A Journey Through Numerical Ranges*

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DEPARTMENT OF MATHEMATICS
LOS ANGELES, CALIFORNIA 90024

April 12, 1973

.ofessor John B. Conway
.hematics Department
.iana University
.oomington, Indiana 47401

Dear John:

Thanks for your interesting preprint. Actually, the "easy" part of your theorem was pointed out to me some time ago by Sarason (see p. 252 of the enclosed paper; see p. 248 4th ℙ for notation). You should also look at two papers by Nikol'skii and Pavlov in the "Soviet Math. Dokl." vol. 10 (1969), pp. 138-141 and 163-166.

Have you thought about which (scalar) Sz.-Nagy-Foias models are similar to normal operators? The question is nontrivial, since sometimes they are similar to unitary operators (see Sz.-N.-F. book); necessary and sufficient for this is that the characteristic function be bounded below in the unit disc. If the outer part is bounded below and the zeros of the inner part are uniformly separated is the model similar to a normal operator? Are these conditions necessary?

Kriete (B. A. M. S. 76 #2 (1970)) wrote a paper on when two (scalar) models are similar (to eachother).

With best regards,

Doug

Douglas N. Clark

DNC:me

Separatum

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ADIUVANTIBUS

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REDIGIT

B. SZ.-NAGY

TOMUS 37
FASC. 3—4

J. B. Conway & P. Y. Wu

On nonorthogonal decompositions of certain contractions

SZEGED, 1975

INSTITUTUM BOLYAIANUM UNIVERSITATIS SZEGEDIENSIS

On nonorthogonal decompositions of certain contractions

By PEI YUAN WU in Bloomington (Indiana, U.S.A.)

John B. Conway &

SZ.-NAGY and FOIAŞ showed in [4] that a contraction T on a separable Hilbert space H is similar to a unitary operator if and only if its characteristic function $\Theta_T(\lambda)$ has a bounded analytic inverse (see also [5], Ch. IX). In the present paper, we give a generalization of this result. We prove that a contraction T is similar to a direct sum of a unitary operator and a contraction of class $C_{\cdot 0}$ if and only if the outer factor of $\Theta_T(\lambda)$ has a bounded analytic inverse. We shall also indicate some interesting consequences.

1. Preliminaries. We only consider non-trivial, complex, separable Hilbert spaces. For completely non-unitary contractions we will use the functional models as developed in [5], Ch. VI.

Let T be a contraction on the Hilbert space H. Denote by $D_T=(1-T^*T)^{1/2}$, $D_{T^*}=(1-TT^*)^{1/2}$ the defect operators and $\mathfrak{D}_T=\overline{D_T H}$, $\mathfrak{D}_{T^*}=\overline{D_{T^*} H}$ the defect spaces of T.

The characteristic function $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}, \Theta_T(\lambda)\}$ of T is the purely contractive analytic function from $\mathfrak{D}_T$ to $\mathfrak{D}_{T^*}$ defined by

$$\Theta_T(\lambda) = [-T+\lambda D_{T^*}(1-\lambda T^*)^{-1} D_T]|\mathfrak{D}_T \quad \text{for} \quad |\lambda| < 1.$$

If T is completely non-unitary, we will consider T in its functional model, i.e. defined by

$$T^*(u \oplus v) = e^{-it}[u(e^{it}) - u(0)] \oplus e^{-it} v(t)$$

on the space

$$H = [H^2(\mathfrak{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathfrak{D}_T)}] \ominus \{\Theta_T u \oplus \Delta_T u:\ u \in H^2(\mathfrak{D}_T)\},$$

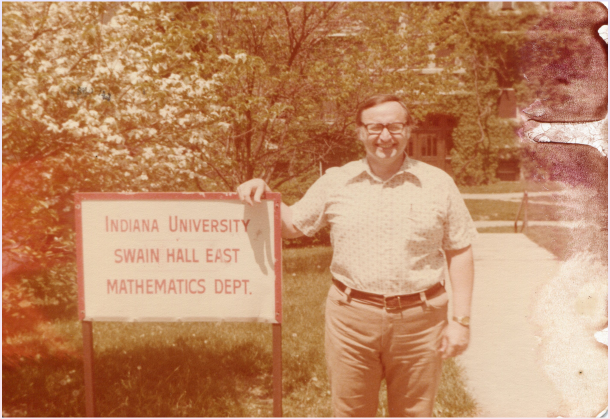
where $\Delta_T(t)=[I-\Theta_T(e^{it})^* \Theta_T(e^{it})]^{1/2}$. Let $\Theta_T(\lambda)=\Theta_2(\lambda)\Theta_1(\lambda)$ be the canonical factorization of $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}, \Theta_T(\lambda)\}$ into the product of its outer factor $\{\mathfrak{D}_T, \mathfrak{F}, \Theta_1(\lambda)\}$ and inner factor $\{\mathfrak{F}, \mathfrak{D}_{T^*}, \Theta_2(\lambda)\}$. Let

$$H_1 = \{\Theta_2 u \oplus v:\ u \in H^2(\mathfrak{F}),\ v \in \overline{\Delta_T L^2(\mathfrak{D}_T)}\} \ominus \{\Theta_T w \oplus \Delta_T w:\ w \in H^2(\mathfrak{D}_T)\}$$

be the induced invariant subspace for T and

$$H_2 = H \ominus H_1 = [H^2(\mathfrak{D}_{T^*}) \ominus \Theta_2 H^2(\mathfrak{F})] \oplus \{0\}$$

its orthogonal complement. Let $T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ be the triangulation of T corresponding



May, 1980
Bloomington, Indiana

May, 1980
Bloomington, Indiana

July, 1988
Durham, New Hampshire
C. Cowen, P. Rosenthal, J. Conway, E. Nordgren, H. Radjavi

August, 1993
Szeged, Hungary
J. Conway, B. Sz.-Nagy

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Lucas' Theorem Refined*

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We prove a refined version of the classical Lucas' theorem: if p is a polynomial with zeros $a_1, \ldots, a_{n+1}$ all having modulus one and ϕ is the Blaschke product whose zeros are those of the derivative p', then the compression of the shift $S(\phi)$ has its numerical range circumscribed about by the $(n+1)$-gon $a_1 \cdots a_{n+1}$ with tangent points the midpoints of the $n+1$ sides of the polygon. This is proved *via* a special matrix representation of $S(\phi)$ and is a generalization of the known case for $n = 2$.

Keywords: Compression of the shift; numerical range; dilation

AMS Subject Classification: 15A60, 47A12

A classical result in complex analysis, variously attributed to Gauss, Lucas, Grace and others and usually called Lucas' theorem, says that zeros of the derivative of a polynomial are all contained in the convex hull of the set of zeros of the polynomial. A more refined assertion for polynomials of degree three, due to Siebeck [5] (*cf.* also [4, p. 9]), is that if the degree-three polynomial p has zeros a_1, a_2 and a_3 and the derivative p' has zeros b_1 and b_2, then there is an ellipse with foci at b_1 and b_2 which is circumscribed about by the triangle $\Delta a_1 a_2 a_3$ with

*Dedicated to John B. Conway, the thesis advisor of the second author and the mathematical grandfather of the first, on his 60th birthday.

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H complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$

A bounded linear operator on H

Def. $W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$

numerical range of A

$w(A) = \sup \{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}$

numerical radius of A

Properties:

(1) U unitary $\Rightarrow W(U^*AU) = W(A)$

(2) $W(A)$ bounded subset of $\mathbb{C}$

(3) $W(aA + bI) = aW(A) + b \quad \forall\, a\,, b \in \mathbb{C}$

(4) $W(Re\ A) = Re\ W(A)\ , \quad W(Im\ A) = Im\ W(A)$

$Re\ A = (A + A^*)/2\ , \quad Im\ A = (A - A^*)/(2i)$

(5) $W(A)$ convex (Toeplitz–Hausdorff, 1918–19)

(6) $\sigma(A) \subseteq \overline{W(A)}$

(7) A normal $\Rightarrow \overline{W(A)} = \sigma(A)^\wedge$

(8) $W(A \oplus B) = (W(A) \cup W(B))^\wedge$

Question 1. Given A, what can we say about $W(A)$?

Question 2. Known $W(A)$, what can we say about A ?

Question 3. Which bdd convex $\triangle \subseteq \mathbb{C}$ is $W(A)$ for some A ?

Ex.1.

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

$\Rightarrow W(A) =$ elliptic disc with foci a, c & minor axis of length $|b|$

Ex.2.

$$A = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$$

$\Rightarrow W(A) =$ polygonal region with some of the a_j's as vertices

Ex.3.

$$A = J_n \equiv \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \quad (n \times n \text{ Jordan block})$$

$\Rightarrow W(A) =$ circular disc centered at 0 & radius

$$\cos\left(\pi/(n+1)\right)$$

Ex.4.

$$A = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & \ddots \end{bmatrix} \quad \text{on } l^2$$

$$\Rightarrow W(A) = \mathbb{D} \equiv \{z \in \mathbb{C} : |z| < 1\}$$

(1) J. Anderson (early 1970s):

Condition for $W(A) =$ circular disc

(2) J. Holbrook (1969):

Inequality of $w(AB)$ for $AB = BA$

(3) J. P. Williams & T. Crimmins (1967):

Condition for $w(A) = \|A\|/2$

J. Anderson (early 1970s):

Thm. A $n \times n$, $W(A) \subseteq \overline{\mathbb{D}}$ & $\#\,(\partial W(A) \cap \partial \mathbb{D}) > n \Rightarrow W(A) = \overline{\mathbb{D}}$

Pf: (Wu, 1999) $W(A) \subseteq \overline{\mathbb{D}} \Leftrightarrow Re\,(e^{-i\theta}A) \leq I_n \quad \forall\, \theta \in \mathbb{R}$

$\therefore p(e^{i\theta}) \equiv \det\,(I_n - Re\,(e^{-i\theta}A)) \geq 0 \quad \forall\, \theta$

$= \bar{a}_n e^{-in\theta} + \cdots + \bar{a}_1 e^{-i\theta} + a_0 + a_1 e^{i\theta} + \cdots + a_n e^{in\theta}$

$= |q(e^{i\theta})|^2$ for some poly. q of deg.$\leq n$ by Riesz-Fejér (1916)

$\#\,(\partial W(A) \cap \partial \mathbb{D}) > n \Rightarrow p(e^{i\theta}) = 0$ for more than n θ's

$\Rightarrow q(e^{i\theta}) = 0$ for more than n θ's

$\therefore$ Fund. thm. of algebra

$\Rightarrow q \equiv 0 \quad \Rightarrow p \equiv 0 \quad \Leftrightarrow W(A) = \overline{\mathbb{D}}$

Cor. $\{z \in \overline{\mathbb{D}} : Re\, z \geq 0\} \neq W(A) \quad \forall$ finite matrix A

Elaborations:

(1) Gau & Wu (2004, 2007):

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_n & \cdots & -a_2 & -a_1 \end{bmatrix} \quad n \times n \text{ companion matrix}$$

$W(A) \supseteq D$ (circular disc at 0), $\#(\partial W(A) \cap \partial D) > n \Rightarrow A = J_n$

(2) Gau & Wu (2006):

A compact, $W(A) \subseteq \overline{\mathbb{D}}$, $\#(\partial W(A) \cap \partial \mathbb{D}) = \infty \Rightarrow W(A) = \overline{\mathbb{D}}$

Idea: "analytic" branch of $d_A(\theta) = \max \overline{W(Re\,(e^{-i\theta}A))}$, $\theta \in \mathbb{R}$

$\because d_A(\theta) \leq 1 \quad \forall \ \theta$

$d_A(\theta) = 1$ for infinitely many θ's $\quad \Rightarrow d_A \equiv 1$

Cor. $\{z \in \overline{\mathbb{D}} : Re\, z \geq 0\} \neq W(A)$ for A compact

(3) Gau & Wu (2008):

A $n \times n$ $(n \geq 3)$, $W(A) \supseteq \overline{\mathbb{D}}$, $\#(\partial W(A) \cap \partial\mathbb{D}) > n$

$\Rightarrow$ $\partial W(A)$ contains at least one arc and at most $n-2$ arcs of $\partial\mathbb{D}$

Ex. $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus diag\,(r, re^{i\theta_0}, \ldots, re^{i(n-3)\theta_0})$

$1 < r < \sec\,(\pi/(n-2))$, $\theta_0 = 2\pi/(n-2)$

$\Rightarrow$ $W(A) \supseteq \overline{\mathbb{D}}$ & $\partial W(A)$ contains $n-2$ arcs of $\partial\mathbb{D}$

(4) A $n \times n$ nilpotent, $W(A) \subseteq \overline{\mathbb{D}}$, $\#(\partial W(A) \cap \partial \mathbb{D}) > n-2$

$\Rightarrow W(A) = \overline{\mathbb{D}}$ & "$n-2$" is sharp:

$$\text{Ex. } A_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ & 0 & 1 & & & 0 \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & 0 \\ & & & & \ddots & 1 \\ & & & & & 0 \end{bmatrix} \quad n \times n \ \ (n \geq 3)$$

$\Rightarrow W(A_n) \subsetneqq \overline{\mathbb{D}}$ & $\#(\partial W(A_n) \cap \partial \mathbb{D}) = n-2$

(5) A $n \times n$ nilpotent, $W(A) \supseteq \overline{\mathbb{D}}$, $\#(\partial W(A) \cap \partial \mathbb{D}) > n-2$

$$\Rightarrow \begin{cases} W(A) = \overline{\mathbb{D}} & \text{if } 2 \leq n \leq 4 \\ \partial W(A) \text{ contains an arc of } \partial \mathbb{D} & \text{if } n \geq 5 \end{cases}$$

(6) A $n \times n$, $W(A) =$ circular disc centered at a

$\Rightarrow$ a is eigenvalue of A with $1 \leq$ geom. multi. $<$ alg. multi.

Cor.1. A $n \times n$ similar to normal $\Rightarrow$ $W(A) \neq$ circular disc

Cor.2. A $n \times n$ nonnegative & irreducible

$\Rightarrow$ $W(A) \neq$ circular disc

Cor.3. A $n \times n$ row stochastic $\Rightarrow$ $W(A) \neq$ circular disc

Note. $A = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$ $n \times n$

$\Rightarrow W(A) = \{z \in \mathbb{C} : |z| \leq \cos(\pi/(n+1))\}$

J. Holbrook (1969):

$$AB = BA \overset{?}{\Rightarrow} w(AB) \leq w(A)\|B\|,\ \|A\|w(B)$$

Known:

(1) $w(AB) \leq 4w(A)w(B)$ & "4" is sharp:

Ex. $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore w(AB) = 1,\ w(A) = w(B) = 1/2$$

(2) $AB = BA \Rightarrow w(AB) \leq 2w(A)w(B)$ & "2" is sharp:

Ex. $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$$w(AB) = w(A) = w(B) = 1/2$$

(3) True if A normal (Holbrook, 1969)

if A isometry (Bouldin, 1971)

if $AB = BA$ & $AB^* = B^*A$ (Holbrook, 1969)

if A, B 2×2 (Holbrook, 1992)

(4) Crabb (1976):

$$AB = BA \Rightarrow w(AB) \leq (\sqrt{2 + 2\sqrt{3}}/2)w(A)\|B\|$$

(5) Müller (1988), Davidson & Holbrook (1988):

Ex. $A = J_9, \;\; B = J_9^3 + J_9^7$

$w(AB) = \|A\| = 1, \;\; \|B\| \geq \sqrt{2}$

$w(A) = w(B) = \cos(\pi/10)$

$\Rightarrow w(AB) > \|A\| w(B)$, but $w(AB) \leq w(A)\|B\|$

Schoch (2002): $A, \; B \quad 4 \times 4$

Wu, Gau & Tsai (2009) :

Thm 1. $A = S(\phi)$, $AB = BA \Rightarrow w(AB) \leq w(A)\|B\|$

But $A = S(\phi)$, $AB = BA \not\Rightarrow w(AB) \leq \|A\|w(B)$

ϕ inner func. ($\phi : \mathbb{D} \to \overline{\mathbb{D}}$ analy. & $|\phi| = 1$ a.e. on $\partial\mathbb{D}$)

Def. $S(\phi)f = P_{H^2 \ominus \phi H^2}(zf(z))$ for $f \in H^2 \ominus \phi H^2$

Ex. $\phi(z) = z^n$ $(n \geq 1)$ Then $S(\phi) \cong J_n$

More generally,

Thm 2. A C_0 contraction with minimal func. ϕ, $w(A) = w(S(\phi))$

(Def. $\|A\| \leq 1$ & $\phi(A) = 0$)

$AB = BA \Rightarrow w(AB) \leq w(A)\|B\|$

Cor. A quadratic, $AB = BA \Rightarrow w(AB) \leq w(A)\|B\|$

$$\uparrow$$

$$A^2 + aA + bI = 0 \text{ for some } a, b \in \mathbb{C}$$

Unknown:

$$A \text{ quadratic}, AB = BA \overset{?}{\Rightarrow} w(AB) \leq \|A\| w(B)$$

Known:

(1) Rao (1994):

$$A^2 = aI, AB = BA \Rightarrow w(AB) \leq \|A\| w(B)$$

(2) Gau, Huang & Wu (2008):

$$A^2 = 0 \text{ or } A^2 = A, AB = BA$$

$$\Rightarrow w(AB) \leq \min\{w(A)\|B\|, \|A\| w(B)\}$$

$$\|A\|/2 \leq w(A) \leq \|A\|$$

(1) Williams & Crimmins (1967):

$\|A\| = 2,\ w(A) = 1,\ \|Ax\| = \|A\|$ for some $\|x\| = 1$

$$\Rightarrow A \cong \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus A' \quad \& \quad W(A) = \overline{\mathbb{D}}$$

Pf: Let $y = (1/2)Ax \quad (x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \oplus 0 \ \&\ y = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus 0)$

Then $\|y\| = 1 \ \&\ x \perp y$

Let $K = \vee\{y, x\}$

Then $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus A'$ on $H = K \bigoplus K^{\perp}$

August, 1978
Helsinki, Finland
James P. Williams

(2) Crabb (1971):

$w(A) \leq 1$, $\|A^n x\| = 2$ for some $n \geq 1$ & $\|x\| = 1$

$$\Rightarrow A \cong \begin{cases} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus A' & \text{if } n = 1 \\ \begin{bmatrix} 0 & \sqrt{2} & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & 1 & \\ & & & & \ddots & \sqrt{2} \\ & & & & & 0 \end{bmatrix} \oplus A' & \text{if } n \geq 2 \end{cases}$$

& $W(A) = \overline{\mathbb{D}}$

Gau & Wu (2009): confirming a conjecture of Drury (2008)

Thm. f inner func., $f(0) = 0$

$w(A) \leq 1$, A has no singular unitary part

$\|f(A)x\| = 2$ for some $\|x\| = 1$

$\Rightarrow A \cong B \oplus A'$, where B similar to $S(\phi)$,

$\phi(z) = zf(z)$ & $\overline{W(A)} = \overline{\mathbb{D}}$

Special cases:

$f(z) = z \Rightarrow$ Williams & Crimmins

$f(z) = z^n$ $(n \geq 2) \Rightarrow$ Crabb

莊康威：

惠我算子, 退休快樂！

John Conway:

Beneficial to Operator Theory,

Happy Retirement !