New estimates of Essential norms of weighted composition operators between Bloch type spaces

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Our Goal

Let $\phi$ be an analytic self-map of $D$, and $u$ be an analytic function on $D$. The weighted composition operator induced by $u$ and $\phi$ is defined by $uC_{\phi}(f)(z) = u(z)f(\phi(z))$. In this talk we give estimates of the essential norms of the weighted composition operators $uC_{\phi}$ between different $\alpha$-Bloch spaces in terms of the $n$-th power of $\phi$. We also give similar characterizations for boundedness and compactness of $uC_{\phi}$ between different $\alpha$-Bloch spaces. This is a joint work with Jasbir Singh Manhas.
Let $\varphi$ be an analytic self-map of $D$, and $u$ be an analytic function on $D$. The weighted composition operator induced by $u$ and $\varphi$ is defined by $uC_{\varphi}(f)(z) = u(z)f(\varphi(z))$. In this talk we give estimates of the essential norms of the weighted composition operators $uC_{\varphi}$ between different $\alpha$-Bloch spaces in terms of the $n$-th power of $\varphi$. We also give similar characterizations for boundedness and compactness of $uC_{\varphi}$ between different $\alpha$-Bloch spaces.

This is a joint work with Jasbir Singh Manhas.
The $\alpha$-Bloch Space: Let $0 < \alpha < \infty$. The $\alpha$-Bloch Space $B^{\alpha}$ consists of analytic functions $f$ in $D$ with $\|f\|_{B^{\alpha}} = \sup_{z \in D} |f'(z)| (1 - |z|^2)^\alpha < \infty$.

$B^{\alpha}$ is a Banach space under the norm $|f(0)| + \|f\|_{B^{\alpha}}$.

When $0 < \alpha < 1$, $B^{\alpha} = \text{Lip}^{1-\alpha}$, the Lipschitz space, which contains analytic functions $f$ in $D$ such that, for all $z, w \in D$, $|f(z) - f(w)| \leq C |z - w|^{1-\alpha}$.

When $\alpha > 1$, $B^{\alpha} = A^{\alpha - 1}(H^{\infty})$, which consists of analytic functions $f$ in $D$ such that $\sup_{z \in D} |f(z)| (1 - |z|^2)^{\alpha - 1} < \infty$. 
• The $\alpha$-Bloch Space:

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\sup_{z \in D} |f(z)|(1 - |z|^2)^{\alpha-1} < \infty.
$$
Bounded and compact operators:

Let $X$ and $Y$ be Banach spaces. Let $B_X$ is the unit ball in $X$; A linear operator $T : X \rightarrow Y$ is **bounded**, if $TB_X$ is bounded in $Y$, $T$ is **compact**, if the closure of $TB_X$ is a compact set in $Y$. 

Two integral operators:

For an analytic function $u$ on $D$, we define two integral operators on $H(D)$ as follows: for every $f \in H(D)$,

$I_u f(z) = \int_{z_0}^z f'(\zeta) u(\zeta) \, d\zeta,$

$J_u f(z) = \int_{z_0}^z f(\zeta) u'(\zeta) \, d\zeta.$
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Essential norms:

The essential norm $\|T\|_e$ of a bounded operator $T$ between Banach spaces $X$ and $Y$ is the distance from $T$ to the space of compact operators from $X$ to $Y$. 

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$$ I_u f(z) = \int_0^z f'(\zeta)u(\zeta) \, d\zeta, \quad J_u f(z) = \int_0^z f(\zeta)u'(\zeta) \, d\zeta. $$
Let $0 < \alpha, \beta < \infty$ and $\varphi$ be an analytic self-map of the unit disk $D$. Then the essential norm of composition operator $C_{\varphi} : B^\alpha \rightarrow B^\beta$ is

$$\|C_{\varphi}\|_e = \lim_{s \to 1} \sup_{|\varphi(z)| > s} |\varphi'(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha}.$$
Motivations

**Theorem**

Let $0 < \alpha, \beta < \infty$ and $\varphi$ be an analytic self-map of the unit disk $D$. Then the essential norm of composition operator $C_\varphi : B^\alpha \to B^\beta$ is

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$$

The result was first proved by Montes-Rodríguez in 1999 for the case $\alpha = \beta = 1$. For the case $0 < \alpha = \beta < \infty$, the result was proved by Montes-Rodríguez in 2000. Contreras and Hernandez-Díaz proved for the general case in 2000. When $0 < \alpha \leq 1$, the result was also proved by MacCluer and Z in 2003. The last three papers actually generalized this result to weighted composition operators.
Wulan, Zheng and Zhu obtained the following result (PAMS 2009).

**Theorem**

Let $\varphi$ be an analytic self-map of $D$. Then $C_\varphi$ is compact on the Bloch space $B$ if and only if

$$\lim_{n \to \infty} \|\varphi^n\|_B = 0.$$
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**Question 1.** Can we have an essential norm formula for $C_\varphi : B \to B$, in terms of $\varphi^n$? How about $C_\varphi : B^\alpha \to B^\beta$?
The question has been answered affirmatively by Z (PAMS 2010):

\textbf{Theorem}

Let \(0 < \alpha, \beta < \infty\). Let \(\varphi\) be an analytic self-map of the unit disk \(D\). Then the essential norm of composition operator \(C_{\varphi} : B^\alpha \rightarrow B^\beta\) is

\[\|C_{\varphi}\|_e = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{n \to \infty} n^{\alpha - 1} \|\varphi^n\|_{B^\beta}.\]
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**Question 2.** Can we generalize this result to the weighted composition operators $uC_\varphi$?
Boundedness

Theorem (Boundedness)

Let $\phi$ be an analytic self map of $D$, let $u$ be analytic on $D$, and let $\alpha$ and $\beta$ be positive real numbers.

(i) If $0 < \alpha < 1$, then $uC_\phi$ maps $B_\alpha$ boundedly into $B_\beta$ if and only if $u \in B_\beta$ and

$$\sup_{n \geq 1} n^{\alpha - 1} \|I_u(\phi^n)\|_{B_\beta} < \infty.$$ 

(ii) If $\alpha > 1$, then $uC_\phi$ maps $B_\alpha$ boundedly into $B_\beta$ if and only if

$$\sup_{n \geq 1} n^{\alpha - 1} \|I_u(\phi^n)\|_{B_\beta} < \infty$$
and

$$\sup_{n \geq 1} n^{\alpha - 1} \|J_u(\phi^n)\|_{B_\beta} < \infty.$$
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$$\sup_{n \geq 1} n^{\alpha - 1} \| l_u(\varphi^n) \|_{B^\beta} < \infty.$$  

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The case $\alpha = 1$

For $\alpha = 1$, the corresponding conditions would be

$$\sup_{n \geq 1} \| I_u(\phi_n) \|_{B^\beta} < \infty,$$
$$\sup_{n \geq 1} \| J_u(\phi_n) \|_{B^\beta} < \infty.$$

Curiously enough, while these conditions are necessary for $u C \phi$ to be bounded from $B$ to $B^\beta$, but they are not sufficient.

Example. Let $1 < \beta < \infty$. Let $u(z) = (1 - z)^{1 - \beta}$, $\phi(z) = z$, and $f(z) = \log(2/(1 - z))$. Then $u \in B^\beta$, and $f \in B$. Easy computations show that, for $\beta > 1$,

$$\| I_u(\phi_n) \|_{B^\beta} \leq 2^\beta \| J_u(\phi_n) \|_{B^\beta} \leq \| u \|_{B^\beta}.$$

However, for $\beta > 1$, we have

$$\| u C \phi(f) \|_{B^\beta} = \infty.$$

Therefore, $u C \phi : B \to B^\beta$ is not bounded.
The case $\alpha = 1$

For $\alpha = 1$, the corresponding conditions would be

$$\sup_{n \geq 1} \| I_u(\varphi^n) \|_{B^\beta} < \infty, \quad \sup_{n \geq 1} \| J_u(\varphi^n) \|_{B^\beta} < \infty.$$

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\sup_{n \geq 1} \| I_u(\varphi^n) \|_{B^\beta} < \infty, \quad \sup_{n \geq 1} \| J_u(\varphi^n) \|_{B^\beta} < \infty.
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Curiously enough, while these conditions are necessary for $uC_\varphi$ to be bounded from $B$ to $B^\beta$, but they are not sufficient.

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\| I_u(\varphi^n) \|_{B^\beta} \leq 2^\beta \quad \| J_u(\varphi^n) \|_{B^\beta} \leq \| u \|_{B^\beta}.
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However, for $\beta > 1$, we have

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\| uC_\varphi(f) \|_{B^\beta} = \infty.
$$

Therefore, $uC_\varphi : B \to B^\beta$ is not bounded.
Theorem
Suppose \( 0 < \alpha < 1 \) and \( 0 < \beta < \infty \) and suppose the weighted composition operator \( uC \phi \) is bounded from \( B^\alpha \) to \( B^\beta \). Then
\[
\|uC \phi\|_e = (e^{2\alpha})^{\alpha} \limsup_{n \to \infty} n^{\alpha - 1} \|I_u(\phi^n)\|_{B^\beta}.
\]

Corollary
Suppose \( 0 < \alpha < 1 \) and \( 0 < \beta < \infty \) and suppose the weighted composition operator \( uC \phi \) is bounded from \( B^\alpha \) to \( B^\beta \). Then \( uC \phi \) is compact from \( B^\alpha \) to \( B^\beta \) if and only if
\[
\limsup_{n \to \infty} n^{\alpha - 1} \|I_u(\phi^n)\|_{B^\beta} = 0.
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The case $0 < \alpha < 1$
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**Theorem**

Suppose $0 < \alpha < 1$ and $0 < \beta < \infty$ and suppose the weighted composition operator $uC_\varphi$ is bounded from $B^\alpha$ to $B^\beta$. Then

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\|uC_\varphi\|_e = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{n \to \infty} n^{\alpha - 1} \|I_u(\varphi^n)\|_{B^\beta}.
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**Corollary**

Suppose $0 < \alpha < 1$ and $0 < \beta < \infty$ and suppose the weighted composition operator $uC_\varphi$ is bounded from $B^\alpha$ to $B^\beta$. Then $uC_\varphi$ is compact from $B^\alpha$ to $B^\beta$ if and only if

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Essential Norms

The case $0 < \alpha < 1$

Theorem

Suppose $0 < \alpha < 1$ and $0 < \beta < \infty$ and suppose the weighted composition operator $uC_\varphi$ is bounded from $B^\alpha$ to $B^\beta$. Then

$$\|uC_\varphi\|_e = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{n \to \infty} n^{\alpha - 1} \|I_u(\varphi^n)\|_{B^\beta}.$$ 

Corollary

Suppose $0 < \alpha < 1$ and $0 < \beta < \infty$ and suppose the weighted composition operator $uC_\varphi$ is bounded from $B^\alpha$ to $B^\beta$. Then $uC_\varphi$ is compact from $B^\alpha$ to $B^\beta$ if and only if

$$\limsup_{n \to \infty} n^{\alpha - 1} \|I_u(\varphi^n)\|_{B^\beta} = 0.$$
The case $\alpha > 1$
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Denote by

$$A = \left( \frac{e}{2\alpha} \right)^{\alpha} \limsup_{n \to \infty} n^{\alpha - 1} \| I_u(\varphi^n) \|_{B^\beta}$$

and

$$B = \left( \frac{e}{2(\alpha - 1)} \right)^{\alpha - 1} \limsup_{n \to \infty} n^{\alpha - 1} \| J_u(\varphi^n) \|_{B^\beta}.$$ 

Then we have the following result.
The case $\alpha > 1$

Denote by

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Then we have the following result.

**Theorem**

Let $\alpha > 1$, $0 < \beta < \infty$. Suppose that the weighted composition operator $uC_\varphi$ is bounded from $B^\alpha$ to $B^\beta$. Then

$$\max \left( \frac{1}{2^{1+\alpha}(3\alpha + 2)} A, \frac{1}{2^{1+\alpha}3\alpha(\alpha + 1)} B \right) \leq \| uC_\varphi \|_e \leq A + B.$$
Corollary

Let $\alpha > 1$, $0 < \beta < \infty$. Suppose that the weighted composition operator $uC\varphi$ is bounded from $B^\alpha$ to $B^\beta$. Then $uC\varphi$ is compact from $B^\alpha$ to $B^\beta$ if and only if the following two conditions are satisfied.

$$\limsup_{n \to \infty} n^{\alpha-1} \| l_u(\varphi^n) \|_{B^\beta} = 0$$

and

$$\limsup_{n \to \infty} n^{\alpha-1} \| J_u(\varphi^n) \|_{B^\beta} = 0.$$
Idea of Proofs
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Recall

Theorem (Boundedness)

Part (i). If $0 < \alpha < 1$, then $uC_{\varphi}$ maps $B^\alpha$ boundedly into $B^\beta$ if and only if $u \in B^\beta$ and

$$\sup_{n \geq 1} n^{\alpha - 1} \| I_u(\varphi^n) \|_{B^\beta} < \infty.$$
Idea of Proof. We are going to use the following theorem by Ohno, Stroethoff and Z in 2003: $uC\varphi$ be bounded from $B^\alpha$ to $B^\beta$ if and only if $u \in B^\beta$ and

$$M = \sup_{z \in D} |u(z)||\varphi'(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} < \infty.$$ 

Let $uC\varphi$ be bounded from $B^\alpha$ to $B^\beta$. Then, from above theorem we have $u \in B^\beta$ (actually, $u = uC\varphi(1) \in B^\beta$). Notice that

$$(I_u(\varphi^n)(z))' = u(z)(\varphi^n(z))' = nu(z)\varphi^{n-1}(z)\varphi'(z).$$

Thus we have, for all $n \geq 1$,

$$n^{\alpha-1} \|I_u(\varphi^n)\|_{B^\beta} = n^{\alpha-1} \sup_{z \in D} n|u(z)||\varphi(z)|^{n-1}|\varphi'(z)|(1 - |z|^2)^\beta$$

$$= \sup_{z \in D} n^{\alpha}|\varphi(z)|^{n-1}(1 - |\varphi(z)|^2)^\alpha|u(z)||\varphi'(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha}$$

$$\leq M \sup_{z \in D} n^{\alpha}|\varphi(z)|^{n-1}(1 - |\varphi(z)|^2)^\alpha$$

$$\leq MK.$$
Conversely, let \( u \in B^\beta \) and \( \sup_{n \geq 1} n^{\alpha-1} \| I_u(\varphi^n) \|_{B^\beta} < \infty \). For any integer \( n \geq 1 \), let

\[
D_n = \{ z \in D : r_n \leq |\varphi(z)| \leq r_{n+1} \},
\]

where

\[
r_n = \begin{cases} 
0, & \text{as } n = 1 \\
\left( \frac{n-1}{n-1+2\alpha} \right)^{1/2}, & \text{as } n \geq 2.
\end{cases}
\]

Let \( m \) and \( k \) be the smallest and largest positive integers such that \( D_m \neq \emptyset \) and \( D_k \neq \emptyset \) (\( k \) could be \( \infty \)). Then we can decompose \( D \) as \( D = \bigcup_{n=m}^k D_n \). An easy exercise in Calculus shows that, there exists a constant \( \delta > 0 \), independent of \( n \), such that

\[
\min_{z \in D_n} n^{\alpha} |\varphi(z)|^{n-1} (1 - |\varphi(z)|^2)^\alpha \geq \delta.
\]
Hence,

\[
\begin{align*}
\sup_{z \in D} |u(z)||\varphi'(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} \\
= \sup_{m \leq n \leq k} \sup_{z \in D_n} |u(z)||\varphi'(z)| \frac{n^\alpha|\varphi(z)|^{n-1}(1 - |z|^2)^\beta}{n^\alpha|\varphi(z)|^{n-1}(1 - |\varphi(z)|^2)^\alpha} \\
\leq \frac{1}{\delta} \sup_{m \leq n \leq k} \sup_{z \in D_n} n^\alpha|u(z)||\varphi(z)|^{n-1}|\varphi'(z)|(1 - |z|^2)^\beta \\
\leq \frac{1}{\delta} \sup_{n \geq 1} \sup_{z \in D} n^\alpha|u(z)||\varphi(z)|^{n-1}|\varphi'(z)|(1 - |z|^2)^\beta \\
\leq \frac{1}{\delta} n^{\alpha-1} \sup_{n \geq 1} \|l_u(\varphi^n)\|_{B^\beta} < \infty.
\end{align*}
\]

Thus by Theorem (OSZ 2003) we know that \( uC \varphi \) is bounded from \( B^\alpha \) to \( B^\beta \).
THANK YOU!