Technical Appendix to Accompany

"Access Pricing in Network Industries with Mixed Oligopoly"

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Part I of this Technical Appendix presents additional numerical solutions to illustrate how industry outcomes change as elements of the benchmark setting change. Part II provides a detailed proof of Lemma 1 in the text.

I. Additional Numerical Solutions: Variations on the Benchmark Setting.

Tables A1 – A12 below illustrate how industry outcomes change when selected elements of the benchmark setting change. The selected elements in question are specified below each table. All parameter values not explicitly identified below each table assume their values in the benchmark setting (where, recall, $u_1 = u_2 = 20$, $c_u = 5$, $c_1 = c_2 = 2$, $F = 50$, $b = 0.5$, and $\gamma = 1$).

Optimal access prices are specified below each of Tables A1 – A6. The corresponding optimal outcomes (both when firm 1 is a public enterprise and when it is a private firm) are recorded in the first column of data in each table.[1] The last column in each table reports the outcomes that arise in the hypothetical setting where: (i) firm 1 is a public enterprise; (ii) the regulator implements the access prices that are optimal when firm 1 is a private firm (w_1^{0*},w_2^{0*}); and (iii) industry suppliers operate even when industry profit is negative.[2] In each setting considered in Tables A1 – A6, industry profit is negative in this hypothetical setting. Consequently, industry profit, consumer surplus, and welfare will all be zero if suppliers decline to operate when industry profit is negative.

Tables A7 – A12 present outcomes under the optimal policy in the presence of mixed duopoly when the regulator is required to set the same input price for both downstream firms. Thus, the data in Tables A7 – A12 correspond to the data in Table 3 in the text.

[1] Thus, the data in the middle column in each of Tables A1 – A6 correspond to the data in the last column of Table 1 in the text.

[2] Thus, the data in the last column in each of Tables A1 – A6 correspond to the data in the last column of Table 2 in the text.

Optimal Outcomes and Outcomes under (w_1^{0*}, w_2^{0*}) in Mixed Duopoly

Table A1

	Optimal Outcomes	Hypothetical Outcomes
p_1	10.209	6.313
p_2	10.209	9.235
X_1	7.791	14.609
X_2	7.791	5.844
Π	0	– 46.976
CS	60.704	110.975
W	60.704	63.999

Outcomes when $u_1 = u_2 = 18$.
$w_1^* = 8.209,\ w_2^* = w_2^{0*} = w_1^{0*} = 4.313$

Table A2

	Optimal Outcomes	Hypothetical Outcomes
p_1	11.321	6.981
p_2	11.321	10.236
X_1	8.679	16.274
X_2	8.679	6.510
Π	0	– 54.250
CS	75.333	137.718
W	75.333	83.468

Outcomes when $F = 75$.
$w_1^* = 9.321,\ w_2^* = w_2^{0*} = w_1^{0*} = 4.981$

Table A3

	Optimal Outcomes	Hypothetical Outcomes
p_1	12.209	8.313
p_2	12.209	11.235
X_1	7.791	14.609
X_2	7.791	5.844
Π	0	– 46.976
CS	60.704	110.975
W	60.704	63.999

Outcomes when $c_u = 7$.
$w_1^* = 10.209,\ w_2^* = w_2^{0*} = w_1^{0*} = 6.313$

Table A4

	Optimal Outcomes	Hypothetical Outcomes
p_1	10.683	6.025
p_2	10.683	9.519
X_1	9.317	17.469
X_2	9.317	6.988
Π	0	– 73.887
CS	86.800	158.680
W	86.800	84.793

Outcomes when $c_1 = c_2 = 3$.
$w_1^* = 7.683,\ w_2^* = w_2^{0*} = w_1^{0*} = 3.025$

Table A5

	Optimal Outcomes	**Hypothetical Outcomes**
p_1	11.390	7.868
p_2	9.825	8.944
X_1	7.045	13.208
X_2	11.741	9.980
Π	0	-45.554
CS	90.060	135.291
W	90.060	89.738

Outcomes when $c_1 = 4$, $c_2 = 2$.
$w_1^* = 7.390$, $w_2^* = 1.954$
$w_1^{0*} = 3.868$, $w_2^{0*} = 1.954$

Table A6

	Optimal Outcomes	**Hypothetical Outcomes**
p_1	11.304	10.315
p_2	9.723	9.352
X_1	3.953	6.794
X_2	15.021	13.539
Π	0	-9.222
CS	94.378	104.980
W	94.378	95.758

Outcomes when $c_1 = 4$, $c_2 = 2$, $b = 0.75$.
$w_1^* = 7.304$, $w_2^* = 3.967$
$w_1^{0*} = 6.315$, $w_2^{0*} = 3.967$

Optimal Outcomes under Mixed Duopoly with No Input Price Discrimination

Table A7

w^*	8.000
p_1	10.000
p_2	12.000
X_1	10.000
X_2	4.000
Π	0
CS	52.000
W	52.000

Outcomes when $u_1 = u_2 = 18$.

Table A8

w^*	9.313
p_1	11.313
p_2	13.485
X_1	10.859
X_2	4.344
Π	0
CS	61.315
W	61.315

Outcomes when $F = 75$.

Table A9

w^*	10.000
p_1	12.000
p_2	14.000
X_1	10.000
X_2	4.000
Π	0
CS	52.000
W	52.000

Outcomes when $c_u = 7$.

Table A10

w^*	7.225
p_1	10.225
p_2	12.668
X_1	12.219
X_2	4.888
Π	0
CS	77.643
W	77.643

Outcomes when $c_1 = c_2 = 3$.

Table A11

w^*	6.568
p_1	10.568
p_2	11.926
X_1	10.790
X_2	6.716
Π	0
CS	78.001
W	78.001

Outcomes when $c_1 = 4$, $c_2 = 2$.

Table A12

w^*	6.696
p_1	10.696
p_2	10.859
X_1	9.793
X_2	8.652
Π	0
CS	85.097
W	85.097

Outcomes when $c_1 = 4$, $c_2 = 2$, $b = 0.75$.

Tables A1 and A7 modify the benchmark setting by reducing $u_1 = u_2$, a measure of how highly consumers value the firms' products, from 20 to 18. When the regulator can set discriminatory access prices in this setting (Table A1), she increases both access prices in order to induce the higher retail prices required to ensure non-negative industry profit in the presence of reduced demand for the firms' products. When the regulator is required to set non-discriminatory access prices in this setting (Table A7), the public enterprise produces substantially more output than the private firm and consumer surplus declines (from 60.704 to 52.00).

Tables A2 – A4 and A8 – A10 modify the benchmark setting by raising industry costs. In Tables A2 and A8, industry fixed cost, F, is increased from 50 to 75. In Tables A3 and A9, the upstream supplier's marginal cost, c_u, is increased from 5 to 7. In Tables A4 and A10, the downstream marginal cost, $c_1 = c_2$, is increased from 2 to 3. In each of these cases, when the regulator can set discriminatory access prices, she increases both access prices in order to induce the higher retail prices required to ensure non-negative industry profit in the presence of higher industry costs. A prohibition on access price discrimination causes a substantial reduction in consumer surplus (e.g., from 75.333 to 61.315 when $F = 75$ in Tables A2 and A8).

Tables A5 and A11 introduce an asymmetry in the downstream suppliers' costs, holding the sum of their costs constant. Specifically, relative to Tables A4 and A10, Tables A5 and A11 consider an increase in firm 1's marginal cost and an identical reduction in firm 2's marginal cost (i.e., $c_1 = 4 > 2 = c_2$). When the regulator can set discriminatory prices in this setting (Table A5), she sets a relatively low access price for the low-cost supplier (firm 2). Doing so ensures firm 2 sets a relatively low price and produces a relatively large amount of output in equilibrium, which increases consumer surplus (from 86.80 to 90.06).

Tables A6 and A12 modify the asymmetric setting in Tables A5 and A11 by increasing b from 0.5 to 0.75. The increased product homogeneity results in the low-cost firm (firm 2) producing a substantially larger fraction of equilibrium output, despite the significant increase in the access price it faces when the regulator can set discriminatory access prices.[3]

II. Detailed Proof of Lemma 1.

Lemma 1. $\Pi(\mathbf{w}) = 0$ *at the solution to [RP]. Furthermore: (i)* $v^* = v^{0*} = 1$ *if* $F = 0$*; and (ii)* $v^* > 1$ *and* $v^{0*} > 1$ *if* $F > 0$*. In addition, for* $i, j \in \{1, 2\}$ $(j \neq i)$*:*

$$w_1^* = c_u + \frac{[\,v^* - 1\,]\,\Delta_1}{2\,v^* - 1}\,; \;\; w_2^* = c_u - \frac{\Delta_2 - b\,v^*\,\Delta_1}{2\,v^* - 1}\,; \;\; w_i^{0*} = c_u - \frac{\Delta_i - b\,v^{0*}\,\Delta_j}{2\,v^{0*} - 1}\,; \tag{1}$$

$$p_i^* = c_u + c_i + \frac{[\,v^* - 1\,]\,\Delta_i}{2\,v^* - 1}\,; \quad p_i^{0*} = c_u + c_i + \frac{[\,v^{0*} - 1\,]\,\Delta_i}{2\,v^{0*} - 1}\,; \tag{2}$$

$$X_i^* = \frac{v^*\,[\,\Delta_i - b\,\Delta_j\,]}{[\,2\,v^* - 1\,]\,[\,1 - b\,]}\,; \qquad X_i^{0*} = \frac{v^{0*}\,[\,\Delta_i - b\,\Delta_j\,]}{[\,2\,v^{0*} - 1\,]\,[\,1 - b\,]}\,. \tag{3}$$

<u>Proof</u>. If downstream supplier $i \in \{1, 2\}$ seeks to maximize its profit plus the fraction $\alpha_i \in [\,0, 1\,]$ of consumer surplus, the supplier's problem, $[Pi]$, is:

$$\underset{p_i}{Maximize} \quad [\,p_i - w_i - c_i\,]\,X_i(\mathbf{p}) + \alpha_i\,[\,U(X_i(\mathbf{p}), X_j(\mathbf{p})) - \sum_{i=1}^{2} p_i\,X_i(\mathbf{p})\,] \tag{4}$$

$$\text{where} \;\; U(x_1, x_2) = \sum_{i=1}^{2} u_i\,x_i - \frac{1}{2\,[\,1 + b\,]}\,\left[\,(x_1)^2 + 2\,b\,x_1\,x_2 + (x_2)^2\,\right]. \tag{5}$$

The necessary conditions for a solution to $[Pi]$ are, for $i, j \in \{1, 2\}$ $(j \neq i)$:

$$X_i(\mathbf{p}) + [\,p_i - w_i - c_i\,]\,\frac{\partial X_i(\mathbf{p})}{\partial p_i}$$
$$+\, \alpha_i \left[\,\frac{\partial U(X_i(\mathbf{p}), X_j(\mathbf{p}))}{\partial p_i} - X_i(\mathbf{p}) - p_i\,\frac{X_i(\mathbf{p})}{\partial p_i} - p_j\,\frac{X_j(\mathbf{p})}{\partial p_i}\,\right] = 0$$

[3] $\frac{X_2}{X_1} \approx 3.80$ when $b = 0.75$ (Table A6) whereas $\frac{X_2}{X_1} \approx 1.67$ when $b = 0.50$ (Table A5).

$$\Leftrightarrow \quad [\,1-\alpha_i\,]\,X_i(\mathbf{p}) + [\,p_i - w_i - c_i\,]\,\frac{\partial X_i(\mathbf{p})}{\partial p_i} \;=\; 0\,. \tag{6}$$

The equivalence in (6) holds because consumer utility maximization ensures $\frac{\partial U(X_i(\mathbf{p}), X_j(\mathbf{p}))}{\partial X_i(\mathbf{p})} = p_i$ and $\frac{\partial U(X_i(\mathbf{p}), X_j(\mathbf{p}))}{\partial X_j(\mathbf{p})} = p_j$. The first of these equalities and (5) imply that for $i, j \in \{1, 2\}$ ($j \neq i$):

$$u_i - \frac{1}{1+b}\,[\,x_i + b\,x_j\,] \;=\; p_i \quad \Rightarrow \quad [\,1+b\,]\,[\,u_i - p_i\,] \;=\; x_i + b\,x_j$$

$$\Rightarrow \quad x_i \;=\; [\,1+b\,]\,[\,u_i - p_i\,] - b\,x_j \;=\; [\,1+b\,]\,[\,u_i - p_i\,] - b\,[\,(\,1+b\,)\,(\,u_j - p_j\,) - b\,x_i\,]$$

$$\Rightarrow \quad x_i \left[\,1-b^2\,\right] \;=\; [\,1+b\,]\,[\,u_i - p_i - b\,(\,u_j - p_j\,)\,]$$

$$\Rightarrow \quad x_i \;=\; \frac{1}{1-b}\,[\,u_i - p_i - b\,(\,u_j - p_j\,)\,]$$

$$\Rightarrow \quad X_i(\mathbf{p}) \;=\; \frac{u_i - b\,u_j}{1-b} - \left[\frac{1}{1-b}\right] p_i + \left[\frac{b}{1-b}\right] p_j\,. \tag{7}$$

(6) and (7) imply:

$$[\,1-\alpha_i\,]\left[\frac{u_i - b\,u_j}{1-b} - \frac{p_i}{1-b} + \frac{b\,p_j}{1-b}\right] + [\,p_i - w_i - c_i\,]\left[-\frac{1}{1-b}\right] \;=\; 0$$

$$\Rightarrow \quad [\,1-\alpha_i\,]\,[\,u_i - b\,u_j - p_i + b\,p_j\,] \;=\; p_i - w_i - c_i$$

$$\Rightarrow \quad [\,2-\alpha_i\,]\,p_i \;=\; w_i + c_i + [\,1-\alpha_i\,]\,[\,u_i - b\,u_j + b\,p_j\,]$$

$$\Rightarrow \quad p_i \;=\; \frac{w_i + c_i}{2-\alpha_i} + \left[\frac{1-\alpha_i}{2-\alpha_i}\right][\,u_i - b\,u_j\,] + \frac{b\,[\,1-\alpha_i\,]}{2-\alpha_i}\,p_j$$

$$= \frac{w_i + c_i}{2-\alpha_i} + \left[\frac{1-\alpha_i}{2-\alpha_i}\right][\,u_i - b\,u_j\,] + \frac{b\,[\,1-\alpha_i\,]}{2-\alpha_i}\left[\frac{w_j + c_j}{2-\alpha_j} + \left(\frac{1-\alpha_j}{2-\alpha_j}\right)(\,u_j - b\,u_i\,) + \frac{b\,(\,1-\alpha_j\,)}{2-\alpha_j}\,p_i\right]$$

$$\Rightarrow \quad p_i\left[1 - \frac{b^2\,(1-\alpha_i)\,(1-\alpha_j)}{(2-\alpha_i)\,(2-\alpha_j)}\right] \;=\; \frac{w_i + c_i}{2-\alpha_i} + \left[\frac{1-\alpha_i}{2-\alpha_i}\right][\,u_i - b\,u_j\,] + \frac{b\,[\,1-\alpha_i\,]}{2-\alpha_i}\left[\frac{w_j + c_j}{2-\alpha_j} + \left(\frac{1-\alpha_j}{2-\alpha_j}\right)(\,u_j - b\,u_i\,)\right]$$

$$\Rightarrow \quad p_i\left[\,(2-\alpha_i)\,(2-\alpha_j) - b^2\,(1-\alpha_i)\,(1-\alpha_j)\,\right]$$

$$= \; [\,w_i + c_i\,]\,[\,2-\alpha_j\,] + [\,1-\alpha_i\,]\,[\,2-\alpha_j\,]\,[\,u_i - b\,u_j\,] + \; b\,[\,1-\alpha_i\,]\,[\,w_j + c_j + (\,1-\alpha_j\,)\,(\,u_j - b\,u_i\,)\,]$$

$$\Rightarrow \quad p_i(\mathbf{w}) \;=\; \frac{1}{D}\{\,[\,2-\alpha_j\,]\,[\,w_i+c_i\,]+b\,[\,1-\alpha_i\,]\,[\,w_j+c_j\,]$$

$$+\;[\,1-\alpha_i\,]\,[\,2-\alpha_j\,]\,[\,u_i-b\,u_j\,]+b\,[\,1-\alpha_i\,]\,[\,1-\alpha_j\,]\,[\,u_j-b\,u_i\,]\,\} \tag{8}$$

where $\mathbf{w} \equiv (w_1, w_2)$ and:

$$D \;\equiv\; [\,2-\alpha_i\,]\,[\,2-\alpha_j\,]-b^2\,[\,1-\alpha_i\,]\,[\,1-\alpha_j\,] \;>\; 0\,. \tag{9}$$

(8) and (9) imply that when $\alpha_i \;=\; \alpha_j \;= 0$:

$$p_i^{0*} \;=\; \frac{1}{4-b^2}\{\,2\,[\,w_i+c_i\,]+b\,[\,w_j+c_j\,]+2\,[\,u_i-b\,u_j\,]+b\,[\,u_j-b\,u_i\,]\,\}$$

$$= \frac{1}{4-b^2}\left[\,(\,2-b^2\,)\,u_i+2\,(\,w_i+c_i\,)-b\,(\,u_j-w_j-c_j\,)\,\right]$$

$$= w_i+c_i+\frac{1}{4-b^2}\left[\,(\,2-b^2\,)\,u_i+2\,(\,w_i+c_i\,)-\left(4-b^2\right)(\,w_i+c_i\,)-b\,(\,u_j-w_j-c_j\,)\,\right]$$

$$= w_i+c_i+\frac{1}{4-b^2}\left[\,(\,2-b^2\,)\,u_i+\left(b^2-2\right)(\,w_i+c_i\,)-b\,(\,u_j-w_j-c_j\,)\,\right]$$

$$= w_i+c_i+\frac{1}{4-b^2}\left[\,\left(2-b^2\right)(\,u_i-w_i-c_i\,)-b\,(\,u_j-w_j-c_j\,)\,\right]. \tag{10}$$

(8) and (9) imply that when $\alpha_1 \;=\; 1$ and $\alpha_2 \;=\; 0$:

$$p_1^* \;=\; \frac{1}{2}\{\,2\,[\,w_1+c_1\,]\,\} \;=\; w_1+c_1\,; \;\text{ and} \tag{11}$$

$$p_2^* \;=\; \frac{1}{2}\{\,[\,2-\alpha_1\,]\,[\,w_2+c_2\,]+b\,[\,1-\alpha_2\,]\,[\,w_1+c_1\,]$$

$$+\;[\,1-\alpha_2\,]\,[\,2-\alpha_1\,]\,[\,u_2-b\,u_1\,]+b\,[\,1-\alpha_2\,]\,[\,1-\alpha_1\,]\,[\,u_1-b\,u_2\,]\,\}$$

$$=\; \frac{1}{2}\,[\,w_2+c_2\;+b\,(\,w_1+c_1\,)+u_2-b\,u_1\,]$$

$$=\; \frac{1}{2}\,[\,u_2+w_2+c_2-b\,(\,u_1-w_1-c_1\,)]\,. \tag{12}$$

(7) and (8) imply:

$$X_i(\mathbf{w}) \;=\; \frac{u_i-b\,u_j}{1-b}-\left[\frac{1}{1-b}\right]p_i(\mathbf{w})+\left[\frac{b}{1-b}\right]p_j(\mathbf{w})\,. \tag{13}$$

It is readily verified that industry profit, given access prices $\mathbf{w}$ is:

$$\Pi(\mathbf{w}) \;=\; [\,p_1(\mathbf{w})-c_u-c_1\,]\,X_1(\mathbf{w})+[\,p_2(\mathbf{w})-c_u-c_2\,]\,X_2(\mathbf{w})-F\,. \tag{14}$$

The regulator's problem, [RP], is:

$$\underset{\mathbf{w}\equiv(w_1,w_2)}{\text{Maximize}} \quad U(X_1(\mathbf{w}), X_2(\mathbf{w})) - \sum_{i=1}^{2} p_i(\mathbf{w})\, X_i(\mathbf{w}) + \gamma\,[\,\pi_i + \pi_j + \pi_u\,] \tag{15}$$

$$\text{subject to: } \Pi(\mathbf{w}) \;\equiv\; \pi_1 + \pi_2 + \pi_u - F \;\geq\; 0\,. \tag{16}$$

Let $\lambda \geq 0$ denote the Lagrange multiplier associated with (16). Also define $v \equiv \gamma + \lambda$. Then (14) – (16) imply that the necessary conditions for a solution to [RP] include, for $i, j \in \{1, 2\}$ ($j \neq i$):

$$\begin{aligned}
&\frac{\partial U(X_i(\mathbf{w}), X_j(\mathbf{w}))}{\partial X_i^*(\mathbf{w})}\,\frac{\partial X_i(\mathbf{w})}{\partial w_i} + \frac{\partial U(X_i(\mathbf{w}), X_j(\mathbf{w}))}{\partial X_j^*(\mathbf{w})}\,\frac{\partial X_j(\mathbf{w})}{\partial w_i} \\
&\quad - \left[p_i(\mathbf{w})\,\frac{\partial X_i(\mathbf{w})}{\partial w_i} + \frac{\partial\, p_i(\mathbf{w})}{\partial w_i}\,X_i(\mathbf{w}) + p_j(\mathbf{w})\,\frac{\partial X_j(\mathbf{w})}{\partial w_i} + \frac{\partial p_j(\mathbf{w})}{\partial w_i}\,X_j(\mathbf{w}) \right] \\
&\quad + v \left[(\, p_i(\mathbf{w}) - c_u - c_i\,)\,\frac{\partial X_i(\mathbf{w})}{\partial w_i} + \frac{\partial\, p_i(\mathbf{w})}{\partial w_i}\,X_i(\mathbf{w}) \right. \\
&\qquad\qquad \left. + \,(\, p_j(\mathbf{w}) - c_u - c_j\,)\,\frac{\partial X_j(\mathbf{w})}{\partial w_i} + \frac{\partial p_j(\mathbf{w})}{\partial w_i}\,X_j(\mathbf{w}) \right] \\
&= -\left[\frac{\partial\, p_i(\mathbf{w})}{\partial w_i}\,X_i(\mathbf{w}) + \frac{\partial p_j(\mathbf{w})}{\partial w_i}\,X_j(\mathbf{w}) \right] \\
&\quad + v \left[(\, p_i(\mathbf{w}) - c_u - c_i\,)\,\frac{\partial X_i(\mathbf{w})}{\partial w_i} + \frac{\partial\, p_i(\mathbf{w})}{\partial w_i}\,X_i(\mathbf{w}) \right. \\
&\qquad\qquad \left. + \,(\, p_j(\mathbf{w}) - c_u - c_j\,)\,\frac{\partial X_j(\mathbf{w})}{\partial w_i} + \frac{\partial p_j(\mathbf{w})}{\partial w_i}\,X_j(\mathbf{w}) \right] \\
&= v \left[(\, p_i(\mathbf{w}) - c_i - c_u\,)\,\frac{\partial X_i(\mathbf{w})}{\partial w_i} + (\, p_j(\mathbf{w}) - c_j - c_u\,)\,\frac{\partial X_j(\mathbf{w})}{\partial w_i} \right] \\
&\quad + [\,v - 1\,] \left[\frac{\partial\, p_i(\mathbf{w})}{\partial w_i}\,X_i(\mathbf{w}) + \frac{\partial p_j(\mathbf{w})}{\partial w_i}\,X_j(\mathbf{w}) \right] \;=\; 0
\end{aligned} \tag{17}$$

where, from (8) and (13):

$$\frac{\partial p_i(\mathbf{w})}{\partial w_i} \;=\; \frac{2 - \alpha_j}{D}\,; \qquad \frac{\partial p_j(\mathbf{w})}{\partial w_i} \;=\; \frac{b\,[\,1 - \alpha_j\,]}{D}\,; \tag{18}$$

$$\begin{aligned}
\frac{\partial X_i(\mathbf{w})}{\partial w_i} \;&=\; -\left[\frac{1}{1-b}\right]\frac{\partial p_i(\mathbf{w})}{\partial w_i} + \left[\frac{b}{1-b}\right]\frac{\partial p_j(\mathbf{w})}{\partial w_i} \;=\; -\left[\frac{1}{1-b}\right]\frac{2-\alpha_j}{D} \\
&\quad + \left[\frac{b}{1-b}\right]\frac{b\,[\,1-\alpha_j\,]}{D} \;=\; \frac{b^2\,[\,1-\alpha_j\,] - [\,2-\alpha_j\,]}{D\,[\,1-b\,]}\,; \;\text{ and}
\end{aligned} \tag{19}$$

$$\frac{\partial X_j(\mathbf{w})}{\partial w_i} = - \left[\frac{1}{1-b} \right] \frac{\partial p_j(\mathbf{w})}{\partial w_i} + \left[\frac{b}{1-b} \right] \frac{\partial p_i(\mathbf{w})}{\partial w_i} = - \left[\frac{1}{1-b} \right] \frac{b\,[\,1-\alpha_j\,]}{D}$$

$$+ \left[\frac{b}{1-b} \right] \frac{2-\alpha_j}{D} = \frac{b\,[\,2-\alpha_j-1+\alpha_j\,]}{D\,[\,1-b\,]} = \frac{b}{D\,[\,1-b\,]} \,. \tag{20}$$

(7) and (18) – (20) imply that (17) can be written as:

$$\begin{aligned}
0 \;=\; & \frac{v}{D\,[\,1-b\,]} \left\{ [\,p_i(\mathbf{w}) - c_i - c_u\,] \left[b^2\,(\,1-\alpha_j\,) - (\,2-\alpha_j\,) \right] + b\,[\,p_j(\mathbf{w}) - c_j - c_u\,] \right\} \\
& + \frac{v-1}{D} \left\{ \frac{[\,2-\alpha_j\,]}{1-b} \,[\,u_i - b\,u_j - p_i(\mathbf{w}) + b\,p_j(\mathbf{w})\,] \right. \\
& \qquad \left. + \frac{b\,[\,1-\alpha_j\,]}{1-b} \,[\,u_j - b\,u_i - p_j(\mathbf{w}) + b\,p_i(\mathbf{w})\,] \right\} \\
=\; & \frac{v}{D\,[\,1-b\,]} \left\{ [\,p_i(\mathbf{w}) - c_i - c_u\,] \left[b^2\,(\,1-\alpha_j\,) - (\,2-\alpha_j\,) \right] + b\,[\,p_j(\mathbf{w}) - c_j - c_u\,] \right\} \\
& + \frac{v-1}{D\,[\,1-b\,]} \left\{ [\,2-\alpha_j\,]\,[\,u_i - b\,u_j\,] + b\,[\,1-\alpha_j\,]\,[\,u_j - b\,u_i\,] \right. \\
& \qquad \left. + \; b\,[\,2-\alpha_j-(1-\alpha_j)\,]\,p_j(\mathbf{w}) + \left[b^2\,(\,1-\alpha_j\,) - (\,2-\alpha_j\,) \right] p_i(\mathbf{w}) \right\} \\
=\; & \frac{v}{D\,[\,1-b\,]} \left\{ [\,p_i(\mathbf{w}) - c_i - c_u\,] \left[b^2\,(\,1-\alpha_j\,) - (\,2-\alpha_j\,) \right] + b\,[\,p_j(\mathbf{w}) - c_j - c_u\,] \right\} \\
& + \frac{v-1}{D\,[\,1-b\,]} \left\{ [\,2-\alpha_j\,]\,[\,u_i - b\,u_j\,] + b\,[\,1-\alpha_j\,]\,[\,u_j - b\,u_i\,] \right. \\
& \qquad \left. + \; b\,p_j(\mathbf{w}) + \left[b^2\,(\,1-\alpha_j\,) - (\,2-\alpha_j\,) \right] p_i(\mathbf{w}) \right\} \\
=\; & \frac{2\,v-1}{D\,[\,1-b\,]} \left\{ b\;p_j(\mathbf{w}) + \left[b^2\,(\,1-\alpha_j\,) - (\,2-\alpha_j\,) \right] p_i(\mathbf{w}) \right\} \\
& - \frac{v}{D\,[\,1-b\,]} \left\{ [\,c_i + c_u\,] \left[b^2\,(\,1-\alpha_j\,) - (\,2-\alpha_j\,) \right] + b\,[\,c_j + c_u\,] \right\} \\
& + \frac{v-1}{D\,[\,1-b\,]} \left\{ [\,2-\alpha_j\,]\,[\,u_i - b\,u_j\,] + b\,[\,1-\alpha_j\,]\,[\,u_j - b\,u_i\,] \right\} .
\end{aligned} \tag{21}$$

Because $b \in [\,0, 1)$ by assumption, (8) and (21) imply:

$$\begin{aligned}
0 \;=\; & \frac{b\,[\,2\,v-1\,]}{D^2} \left\{ [\,2-\alpha_i\,]\,[\,w_j + c_j\,] + b\,[\,1-\alpha_j\,]\,[\,w_i + c_i\,] \right\} \\
& + \frac{b\,[\,2\,v-1\,]}{D^2} \left\{ [\,1-\alpha_j\,]\,[\,2-\alpha_i\,]\,[\,u_j - b\,u_i\,] + b\,[\,1-\alpha_j\,]\,[\,1-\alpha_i\,]\,[\,u_i - b\,u_j\,] \right\}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{2\,v-1}{D^2}\left[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,\right]\{\,[\,2-\alpha_j\,]\,[\,w_i+c_i\,]+b\,[\,1-\alpha_i\,]\,[\,w_j+c_j\,]\,\}\\
&+ \frac{[\,2\,v-1\,]\,[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]}{D^2}\{\,[\,1-\alpha_i\,]\,[\,2-\alpha_j\,]\,[\,u_i-b\,u_j\,]\\
&\qquad\qquad\qquad\qquad + b\,[\,1-\alpha_i\,]\,[\,1-\alpha_j\,]\,[\,u_j-b\,u_i\,]\,\}\\
&- \frac{v}{D}\left\{\,[\,c_i+c_u\,]\left[\,b^2\,(\,1-\alpha_j\,)\;-\;(\,2-\alpha_j\,)\,\right]+b\,[\,c_j+c_u\,]\,\right\}\\
&+ \frac{v-1}{D}\{\,[\,2-\alpha_j\,]\,[\,u_i-b\,u_j\,]+b\,[\,1-\alpha_j\,]\,[\,u_j-b\,u_i\,]\,\}\\
=\;& \frac{b\,[\,2\,v-1\,]\,[\,2-\alpha_i\,]}{D^2}\,w_j+\frac{b\,[\,2\,v-1\,]\,b\,[\,1-\alpha_j\,]}{D^2}\,w_i\\
&+ \frac{b\,[\,2\,v-1\,]\,[\,2-\alpha_i\,]}{D^2}\,c_j+\frac{b\,[\,2\,v-1\,]\,b\,[\,1-\alpha_j\,]}{D^2}\,c_i\\
&+ \frac{b\,[\,2\,v-1\,]\,[\,1-\alpha_j\,]\,[\,2-\alpha_i\,]}{D^2}\,[\,u_j-b\,u_i\,]+\frac{b\,[\,2\,v-1\,]\,b\,[\,1-\alpha_j\,]\,[\,1-\alpha_i\,]}{D^2}\,[\,u_i-b\,u_j\,]\\
&+ \frac{[\,2\,v-1\,]\,[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\,[\,2-\alpha_j\,]}{D^2}\,w_i\\
&+ \frac{[\,2\,v-1\,]\,[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\,b\,[\,1-\alpha_i\,]}{D^2}\,w_j\\
&+ \frac{[\,2\,v-1\,]\,[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\,[\,2-\alpha_j\,]}{D^2}\,c_i\\
&+ \frac{[\,2\,v-1\,]\,[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\,b\,[\,1-\alpha_i\,]}{D^2}\,c_j\\
&+ \frac{[\,2\,v-1\,]\;[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\;[\,1-\alpha_i\,]\,[\,2-\alpha_j\,]}{D^2}\,[\,u_i-b\,u_j\,]\\
&+ \frac{[\,2\,v-1\,]\;[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\;b\,[\,1-\alpha_i\,]\,[\,1-\alpha_j\,]}{D^2}\,[\,u_j-b\,u_i\,]\\
&- \frac{v\,[\,b^2\,(\,1-\alpha_j\,)\;-\;(\,2-\alpha_j\,)\,]}{D}\,c_i-\frac{v\,b}{D}\,c_j\\
&- \frac{v\,[\,b^2\,(\,1-\alpha_j\,)\;-\;(\,2-\alpha_j\,)\,]}{D}\,c_u-\frac{v\,b}{D}\,c_u\\
&+ \frac{[\,v-1\,]\,[\,2-\alpha_j\,]}{D}\,[\,u_i-b\,u_j\,]+\frac{[\,v-1\,]\,b\,[\,1-\alpha_j\,]}{D}\,[\,u_j-b\,u_i\,]\,. \qquad (22)
\end{aligned}
$$

(22) can be written as:

$$
\begin{aligned}
0 \;=\;& \left\{ \frac{b\,[\,2\,v-1\,]\,[\,2-\alpha_i\,]}{D^2} + \frac{[\,2\,v-1\,]\,[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\,b\,[\,1-\alpha_i\,]}{D^2} \right\} w_j \\
&+ \left\{ \frac{b\,[\,2\,v-1\,]\,b\,[\,1-\alpha_j\,]}{D^2} + \frac{[\,2\,v-1\,]\,[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\,[\,2-\alpha_j\,]}{D^2} \right\} w_i \\
&+ \left\{ \frac{b\,[\,2\,v-1\,]\,[\,2-\alpha_i\,]}{D^2} + \frac{[\,2\,v-1\,]\,[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\,b\,[\,1-\alpha_i\,]}{D^2} - \frac{v\,b}{D} \right\} c_j \\
&+ \left\{ \frac{b\,[\,2\,v-1\,]\,b\,[\,1-\alpha_j\,]}{D^2} + \frac{[\,2\,v-1\,]\,[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\,[\,2-\alpha_j\,]}{D^2} \right. \\
&\qquad\qquad \left. - \frac{v\,[\,b^2\,(\,1-\alpha_j\,)\,-\,(\,2-\alpha_j\,)\,]}{D} \right\} c_i \\
&- \left\{ \frac{v\,[\,b^2\,(\,1-\alpha_j\,)\,-\,(\,2-\alpha_j\,)\,]}{D} + \frac{v\,b}{D} \right\} c_u \\
&+ \left\{ \frac{[\,2\,v-1\,]\,[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\,[\,1-\alpha_i\,]\,[\,2-\alpha_j\,]}{D^2} + \frac{[\,v-1\,]\,[\,2-\alpha_j\,]}{D} \right. \\
&\qquad\qquad \left. + \frac{b\,[\,2\,v-1\,]\,b\,[\,1-\alpha_j\,]\,[\,1-\alpha_i\,]}{D^2} \right\} [\,u_i - b\,u_j\,] \\
&+ \left\{ \frac{[\,2\,v-1\,]\,[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\,b\,[\,1-\alpha_i\,]\,[\,1-\alpha_j\,]}{D^2} + \frac{[\,v-1\,]\,b\,[\,1-\alpha_j\,]}{D} \right. \\
&\qquad\qquad \left. + \frac{b\,[\,2\,v-1\,]\,[\,1-\alpha_j\,]\,[\,2-\alpha_i\,]}{D^2} \right\} [\,u_j - b\,u_i\,]\,. \qquad (23)
\end{aligned}
$$

The numerators of the first and third terms in the coefficient on $u_i - b\,u_j$ in (23) sum to:

$$
\begin{aligned}
&[\,2\,v-1\,]\left[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,\right][\,1-\alpha_i\,]\,[\,2-\alpha_j\,] + b\,[\,2\,v-1\,]\,b\,[\,1-\alpha_j\,]\,[\,1-\alpha_i\,] \\
&\quad = [\,2\,v-1\,]\,[\,1-\alpha_i\,]\left\{\,[\,2-\alpha_j\,]\left[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,\right] + b^2\,[\,1-\alpha_j\,]\,\right\} \\
&\quad = [\,2\,v-1\,]\,[\,1-\alpha_i\,]\left\{\,b^2\,[\,1-\alpha_j\,]\,[\,3-\alpha_j\,] - [\,2-\alpha_j\,]^2\,\right\}. \qquad (24)
\end{aligned}
$$

The numerators of the first and third terms in the coefficient on $u_j - b\,u_i$ in (23) sum to:

$$
\begin{aligned}
&[\,2\,v-1\,]\left[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,\right] b\,[\,1-\alpha_i\,]\,[\,1-\alpha_j\,] + b\,[\,2\,v-1\,]\,[\,1-\alpha_j\,]\,[\,2-\alpha_i\,] \\
&\quad = b\,[\,2\,v-1\,]\,[\,1-\alpha_j\,]\left\{\,[\,1-\alpha_i\,]\left[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,\right] + 2-\alpha_i\,\right\}. \qquad (25)
\end{aligned}
$$

(23), (24), and (25) imply:

$$\frac{[\,2\,v-1\,]\,\{\,2-\alpha_i+[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\,[\,1-\alpha_i\,]\,\}}{D^2}\,b\,w_j$$

$$+\ \frac{[\,2\,v-1\,]\,\{\,b^2\,[\,1-\alpha_j\,]+[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\,[\,2-\alpha_j\,]\,\}}{D^2}\,w_i$$

$$+\ \left\{\frac{[\,2\,v-1\,]\,\{\,2-\alpha_i+[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\,[\,1-\alpha_i\,]\,\}}{D^2}-\frac{v}{D}\right\}b\,c_j$$

$$+\ \left\{\frac{[\,2\,v-1\,]\,\{\,b^2\,[\,1-\alpha_j\,]+[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,]\,[\,2-\alpha_j\,]\,\}}{D^2}\right.$$
$$\left.-\ \frac{v\,[\,b^2\,(\,1-\alpha_j\,)\ -\ (\,2-\alpha_j\,)\,]}{D}\right\}c_i$$

$$-\ \frac{v\,[\,b^2\,(\,1-\alpha_j\,)\ -\ (\,2-\alpha_j\,)+b\,]}{D}\,c_u$$

$$+\ \left\{\frac{[\,2\,v-1\,]\,[\,1-\alpha_i\,]\left[\,b^2\,(\,1-\alpha_j\,)\,(\,3-\alpha_j\,)-(\,2-\alpha_j\,)^2\,\right]}{D^2}+\frac{[\,v-1\,]\,[\,2-\alpha_j\,]}{D}\right\}[\,u_i-b\,u_j\,]$$

$$+\ \left\{\frac{[\,2\,v-1\,]\,[\,1-\alpha_j\,]\,[\,2-\alpha_i+(\,1-\alpha_i\,)\,(\,b^2\,[\,1-\alpha_j\,]-[\,2-\alpha_j\,]\,)\,]}{D^2}\right.$$
$$\left.+\ \frac{[\,v-1\,]\,[\,1-\alpha_j\,]}{D}\right\}b\,[\,u_j-b\,u_i\,]\ =\ 0\,. \tag{26}$$

Multiplying (26) by D^2 provides:

$$\begin{aligned}
&\gamma_1\,b\,w_j+\gamma_2\,w_i+[\,\gamma_1-\alpha\,D\,]\,b\,c_j+\left[\,\gamma_2-v\,D\left(b^2\,[\,1-\alpha_j\,]-[\,2-\alpha_j\,]\right)\right]c_i\\
&\qquad-\ v\,D\left[\,b^2\,(\,1-\alpha_j\,)\ -\ (\,2-\alpha_j\,)+b\,\right]c_u\\
&\qquad+\ [\,(\,1-\alpha_i\,)\,\gamma_2+(\,v-1\,)\,(\,2-\alpha_j\,)\,D\,]\,[\,u_i-b\,u_j\,]\\
&\qquad+\ [\,\gamma_1+(\,v-1\,)\,D\,]\,[\,1-\alpha_j\,]\,b\,[\,u_j-b\,u_i\,]\ =\ 0\,,\quad\text{and}
\end{aligned}\tag{27}$$

$$\begin{aligned}
&\gamma_3\,b\,w_i+\gamma_4\,w_j+[\,\gamma_3-v\,D\,]\,b\,c_i+\left[\,\gamma_4-v\,D\left(b^2\,[\,1-\alpha_i\,]-[\,2-\alpha_i\,]\right)\right]c_j\\
&\qquad-\ v\,D\left[\,b^2\,(\,1-\alpha_i\,)\ -\ (\,2-\alpha_i\,)+b\,\right]c_u\\
&\qquad+\ [\,(\,1-\alpha_j\,)\,\gamma_4+(\,v-1\,)\,(\,2-\alpha_i\,)\,D\,]\,[\,u_j-b\,u_i\,]\\
&\qquad+\ [\,\gamma_3+(\,v-1\,)\,D\,]\,[\,1-\alpha_i\,]\,b\,[\,u_i-b\,u_j\,]\ =\ 0\,,
\end{aligned}\tag{28}$$

where:

$$
\begin{aligned}
\gamma_1 &= [\,2\,v-1\,]\left\{\,2-\alpha_i+\left[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,\right][\,1-\alpha_i\,]\,\right\},\\
\gamma_2 &= [\,2\,v-1\,]\left\{\,b^2\,[\,1-\alpha_j\,]+\left[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,\right][\,2-\alpha_j\,]\,\right\}\\
&= [\,2\,v-1\,]\left\{\,b^2\,[\,1-\alpha_j\,]+b^2\,[\,1-\alpha_j\,][\,2-\alpha_j\,]-[\,2-\alpha_j\,]^2\,\right\}\\
&= [\,2\,v-1\,]\left\{\,b^2\,[\,1-\alpha_j\,][\,3-\alpha_j\,]-[\,2-\alpha_j\,]^2\,\right\},\\
\gamma_3 &= [\,2\,v-1\,]\left\{\,2-\alpha_j+\left[\,b^2\,(\,1-\alpha_i\,)-(\,2-\alpha_i\,)\,\right][\,1-\alpha_j\,]\,\right\},\ \text{ and}\\
\gamma_4 &= [\,2\,v-1\,]\left\{\,b^2\,[\,1-\alpha_i\,][\,3-\alpha_i\,]-[\,2-\alpha_i\,]^2\,\right\}. \qquad (29)
\end{aligned}
$$

Multiplying (27) by γ_4 provides:

$$
\begin{aligned}
&\gamma_4\,\gamma_1\,b\,w_j+\gamma_4\,\gamma_2\,w_i+\gamma_4\,[\,\gamma_1-v\,D\,]\,b\,c_j+\gamma_4\left[\,\gamma_2-v\,D\left(b^2\,[\,1-\alpha_j\,]-[\,2-\alpha_j\,]\right)\right]c_i\\
&\quad-\gamma_4\,v\,D\left[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)+b\,\right]c_u\\
&\quad+\gamma_4\,[\,(\,1-\alpha_i\,)\,\gamma_2+(\,v-1\,)\,(\,2-\alpha_j\,)\,D\,]\,[\,u_i-b\,u_j\,]\\
&\quad+\gamma_4\,[\,\gamma_1+(\,v-1\,)\,D\,]\,[\,1-\alpha_j\,]\,b\,[\,u_j-b\,u_i\,]\;=\;0\,. \qquad (30)
\end{aligned}
$$

Multiplying (28) by $\gamma_1\,b$ provides:

$$
\begin{aligned}
&\gamma_1\,b\,\gamma_3\,b\,w_i+\gamma_1\,b\,\gamma_4\,w_j+\gamma_1\,b\,[\,\gamma_3-v\,D\,]\,b\,c_i+\gamma_1\,b\left[\,\gamma_4-v\,D\left(b^2\,[\,1-\alpha_i\,]-[\,2-\alpha_i\,]\right)\right]c_j\\
&\quad-\gamma_1\,b\,v\,D\left[\,b^2\,(\,1-\alpha_i\,)-(\,2-\alpha_i\,)+b\,\right]c_u\\
&\quad+\gamma_1\,b\,[\,(\,1-\alpha_j\,)\,\gamma_4+(\,v-1\,)\,(\,2-\alpha_i\,)\,D\,]\,[\,u_j-b\,u_i\,]\\
&\quad+\gamma_1\,b\,[\,\gamma_3+(\,v-1\,)\,D\,]\,[\,1-\alpha_i\,]\,b\,[\,u_i-b\,u_j\,]\;=\;0\,. \qquad (31)
\end{aligned}
$$

Multiplying (27) by $\gamma_3\,b$ provides:

$$
\begin{aligned}
&\gamma_3\,b\gamma_1\,b\,w_j+\gamma_3\,b\gamma_2\,w_i+\gamma_3\,b\,[\,\gamma_1-v\,D\,]\,b\,c_j+\gamma_3\,b\left[\,\gamma_2-v\,D\left(b^2\,[\,1-\alpha_j\,]-[\,2-\alpha_j\,]\right)\right]c_i\\
&\quad-\gamma_3\,b\,v\,D\left[\,b^2\,(\,1-\alpha_j\,)\,-\,(\,2-\alpha_j\,)+b\,\right]c_u\\
&\quad+\gamma_3\,b\,[\,(\,1-\alpha_i\,)\,\gamma_2+(\,v-1\,)\,(\,2-\alpha_j\,)\,D\,]\,[\,u_i-b\,u_j\,]\\
&\quad+\gamma_3\,b\,[\,\gamma_1+(\,v-1\,)\,D\,]\,[\,1-\alpha_j\,]\,b\,[\,u_j-b\,u_i\,]\;=\;0\,,\quad\text{and} \qquad (32)
\end{aligned}
$$

Multiplying (28) by γ_2 provides:

$$
\begin{aligned}
&\gamma_2\,\gamma_3\,b\,w_i+\gamma_2\gamma_4\,w_j+\gamma_2\,[\,\gamma_3-v\,D\,]\,b\,c_i+\gamma_2\left[\,\gamma_4-v\,D\left(b^2\,[\,1-\alpha_i\,]-[\,2-\alpha_i\,]\right)\right]c_j\\
&\quad-\gamma_2 v\,D\left[\,b^2\,(\,1-\alpha_i\,)\,-\,(\,2-\alpha_i\,)+b\,\right]c_u
\end{aligned}
$$

$$+\ \gamma_2\,[\,(\,1-\alpha_j\,)\,\gamma_4+(\,v-1\,)\,(\,2-\alpha_i\,)\,D\,]\,[\,u_j-b\,u_i\,]$$

$$+\ \gamma_2\,[\,\gamma_3+(\,v-1\,)\,D\,]\,[\,1-\alpha_i\,]\,b\,[\,u_i-b\,u_j\,]\ =\ 0\,, \tag{33}$$

Subtracting (31) from (30) provides:

$$[\,\gamma_4\,\gamma_2-\gamma_1\,b\,\gamma_3\,b\,]\ w_i+\left\{\,\gamma_4\,[\,\gamma_1-v\,D\,]\,b-\gamma_1\,b\left[\,\gamma_4-v\,D\left[\,b^2\,(\,1-\alpha_i\,)-(\,2-\alpha_i\,)\,\right]\,\right]\,\right\}\ c_j$$

$$+\ \left\{\,\gamma_4\left[\,\gamma_2-v\,D\left(\,b^2\,[\,1-\alpha_j\,]-[\,2-\alpha_j\,]\,\right)\,\right]-\gamma_1\,b\,[\,\gamma_3-v\,D\,]\,b\,\right\}c_i$$

$$-\ \left\{\,\gamma_4\,v\,D\left[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)+b\,\right]-\gamma_1\,b\,v\,D\left[\,b^2\,(\,1-\alpha_i\,)-(\,2-\alpha_i\,)+b\,\right]\,\right\}c_u$$

$$+\ \{\,\gamma_4\,[\,(\,1-\alpha_i\,)\,\gamma_2+(\,v-1\,)\,(\,2-\alpha_j\,)\,D\,]-\gamma_1\,b\,[\,\gamma_3+(\,v-1\,)\,D\,]\,b\,[\,1-\alpha_i\,]\,\}\,[\,u_i-b\,u_j\,]$$

$$+\ \{\,\gamma_4\,[\,\gamma_1+(\,v-1\,)\,D\,]\ b\,[\,1-\alpha_j\,]-\gamma_1\,b\,[\,(\,1-\alpha_j\,)\,\gamma_4+(\,v-1\,)\,(\,2-\alpha_i\,)\,D\,]\,\}$$

$$\cdot\,[\,u_j-b\,u_i\,]\ =\ 0\,. \tag{34}$$

Subtracting (32) from (33) provides:

$$[\gamma_2\,\gamma_4\ -\gamma_3\,b\,\gamma_1\,b\,]\,w_j+\left\{\gamma_2\,[\,\gamma_3-v\,D\,]\,b-\gamma_3\,b\left[\,\gamma_2-v\,D\left(\,b^2\,[\,1-\alpha_j\,]-[\,2-\alpha_j\,]\,\right)\,\right]\right\}c_i$$

$$+\ \left\{\gamma_2\left[\,\gamma_4-v\,D\left(\,b^2\,[\,1-\alpha_i\,]-[\,2-\alpha_i\,]\,\right)\,\right]-\gamma_3\,b\,[\,\gamma_1-v\,D\,]\,b\right\}\ c_j$$

$$-\ \left\{\gamma_2\,v\,D\left[\,b^2\,(\,1-\alpha_i\,)\ -\ (\,2-\alpha_i\,)+b\,\right]\ -\gamma_3\,b\,v\,D\left[\,b^2\,(\,1-\alpha_j\,)\ -\ (\,2-\alpha_j\,)+b\,\right]\right\}c_u$$

$$+\ \{\,\gamma_2\,[\,(\,1-\alpha_j\,)\,\gamma_4+(\,v-1\,)\,(\,2-\alpha_i\,)\,D\,]-\gamma_3\,b\,[\,\gamma_1+(\,v-1\,)\,D\,]\,[\,1-\alpha_j\,]\,b\}\,[\,u_j-b\,u_i\,]$$

$$+\ \{\,\gamma_2\,[\,\gamma_3+(\,v-1\,)\,D\,]\,[\,1-\alpha_i\,]\,b\ -\gamma_3\,b\,[\,(\,1-\alpha_i\,)\,\gamma_2+(\,v-1\,)\,(\,2-\alpha_j\,)\,D\,]\,\}$$

$$\cdot\,[\,u_i-b\,u_j\,]\ =\ 0\,. \tag{35}$$

(34) implies:

$$\left[\,\gamma_4\,\gamma_2-\gamma_1\,b^2\,\gamma_3\,\right]w_i+\left\{\,\gamma_1\ \left[\,b^2\,(\,1-\alpha_i\,)-(\,2-\alpha_i\,)\,\right]-\gamma_4\,\right\}\ b\,v\,D\,c_j$$

$$+\ \left\{\,\gamma_4\,\gamma_2-\gamma_4\,v\,D\left[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,\right]-\gamma_1\,b^2\,\gamma_3+\gamma_1\,b^2\,v\,D\,\right\}c_i$$

$$-\ \left\{\,\gamma_4\left[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)+b\,\right]-\gamma_1\,b\left[\,b^2\,(\,1-\alpha_i\,)-(\,2-\alpha_i\,)+b\,\right]\,\right\}v\,D\,c_u$$

$$+\ \ \left\{\,\left[\,\gamma_2\,\gamma_4-b^2\,\gamma_1\,\gamma_3\,\right][\,1-\alpha_i\,]+\left[\,\gamma_4\,(\,2-\alpha_j\,)-b^2\,\gamma_1\,(\,1-\alpha_i\,)\,\right][\,v-1\,]\,D\,\right\}[\,u_i-b\,u_j\,]$$

$$+\ [\,\gamma_4\,(\,1-\alpha_j\,)-\gamma_1\,(\,2-\alpha_i\,)\,]\,[\,v-1\,]\,D\,b\,[\,u_j-b\,u_i\,]\ =\ 0$$

$$\Rightarrow\quad\left[\,b^2\,\gamma_1\,\gamma_3-\gamma_2\,\gamma_4\,\right]w_i\ =\ T_i\,,\quad\text{and} \tag{36}$$

(35) implies:

$$\left[\gamma_2\,\gamma_4\ -\gamma_3\,b^2\,\gamma_1\,\right]\ w_j+\left\{\,\gamma_3\ \left[\,b^2\,(\,1-\alpha_j\,)-(\,2-\alpha_j\,)\,\right]-\gamma_2\,\right\}b\,v\,D\,c_i$$

$$+\ \left\{\,\gamma_2\,\gamma_4-\gamma_2\,v\,D\left[\,b^2\,(\,1-\alpha_i\,)-(\,2-\alpha_i\,)\,\right]-\ \gamma_3\,b^2\,\gamma_1+\gamma_3\,b^2\,v\,D\,\right\}\ c_j$$

$$- \ \left\{ \gamma_2 \left[b^2 (1 - \alpha_i) \ - \ (2 - \alpha_i) + b \right] \ - \gamma_3 \, b \left[b^2 (1 - \alpha_j) \ - \ (2 - \alpha_j) + b \right] \right\} \ v \, D \, c_u$$

$$+ \ \left\{ \left[\gamma_2 \gamma_4 - \ b^2 \gamma_3 \gamma_1 \right] \left[1 - \alpha_j \right] + \left[\gamma_2 (2 - \alpha_i) \ - b^2 \gamma_3 (1 - \alpha_j) \right] \left[v - 1 \right] D \right\} \left[u_j - b \, u_i \right]$$

$$+ \ \left[\gamma_2 (1 - \alpha_i) - \gamma_3 (2 - \alpha_j) \right] \left[v - 1 \right] D \, b \left[u_i - b \, u_j \right] \ = \ 0$$

$$\Rightarrow \quad \left[b^2 \gamma_1 \gamma_3 - \gamma_2 \gamma_4 \right] w_j \ = \ T_j \, , \tag{37}$$

where

$$T_i \ \equiv \ \left[\gamma_1 \theta_i - \gamma_4 \right] v \, b \, D \, c_j + \left[\gamma_4 \gamma_2 - \gamma_4 v \, D \, \theta_j \ - \ \gamma_1 b^2 \gamma_3 + \gamma_1 b^2 v \, D \right] c_i$$

$$- \ \left[\gamma_4 (\theta_j + b) - \gamma_1 b (\theta_i + b) \right] v \, D \, c_u$$

$$+ \ \left\{ \left[\gamma_2 \gamma_4 - b^2 \gamma_1 \gamma_3 \right] \left[1 - \alpha_i \right] + \left[\gamma_4 (2 - \alpha_j) - b^2 \gamma_1 (1 - \alpha_i) \right] \left[v - 1 \right] D \right\} \left[u_i - b \, u_j \right]$$

$$+ \ \left[\gamma_4 (1 - \alpha_j) - \gamma_1 (2 - \alpha_i) \right] \left[v - 1 \right] D \, b \left[u_j - b \, u_i \right] \quad \text{and}$$

$$T_j \ \equiv \ \left[\gamma_3 \theta_j - \gamma_2 \right] v \, b \, D \, c_i + \left[\gamma_4 \gamma_2 - \gamma_2 v \, D \, \theta_i \ - \ \gamma_1 b^2 \gamma_3 + \gamma_3 b^2 v \, D \right] c_j$$

$$- \ \left[\gamma_2 (\theta_i + b) - \gamma_3 b (\theta_j + b) \right] v \, D \, c_u$$

$$+ \ \left\{ \left[\gamma_2 \gamma_4 - b^2 \gamma_1 \gamma_3 \right] \left[1 - \alpha_j \right] + \left[\gamma_2 (2 - \alpha_i) - b^2 \gamma_3 (1 - \alpha_j) \right] \left[v - 1 \right] D \right\} \left[u_j - b \, u_i \right]$$

$$+ \ \left[v - 1 \right] D \, b \left[u_i - b \, u_j \right]$$

$$\text{where} \ \ \theta_i \ \equiv \ b^2 \left[1 - \alpha_i \right] - \left[2 - \alpha_i \right] \quad \text{and} \quad \theta_j \ \equiv \ b^2 \left[1 - \alpha_j \right] - \left[2 - \alpha_j \right] . \tag{38}$$

(36) and (37) imply the optimal access prices are:

$$w_i \ = \ \frac{T_i}{b^2 \gamma_1 \gamma_3 - \gamma_2 \gamma_4} \quad \text{and} \quad w_j \ = \ \frac{T_j}{b^2 \gamma_1 \gamma_3 - \gamma_2 \gamma_4} \, . \tag{39}$$

Case 1. Firm 1 is a public enterprise (so $\alpha_1 \ = \ 1$ and $\alpha_2 \ = \ 0$).

(9), (29), and (38) imply that in this case:

$$\gamma_1 \ = \ \gamma_3 \ = \ \left[2 \, v - 1 \right]; \quad \gamma_2 \ = \ \left[2 \, v - 1 \right] \left[3 \, b^2 - 4 \right]; \quad \gamma_4 \ = \ - \left[2 \, v - 1 \right];$$

$$\theta_i \ = \ -1 \, ; \quad \theta_j \ = \ b^2 - 2 \, ; \quad \text{and} \quad D \ = \ 2 \, . \tag{40}$$

(38), (39), and (40) imply that in this case, for $v = v^*$:

$$w_1^* \ = \ \frac{1}{b^2 \left[2 \, v - 1 \right]^2 + \left[2 \, v - 1 \right]^2 \left[3 \, b^2 - 4 \right]}$$

$$\cdot \left\{ \left[- (2 \, v - 1) + 2 \, v - 1 \right] 2 \, v \, b \, c_j \right.$$

$$+ \ \left[- \left[2 \, v - 1 \right]^2 \left[3 \, b^2 - 4 \right] + \ \left[2 \, v - 1 \right] 2 \, v \left[b^2 - 2 \right] - \left[2 \, v - 1 \right]^2 \, b^2 + 2 \left[2 \, v - 1 \right] \, b^2 v \right] c_1$$

$$- \Big[- (2v-1)\big(b^2-2+b\big) - (2v-1)\,b\,(-1+b) \Big]\, v\, 2\, c_u$$

$$+ \left[- (2v-1)\, 2 \right] [v-1]\, 2\, [u_1 - b\, u_2]$$

$$+ \left[- (2v-1) - (2v-1) \right] [v-1]\, 2\, b\, [u_2 - b\, u_1] \}$$

$$= \frac{1}{4\,[b^2-1]\,[2v-1]^2} \{ \Big[4v\,(2v-1)\big(b^2-1\big) - 4\,(2v-1)^2 \big(b^2-1\big) \Big]\, c_1$$

$$+ 4v \left[b^2-1\right] [2v-1]\, c_u - 4\,[2v-1]\,[v-1]\,[u_1 - b\, u_2]$$

$$- 4\,[2v-1]\,[v-1]\, b\, [u_2 - b\, u_1] \}$$

$$= \frac{1}{4\,[b^2-1]\,[2v-1]^2} \{ 4\,[2v-1] \left[b^2-1\right] [v - (2v-1)]\, c_1 + 4v \left[b^2-1\right] [2v-1]\, c_u$$

$$- 4\,[2v-1]\,[v-1]\,[u_1 - b\, u_2] - 4\,[2v-1]\,[v-1]\, b\, [u_2 - b\, u_1] \}$$

$$= \frac{1}{4\,[b^2-1]\,[2v-1]^2} \{ 4\,[1-v]\,[2v-1] \left[b^2-1\right] c_1 + 4v \left[b^2-1\right] [2v-1]\, c_u$$

$$- 4\,[2v-1]\,[v-1]\,[u_1 - b\, u_2 + b\,(u_2 - b\, u_1)] \}$$

$$= \frac{[b^2-1]\,[(1-v)\, c_1 + v\, c_u] + [1-v]\,[b\,(u_2 - b\, u_1) + u_1 - b\, u_2]}{[b^2-1]\,[2v-1]}$$

$$= \frac{[1-v]\, c_1 + v\, c_u}{2v-1} + \left[\frac{1-v}{2v-1} \right] \frac{[1-b^2]\, u_1}{b^2-1} = \frac{[1-v]\, c_1 + v\, c_u - [1-v]\, u_1}{2v-1}$$

$$= \frac{[1-v]\,[c_1 - u_1] + v\, c_u}{2v-1} = c_u + \frac{v-1}{2v-1}\, \Delta_1 \,. \tag{41}$$

(38), (39), and (40) also imply that in this case, for $v = v^*$:

$$w_2^* = \frac{1}{4\,[b^2-1]\,[2v-1]^2}$$

$$\cdot \{ \left[1 - b^2\right] 2\,[2v-1]\, v\, b\, 2\, c_1 + 4 \left[b^2-1\right] [2v-1]\, c_2$$

$$- \Big[(2v-1)\big(3b^2-4\big)(b-1) - (2v-1)\, b \big(b^2-2+b\big) \Big]\, v\, 2\, c_u$$

$$+ \{ 4 \left[1 - b^2\right] [2v-1] + 4 \left[b^2-2\right] [v-1] \} [2v-1]\, [u_2 - b\, u_1]$$

$$- [2v-1]\, 2\, [v-1]\, 2\, b\, [u_1 - b\, u_2] \}$$

$$= \frac{1}{4\,[b^2-1]\,[2v-1]} \{ \left[1 - b^2\right] 2\, v\, b\, 2\, c_1 + 4 \left[b^2-1\right] c_2$$

$$- \Big[\big(3b^2-4\big)(b-1) - b \big(b^2-2+b\big) \Big]\, v\, 2\, c_u$$

$$+ \ \{ \left[1 - b^2 \right] [2v - 1] + \left[b^2 - 2 \right] [v - 1] \} \, 4 [u_2 - b u_1]$$

$$- [v - 1] \, 4 b [u_1 - b u_2] \}$$

$$= \frac{1}{4 [b^2 - 1] [2v - 1]} \{ \left[1 - b^2 \right] 2 v b 2 c_1 + 4 \left[b^2 - 1 \right] c_2$$

$$- \ \left[\left(3 b^2 - 4 \right) (b - 1) - b \left(b^2 - 2 + b \right) \right] v \, 2 c_u$$

$$+ \ \left(\left[1 - b^2 \right] [2v - 1] + 2 \left[b^2 - 1 \right] [v - 1] \right) \, 4 u_2$$

$$- \left(\left[1 - b^2 \right] [2v - 1] + \left[b^2 - 1 \right] [v - 1] \right) 4 b u_1 \}$$

$$= \frac{1}{4 [b^2 - 1] [2v - 1]} \{ \left[1 - b^2 \right] 2 v b 2 c_1 + 4 \left[b^2 - 1 \right] c_2$$

$$- \ \left[\left(3 b^2 - 4 \right) (b - 1) - \ b \left(b^2 - 2 + b \right) \right] v \, 2 c_u$$

$$+ \ 4 \left[1 - b^2 \right] u_2 - \ v \, 4 b \ \left[1 - b^2 \right] u_1 \} . \tag{42}$$

Observe that:

$$\left[3 b^2 - 4 \right] [b - 1] - b \left[b^2 - 2 + b \right] \ = \ 3 b^3 - 3 b^2 - 4 b + 4 - b^3 - b^2 + 2 b$$

$$= \ 2 b^3 - 4 b^2 - 2 b + 4 \ = \ [4 - 2 b] \left[1 - b^2 \right] . \tag{43}$$

(42) and (43) imply that for $v = v^*$:

$$w_2^* \ = \ \frac{1}{4 [b^2 - 1] [2v - 1]} \{ \left[1 - b^2 \right] 2 b v 2 c_1 + 4 \left[b^2 - 1 \right] c_2 - [4 - 2 b] \left[1 - b^2 \right] v \, 2 c_u$$

$$+ \ 4 \left[1 - b^2 \right] u_2 - \ v \, 4 b \left[1 - b^2 \right] u_1 \}$$

$$= \ \frac{c_2 - b v c_1 + \ v [2 - b] c_u - u_2 + b v u_1}{2v - 1} \tag{44}$$

$$= \ \frac{c_2 - b v c_1 + 2 v c_u - b v c_u - u_2 + b v u_1}{2v - 1}$$

$$= \ \frac{c_2 - b v c_1 + [2v - 1 + 1] c_u - b v c_u - u_2 + b v u_1}{2v - 1}$$

$$= \ c_u + \frac{c_2 - b v c_1 + c_u - b v c_u - u_2 + b v u_1}{2v - 1} \ = \ c_u - \frac{\Delta_2 - b v \Delta_1}{2v - 1} . \tag{45}$$

Case 2. Firm 1 is a private firm (so $\alpha_1 \ = \ \alpha_2 \ = \ 0$).

(9), (29), and (38) imply that in this case:

$$\gamma_1 \ = \ \gamma_3 \ = \ [2v - 1] b^2 \ ; \quad \gamma_2 \ = \gamma_4 \ = \ [2v - 1] \left[3 b^2 - 4 \right] ;$$

$$\theta_i \;=\; \theta_j \;=\; b^2 - 2\,; \quad \text{and} \quad D \;=\; 4 - b^2\,.$$

Therefore:

$$b^2\,\gamma_1\,\gamma_3 - \gamma_2\,\gamma_4 \;=\; [\,2\,v - 1\,]^2 \left[\,b^6 - \left(\,3\,b^2 - 4\,\right)^2\,\right]$$

$$= \; [\,2\,v - 1\,]^2 \left[\,b^6 - 9\,b^4 - 16 + 24\,b^2\,\right]. \tag{46}$$

(38), (39), and (46) imply that in this case, for $v = v^{0*}$:

$$T_i \;=\; \left\{\; [\,2\,v - 1\,]\,b^2 \left[\,b^2 - 2\,\right] - [\,2\,v - 1\,] \left[\,3\,b^2 - 4\,\right] \right\} v\,b \left[\,4 - b^2\,\right] c_j$$

$$+ \; \Big\{ [\,2\,v - 1\,]^2 \left[\,3\,b^2 - 4\,\right]^2 - [\,2\,v - 1\,] \left[\,3\,b^2 - 4\,\right] v \left[\,4 - b^2\,\right] \left[\,b^2 - 2\,\right]$$

$$- \; [\,2\,v - 1\,]^2\,b^4\,b^2 \; + [\,2\,v - 1\,]\,b^4\,v \;\left[\,4 - b^2\,\right] \Big\} \; c_i$$

$$- \; \left\{ [\,2\,v - 1\,] \left[\,3\,b^2 - 4\,\right] \left[\,b^2 - 2 + b\,\right] - \; [\,2\,v - 1\,]\,b^2\,b \left[\,b^2 - 2 + b\,\right] \right\} v \left[\,4 - b^2\,\right] \; c_u$$

$$+ \; \Big\{ \left[\,(\,2\,v - 1\,)^2 \left(\,3\,b^2 - 4\,\right)^2 - b^2\,(\,2\,v - 1\,)^2\;b^4\,\right]$$

$$+ \left[\,2\,(\,2\,v - 1\,) \left(\,3\,b^2 - 4\,\right) - b^2\,(\,2\,v - 1\,)\,b^2\,\right] [\,v - 1\,] \left[\,4 - b^2\,\right] \Big\} \, [\,u_i - b\,u_j\,]$$

$$+ \; \left[\,(\,2\,v - 1\,) \left(\,3\,b^2 - 4\,\right) - 2\,(\,2\,v - 1\,)\,b^2\,\right] [\,v - 1\,] \left[\,4 - b^2\,\right] b\,[\,u_j - b\,u_i\,]$$

$$= \; [\,2\,v - 1\,] \left[\,b^4 - 5\,b^2 + 4\,\right] v\,b \left[\,4 - b^2\,\right] \; c_j$$

$$+ \; [\,2\,v - 1\,] \Big\{ [\,2\,v - 1\,] \left[\left(\,3\,b^2 - 4\right)^2 - \; b^4\,b^2 \right] + v \left[\,4 - b^2\,\right] \left[\,b^4 - \left(\,3\,b^2 - 4\,\right) \left(\,b^2 - 2\,\right)\,\right] \Big\} \, c_i$$

$$- \, [\,2\,v - 1\,] \left[\,b^2 - 2 + b\,\right] \left[\,3\,b^2 - 4 \; - \; b^3\,\right] v \left[\,4 - b^2\,\right] c_u$$

$$+ \; [\,2\,v - 1\,] \left\{ [\,2\,v - 1\,] \left[\left(\,3\,b^2 - 4\,\right)^2 - \; b^6 \right] + \left[\,6\,b^2 - 8 - b^4\,\right] [\,v - 1\,] \left[\,4 - b^2\,\right] \right\} [\,u_i - b\,u_j\,]$$

$$- \; [\,2\,v - 1\,] \left[\,4 - b^2\,\right]^2 [\,v - 1\,]\,b\,[\,u_j - b\,u_i\,]$$

$$= \; [\,2\,v - 1\,] \left[\,1 - b^2\,\right] \left[\,4 - b^2\,\right]^2 v\,b\,c_j$$

$$+ \, [\,2\,v - 1\,] \Big\{ [\,2\,v - 1\,] \left[\,3\,b^2 - 4 + b^3\,\right] \left[\,3\,b^2 - 4 - b^3\,\right] - 2\,v \left[\,4 - b^2\,\right]^2 \left[\,1 - b^2\,\right] \Big\} \, c_i$$

$$- \; [\,2\,v - 1\,] \left[\,b^2 - 2 + b\,\right] \left[\,3\,b^2 - 4 \; - \; b^3\,\right] v \left[\,4 - b^2\,\right] c_u$$

$$+ [\,2\,v - 1\,] \left\{ [\,2\,v - 1\,] \left[\,3\,b^2 - 4 + b^3\,\right] \left[\,3\,b^2 - 4 - b^3\,\right] + [\,1 - v\,] \left[\,2 - b^2\,\right] \left[\,4 - b^2\,\right]^2 \right\} [\,u_i - b\,u_j\,]$$

$$+ \; [\,2\,v - 1\,] \left[\,4 - b^2\,\right]^2 [\,1 - v\,]\,b\,[\,u_j - b\,u_i\,] \qquad \text{and} \tag{47}$$

$$T_j \;=\; \left\{\; [\,2\,v - 1\,]\,b^2 \left[\,b^2 - 2\,\right] - [\,2\,v - 1\,] \left[\,3\,b^2 - 4\,\right] \right\} \; v\,b \left[\,4 - b^2\,\right] c_i$$

$$+ \ \Big\{ [\,2\,v-1\,]^2 \left[\,3\,b^2-4\,\right]^2 - [\,2\,v-1\,] \left[\,3\,b^2-4\,\right]\, v \left[\,4-b^2\,\right] \left[\,b^2-2\,\right]$$

$$- \ \left[\,2\,v-1\,\right]^2\, b^4\, b^2 \, + [\,2\,v-1\,]\, b^4\, v \left[\,4-b^2\,\right] \Big\}\ c_j$$

$$- \ \left\{ [\,2\,v-1\,] \left[\,3\,b^2-4\,\right] \left[\,b^2-2+b\,\right] - \ [\,2\,v-1\,]\, b^2\, b \left[\,b^2-2+b\,\right] \right\}\ v \left[\,4-b^2\,\right] c_u$$

$$+ \ \Big\{ \ [\,2\,v-1\,]^2 \left[\,3\,b^2-4\,\right]^2 - b^2 \ [\,2\,v-1\,]^2 \ b^4$$

$$+ \ \left[\,2\,(\,2\,v-1\,)\left(\,3\,b^2-4\,\right) - b^2\,(\,2\,v-1\,)\,b^2\,\right] [\,v-1\,] \left[\,4-b^2\,\right] \Big\}\, [\,u_j - b\,u_i\,]$$

$$+ \ \left[\,(\,2\,v-1\,)\left(\,3\,b^2-4\,\right) - 2\,(\,2\,v-1\,)\,b^2\,\right] [\,v-1\,] \left[\,4-b^2\,\right]\ b\,[\,u_i - b\,u_j\,]$$

$$= \ [\,2\,v-1\,] \left[\,b^4-5\,b^2+4\,\right] v\, b\ \left[\,4-b^2\,\right] c_i$$

$$+ \ [\,2\,v-1\,] \Big\{ [\,2\,v-1\,] \Big[\left(\,3\,b^2-4\,\right)^2 - b^4\,b^2 \Big] + v \left[\,4-b^2\,\right] \left[\,b^4 - \left(\,3\,b^2-4\,\right)\left(b^2-2\right)\,\right] \Big\}\ c_j$$

$$- \ [\,2\,v-1\,] \left[\,b^2-2+b\,\right] \left[\,3\,b^2-4\,-b^3\,\right] v \left[\,4-b^2\,\right]\ c_u$$

$$+ \ \ [\,2\,v-1\,] \{ [\,2\,v-1\,] \Big[\left(\,3\,b^2-4\,\right)^2 - \ b^6 \Big] \ + \left[\,6\,b^2-8\,-b^4\,\right] [\,v-1\,] \left[\,4-b^2\,\right] \} [\,u_j - b\,u_i\,]$$

$$- \ [\,2\,v-1\,] \left[\,4-b^2\,\right]^2 [\,v-1\,]\, b\, [\,u_i - b\,u_j\,]$$

$$= \ [\,2\,v-1\,] \left[\,1-b^2\,\right] \left[\,4-b^2\,\right]^2 v\, b\ c_i$$

$$+ \ [\,2\,v-1\,] \Big\{ [\,2\,v-1\,] \left[\,3\,b^2-4+b^3\,\right] \left[3\,b^2-4-b^3\,\right] - 2\,v \left[\,4-b^2\,\right]^2 \left[\,1-b^2\,\right] \Big\}\ c_j$$

$$- \ \ [\,2\,v-1\,] \left[\,b^2-2+b\,\right] \left[\,3\,b^2-4\ -\ b^3\,\right] v \left[\,4-b^2\,\right]\ c_u$$

$$+ \ [\,2\,v-1\,] \{ [\,2\,v-1\,] \left[\,3\,b^2-4+b^3\,\right] \left[\,3\,b^2-4-b^3\,\right] + [1-v\,] \left[\,2-b^2\,\right] \left[\,4-b^2\,\right]^2 \} [\,u_j - b\,u_i\,]$$

$$+ \ \ [\,2\,v-1\,] \left[\,4-b^2\,\right]^2 [\,1-v\,]\ b\,[\,u_i - b\,u_j\,]\,. \qquad (48)$$

(39), (46), and (47) imply that for $v = v^{0*}$:

$$w_i^{0*} \ = \ \frac{1}{[\,2\,v-1\,]^2 \left[\,b^6 - (\,3\,b^2-4\,)^2\,\right]}$$

$$\cdot \Big\{ [\,2\,v-1\,] \left[\,1-b^2\,\right] \left[\,4-b^2\,\right]^2 v\, b\, c_j$$

$$+ [\,2\,v-1\,] \left[\,(\,2\,v-1\,)\left(\,3\,b^2-4+b^3\,\right)\left(\,3\,b^2-4-b^3\,\right) - 2\,v\left(\,4-b^2\,\right)\left(\,1-b^2\,\right)\,\right] c_i$$

$$- [\,2\,v-1\,] \left[\,b^2-2+b\,\right] \left[\,3\,b^2-4\ -\ b^3\,\right] v \left[\,4-b^2\,\right]\ c_u$$

$$+ \ [\,2\,v-1\,] \Big[(\,2\,v-1\,)\left(3\,b^2-4+b^3\right)\left(3\,b^2-4-b^3\right) \ + (\,1-v\,)\left(2-b^2\right)\left(4-b^2\right)^2 \Big]\ [\,u_i - b\,u_j\,]$$

$$+ [\,2\,v-1\,]\,\big[\,4-b^2\,\big]^2\,[\,1-v\,]\;b\,[\,u_j-b\,u_i\,]\,\Big\}$$

$$= \frac{1}{[\,2\,v-1\,]\,[\,3\,b^2-4+b^3\,]\,[\,b^3-(3\,b^2-4)\,]}$$

$$\cdot \Big\{\,\big[\,1-b^2\,\big]\,\big[\,4-b^2\,\big]^2\,v\,b\,c_j$$

$$+[\,2\,v-1\,]\,\big[\,(\,2\,v-1\,)\,\big(\,3\,b^2-4+b^3\,\big)\,\big(\,3\,b^2-4-b^3\,\big)-2\,v\,\big(\,4-b^2\,\big)\,\big(\,1-b^2\,\big)\,\big]\,c_i$$

$$-\big[\,b^2-2+b\,\big]\,\big[\,3\,b^2-4\,-\,b^3\,\big]\,v\,\big[\,4-b^2\,\big]\;c_u$$

$$+\,\{\,[\,2\,v-1\,]\,\big[\,3\,b^2-4+b^3\,\big]\,\big[\,3\,b^2-4-b^3\,\big]\,+[\,1-v\,]\,\big[\,2-b^2\,\big]\,\big[\,4-b^2\,\big]^2\,\}\;[\,u_i-b\,u_j\,]$$

$$+\;\big[\,4-b^2\big]^2\,[\,1-v\,]\;b\,[\,u_j-b\,u_i\,]\,\Big\}$$

$$= \frac{[\,1-b^2\,]\,[4-b^2]^2\,v\,b}{[\,2\,v-1\,]\,[\,3\,b^2-4+b^3\,]\,[\,b^3-(3\,b^2-4)\,]}\;c_j$$

$$-\left[\,1\,+\,\frac{2\,v\,[\,4-b^2\,]^2\,[\,1-b^2\,]}{[\,2\,v-1\,]\,[\,3\,b^2-4+b^3\,]\,[\,b^3-(3\,b^2-4)\,]}\right]c_i$$

$$+\;\frac{[\,b^2-2+b\,]\,v\,[\,4-b^2\,]}{[\,2\,v-1\,]\,[\,3\,b^2-4+b^3\,]}\,c_u$$

$$+\;\left[\,-1\,+\,\frac{[\,1-v\,]\,[\,2-b^2\,]\,[\,4-b^2\,]^2}{[\,2\,v-1\,]\,[\,3\,b^2-4+b^3\,]\,[\,b^3-(3\,b^2-4)\,]}\right][\,u_i-b\,u_j\,]$$

$$+\;\frac{[\,4-b^2\,]^2\,[\,1-v\,]\,b}{[\,2\,v-1\,]\,[\,3\,b^2-4+b^3\,]\,[\,b^3-(3\,b^2-4)\,]}\;\;[\,u_j-b\,u_i\,]\,. \tag{49}$$

Observe that:

$$\big[\,1-b^2\,\big]\,\big[\,4-b^2\,\big]^2 \;=\; -\,\big[\,b^4-8\,b^2+16\,\big]\,\big[\,b^2-1\,\big] \;=\; -\,b^6+8\,b^4-16\,b^2+b^4-8\,b^2+16$$

$$=\; -\,\big[\,b^6-9\,b^4+24\,b^2-16\,\big]\,,\;\text{ and} \tag{50}$$

$$\big[\,3\,b^2-4+b^3\,\big]\,\big[\,b^3-\big(3\,b^2-4\big)\,\big] \;=\; \big[\,b^3-\big(3\,b^2-4\big)\,\big]\,\big[\,b^3+\big(3\,b^2-4\big)\,\big]$$

$$=\; b^6-\big(3\,b^2-4\big)^2 \;=\; b^6-9\,b^4+24\,b^2-16\,. \tag{51}$$

(50) and (51) imply:

$$\frac{[\,1-b^2\,]\,[\,4-b^2\,]^2}{[\,3\,b^2-4+b^3\,]\,[\,b^3-(3\,b^2-4)\,]} \;=\; -\,1 \tag{52}$$

$$\Rightarrow \quad \frac{[\,2-b^2\,]\,[\,4-b^2\,]^2}{[\,3\,b^2-4+b^3\,]\,[\,b^3-(3\,b^2-4)\,]} \;=\; -\left[\frac{2-b^2}{1-b^2}\right] \tag{53}$$

$$\Rightarrow \quad \frac{[\,4-b^2\,]^2}{[\,3\,b^2-4+b^3\,]\,[\,b^3-(3\,b^2-4)\,]} \;=\; -\left[\frac{1}{1-b^2}\right]. \tag{54}$$

Also observe that:

$$[\,b+2\,]\left[\,b^2+b-2\,\right] \;=\; b^3+b^2-2\,b+2\,b^2+2\,b-4 \;=\; b^3+3\,b^2-4$$

$$\Rightarrow \quad \frac{[\,b^2-2+b\,]\,[\,2+b\,]}{3\,b^2-4+b^3} \;=\; 1 \quad \text{and} \quad \frac{[\,b^2-2+b\,]\,[\,4-b^2\,]}{3\,b^2-4+b^3} \;=\; 2-b\,. \tag{55}$$

(49), (52), (53), (54), and (55) imply that for $v = v^{0*}$:

$$\begin{aligned}
w_i^{0*} \;=\; & -\frac{b\,v}{2\,v-1}\;c_j - \left[1\;-\;\frac{2\,v}{2\,v-1}\right]c_i + \frac{v\,[\,2-b\,]}{2\,v-1}\,c_u \\
& + \left[-1-\left(\frac{2-b^2}{1-b^2}\right)\left(\frac{1-v}{2\,v-1}\right)\right][\,u_i-b\,u_j\,] - \left[\frac{1}{1-b^2}\right]\frac{[\,1-v\,]\,b}{2\,v-1}\;[\,u_j-b\,u_i\,] \\
=\; & -\left[\frac{b\,v}{2\,v-1}\right]c_j - \left[1\;-\;\frac{2\,v}{2\,v-1}\right]c_i + \frac{v\,[\,2-b\,]}{2\,v-1}\,c_u \\
& + \left[-1+\left(\frac{b^2}{1-b^2}-\frac{2-b^2}{1-b^2}\right)\frac{1-v}{2\,v-1}\right]u_i \\
& + \left[b+b\left(\frac{2-b^2}{1-b^2}\right)\frac{1-v}{2\,v-1}-\left(\frac{1}{1-b^2}\right)\frac{(\,1-v\,)\,b}{2\,v-1}\right]u_j \\
=\; & -\left[\frac{b\,v}{2\,v-1}\right]c_j - \left[1\;-\;\frac{2\,v}{2\,v-1}\right]c_i + \frac{v\,[\,2-b\,]}{2\,v-1}\,c_u \\
& + \left[-1+2\left(\frac{b^2-1}{1-b^2}\right)\frac{1-v}{2\,v-1}\right]u_i + \left[b+\left(\frac{b-b^3}{1-b^2}\right)\frac{1-v}{2\,v-1}\right]u_j \\
=\; & -\left[\frac{b\,v}{2\,v-1}\right]c_j - \left[1\;-\;\frac{2\,v}{2\,v-1}\right]c_i + \frac{v\,[\,2-b\,]}{2\,v-1}\,c_u \\
& + \left[-1-\frac{2\,(\,1-v\,)}{2\,v-1}\right]u_i + \left[b+b\left(\frac{1-v}{2\,v-1}\right)\right]u_j \\
=\; & -\left[\frac{b\,v}{2\,v-1}\right]c_j - \left[-\,\frac{1}{2\,v-1}\right]c_i + \frac{2\,v}{2\,v-1}\,c_u - \frac{b\,v}{2\,v-1}\,c_u
\end{aligned}$$

$$- \left[1 + \frac{2\,(\,1 - v\,)}{2\,v - 1} \right] u_i + \frac{v\,b}{2\,v - 1}\, u_j$$

$$= \; - \left[\frac{b\,v}{2\,v - 1} \right] c_j + \frac{1}{2\,v - 1}\, c_i + \frac{2\,v - 1 + 1}{2\,v - 1}\, c_u - \frac{b\,v}{2\,v - 1}\, c_u - \frac{1}{2\,v - 1}\, u_i + \frac{b\,v}{2\,v - 1}\, u_j$$

$$= \; - \left[\frac{b\,v}{2\,v - 1} \right] c_j + \frac{1}{2\,v - 1}\, c_i + c_u + \frac{1}{2\,v - 1}\, c_u - \frac{b\,v}{2\,v - 1}\, c_u - \frac{1}{2\,v - 1}\, u_i + \frac{b\,v}{2\,v - 1}\, u_j$$

$$= \; c_u - \frac{\Delta_i - b\,v\,\Delta_j}{2\,v - 1}\,. \tag{56}$$

(39), (46), and (48) imply that for $v = v^{0*}$:

$$w_j^{0*} \; = \; \frac{1}{[\,2\,v - 1\,]^2 \left[\, b^6 - (\,3\,b^2 - 4\,)^2 \,\right]}$$

$$\cdot \Big\{ [\,2\,v - 1\,] \left[\,1 - b^2\,\right] \left[\,4 - b^2\,\right]^2 v\,b\,c_i$$

$$+ \; [\,2\,v - 1\,] \Big\{ [\,2\,v - 1\,] \left[\,3\,b^2 - 4 + b^3\,\right] \left[\,3\,b^2 - 4 - b^3\,\right] - 2\,v \left[\,4 - b^2\,\right]^2 \left[\,1 - b^2\,\right] \Big\}\; c_j$$

$$- \; [\,2\,v - 1\,] \left[\,b^2 - 2 + b\,\right] \left[\,3\,b^2 - 4 \, - \, b^3\,\right] v \left[\,4 - b^2\,\right] \, c_u$$

$$+ \; [\,2\,v - 1\,] \{ [\,2\,v - 1\,] \left[\,3\,b^2 - 4 + b^3\right] \left[\,3\,b^2 - 4 - b^3\,\right] + \left[\,1 - v\,\right] \left[\,2 - b^2\,\right] \left[\,4 - b^2\,\right]^2 \} \; [\,u_j - b\,u_i\,]$$

$$+ \; [\,2\,v - 1\,] \left[\,4 - b^2\,\right]^2 [\,1 - v\,] \; b\,[\,u_i - b\,u_j\,] \Big\}$$

$$= \; \frac{[\,1 - b^2\,]\,[\,4 - b^2\,]^2\,v\,b}{[\,2\,v - 1\,]\,[\,3\,b^2 - 4 + b^3\,]\,[\,b^3 - (3\,b^2 - 4)\,]} \; c_i$$

$$- \; \left[1 \, + \, \frac{2\,v\,[\,4 - b^2\,]^2\,[\,1 - b^2\,]}{[\,2\,v - 1\,]\,[\,3\,b^2 - 4 + b^3\,]\,[\,b^3 - (3\,b^2 - 4)\,]} \right] c_j$$

$$+ \; \frac{[\,b^2 - 2 + b\,]\,v\,[\,4 - b^2\,]}{[\,2\,v - 1\,]\,[\,3\,b^2 - 4 + b^3\,]}\; c_u$$

$$+ \; \left[-1 \, + \, \frac{[\,1 - v\,]\,[\,2 - b^2\,]\,[\,4 - b^2\,]^2}{[\,2\,v - 1\,]\,[\,3\,b^2 - 4 + b^3\,]\,[\,b^3 - (3\,b^2 - 4)\,]} \right] [\,u_j - b\,u_i\,]$$

$$+ \; \frac{[\,4 - b^2\,]^2\,[\,1 - v\,]\,b}{[\,2\,v - 1\,]\,[\,3\,b^2 - 4 + b^3\,]\,[\,b^3 - (3\,b^2 - 4)\,]}\; [\,u_i - b\,u_j\,]\,. \tag{57}$$

(52), (53), (54), (55), and (57) imply that for $v = v^{0*}$:

$$
\begin{aligned}
w_j^{0*} \;=\; & -\left[\frac{b\,v}{2\,v-1}\right] c_i - \left[1 - \frac{2\,v}{2\,v-1}\right] c_j + \frac{v\,[\,2-b\,]}{2\,v-1}\,c_u \\
& + \left[-1 - \left(\frac{2-b^2}{1-b^2}\right)\left(\frac{1-v}{2\,v-1}\right)\right][\,u_j - b\,u_i\,] \\
& - \left[\frac{1}{1-b^2}\right]\frac{[\,1-v\,]\,b}{2\,v-1}\,[\,u_i - b\,u_j\,]. \\
= & -\left[\frac{b\,v}{2\,v-1}\right] c_i - \left[1 - \frac{2\,v}{2\,v-1}\right] c_j + \frac{2\,v}{2\,v-1}\,c_u - \frac{b\,v}{2\,v-1}\,c_u \\
& - \left[\frac{1}{1-b^2}\right]\frac{[\,1-v\,]\,b}{[\,2\,v-1\,]}\,[\,u_i\,] - \left[1 + \left(\frac{2-b^2}{1-b^2}\right)\left(\frac{1-v}{2\,v-1}\right)\right][\,-b\,u_i\,] \\
& + \left[-1 - \left(\frac{2-b^2}{1-b^2}\right)\left(\frac{1-v}{2\,v-1}\right)\right] u_j + \left[\frac{1}{1-b^2}\right]\frac{[\,1-v\,]\,b}{2\,v-1}\,b\,u_j \\
= & -\left[\frac{b\,v}{2\,v-1}\right] c_i - \left[1 - \frac{2\,v}{2\,v-1}\right] c_j + \frac{2\,v}{2\,v-1}\,c_u - \frac{b\,v}{2\,v-1}\,c_u \\
& + \left[b + \left(\frac{2-b^2}{1-b^2}\right)\frac{b\,(\,1-v\,)}{2\,v-1} - \left(\frac{1}{1-b^2}\right)\frac{(\,1-v\,)\,b}{2\,v-1}\right] u_i \\
& + \left[\left(\frac{1}{1-b^2}\right)\frac{(\,1-v\,)\,b^2}{2\,v-1} - 1 - \left(\frac{2-b^2}{1-b^2}\right)\frac{1-v}{2\,v-1}\right] u_j \\
= & -\left[\frac{b\,v}{2\,v-1}\right] c_i - \left[-\frac{1}{2\,v-1}\right] c_j + \frac{2\,v-1+1}{2\,v-1}\,c_u - \left[\frac{b\,v}{2\,v-1}\right] c_u \\
& + \left[1 + \left(\frac{2-b^2}{1-b^2} - \frac{1}{1-b^2}\right)\frac{1-v}{2\,v-1}\right] b\,u_i \\
& + \left[\left(\frac{b^2}{1-b^2} - \frac{2-b^2}{1-b^2}\right)\frac{1-v}{2\,v-1} - 1\right] u_j \\
= & -\left[\frac{b\,v}{2\,v-1}\right] c_i - \left[-\frac{1}{2\,v-1}\right] c_j + c_u + \frac{1}{2\,v-1}\,c_u - \frac{b\,v}{2\,v-1}\,c_u \\
& + \left[1 + \frac{1-v}{2\,v-1}\right] b\,u_i + \left[-2\left(\frac{1-v}{2\,v-1}\right) - 1\right] u_j \\
= & -\left[\frac{b\,v}{2\,v-1}\right] c_i - \left[-\frac{1}{2\,v-1}\right] c_j + c_u + \frac{1}{2\,v-1}\,c_u - \frac{b\,v}{2\,v-1}\,c_u
\end{aligned}
$$

$$+\ \frac{b\,v}{2\,v-1}\,u_i - \frac{1}{2\,v-1}\,u_j \ = \ c_u - \frac{\Delta_j - b\,v\,\Delta_i}{2\,v-1}\,. \tag{58}$$

(56) and (58) imply that when $v^{0*} \neq \frac{1}{2}$:

$$w_1^{0*} \ = \ c_u - \frac{\Delta_1 - b\,v^{0*}\,\Delta_2}{2\,v^{0*}-1} \ \text{ and } \ w_2^{0*} \ = \ c_u - \frac{\Delta_2 - b\,v^{0*}\,\Delta_1}{2\,v^{0*}-1}\,. \tag{59}$$

(10) and (59) imply that for $i, j \in \{1,2\}$ $(j \neq i)$ and for $v = v^{0*}$:

$$\begin{aligned}
p_i^{0*} \ &= \ w_i^{0*} + c_i + \frac{1}{4-b^2}\left[\,\left(2-b^2\right)\left(\,u_i - w_i^{0*} - c_i\,\right) - b\left(\,u_j - w_j^{0*} - c_j\,\right)\,\right] \\
&= \ c_i + \frac{1}{4-b^2}\left[\,\left(2-b^2\right)(\,u_i - c_i\,) - b\,(\,u_j - c_j\,) + \left(4-b^2-2+b^2\right)w_i^{0*} + b\,w_j^{0*}\,\right] \\
&= \ c_i + \frac{1}{4-b^2}\left[\,\left(2-b^2\right)(\,u_i - c_u - c_i\,) - b\,(\,u_j - c_u - c_j\,) + 2\,w_i^{0*} + b\,w_j^{0*}\right. \\
&\qquad\qquad \left. +\ c_u\left(2-b^2-b\right)\,\right] \\
&= \ c_i + \frac{1}{4-b^2}\left[\,\left(2-b^2\right)\Delta_i - b\,\Delta_j + 2\left(c_u - \frac{\Delta_i - b\,v\,\Delta_j}{2\,v-1}\right)\right. \\
&\qquad\qquad \left. +\ b\left(c_u - \frac{\Delta_j - b\,v\,\Delta_i}{2\,v-1}\right) + c_u\left(2-b^2-b\right)\,\right] \\
&= \ c_i + \frac{Z_i}{[\,4-b^2\,]\,[\,2\,v-1\,]}
\end{aligned} \tag{60}$$

where

$$\begin{aligned}
Z_i \ &\equiv \ \left[\,2-b^2\,\right][\,2\,v-1\,]\,\Delta_i - b\,[\,2\,v-1\,]\,\Delta_j + 2\,[\,c_u\,(2\,v-1) - \Delta_i + b\,v\,\Delta_j\,] \\
&\quad +\ b\,[\,c_u\,(2\,v-1) - \Delta_j + b\,v\,\Delta_i\,] + [\,2\,v-1\,]\left[\,2-b^2-b\,\right]c_u \\
&= \ \left[\,\left(2-b^2\right)(2\,v-1) - 2 + b^2\,v\,\right]\Delta_i - [\,b\,(2\,v-1) - 2\,b\,v + b\,]\,\Delta_j \\
&\quad +\ \left[\,4\,v - 2 + 2\,b\,v - b + (2\,v-1)\left(2-b^2-b\right)\,\right]c_u \\
&= \ \Delta_i\left[\,4\,v - 2 - 2\,b^2\,v + b^2 - 2 + b^2\,v\,\right] \\
&\quad +\ c_u\left[\,4\,v - 2 + 2\,b\,v - b + 4\,v - 2\,b\,v - 2\,b^2\,v - 2 + b + b^2\,\right] \\
&= \ \Delta_i\left[\,4\,v - 4 - b^2\,v + b^2\,\right] + c_u\left[\,8\,v - 2\,b^2\,v + b^2 - 4\,\right] \\
&= \ \Delta_i\left[\,4-b^2\,\right][\,v-1\,] + c_u\left[\,4-b^2\,\right][\,2\,v-1\,]\,.
\end{aligned} \tag{61}$$

(60) and (61) imply that for $i \in \{1,2\}$:

$$p_i^{0*} \;=\; c_u + c_i + \frac{[\,v^{0*} - 1\,]\,\Delta_i}{2\,v^{0*} - 1}\,. \tag{62}$$

(11) and (41) imply:

$$p_1^* \;=\; w_1^* + c_1 \;=\; c_u + c_1 + \frac{[\,v^* - 1\,]\,\Delta_1}{2\,v^* - 1}\,. \tag{63}$$

(12), (41), and (45) imply:

$$\begin{aligned}
p_2^* \;&=\; \frac{1}{2}\,[\,u_2 + w_2^* + c_2 - b\,(\,u_1 - w_1^* - c_1\,)] \\
&=\; \frac{1}{2}\left[\,u_2 + c_u - \frac{\Delta_2 - b\,v^*\,\Delta_1}{2\,v^* - 1} + c_2 - b\left(\,u_1 - c_u - \frac{v^* - 1}{2\,v^* - 1}\,\Delta_1 - c_1\,\right)\right] \\
&=\; \frac{1}{2\,[\,2\,v^* - 1\,]}\,[\,(u_2 + c_u + c_2)\,(2\,v^* - 1) - \Delta_2 + b\,v^*\,\Delta_1 \\
&\qquad\qquad -\; b\,(u_1 - c_u - c_1)\,(2\,v^* - 1) + b\,(v^* - 1)\,\Delta_1\,] \\
&=\; \frac{1}{2\,[\,2\,v^* - 1\,]}\,[\,(u_2 - c_u - c_2)\,(2\,v^* - 1) + 2\,(c_u + c_2)\,(2\,v^* - 1) - \Delta_2 + b\,v^*\,\Delta_1 \\
&\qquad\qquad -\; b\,\Delta_1\,(2\,v^* - 1) + b\,(v^* - 1)\,\Delta_1\,] \\
&=\; \frac{1}{2\,[\,2\,v^* - 1\,]}\,[\,(2\,v^* - 1)\,\Delta_2 + 2\,(c_u + c_2)\,(2\,v^* - 1) - \Delta_2 \\
&\qquad\qquad -\; b\,(2\,v^* - 1)\,\Delta_1 + b\,(2\,v^* - 1)\,\Delta_1\,] \\
&=\; \frac{1}{2\,[\,2\,v^* - 1\,]}\,[\,2\,(v^* - 1)\,\Delta_2 + 2\,(c_u + c_2)\,(2\,v^* - 1)\,] \;=\; c_u + c_2 + \frac{[\,v^* - 1\,]\,\Delta_2}{2\,v^* - 1}\,.
\end{aligned} \tag{64}$$

(13) and (62) imply that for $i, j \in \{1, 2\}$ $(i \neq j)$:

$$\begin{aligned}
X_i^{0*} \;&=\; \frac{1}{1 - b}\,\left[\,u_i - b\,u_j - p_i^{0*} + b\,p_j^{0*}\,\right] \\
&=\; \frac{1}{1 - b}\left[\,u_i - b\,u_j - \left(c_u + c_i + \frac{[\,v^{0*} - 1\,]\,\Delta_i}{2\,v - 1}\right) + b\left(c_u + c_j + \frac{[\,v^{0*} - 1\,]\,\Delta_j}{2\,v - 1}\right)\right] \\
&=\; \frac{1}{1 - b}\left[\,u_i - c_u - c_i - \frac{[\,v^{0*} - 1\,]\,\Delta_i}{2\,v^{0*} - 1} - b\left(u_j - c_u - c_j - \frac{[\,v^{0*} - 1\,]\,\Delta_j}{2\,v^{0*} - 1}\right)\right] \\
&=\; \frac{1}{1 - b}\left[\,\Delta_i - \frac{[\,v^{0*} - 1\,]\,\Delta_i}{2\,v^{0*} - 1} - b\,\Delta_j + b\,\frac{[\,v^{0*} - 1\,]\,\Delta_j}{2\,v^{0*} - 1}\,\right] \\
&=\; \frac{1}{[\,1 - b\,]\,[\,2\,v^{0*} - 1\,]}\,\left[\,\Delta_i\left(2\,v^{0*} - 1\right) - \left[\,v^{0*} - 1\,\right]\Delta_i - b\,\Delta_j\left(2\,v^{0*} - 1\right) + b\left(v^{0*} - 1\right)\Delta_j\,\right]
\end{aligned}$$

$$= \frac{1}{[1-b][2v^{0*}-1]}\left[v^{0*}\Delta_i - b\,v^{0*}\Delta_j\right] = \frac{v^{0*}[\Delta_i - b\,\Delta_j]}{[1-b][2v^{0*}-1]}. \tag{65}$$

(13), (63), and (64) imply that for $i, j \in \{1,2\}$ $(i \neq j)$:

$$\begin{aligned}
X_i^* &= \frac{1}{1-b}\left[u_i - b\,u_j - p_i^* + b\,p_j^*\right] \\
&= \frac{1}{1-b}\left[u_i - b\,u_j - \left(c_u + c_i + \frac{[v^*-1]\Delta_i}{2v^*-1}\right) + b\left(c_u + c_j + \frac{[v^*-1]\Delta_j}{2v^*-1}\right)\right] \\
&= \frac{1}{1-b}\left[u_i - c_u - c_i - \frac{[v^*-1]\Delta_i}{2v^*-1} - b\left(u_j - c_u - c_j - \frac{[v^*-1]\Delta_j}{2v^*-1}\right)\right] \\
&= \frac{1}{1-b}\left[\Delta_i - \frac{[v^*-1]\Delta_i}{2v^*-1} - b\left(\Delta_j - \frac{[v^*-1]\Delta_j}{2v^*-1}\right)\right] \\
&= \frac{1}{[1-b][2v^*-1]}\left[\Delta_i(2v^*-1-v^*+1) - b\,\Delta_j(2v^*-1-v^*+1)\right] \\
&= \frac{v^*[\Delta_i - b\,\Delta_j]}{[1-b][2v^*-1]}.
\end{aligned} \tag{66}$$

(14), (63), (64), and (66) imply that when firm 1 is a public enterprise, industry profit is:

$$\begin{aligned}
\Pi(\mathbf{w}) &= [p_1^* - c_u - c_1]X_1^* + [p_2^* - c_u - c_2]X_2^* - F \\
&= \frac{[v^*-1]\Delta_1}{2v^*-1}\,\frac{v^*[\Delta_1 - b\,\Delta_2]}{[1-b][2v^*-1]} + \frac{[v^*-1]\Delta_2}{2v^*-1}\,\frac{v^*[\Delta_2 - b\,\Delta_1]}{[1-b][2v^*-1]} - F \\
&= \frac{v^*[v^*-1]}{[1-b][2v^*-1]^2}\{\Delta_1[\Delta_1 - b\,\Delta_2] + \Delta_2[\Delta_2 - b\,\Delta_1]\} - F \\
&= g(v^*)\,H - F
\end{aligned} \tag{67}$$

$$\text{where } g(v) \equiv \frac{v[v-1]}{[1-b][2v-1]^2} \gtreqless 0 \Leftrightarrow v \gtreqless 1 \text{ and} \tag{68}$$

$$H \equiv \Delta_1[\Delta_1 - b\,\Delta_2] + \Delta_2[\Delta_2 - b\,\Delta_1] = [\Delta_1]^2 - 2b\,\Delta_1\Delta_2 + [\Delta_2]^2 > 0. \tag{69}$$

(14), (62), and (65) imply that when firm 1 is a private firm, industry profit is:

$$\begin{aligned}
\Pi(\mathbf{w}) &= \left[p_1^{0*} - c_u - c_1\right]X_1^{0*} + \left[p_2^{0*} - c_u - c_2\right]X_2^{0*} - F \\
&= \frac{[v^{0*}-1]\Delta_1}{2v^{0*}-1}\,\frac{v^{0*}[\Delta_1 - b\,\Delta_2]}{[1-b][2v^{0*}-1]} + \frac{[v^{0*}-1]\Delta_2}{2v^{0*}-1}\,\frac{v^{0*}[\Delta_2 - b\,\Delta_1]}{[1-b][2v^{0*}-1]} - F
\end{aligned}$$

$$= \frac{v^{0*}[v^{0*}-1]}{[1-b][2v^{0*}-1]^2}\{\Delta_1[\Delta_1 - b\Delta_2] + \Delta_2[\Delta_2 - b\Delta_1]\} - F$$

$$= g(v^{0*})H - F. \tag{70}$$

Case (i). $\gamma \in [0,1)$.

Suppose $\lambda = 0$ at the solution to [RP]. Then (16) and (68) imply that for $v \in \{v^*, v^{0*}\}$:

$$v = \gamma \in [0,1) \Rightarrow g(v) < 0 \Rightarrow \Pi(\mathbf{w}) < 0. \tag{71}$$

(71) implies (16) is violated. Therefore, by contradiction, $\lambda > 0$, and so $\Pi(\mathbf{w}) = 0$ by complementary slackness. Consequently, (67) and (70) imply:

$$g(v) = \frac{F}{H}. \tag{72}$$

Case (ii). $\gamma = 1$.

If $\Pi(\mathbf{w}) > 0$, then (16) implies that $g(v) > \frac{F}{H} \geq 0$ for $v \in \{v^*, v^{0*}\}$. Therefore, $v > 1$ from (68). $\lambda > 0$ because $\gamma = 1$ and $v = \gamma + \lambda > 1$. Therefore, $\Pi(\mathbf{w}) = 0$ by complementary slackness. This contradiction ensures $\Pi(\mathbf{w}) = 0$, so (72) holds.

(72) implies that if $F = 0$, then $g(v) = \frac{F}{H} = 0$, so $v = 1$, from (68). (72) also implies that if $F > 0$, then $g(v) = \frac{F}{H} > 0$, so $v > 1$, from (68). ■