New estimates of Essential norms of weighted composition operators between Bloch type spaces

Ruhan Zhao

SUNY-Brockport, NY, USA

March 19, 2011

Our Goal

Our Goal

Let φ be an analytic self-map of D, and u be an analytic function on D. The weighted composition operator induced by u and φ is defined by $uC_\varphi(f)(z) = u(z)f(\varphi(z))$. In this talk we give estimates of the essential norms of the weighted composition operators uC_φ between different α-Bloch spaces in terms of the n-th power of φ. We also give similar characterizations for boundedness and compactness of uC_φ between different α-Bloch spaces.

This is a joint work with Jasbir Singh Manhas.

Definitions

Definitions

- **The α-Bloch Space:**

Let $0 < \alpha < \infty$. The α-Bloch Space B^α consists of analytic functions f in D with

$$\|f\|_{B^\alpha} = \sup_{z \in D} |f'(z)|(1 - |z|^2)^\alpha < \infty.$$

Definitions

- **The α-Bloch Space:**

Let $0 < \alpha < \infty$. The α-Bloch Space B^α consists of analytic functions f in D with

$$\|f\|_{B^\alpha} = \sup_{z \in D} |f'(z)|(1 - |z|^2)^\alpha < \infty.$$

- B^α is a Banach space under the norm $|f(0)| + \|f\|_{B^\alpha}$.

Definitions

• The α-Bloch Space:

Let $0 < \alpha < \infty$. The α-Bloch Space B^α consists of analytic functions f in D with

$$\|f\|_{B^\alpha} = \sup_{z \in D} |f'(z)|(1 - |z|^2)^\alpha < \infty.$$

• B^α is a Banach space under the norm $|f(0)| + \|f\|_{B^\alpha}$.
• When $0 < \alpha < 1$, $B^\alpha = \mathrm{Lip}_{1-\alpha}$, the Lipschitz space, which contains analytic functions f in D such that, for all $z, w \in D$,

$$|f(z) - f(w)| \leq C|z - w|^{1-\alpha}.$$

Definitions

• The α-Bloch Space:

Let $0 < \alpha < \infty$. The α-Bloch Space B^α consists of analytic functions f in D with

$$\|f\|_{B^\alpha} = \sup_{z \in D} |f'(z)|(1 - |z|^2)^\alpha < \infty.$$

• B^α is a Banach space under the norm $|f(0)| + \|f\|_{B^\alpha}$.
• When $0 < \alpha < 1$, $B^\alpha = \mathrm{Lip}_{1-\alpha}$, the Lipschitz space, which contains analytic functions f in D such that, for all $z, w \in D$,

$$|f(z) - f(w)| \le C|z - w|^{1-\alpha}.$$

• When $\alpha > 1$, $B^\alpha = A^{-(\alpha-1)}$ ($H^\infty_{\alpha-1}$), which consists of analytic functions f in D such that

$$\sup_{z \in D} |f(z)|(1 - |z|^2)^{\alpha-1} < \infty.$$

Bounded and compact operators:

Let X and Y be Banach spaces. Let B_X is the unit ball in X; A linear operator $T : X \to Y$ is **bounded**, if TB_X is bounded in Y, T is **compact**,if the closure of TB_X is a compact set in Y.

Bounded and compact operators:

Let X and Y be Banach spaces. Let B_X is the unit ball in X; A linear operator $T : X \to Y$ is **bounded**, if TB_X is bounded in Y, T is **compact**,if the closure of TB_X is a compact set in Y.

Essential norms:

The essential norm $\|T\|_e$ of a bounded operator T between Banach spaces X and Y is the distance from T to the space of compact operators from X to Y.

Bounded and compact operators:

Let X and Y be Banach spaces. Let B_X is the unit ball in X; A linear operator $T : X \to Y$ is **bounded**, if TB_X is bounded in Y, T is **compact**,if the closure of TB_X is a compact set in Y.

Essential norms:

The essential norm $\|T\|_e$ of a bounded operator T between Banach spaces X and Y is the distance from T to the space of compact operators from X to Y.

Two integral operators:

For an analytic function u on D, we define two integral operators on $H(D)$ as follows: for every $f \in H(D)$,

$$I_u f(z) = \int_0^z f'(\zeta)u(\zeta)\,d\zeta, \qquad J_u f(z) = \int_0^z f(\zeta)u'(\zeta)\,d\zeta.$$

Motivations

Theorem

Let $0 < \alpha, \beta < \infty$ and φ be an analytic self-map of the unit disk D. Then the essential norm of composition operator $C_\varphi : B^\alpha \to B^\beta$ is

$$\|C_\varphi\|_e = \lim_{s \to 1} \sup_{|\varphi(z)| > s} |\varphi'(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha}.$$

Motivations

Theorem

Let $0 < \alpha, \beta < \infty$ and φ be an analytic self-map of the unit disk D. Then the essential norm of composition operator $C_\varphi : B^\alpha \to B^\beta$ is

$$\|C_\varphi\|_e = \lim_{s\to 1} \sup_{|\varphi(z)|>s} |\varphi'(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}.$$

The result was first proved by Montes-Rodríguez in 1999 for the case $\alpha = \beta = 1$. For the case $0 < \alpha = \beta < \infty$, the result was proved by Montes-Rodríguez in 2000. Contreras and Hernandez-Díaz proved for the general case in 2000. When $0 < \alpha \leq 1$, the result was also proved by MacCluer and Z in 2003. The last three papers actually generalized this result to weighted composition operators.

Wulan, Zheng and Zhu obtained the following result (PAMS 2009).

Theorem

Let φ be an analytic self-map of D. Then C_φ is compact on the Bloch space B if and only if

$$\lim_{n\to\infty} \|\varphi^n\|_B = 0.$$

Wulan, Zheng and Zhu obtained the following result (PAMS 2009).

Theorem

Let φ be an analytic self-map of D. Then C_φ is compact on the Bloch space B if and only if

$$\lim_{n\to\infty} \|\varphi^n\|_B = 0.$$

Question 1. Can we have an essential norm formula for $C_\varphi : B \to B$, in terms of φ^n? How about $C_\varphi : B^\alpha \to B^\beta$?

The question has been answered affirmatively by Z (PAMS 2010):

Theorem

Let $0 < \alpha, \beta < \infty$. Let φ be an analytic self-map of the unit disk D. Then the essential norm of composition operator $C_\varphi : B^\alpha \to B^\beta$ is

$$\|C_\varphi\|_e = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{n\to\infty} n^{\alpha-1}\|\varphi^n\|_{B^\beta}.$$

The question has been answered affirmatively by Z (PAMS 2010):

Theorem

Let $0 < \alpha, \beta < \infty$. Let φ be an analytic self-map of the unit disk D. Then the essential norm of composition operator $C_\varphi : B^\alpha \to B^\beta$ is

$$\|C_\varphi\|_e = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{n\to\infty} n^{\alpha-1}\|\varphi^n\|_{B^\beta}.$$

Question 2. Can we generalize this result to the weighted composition operators uC_φ?

Boundedness

Boundedness

Theorem (Boundedness)

Let φ be an analytic self map of D, let u be analytic on D, and let α and β be positive real numbers.

(i) If $0 < \alpha < 1$, then uC_φ maps B^α boundedly into B^β if and only if $u \in B^\beta$ and
$$\sup_{n\geq 1} n^{\alpha-1} \| I_u(\varphi^n) \|_{B^\beta} < \infty.$$

(ii) If $\alpha > 1$, then uC_φ maps B^α boundedly into B^β if and only if
$$\sup_{n\geq 1} n^{\alpha-1} \| I_u(\varphi^n) \|_{B^\beta} < \infty$$
and
$$\sup_{n\geq 1} n^{\alpha-1} \| J_u(\varphi^n) \|_{B^\beta} < \infty.$$

The case $\alpha = 1$

The case $\alpha = 1$

For $\alpha = 1$, the corresponding conditions would be

$$\sup_{n \geq 1} \|I_u(\varphi^n)\|_{B^\beta} < \infty, \qquad \sup_{n \geq 1} \|J_u(\varphi^n)\|_{B^\beta} < \infty.$$

Curiously enough, while these conditions are necessary for uC_φ to be bounded from B to B^β, but they are not sufficient.

The case $\alpha = 1$

For $\alpha = 1$, the corresponding conditions would be

$$\sup_{n \geq 1} \|I_u(\varphi^n)\|_{B^\beta} < \infty, \qquad \sup_{n \geq 1} \|J_u(\varphi^n)\|_{B^\beta} < \infty.$$

Curiously enough, while these conditions are necessary for uC_φ to be bounded from B to B^β, but they are not sufficient.

Example. Let $1 < \beta < \infty$. Let $u(z) = (1-z)^{1-\beta}$, $\varphi(z) = z$, and $f(z) = \log(2/(1-z))$. Then $u \in B^\beta$, and $f \in B$. Easy computations show that, for $\beta > 1$,

$$\|I_u(\varphi^n)\|_{B^\beta} \leq 2^\beta \quad \|J_u(\varphi^n)\|_{B^\beta} \leq \|u\|_{B^\beta}.$$

However, for $\beta > 1$, we have

$$\|uC_\varphi(f)\|_{B^\beta} = \infty.$$

Therefore, $uC_\varphi : B \to B^\beta$ is not bounded.

Essential Norms

Essential Norms

The case $0 < \alpha < 1$

Essential Norms

The case $0 < \alpha < 1$

Theorem

Suppose $0 < \alpha < 1$ and $0 < \beta < \infty$ and suppose the weighted composition operator uC_φ is bounded from B^α to B^β. Then

$$\|uC_\varphi\|_e = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{n\to\infty} n^{\alpha-1}\|I_u(\varphi^n)\|_{B^\beta}.$$

Essential Norms

The case $0 < \alpha < 1$

Theorem

Suppose $0 < \alpha < 1$ and $0 < \beta < \infty$ and suppose the weighted composition operator uC_φ is bounded from B^α to B^β. Then

$$\|uC_\varphi\|_e = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{n\to\infty} n^{\alpha-1}\|I_u(\varphi^n)\|_{B^\beta}.$$

Corollary

Suppose $0 < \alpha < 1$ and $0 < \beta < \infty$ and suppose the weighted composition operator uC_φ is bounded from B^α to B^β. Then uC_φ is compact from B^α to B^β if and only if

$$\limsup_{n\to\infty} n^{\alpha-1}\|I_u(\varphi^n)\|_{B^\beta} = 0.$$

The case $\alpha > 1$

The case $\alpha > 1$

Denote by

$$A = \left(\frac{e}{2\alpha}\right)^{\alpha} \limsup_{n\to\infty} n^{\alpha-1} \| I_u(\varphi^n) \|_{B^\beta}$$

and

$$B = \left(\frac{e}{2(\alpha-1)}\right)^{\alpha-1} \limsup_{n\to\infty} n^{\alpha-1} \| J_u(\varphi^n) \|_{B^\beta}.$$

Then we have the following result.

The case $\alpha > 1$

Denote by

$$A = \left(\frac{e}{2\alpha}\right)^{\alpha} \limsup_{n\to\infty} n^{\alpha-1} \|I_u(\varphi^n)\|_{B^\beta}$$

and

$$B = \left(\frac{e}{2(\alpha-1)}\right)^{\alpha-1} \limsup_{n\to\infty} n^{\alpha-1} \|J_u(\varphi^n)\|_{B^\beta}.$$

Then we have the following result.

Theorem

Let $\alpha > 1$, $0 < \beta < \infty$. Suppose that the weighted composition operator uC_φ is bounded from B^α to B^β. Then

$$\max\left(\frac{1}{2^{1+\alpha}(3\alpha+2)}A,\ \frac{1}{2^{1+\alpha}3\alpha(\alpha+1)}B\right) \leq \|uC_\varphi\|_e \leq A + B.$$

Corollary

Let $\alpha > 1$, $0 < \beta < \infty$. Suppose that the weighted composition operator uC_φ is bounded from B^α to B^β. Then uC_φ is compact from B^α to B^β if and only if the following two conditions are satisfied.

$$\limsup_{n\to\infty} n^{\alpha-1}\|I_u(\varphi^n)\|_{B^\beta} = 0$$

and

$$\limsup_{n\to\infty} n^{\alpha-1}\|J_u(\varphi^n)\|_{B^\beta} = 0.$$

Idea of Proofs

Idea of Proofs

Recall

Theorem (Boundedness)

Part (i). If $0 < \alpha < 1$, then uC_φ maps B^α boundedly into B^β if and only if $u \in B^\beta$ and

$$\sup_{n\geq 1} n^{\alpha-1}\|I_u(\varphi^n)\|_{B^\beta} < \infty.$$

Idea of Proof. We are going to use the following theorem by Ohno, Stroethoff and Z in 2003: uC_φ be bounded from B^α to B^β if and only if $u \in B^\beta$ and

$$M = \sup_{z \in D} |u(z)||\varphi'(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} < \infty.$$

Let uC_φ be bounded from B^α to B^β. Then, from above theorem we have $u \in B^\beta$ (actually, $u = uC_\varphi(1) \in B^\beta$). Notice that

$$(I_u(\varphi^n)(z))' = u(z)(\varphi^n(z))' = nu(z)\varphi^{n-1}(z)\varphi'(z).$$

Thus we have, for all $n \geq 1$,

$$\begin{aligned}
&n^{\alpha-1}\|I_u(\varphi^n)\|_{B^\beta} \\
&\quad = n^{\alpha-1} \sup_{z \in D} n|u(z)||\varphi(z)|^{n-1}|\varphi'(z)|(1-|z|^2)^\beta \\
&\quad = \sup_{z \in D} n^\alpha |\varphi(z)|^{n-1}(1-|\varphi(z)|^2)^\alpha |u(z)||\varphi'(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} \\
&\quad \leq M \sup_{z \in D} n^\alpha |\varphi(z)|^{n-1}(1-|\varphi(z)|^2)^\alpha \\
&\quad \leq MK.
\end{aligned}$$

Conversely, let $u \in B^\beta$ and $\sup\limits_{n\geq 1} n^{\alpha-1}\|I_u(\varphi^n)\|_{B^\beta} < \infty$. For any integer $n \geq 1$, let

$$D_n = \{z \in D : \ r_n \leq |\varphi(z)| \leq r_{n+1}\},$$

where

$$r_n = \begin{cases} 0, & \text{as } n = 1 \\ \left(\frac{n-1}{n-1+2\alpha}\right)^{1/2}, & \text{as } n \geq 2. \end{cases}$$

Let m and k be the smallest and largest positive integers such that $D_m \neq \emptyset$ and $D_k \neq \emptyset$ (k could be ∞). Then we can decompose D as $D = \cup_{n=m}^{k} D_n$. An easy exercise in Calculus shows that, there exists a constant $\delta > 0$, independent of n, such that

$$\min_{z\in D_n} n^\alpha |\varphi(z)|^{n-1}(1-|\varphi(z)|^2)^\alpha \geq \delta.$$

Hence,

$$\begin{aligned}
&\sup_{z\in D} |u(z)||\varphi'(z)|\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)^2|)^\alpha} \\
&= \sup_{m\le n\le k}\sup_{z\in D_n} |u(z)||\varphi'(z)|\frac{n^\alpha|\varphi(z)|^{n-1}(1-|z|^2)^\beta}{n^\alpha|\varphi(z)|^{n-1}(1-|\varphi(z)^2|)^\alpha} \\
&\le \frac{1}{\delta}\sup_{m\le n\le k}\sup_{z\in D_n} n^\alpha|u(z)||\varphi(z)|^{n-1}|\varphi'(z)|(1-|z|^2)^\beta \\
&\le \frac{1}{\delta}\sup_{n\ge 1}\sup_{z\in D} n^\alpha|u(z)||\varphi(z)|^{n-1}|\varphi'(z)|(1-|z|^2)^\beta \\
&\le \frac{1}{\delta} n^{\alpha-1}\sup_{n\ge 1}\|I_u(\varphi^n)\|_{B^\beta} < \infty.
\end{aligned}$$

Thus by Theorem (OSZ 2003) we know that uC_φ is bounded from B^α to B^β.

Some references

1. M. Contreras and A. Hernandez-Díaz, Weighted composition operators in weighted Banach spaces of analytic functions, *J. Australian Math. Soc.*, **69** (2000), 41–60.
2. B. MacCluer and R. Zhao, Essential norms of weighted composition operators between Bloch-type spaces, *Rocky Mountain J. Math.*, **33** (2003), 1437–1458.
3. A. Montes-Rodríguez, The essential norm of a composition operator on Bloch spaces, *Pacific J. Math.*, **188** (1999), 339–351.
4. A. Montes-Rodríguez, Weighted composition operators on weighted Banach spaces of analytic functions, *J. London Math. Soc.* (2), **61** (2000), 872–844.
5. H. Wulan , D. Zheng and K. Zhu, Composition operators on BMOA and the Bloch space, *Proc. Amer. Math. Soc.*, **137** (2009), 3861–3868.
6. R. Zhao, Essential norms of composition operators between Bloch type spaces, *Proc. Amer. Math. Soc.*, **138** (2010), 2537–2546.

THANK YOU!