Operator Identities for Subnormal Tuples of Operators

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1.

A k-tuple of operators $\mathbb{S} = (S_1, \cdots, S_k)$ on a Hilbert space $\mathcal{H}$ is said to be subnormal if there is a commuting k-tuple $\mathbb{N} = (N_1, \cdots, N_k)$ of normal operators on a Hilbert space $\mathcal{H}_0$ containing $\mathcal{H}$ as a subspace, such that

$$S_j = N_j|_{\mathcal{H}}.$$

In this case $\mathbb{N}$ is said to be a normal extension of $\mathbb{S}$. A normal extension is said to be minimal if there is no proper subspace of $\mathcal{H}_0 \ominus \mathcal{H}$ which reduces $\mathbb{N}$. The minimal normal extension (m.n.e.) of a subnormal tuple of operators exists and is unique under unitary equivalence. There are several mathematicians studying subnormal k-tuples of operators such as A. Athevale, J. B. Conway, R. E. Curto, J. Eschmeier, J. Gleason, M. Putinar, J. D. Pincus, D. Xia and D. Zheng.

Let M be the closure of $\bigvee_{i,j}[S_i^*, S_j]\mathcal{H}$. Then M is said to be the defect space of $\mathbb{S}$. Xia proved that

$$S_j^* M \subset M.$$

It is evident that $[S_i^*, S_j]M \subset M$. Xia introduced the operators

$$C_{ij} \overset{def}{=} [S_i^*, S_j]|_M \quad \text{and} \quad \Lambda_j \overset{def}{=} (S_j^*|_M)^*$$

in $L(M)$ and proved that $\{C_{ij}, \Lambda_j\}$ is a complete unitary invariant for S.

In the first part of this talk, the formulas for calculating the product of resolvents, a kind of Lifschitz-Brodski kernel:

$$P_M \prod_{i=1}^{m} (N_{p_i}^* - \bar{w}_i)^{-1} \prod_{j=1}^{n} (N_{q_j} - z_j)^{-1}|_M \tag{1}$$

will be given, where $\mathbb{N}$ is the m.n.e. of a subnormal k-tuple of $\mathbb{S} = (S_1, \cdots, S_k)$, $1 \leq p_i, q_j \leq k$, M is the defect space, P_M is the projection from $\mathcal{H}_0$ to M, $z_i \in \rho(N_{q_i})$ and $w_j \in \rho(N_{p_j})$. If $z_i \in \rho(S_{q_i})$ and $w_j \in \rho(S_{p_j})$ then (1) equals to

$$P_M \prod_{i=1}^{m} (S_{p_i}^* - \bar{w}_i)^{-1} \prod_{j=1}^{n} (S_{q_j} - z_j)^{-1}|_M. \tag{2}$$

Notice that if $\mathbb{S}$ is pure, i.e. if there is no proper subspace $F \subset \mathcal{H}$ reducing $\mathbb{S}$ such that $\mathbb{S}|_F$ is normal, then

$$\mathcal{H} = \text{closure of} \quad \bigvee \{\prod_{j=1}^{k} (S_j - z_j)^{-1} \alpha : z_j \in \rho(S_j), \alpha \in M\}.$$

Thus the calculation of (2) provides a way of calculation of the inner product of any two vectors in $\mathcal{H}$.

In the second part of this talk, we generalize the formula for (2) to the formula for

$$P_{\mathcal{K}} \prod_{i=1}^{m} (T_{p_i}^* - \bar{w}_i)^{-1} \prod_{j=1}^{n} (T_{q_j}^* - z_j)^{-1}|_{\mathcal{K}} \tag{3}$$

where $\mathbb{T} = (T_1, \cdots, T_k)$ is a commuting k-tuple of operators on a Hilbert space $\mathcal{H}$, $\mathcal{K}$ is the minimal subspace which contains the defect space of $\mathbb{T}$ and invariant with respect to T_j^* and $[T_i^*, T_j]$, for $i, j = 1, 2, \cdots, k$.

2.

Let us review the related theory in the case of single subnormal operator S on $\mathcal{H}$. Let N be the m.n.e. of S on $\mathcal{H}_0$, M be the defect space of S. Let

$$C = P_M[S^*, S]|_M, \qquad \Lambda = (S^*|_M)^*.$$

Let $E(\cdot)$ be the spectral measure of N. Define a positive L(M)-valued measure on $\sigma(N)$:

$$e(F) \stackrel{def}{=} P_M E(F)|_M,$$

where F is any Borel set in $\sigma(N)$. In this single subnormal operator case the Lifschitz-Brodski kernel (1) becomes

$$\begin{aligned} S(z, w) &\stackrel{def}{=} P_M (N^* - \bar{w})^{-1} (N - z)^{-1}|_M \\ &= \int_{\sigma(N)} \frac{e(du)}{(\bar{u} - \bar{w})(u - z)}, \quad z, w \in \rho(N). \end{aligned}$$

Xia introduced a mosaic:

$$\begin{aligned} \mu(z) &\stackrel{def}{=} P_M (N - S P_{\mathcal{H}})(N - z)^{-1}|_M, \\ &= \int_{\sigma(N)} \frac{u - \Lambda}{u - z} e(du), \qquad z \in \rho(N). \end{aligned}$$

Then $\mu(z)$ is analytic on $\rho(N)$ and is idempotent: $\mu(z)^2 = \mu(z)$. Besides, $\mu(z) = 0$ for $z \in \rho(S)$.

Let $Q(z,w) \stackrel{def}{=} (\bar{w} - \Lambda^*)(z - \Lambda) - C$ and

$$R(z) \stackrel{def}{=} C(z-\Lambda)^{-1} + \Lambda^*, \qquad z \in \rho(\Lambda).$$

Then $Q(z,w) = (R(z) - \bar{w})(\Lambda - z)$ and

$$[R(z), \mu(z)] = o, \qquad z \in \rho(\Lambda) \cap \rho(N).$$

The Lifschitz-Brodski kernel can be expressed as

$$S(z,w) = (I - \mu(w)^*)Q(z,w)^{-1} - Q(z,w)^{-1}\mu(z), \tag{4}$$

if $z, w \in \rho(N)$ and $Q(z,w)$ is invertible. For $z, w \in \rho(S)$, the kernel (2) becomes

$$S(z,w) = Q(z,w)^{-1}, \qquad z, w \in \rho(S).$$

Especially, if $\dim M < \infty$, this pair $\{C, \Lambda\}$ is a pair of matrices and is a very useful tool for studying S. For example, in this case

$$\sigma(N) \subset \{z : \det Q(z,z) = 0\},$$

and $\sigma(S) \backslash \sigma(N)$ covered by a union of quadrature domains in Riemann surfaces. If $\sigma(S) \backslash \sigma(N)$ is a quadrature domain $D \subset \mathbb{C}$, then

$$R(z)\mu(z) = \mu(z)R(z) = S(z)\mu(z),$$

where $S(\cdot)$ is the Schwartz function of D. Besides, the mosaic $\mu(z)$ is the parallel projection to the eigen-space of the matrix $R(z)$ corresponding to the eigenvalue $S(z)$.

3.

Let $\mathbb{S} = (S_1, \cdots, S_k)$ be a k-tuple of subnormal operators on a Hilbert space $\mathcal{H}$ with m.n.e. $\mathbb{N} = (N_1, \cdots, N_k)$ on $\mathcal{H}_0 \supset \mathcal{H}$. Let $E(\cdot)$ be the spectral measure of $\mathbb{N}$ and $e(\cdot) = P_M E(\cdot)|_M$.

We have generalized the mosaic $\mu(\cdot)$ and $R(\cdot)$ into the k-tuple case.For any $l_1, \cdots, l_n \in \{1, 2, \cdots, k\}$ and $1 \le i < m \le n$, define

$$\mu_{l_i, \cdots, l_m}(z_i, \cdots, z_m) \stackrel{def}{=} P_M(N_{l_i} - S_{l_i}P_{\mathcal{H}}) \prod_{j=i}^{m} (N_{l_j} - z_j)^{-1}|_M$$
$$= \int_{sp(\mathbb{N})} \frac{(u_{l_i} - \Lambda_{l_i})e(du)}{\prod_{j=i}^{m}(u_{l_j} - z_j)}$$

for $z_j \in \rho(N_{l_j})$, where $u = (u_1, \cdots, u_m)$. Let $L = \{l_1, \cdots, l_n\}$. Define the mosaic of $\mathbb{S}$:

$$\mu_L(z) = \begin{pmatrix} \mu_{l_1}(z_1) & \mu_{l_1,l_2}(z_1,z_2) & \mu_{l_1,l_2,l_3}(z_1,z_2,z_3)\cdots & \mu_{l_1,\cdots,l_n}(z_1,\cdots,z_n) \\ 0 & \mu_{l_2}(z_2) & \mu_{l_2,l_3}(z_2,z_3) \quad \cdots & \mu_{l_2,\cdots,l_n}(z_2,\cdots,z_n) \\ 0 & 0 & \mu_{l_3}(z_3) \quad \cdots & \mu_{l_3,\cdots,l_n}(z_3,\cdots,z_n) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \quad \cdots & \mu_{l_n}(z_n) \end{pmatrix}.$$

where $z = (z_1, \cdots, z_n)$, $z_j \in \rho(N_{l_j})$. Then $\mu_L(z)$ is analytic on $\rho_{l_1}(N_{l_1}) \times \cdots \times \rho_{l_n}(N_{l_n})$, and $\mu_L(z)^2 = \mu_L(z)$.

If $z_j \in \rho(S_{l_j})$, $j = 1, 2, \cdots, n$, then $\mu_L(z) = 0$. Define

$$\mu_L^\dagger(z) = \begin{pmatrix} I-\mu_{l_n}(z_n)^* & -\mu_{l_n,l_{n-1}}(z_n,z_{n-1})^* & \cdots & -\mu_{l_n,\cdots,l_1}(z_n,\cdots,z_1)^* \\ 0 & I-\mu_{l_{n-1}}(z_{n-1}) & \cdots & -\mu_{l_{n-1},\cdots,l_1}(z_{n-1},\cdots,z_1)^* \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I-\mu_{l_1}(z_1)^* \end{pmatrix}.$$

Then $\mu_L^\dagger(z)^2 = \mu_L^\dagger(z)$, $\mu_L^\dagger(z)$ is analytic on $\rho_{l_1}(N_{l_1}) \times \cdots \times \rho_{l_n}(N_{l_n})$ and $\mu_L^\dagger(z) = I$ for $z \in \rho_{l_1}(S_{l_1}) \times \cdots \times \rho_{l_n}(S_{l_n})$.

For $m, l \in \{1, 2, \cdots, k\}$, define

$$Q_{ml}(z,w) = (\Lambda_m^* - \bar{w})(\Lambda_l - z) - C_{ml},$$

$$R_{ml}(z) = C_{ml}(z-\Lambda_l)^{-1} + \Lambda_m^*, \qquad z \in \rho(\Lambda_l).$$

Then $[R_{ml_1}(z), R_{ml_2}(z)] = 0$.

For $m, l_1, \cdots, l_n \in \{1, 2, \cdots, k\}$, $L = \{l_1, \cdots, l_n\}$, $z_j \in \rho(\Lambda_{l_j})$, denote $\hat{\Lambda}_j \stackrel{def}{=} (\Lambda_{l_j} - z_j)^{-1}$ and

$$R_{mL}(z) = \begin{pmatrix} R_{ml_1}(z_1) & -C_{ml_1}\prod_{j=1}^{2}\hat{\Lambda}_j & -C_{ml_1}\prod_{j=1}^{3}\hat{\Lambda}_j & \cdots & -C_{ml_1}\prod_{j=1}^{n}\hat{\Lambda}_j \\ 0 & -C_{ml_2}(z_2) & -C_{ml_2}\prod_{j=2}^{3}\hat{\Lambda}_j & \cdots & -C_{ml_2}\prod_{j=2}^{n}\hat{\Lambda}_j \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & R_{ml_n}(z_n) \end{pmatrix}, \tag{5}$$

where $z = (z_1, \cdots, z_n) \in \rho_{l_1}(\Lambda_{l_1}) \times \cdots \times \rho_{l_n}(\Lambda_{l_n})$. We have proved that

$$[R_{mL}(z), \mu_L(z)] = 0,$$

for $z \in (\rho_{l_1}(\Lambda_{l_1}) \times \cdots \times \rho_{l_n}(\Lambda_{l_n})) \bigcap (\rho_{l_1}(N_{l_1}) \times \cdots \times \rho_{l_n}(N_{l_n}))$.

4.

Let us denote (1) by $S_{P,Q}(z,w)$, where $P=(p_1,\cdots,p_m)$, $Q=(q_1,\cdots,q_n)$, $z=(z_1,\cdots,z_n)$ and $w=(w_1,\cdots,w_m)$. This kernel is Hermitian, i.e.

$$S_{P,Q}(z,w)^* = S_{Q,P}(w,z).$$

The kernel $S_{P,Q}(z,w)$ can be written as an integral

$$\int_{sp(\mathbb{N})} \frac{e(du)}{\prod_{i=1}^{m}(\bar{u}_{p_i}-\bar{w}_i)\prod_{j=1}^{n}(u_{q_j}-z_j)}.$$

To study $S_{P,Q}(z,w)$, let $P_j=(p_1,\cdots,p_j)$ and $Q_j=(q_1,\cdots,q_j)$. Denote $S_{P_j,Q_j}(z_1,\cdots,z_j,w_1,\cdots,w_j)$ simply by S_{P_j,Q_j}. We form a matrix with entries in L(M):

$$\mathfrak{S}_{P,Q} = \begin{pmatrix} S_{P_m,Q_1} & S_{P_m,Q_2} & \cdots & S_{P_m,Q_n} \\ S_{P_{m-1},Q_1} & S_{P_{m-1},Q_2} & \cdots & S_{P_{m-1},Q_n} \\ \cdots & \cdots & \cdots & \cdots \\ S_{P_2,Q_1} & S_{P_2,Q_2} & \cdots & S_{P_2,Q_n} \\ S_{P_1,Q_1} & S_{P_1,Q_2} & \cdots & S_{P_1,Q_n} \end{pmatrix}. \tag{6}$$

Thus $\mathfrak{S}_{P,Q}$ is the object for study. In the previous paper we have studied the $(m+n)\times(m+n)$ matrix defined by the block matrices

$$\mathfrak{I}_{P,Q} \overset{def}{=} \begin{pmatrix} \mu_P^\dagger & \mathfrak{S}_{P,Q} \\ 0 & \mu_Q \end{pmatrix}$$

where 0 in this block matrix is a $n\times m$ matrix with all entries zero. We have proved that $\mathfrak{I}_{P,Q}$ is also idempotent, i.e.

$$\mathfrak{I}_{P,Q}^2 = \mathfrak{I}_{P,Q},$$

which means

$$\mathfrak{S}_{P,Q} = \mu_P^\dagger \mathfrak{S}_{P,Q} + \mathfrak{S}_{P,Q}\mu_Q. \tag{7}$$

In the case $k=1$, it is

$$S(z,w) = (I-\mu(w)^*)S(z,w) + S(z,w)\mu(z) \tag{8}$$

Comparing (4) and (8), it suggests us in the right side of (7), we may replace $\mathfrak{S}_{P,Q}$ by some rational function os z and $\bar{w}$ with coefficients constructed by C_{ij}, Λ_j and Λ_j^*.

For $w\in\rho(\Lambda_m)$, if $R_{q_jm}(w)^*-z_j$ is invertible, $j=1,2,\cdots,n$, define

$$X_{m,Q} \overset{def}{=} X_{m,Q}(z,w) = (\prod_{j=1}^{n}(R_{q_jm}(w)^*-z_j)^{-1})(\Lambda_m^*-\bar{w})^{-1},$$

where $Q = \{q_1, \cdots, q_n\}$, $z = \{z_1, \cdots, z_n\}$. For $P = \{p_1, \cdots, p_m\}$, let $P_i = \{p_1, \cdots, p_i\}$ and $Q_j = \{q_1, \cdots, q_j\}$ where $i \le m$, $j \le n$, define $X_{P_1,Q} = X_{p_1,Q}$ and

$$\begin{pmatrix} X_{P_j,Q_1} & X_{P_j,Q_2} & \cdots & X_{P_j,Q_n} \end{pmatrix}$$
$$\overset{def}{=} \begin{pmatrix} X_{P_1,Q_1} & X_{P_1,Q_2} & \cdots & X_{P_1,Q_n} \end{pmatrix} (R_{p_2,Q_n} - \bar{w}_2)^{-1} \cdots (R_{p_j,Q_n} - \bar{w}_n)^{-1}.$$

for $j \ge 2$, where X_{P_j,Q_i} stands for

$$X_{\{p_1,\cdots,p_j\},\{q_1,\cdots,q_i\}}(z_1, \cdots, z_j; w_1, \cdots, w_i).$$

Let $\mathfrak{X}_{P,Q} = \mathfrak{X}_{P,Q}(z_1, \cdots, z_n; w_1, \cdots, w_m)$ be the matrix

$$\begin{pmatrix} X_{P_m,Q_1} & X_{P_m,Q_2} & \cdots & X_{P_m,Q_n} \\ X_{P_{m-1},Q_1} & X_{P_{m-1},Q_2} & \cdots & X_{P_{m-1},Q_n} \\ \cdots & \cdots & \cdots & \cdots \\ X_{P_2,Q_1} & X_{P_2,Q_2} & \cdots & X_{P_2,Q_n} \\ X_{P_1,Q_1} & X_{P_1,Q_2} & \cdots & X_{P_1,Q_n} \end{pmatrix}. \tag{9}$$

Theorem 1. *Let $\mathbb{S} = (S_1, \cdots, S_k)$ be a pure subnormal k-tuple of operators on a separable Hilbert space $\mathcal{H}$ with minimal normal extension $\mathbb{N} = (N_1, \cdots, N_k)$ on $\mathcal{H}_0 \supset \mathcal{H}$. For sets of integers $P = \{p_1, \cdots, p_m\}$, $Q = \{q_1, \cdots, q_n\}$ satisfying $P \bigcup Q \subset \{1, 2, \cdots, k\}$, if $z_i \in \rho(\Lambda_{q_i}) \cap \rho(N_{q_i})$ and $w_j \in \rho(\Lambda_{p_j}) \cap \rho(N_{p_j})$ satisfying the condition that $Q_{p_j,q_i}(z_i, w_j)$ are invertible, $j = 1, 2, \cdots, m$ and $i = 1, 2, \cdots, n$, then*

$$\mathfrak{S}_{P,Q} = \mu_P^\dagger \mathfrak{X}_{P,Q} - \mathfrak{X}_{P,Q}\mu_Q,$$

where $\mathfrak{S}_{P,Q}$ stands for $\mathfrak{S}_{P,Q}(z, w)$, $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_m)$.

If also $z_i \in \rho(S_{q_i})$ and $w_j \in \rho(S_{p_j})$, then

$$\mathfrak{S}_{P,Q} = \mathfrak{X}_{P,Q}. \tag{10}$$

5.

Let $\mathbb{T} = (T_1, \cdots, T_k)$ be a commuting k-tuple of operators, i.e. $[T_i, T_j] = 0$, on a Hilbert space $\mathcal{H}$. Let

$$M = M_{\mathbb{T}} \overset{def}{=} \text{closure of} \bigvee \{[T_i^*, T_j]\mathcal{H} : i, j = 1, 2, \cdots, k\}$$

be the defect space of $\mathbb{T}$. In general, $T^*M \subset M$ may not be true. M. Putinar in the case of $\mathbb{T} = (T_1)$, where T_1 is a hyponormal operator and later J. Eschmeier and M. Putinar for commuting k-tuple of operators $\mathbb{T} = (T_1, \cdots, T_k)$ case introduced

$$\mathcal{K} \overset{def}{=} \text{closure of} \bigvee \{T_1^{*m_1} T_2^{*m_2} \cdots T_k^{*m_k} M_{\mathbb{T}} : m_1, \cdots, m_k = 0, 1, 2, \cdots\}.$$

Then $T_j^*\mathcal{K} \subset \mathcal{K}$. Similar to the operators C_{ij} and Λ_i in §2, define operators

$$C_{ij} \overset{def}{=} [T_i^*, T_j]|_{\mathcal{K}} \text{ and } \Lambda_j \overset{def}{=} (T_j^*|_{\mathcal{K}})^*.$$

Then we can define $R_{m,L}(z)$ for $\mathbb{T}$ as in (5).

Lemma 2. *For* $z_j \in \rho(\Lambda_j)$, $j = 1, 2, \cdots, n$,

$$[R_{m,L}(z), R_{m',L}(z)] = 0,$$

for $m, m' = 1, 2, \cdots, k$.

We define

$$S_{P,Q}(z, w) = P|_{\mathcal{K}} \prod_{i=1}^{m} (T_{p_i}^* - \bar{w}_i)^{-1} \prod_{j=1}^{n} (T_{q_j}^* - z_j)^{-1}|_{\mathcal{K}}$$

for $P = (p_1, \cdots, p_m)$, $Q = (q_1, \cdots, q_n)$, $z = (z_1, \cdots, z_n)$, $w = (w_1, \cdots, w_m)$, $z_j \in \rho(T_{q_j})$ and $w_i \in \rho(T_{p_i})$. We keep the notation $\mathfrak{S}_{P,Q}$ in (6) and $\mathfrak{X}_{P,Q}$ in (9) for the commuting k-tuple of operators $\mathbb{T}$.

Theorem 3. *Let* $\mathbb{T} = \{T_1, \cdots, T_k\}$ *be a commuting k-tuple of operators on* $\mathcal{H}$. *Let* $P = \{p_1, \cdots, p_m\}$, $Q = \{q_1, \cdots, q_n\}$ *satisfying* $P \bigcup Q \subset \{1, 2, \cdots, k\}$. *If* $z = (z_1, \cdots, z_n)$ *and* $w = (w_1, \cdots, w_m)$ *satisfy the condition that* $z_i \in \rho(T_{q_i})$, $i = 1, 2, \cdots, n$ *and* $w_j \in \rho(T_{p_j})$, $j = 1, 2, \cdots, m$. *Then*

$$\mathfrak{S}_{P,Q}(z, w) = \mathfrak{X}_{P,Q}(z, w). \tag{11}$$

Therefore (10) is a special case of (11).