B_\sigma^2(\mathbb{B}_n)$

Theorem (Characterization of Carleson Measures for Besov-Sobolev Spaces, Volberg, BDW Amer. J. Math. to appear))

Let μ be a positive Borel measure in $\mathbb{B}_n$. Then the following conditions are equivalent:

(a) *μ is a $B_2^\sigma(\mathbb{B}_n)$-Carleson measure;*

(b) *$T_{\mu,2\sigma} : L^2(\mathbb{B}_n;\mu) \to L^2(\mathbb{B}_n;\mu)$ is bounded;*

(c) *There is a constant C such that*

  (i) *$\|T_{\mu,2\sigma}\chi_Q\|^2_{L^2(\mathbb{B}_n;\mu)} \leq C\,\mu(Q)$ for all Δ-cubes Q;*

  (ii) *$\mu(B_\Delta(x,r)) \leq C\,r^{2\sigma}$ for all balls $B_\Delta(x,r)$ that intersect $\mathbb{C}^n \setminus \mathbb{B}_n$.*

Above, the sets B_Δ are balls measured with respect to the metric Δ and the set Q is a "cube" defined with respect to the metric Δ.

Remarks about Characterization of Carleson Measures

- We have already proved that $(a) \Leftrightarrow (b)$, and it is trivial $(b) \Rightarrow (c)$.
- It only remains to prove that $(c) \Rightarrow (b)$.
  - The proof of this Theorem follows from the T(1)-Theorem for Bergman-type operators.
  - In a neighborhood of the sphere $\partial\mathbb{B}_n$ the metric Δ looks a Euclidean-type quasi-metric. For example when $n = 2$ we have that

$$\Delta(x, y) \approx |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|^2 + |x_4 - y_4|^2$$

  - The method of proof of the Euclidean Bergman-type T(1) theorem can then be modified to case of Calderón-Zygmund operators with respect to a quasi-metric (essentially verbatim).
- It is possible to show that the T(1) condition reduces to the simpler conditions in certain cases.
- An alternate proof of this Theorem was recently given by Hytönen and Martikainen. Their proof used a non-homogeneous T(b)-Theorem on metric spaces.

Checking the Theorem for Carleson Measures of H^2

Definition (Carleson Measures for $H^2(\mathbb{C}_+)$)

A measure μ is a Carleson measure for $H^2(\mathbb{C}_+)$ (or $H^2(\mathbb{D})$) if

$$\int_{\mathbb{C}_+} |f(z)|^2 d\mu(z) \leq C(\mu)^2 \|f\|^2_{H^2(\mathbb{C}_+)} \quad \forall f \in H^2(\mathbb{C}_+).$$

As is well-known this function-theoretic condition happens if and only if the follow geometric condition is satisfied for all tents $T(I)$ over $I \subset \mathbb{R}$:

$$\mu(T(I)) \leq C|I| \quad \forall I \subset \mathbb{R}.$$

If $I \subset \mathbb{R}$ is an interval, then $T(I)$ will be a cube in $\mathbb{R}^2$. Restricting the integral to $T(I)$ and using standard estimates for the kernel one sees:

$$\frac{\mu(T(I))^3}{|I|^2} \leq C(\mu)\mu(T(I)) \Rightarrow \mu(T(I)) \leq C(\mu)|I|$$

Checking the Theorem for Carleson Measures of H^2

We want to show that if we know that μ satisfies the geometric Carleson condition, then

$$\| T_{\mu,\frac{1}{2}}\chi_Q\|^2_{L^2(\mathbb{C}_+;\mu)} \leq C\,\mu(Q)$$

Observe that the function

$$F_{Q,\mu}(z) := \int_{\mathbb{R}^2} \frac{\chi_Q(\xi)}{\xi - z} d\mu(\xi) \in H^2(\mathbb{C}_-)$$

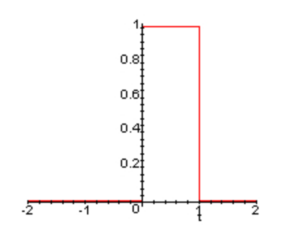
with norm a constant multiple of $\sqrt{\mu(Q)}$.

Since the function then belongs to $H^2(\mathbb{C}_-)$, an application of the Carleson Embedding property (or equivalently, the geometric condition for Carleson measures) gives that

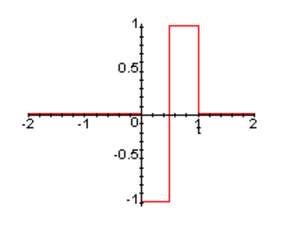
$$\| T_{\mu,\frac{1}{2}}\chi_Q\|^2_{L^2(\mathbb{C}_+\mu)} = \int_{\mathbb{C}_+} |F_{Q,\mu}(z)|^2 d\mu(z) \leq C(\mu)\mu(Q).$$

The Haar Basis for $L^2(\mathbb{R})$

- Let $h^1(x) := \mathbf{1}_{[0,1)}(x)$ and let $h^0(x) := -\mathbf{1}_{[0,1/2)}(x) + \mathbf{1}_{[1/2,1)}(x)$



$h^1(x)$



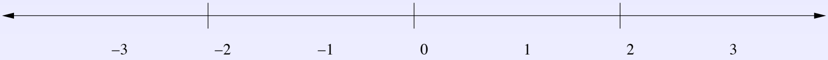
$h^0(x)$

- Let

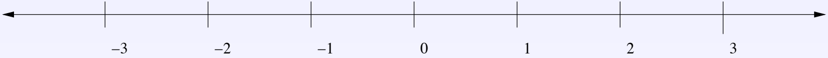
$$\mathcal{D} := \{2^{-k}(j + [0,1)) : j \in \mathbb{Z}, k \in \mathbb{Z}\}$$

i.e., the usual dyadic grid in $\mathbb{R}$.

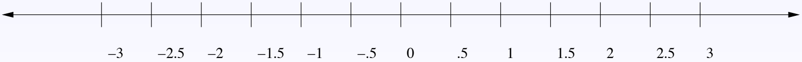
The Dyadic Grid in $\mathbb{R}$



Dyadic intervals of length 2



Dyadic intervals of length 1



Dyadic intervals of length $\frac{1}{2}$

A Wavelet Basis for $L^2(\mathbb{R})$

- For $I \in \mathcal{D}$ and let h_I be the version of h^0 "adapted" to the interval I
- $\{h_I : I \in \mathcal{D}\}$ is the Haar wavelet basis for $L^2(\mathbb{R})$.

$$\langle f, h_I \rangle_{L^2(\mathbb{R})} := \int_{\mathbb{R}} f(x) h_I(x) dx$$

$$f(x) = \sum_{I \in \mathcal{D}} \langle f, h_I \rangle_{L^2(\mathbb{R})} h_I(x)$$

$$\|f\|^2_{L^2(\mathbb{R})} = \sum_{I \in \mathcal{D}} |\langle f, h_I \rangle_{L^2(\mathbb{R})}|^2$$

Showing the the Calderón-Zygmund Operator is Bounded

Expand f and g in the Haar basis:

$$f = \sum_{I\in\mathcal{D}} \langle f, h_I\rangle_{L^2(\mathbb{R})} h_I \quad g = \sum_{I\in\mathcal{D}} \langle g, h_I\rangle_{L^2(\mathbb{R})} h_I$$

$$\begin{aligned}\langle Tf, g\rangle_{L^2(\mathbb{R})} &= \sum_{I,J\in\mathcal{D}} \langle f, h_I\rangle_{L^2(\mathbb{R})}\langle g, h_J\rangle_{L^2(\mathbb{R})}\langle Th_I, h_J\rangle_{L^2(\mathbb{R})} \\ &= \sum_{I=J} + \sum_{I\subset J} + \sum_{J\subset I} \left(\langle f, h_I\rangle_{L^2(\mathbb{R})}\langle g, h_J\rangle_{L^2(\mathbb{R})}\langle Th_I, h_J\rangle_{L^2(\mathbb{R})}\right).\end{aligned}$$

- The diagonal piece is easy, and contributes obvious estimates;
- The other two pieces are dual to each other and reduce to controlling operators like

$$\sum_{I\in\mathcal{D}} \langle T\chi_I, h_I\rangle_{L^2(\mathbb{R})} \left\langle f, \frac{\chi_I}{|I|}\right\rangle_{L^2(\mathbb{R})} \langle g, h_I\rangle_{L^2(\mathbb{R})}.$$

Littlewood-Paley Decomposition

- Construct two independent dyadic lattices $\mathcal{D}_1$ and $\mathcal{D}_2$.
- There are special unit cubes Q^0 and R^0 of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively that contain supp μ deep inside them.
- Define expectation operators Δ_Q (Haar function on Q) and Λ (average on Q^0), then we have for every $\varphi \in L^2(\mathbb{R}^d;\mu)$

$$\varphi = \Lambda\varphi + \sum_{Q\in\mathcal{D}_1} \Delta_Q\varphi,$$

the series converges in $L^2(\mathbb{R}^d;\mu)$. Moreover,

$$\|\varphi\|^2_{L^2(\mathbb{R}^d;\mu)} = \|\Lambda\varphi\|^2_{L^2(\mathbb{R}^d;\mu)} + \sum_{Q\in\mathcal{D}_1} \|\Delta_Q\varphi\|^2_{L^2(\mathbb{R}^d;\mu)}.$$

Good and Bad Decomposition

We fix the decomposition of f and g into good and bad parts:

$$f = f_{good} + f_{bad}\,, \text{ where } f_{good} = \Lambda f + \sum_{Q\in\mathcal{D}_1\cap\mathcal{G}_1} \Delta_Q f$$

$$g = g_{good} + g_{bad}\,, \text{ where } g_{good} = \Lambda g + \sum_{R\in\mathcal{D}_2\cap\mathcal{G}_2} \Delta_R g.$$

It turns out that for any fixed $Q \in \mathcal{D}_1$,

$$\mathbb{P}\{Q \text{ is bad}\} \leq \delta^2$$

$$\mathbb{E}(\|f_{bad}\|_{L^2(\mathbb{R}^d;\mu)}) \leq \delta\|f\|_{L^2(\mathbb{R}^d;\mu)}.$$

Similar statements for g hold as well.

Reduction to Controlling The Good Part

- Using the decomposition above, we have

$$\langle T_{\mu,m} f, g\rangle_{L^2(\mathbb{R}^d;\mu)} = \langle T_{\mu,m} f_{good}, g_{good}\rangle_{L^2(\mathbb{R}^d;\mu)} + R(f,g)$$

- Using the construction above, we have that

$$\mathbb{E}|R_\omega(f,g)| \leq 2\,\delta \|T\|_{L^2(\mathbb{R}^d;\mu)\to L^2(\mathbb{R}^d;\mu)} \|f\|_{L^2(\mathbb{R}^d;\mu)} \|g\|_{L^2(\mathbb{R}^d;\mu)}.$$

- Choosing δ small enough ($< \frac{1}{4}$) we only need to show that

$$\left|\langle T_{\mu,m} f_{good}, g_{good}\rangle_{L^2(\mathbb{R}^d;\mu)}\right| \leq C(\tau, m, A, d) \|f\|_{L^2(\mathbb{R}^d;\mu)} \|g\|_{L^2(\mathbb{R}^d;\mu)}.$$

- This will then give

$$\|T\|_{L^2(\mathbb{R}^d;\mu)\to L^2(\mathbb{R}^d;\mu)} \leq 2C(\tau, m, A, d).$$

Sketch of Proof

Estimating The Good Part

- We then decompose the

$$\langle T_{\mu,m} f_{good}, g_{good} \rangle_{L^2(\mathbb{R}^d;\mu)} = A_1 + A_2 + A_3$$

- The term A_1 is the diagonal part of the sum. This is the easiest part.
- The term A_2 is the long-range interaction part. The second easiest part
  - Here we use the Calderón-Zygmund Estimates and the hypothesis that we can capture all the non-Ahlfors balls in the open set H.
- The term A_3 is the short-range interaction part.
  - Here we use the T(1) hypothesis and reduce the estimates to paraproducts.
- These all then imply that

$$\left| \langle T_{\mu,m} f_{good}, g_{good} \rangle_{L^2(\mathbb{R}^d;\mu)} \right| \leq C(\tau, m, A, d) \| f \|_{L^2(\mathbb{R}^d;\mu)} \| g \|_{L^2(\mathbb{R}^d;\mu)}.$$

Internet Analysis Seminar Announcement

2nd Internet Analysis Seminar will take place during the Fall 2011 – Spring 2012 Academic Year.

- Phase I (October – February), approximately fifteen weekly, electronic lectures will be provided via a public website.
- Phase II (March – May), participants from Phase I will apply to work through a more advanced project.
- Phase III consists of a final one-week workshop held in June, during which, participants will present their projects.

Topic: Multiparameter Harmonic Analysis or Non-Homogeneous Harmonic Analysis Details available on:

http://internetanalysisseminar.gatech.edu/

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Thank You!