Technical Appendix to Accompany

"On the Merits of Antitrust Liability in Regulated Industries"

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This Technical Appendix has two parts. Technical Appendix A begins with Conclusion 1, which provides a formal statement of the regulator's problem, [RP]. Conclusion 2 then identifies conditions under which the profitability constraint (PC) and the behavioral constraint (BC) bind at the solution to [RP] and the solution is unique. Next, Conclusion 3 identifies conditions under which the regulated vertically-integrated firm (V) and the competitor (E) both produce strictly positive output in equilibrium. The remainder of the analysis in Technical Appendix A provides the proofs of the formal conclusions in the paper.

Technical Appendix B identifies conditions under which the behavioral constraint (BC) does not bind at the solution to [RP] and characterizes the optimal regulatory policy in this case.

Technical Appendix A

To begin, define $\underline{d} \equiv d(\underline{\alpha})$, $\overline{d} \equiv d(\overline{\alpha})$, and $\underline{c} \equiv \underline{\alpha}\, c_H + [1-\underline{\alpha}]\, c_L$.

Conclusion 1. The regulator's problem [RP] is the following:

$$\underset{w \geq 0,\, r \in \left[\frac{1}{2}, 1\right],\, D_R \leq \overline{D}_R}{Maximize} \quad W$$

subject to:

$$g(w,r) \;\equiv\; \frac{w}{3\,b}\,[\,a + 3\,u + c_v - 2\,\underline{c}\,] - \frac{2\,w^2}{3\,b} - \phi \;\geq\; 0\,, \quad \text{and} \tag{1}$$

$$h\,(w,r) \;\equiv\; -\frac{1}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\,[\,2\,a + 2\,u + c_L + c_H - 4\,c_v - 4\,w\,] + [\,2\,r - 1\,]\,D_R + \left[\,\overline{d} - \underline{d}\,\right] D_C \;\geq\; 0\,, \tag{2}$$

where

$$W \;\equiv\; \frac{\alpha}{18\,b}\,[\,2\,a - w - u - c_H - c_v\,]^2 + \left[\frac{1-\alpha}{18\,b}\right][\,2\,a - w - u - c_L - c_v\,]^2 - k\left[r - \frac{1}{2}\right]^2 + [\,1 - r\,]\,D_R\,[\,1 - f_R\,] + [\,1 - f_C\,]\,\underline{d}\;D_C\,, \quad \text{and} \tag{3}$$

$$\phi \;\equiv\; \frac{u}{3\,b}\,[\,a + u + c_v - 2\,\underline{c}\,] + [\,1 - r\,]\,D_R + \underline{d}\;D_C + F_u \;>\; 0\,. \tag{4}$$

Proof. The market demand curve is:

$$P(X) \; = \; a - b\,X \; = \; a - b\,[\,x_e + x_v\,]\,. \tag{5}$$

V's profit, given realized costs and abstracting from regulatory and court penalties, is:

$$\widehat{\pi}_v \; = \; [\,w - u\,]\,x_e + [\,P(X) - u - c_v\,]\,x_v - F_u - F_d\,. \tag{6}$$

E's profit, given realized costs and abstracting from regulatory and court penalties, is:

$$\widehat{\pi}_e \; = \; [\,P(X) - w - c_i\,]\,x_e - F_e\,. \tag{7}$$

Because expected regulatory and court penalties do not vary with realized outputs, (5) and (7) imply that E's (interior) profit-maximizing choice of x_e is determined by:

$$\frac{\partial \widehat{\pi}_e}{\partial x_e} \; = \; a - b\,[\,x_e + x_v\,] - w - c_i - b\,x_e \; = \; 0$$

$$\Rightarrow \quad x_e \; = \; \max\left\{ 0,\, \frac{1}{2\,b}\,[\,a - w - c_i - b\,x_v\,] \right\}. \tag{8}$$

Similarly, (5) and (6) imply that V's (interior) profit-maximizing choice of x_v is determined by:

$$\frac{\partial \widehat{\pi}_v}{\partial x_v} \; = \; a - b\,[\,x_e + x_v\,] - u - c_v - b\,x_v \; = \; 0$$

$$\Rightarrow \quad x_v \; = \; \max\left\{ 0,\, \frac{1}{2\,b}\,[\,a - u - c_v - b\,x_e\,] \right\}. \tag{9}$$

(8) and (9) imply that if $x_v > 0$, then:

$$x_v \; = \; \frac{1}{2\,b}\,[\,a - u - c_v\,] - \frac{1}{2}\max\left\{ 0,\, \frac{1}{2\,b}\,[\,a - w - c_i - b\,x_v\,] \right\}$$

$$\Rightarrow \quad x_v \; = \; \frac{1}{2\,b}\,[\,a - u - c_v\,] \quad \text{when } \; x_e \; = \; 0\,.$$

When $x_e > 0$:

$$x_v \; = \; \frac{1}{2\,b}\,[\,a - u - c_v\,] - \frac{1}{4\,b}\,[\,a - w - c_i - b\,x_v\,]$$

$$\Rightarrow \quad \frac{3}{4}\,x_v \; = \; \frac{1}{4\,b}\,[\,2\,a - 2\,u - 2\,c_v - a + w + c_i\,]$$

$$\Rightarrow \quad x_v \; = \; \frac{1}{3\,b}\,[\,a + w + c_i - 2\,u - 2\,c_v\,]\,. \tag{10}$$

(8) and (10) imply that when $x_e > 0$ and $x_v > 0$ in equilibrium:

$$x_e \; = \; \frac{1}{2\,b}\,[\,a - w - c_i\,] - \frac{1}{6\,b}\,[\,a + w + c_i - 2\,u - 2\,c_v\,]$$

$$= \frac{1}{6b}[3a - 3w - 3c_i - a - w - c_i + 2u + 2c_v] = \frac{1}{3b}[a + u + c_v - 2w - 2c_i]. \quad (11)$$

(10) and (11) imply that when $x_e > 0$ and $x_v > 0$ in equilibrium:

$$X = x_e + x_v = \frac{1}{3b}[2a - w - u - c_i - c_v] \quad (12)$$

$$\Rightarrow P(X) = a - \frac{1}{3}[2a - w - u - c_i - c_v] = \frac{1}{3}[a + w + u + c_i + c_v] \quad (13)$$

$$\Rightarrow P(X) - u - c_v = \frac{1}{3}[a + w + c_i - 2u - 2c_v]. \quad (14)$$

(6), (10), (11), and (14) provide:

$$\widehat{\pi}_v = \frac{w-u}{3b}[a + u + c_v - 2w - 2c_i] + \frac{1}{9b}[a + w + c_i - 2u - 2c_v]^2 - F_u - F_d. \quad (15)$$

Then (15) implies that V's expected profit when it undertakes the competitive action is:

$$\begin{aligned}\underline{\pi}_v = {} & \underline{\alpha}\left\{\frac{w-u}{3b}[a + u + c_v - 2w - 2c_H] + \frac{1}{9b}[a + w + c_H - 2u - 2c_v]^2\right\} \\ & + [1-\underline{\alpha}]\left\{\frac{w-u}{3b}[a + u + c_v - 2w - 2c_L] + \frac{1}{9b}[a + w + c_L - 2u - 2c_v]^2\right\} \\ & - F_u - F_d - [1-r]D_R - \underline{d}\, D_C. \quad (16)\end{aligned}$$

(15) also implies that V's expected profit when it undertakes the anticompetitive action is:

$$\begin{aligned}\overline{\pi}_v = {} & \overline{\alpha}\left\{\frac{w-u}{3b}[a + u + c_v - 2w - 2c_H] + \frac{1}{9b}[a + w + c_H - 2u - 2c_v]^2\right\} \\ & + [1-\overline{\alpha}]\left\{\frac{w-u}{3b}[a + u + c_v - 2w - 2c_L] + \frac{1}{9b}[a + w + c_L - 2u - 2c_v]^2\right\} \\ & - F_u - F_d - r\,D_R - \underline{d}\, D_C. \quad (17)\end{aligned}$$

From (12), consumers' surplus when E unit downstream cost is c_i is:

$$S(c_i) = \frac{1}{2}X[a - (a - bX)] = \frac{b}{2}X^2.$$

Therefore, from (12), when $x_e > 0$ and $x_v > 0$ in equilibrium:

$$S(c_i) = \frac{1}{2}X[a - (a - bX)] = \frac{b}{2}X^2 = \frac{1}{18b}[2a - w - u - c_i - c_v]^2. \quad (18)$$

Let $\underline{\pi}_v^u$ denote V's upstream profit when it undertakes the competitive action. From (16):

$$
\begin{aligned}
\underline{\pi}_v^u \;=\; & \underline{\alpha}\left[\frac{w-u}{3\,b}\right][\,a+u+c_v-2\,w-2\,c_H\,] \;+\; [\,1-\underline{\alpha}\,]\left[\frac{w-u}{3\,b}\right][\,a+u+c_v-2\,w-2\,c_L\,] \\
& \qquad - \;[\,1-r\,]\,D_R - \underline{d}\;D_C - F_u \\
=\; & \frac{w-u}{3\,b}\;[\,a+u+c_v-2\,w\,] - \frac{2\,[\,w-u\,]}{3\,b}\,[\;\underline{\alpha}\;c_H + (\,1-\underline{\alpha}\,)\,c_L\,] \\
& \qquad - \;[\,1-r\,]\,D_R - \underline{d}\;D_C - \;F_u \\
=\; & \frac{w-u}{3\,b}\,[\,a+u+c_v-2\,w-2\,\underline{c}\,] - [\,1-r\,]\,D_R - \underline{d}\;D_C - F_u \\
=\; & \frac{w}{3\,b}\,[\,a+u+c_v-2\,\underline{c}\,] - \frac{2\,w^2}{3\,b} \\
& \qquad - \;\frac{u}{3\,b}\,[\,a+u+c_v-2\,w-2\,\underline{c}\,] - [\,1-r\,]\,D_R - \underline{d}\;D_C - F_u \\
=\; & \frac{w}{3\,b}\,[\,a+u+c_v-2\,\underline{c}\,] + \frac{2\,u\,w}{3\,b} - \frac{2\,w^2}{3\,b} \\
& \qquad - \;\frac{u}{3\,b}\,[\,a+u+c_v-2\,\underline{c}\,] - [\,1-r\,]\,D_R - \underline{d}\;D_C - F_u \\
=\; & \frac{w}{3\,b}\,[\,a+3\,u+c_v-2\,\underline{c}\,] - \frac{2\,w^2}{3\,b} - \;\phi\,. \qquad (19)
\end{aligned}
$$

The inequality in (4) holds when both firms produce strictly positive output in equilibrium.[1] Therefore, (11) implies $a+u+c_v-2\,\underline{c}\;\geq\;0$.

(19) implies that V's profitability constraint (PC) is:

$$
\frac{w}{3\,b}\,[\,a+3\;u+c_v-2\,\underline{c}\,] - \frac{2\,w^2}{3\,b} - \phi \;\geq\; 0\,. \qquad (20)
$$

From (16), V's expected profit when it undertakes the competitive action is:

$$
\begin{aligned}
\underline{\pi}_v \;=\; & \frac{\underline{\alpha}}{9\,b}\,\{\,3\,w\,[\,a+u+c_v-2\,c_H\,] - 6\,w^2 - 3\,u\,[\,a+u+c_v-2\,c_H-2\,w\,] \\
& \qquad + \;w^2 + 2\,w\,[\,a+c_H-2\,u-2\,c_v\,] + [\,a+c_H-2\,u-2\,c_v\,]^2\,\} \\
& + \;\frac{1-\underline{\alpha}}{9\,b}\,\{\,3\,w\,[\,a+u+c_v-2\,c_L\,] - 6\,w^2 - 3\,u\,[\,a+u+c_v-2\,c_L-2\,w\,] \\
& \qquad + \;w^2 + 2\,w\,[\,a+c_L-2\,u-2\,c_v\,] + [\,a+c_L-2\,u-2\,c_v\,]^2\,\} \\
& - \;F_u - F_d - \;[\,1-r\,]\,D_R - \underline{d}\;D_C
\end{aligned}
$$

[1] Conclusion 3 below provides sufficient conditions. The conditions are assumed to hold throughout the ensuing analysis.

$$
\begin{aligned}
= \; & \frac{\underline{\alpha}}{9\,b} \{ \, w \, [\, 3\,a + 3\,u + 3\,c_v - 6\,c_H \,] - 6\,w^2 + 6\,w\,u - 3\,u\,[\, a + u + c_v - 2\,c_H \,] \\
& \qquad + \; w^2 + w\,[\, 2\,a + 2\,c_H - 4\,u - 4\,c_v \,] + [\, a + c_H - 2\,u - 2\,c_v \,]^2 \, \} \\
& + \; \frac{1 - \underline{\alpha}}{9\,b} \{ \, w \, [\, 3\,a + 3\,u + 3\,c_v - 6\,c_L \,] - 6\,w^2 + 6\,w\,u - 3\,u\,[\, a + u + c_v - 2\,c_L \,] \\
& \qquad + \; w^2 + w\,[\, 2\,a + 2\,c_L - 4\,u - 4\,c_v \,] + [\, a + c_L - 2\,u - 2\,c_v \,]^2 \, \} \\
& - \, F_u - F_d - \, [\, 1 - \underline{r} \,] \, D_R - \underline{d} \; D_C \\
= \; & \frac{\underline{\alpha}}{9\,b} \{ \, w \, [\, 5\,a + 5\,u - c_v - 4\,c_H \,] - 3\,u\,[\, a + u + c_v - 2\,c_H \,] \\
& \qquad \qquad - \; 5\,w^2 + [\, a + c_H - 2\,u - 2\,c_v \,]^2 \, \} \\
& + \; \frac{1 - \underline{\alpha}}{9\,b} \{ \, w \, [\, 5\,a + 5\,u - c_v - 4\,c_L \,] - 3\,u\,[\, a + u + c_v - 2\,c_L \,] \\
& \qquad \qquad - \; 5\,w^2 + [\, a + c_L - 2\,u - 2\,c_v \,]^2 \, \} \\
& - \, F_u - F_d - \, [\, 1 - r \,] \, D_R - \underline{d} \; D_C \\
= \; & \frac{1}{9\,b} \{ \, w \, [\, 5\,a + 5\,u - c_v - 4\,\underline{c} \,] - 3\,u\,[\, a + u + c_v - 2\,\underline{c} \,] - 5\,w^2 \\
& \qquad + \; \underline{\alpha}\,[\, a + c_H - 2\,u - 2\,c_v \,]^2 + [\, 1 - \underline{\alpha} \,]\,[\, a + c_L - 2\,u - 2\,c_v \,]^2 \, \} \\
& - \, F_u - F_d - \, [\, 1 - r \,] \, D_R - \underline{d} \; D_C \, . \qquad (21)
\end{aligned}
$$

(21) implies that V's expected profit when it undertakes the competitive action is:

$$
\underline{\pi}_v \; = \; A_0 + A_1\,w + A_2\,w^2 \, , \qquad (22)
$$

where:

$$
\begin{aligned}
A_0 \; \equiv \; & \frac{\underline{\alpha}}{9\,b}\,[\, a + c_H - 2\,u - 2\,c_v \,]^2 + \frac{1 - \underline{\alpha}}{9\,b}\,[\, a + c_L - 2\,u - 2\,c_v \,]^2 \\
& - \; \frac{u}{3\,b}\,[\, a + u + c_v - 2\,\underline{c} \,] - F_u - F_d - [\, 1 - r \,] \, D_R - \underline{d} \; D_C \, ; \qquad (23)
\end{aligned}
$$

$$
A_1 \; \equiv \; \frac{1}{9\,b}\,[\, 5\,a + 5\,u - c_v - 4\,\underline{c} \,] \, ; \; \text{ and } \; A_2 \; \equiv \; - \frac{5}{9\,b} \, . \qquad (24)
$$

Analogous calculations using (17) reveal that V's expected profit when it undertakes the anticompetitive action is:

$$
\overline{\pi}_v \; = \; B_0 + B_1\,w + B_2\,w^2 \qquad (25)
$$

where:

$$
\begin{aligned}
B_0 \; \equiv \; & \frac{\overline{\alpha}}{9\,b}\,[\, a + c_H - 2\,u - 2\,c_v \,]^2 + \frac{1 - \overline{\alpha}}{9\,b}\,[\, a + c_L - 2\,u - 2\,c_v \,]^2 \\
& - \; \frac{u}{3\,b}\,[\, a + u + c_v - 2\,\overline{c} \,] - \; F_u - F_d - r \; D_R - \overline{d} \; D_C ; \qquad (26)
\end{aligned}
$$

$$B_1 \;\equiv\; \frac{1}{9\,b}\,[\,5\,a + 5\,u - c_v - 4\,\overline{c}\,]\,; \;\; \text{and} \;\; B_2 \;\equiv\; -\frac{5}{9\,b}\,. \tag{27}$$

From (22), (24), (25), and (27):

$$\underline{\pi}_v - \overline{\pi}_v \;=\; A_0 - B_0 + [\,A_1 - B_1\,]\,w + [\,A_2 - B_2\,]\,w^2 = \; A_0 - B_0 + [\,A_1 - B_1\,]\,w\,. \tag{28}$$

From (23) and (26):

$$\begin{aligned}
A_0 - B_0 \;=\; & \frac{\underline{\alpha}}{9\,b}\,[\,a + c_H - 2\,u - 2\,c_v\,]^2 + \frac{1-\underline{\alpha}}{9\,b}\,[\,a + c_L - 2\,u - 2\,c_v\,]^2 \\
& - \frac{\overline{\alpha}}{9\,b}\,[\,a + c_H - 2\,u - 2\,c_v\,]^2 - \frac{1-\overline{\alpha}}{9\,b}\,[\,a + c_L - 2\,u - 2\,c_v\,]^2 \\
& + \frac{u}{3\,b}\,[\,a + u + c_v - 2\,\overline{c}\,] - \frac{u}{3\,b}\,[\,a + u + c_v - 2\,\underline{c}\,] \\
& + \; r\;D_R + \overline{d}\;D_C - [\,1 - r\,]\,D_R - \underline{d}\;D_C \\
=\; & -[\,\overline{\alpha} - \underline{\alpha}\,]\,\frac{1}{9\,b}\,[\,a + c_H - 2\,u - 2\,c_v\,]^2 + [\,\overline{\alpha} - \underline{\alpha}\,]\;\frac{1}{9\,b}\,[\,a + c_L - 2\,u - 2\,c_v\,]^2 \\
& - \frac{2\,u}{3\,b}\,[\,\overline{c}\; - \;\underline{c}\,] + \;[\,2\,r - 1\,]\,D_R + \left[\;\overline{d} - \underline{d}\,\right]\,D_C \\
=\; & [\,\overline{\alpha} - \underline{\alpha}\,]\,\frac{1}{9\,b}\,\left[\,(\,a + c_L - 2\,u - 2\,c_v\,)^2 - (\,a + c_H - 2\,u - 2\,c_v\,)^2\,\right] \\
& - \frac{2\,u}{3\,b}\,[\;\overline{\alpha}\,c_H + (\,1 - \overline{\alpha}\,)\,c_L \; - \; \underline{\alpha}\,c_H + (\,1 - \underline{\alpha}\,)\,c_L\,] \\
& + [\,2\,r - 1\,]\,D_R + \left[\;\overline{d} - \underline{d}\,\right]\,D_C \\
=\; & [\,\overline{\alpha} - \underline{\alpha}\,]\,\frac{1}{9\,b}\,\{\,[\,a - 2\,u - 2\,c_v\,]^2 + 2\,c_L\,[\,a - 2\,u - 2\,c_v\,] + (c_L)^2 \\
& \qquad - [\,a - 2\,u - 2\,c_v\,]^2 - 2\,c_H\,[\,a - 2\,u - 2\,c_v\,] - (c_H)^2\,\} \\
& - \frac{2\,u}{3\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,] + \;[\,2\,r - 1\,]\,D_R + \left[\;\overline{d} - \underline{d}\,\right]\,D_C \\
=\; & -[\,\overline{\alpha} - \underline{\alpha}\,]\,\frac{1}{9\,b}\,\{\,2\,[\,c_H - c_L\,]\,[\,a - 2\,u - 2\,c_v\,] + (c_H)^2 - (c_L)^2\,\} \\
& - \frac{6\,u}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,] + \;[\,2\,r - 1\,]\,D_R + \left[\;\overline{d} - \underline{d}\,\right]\,D_C \\
=\; & -[\,\overline{\alpha} - \underline{\alpha}\,]\,\frac{1}{9\,b}\,\{\,2\,[\,c_H - c_L\,]\,[\,a - 2\,u - 2\,c_v\,] + [\,c_H - c_L\,]\,[\,c_H + c_L\,]\,\}
\end{aligned}$$

$$- \frac{6\,u}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,] + [\,2\,r - 1\,]\,D_R + \left[\,\overline{d} - \underline{d}\,\right] D_C$$

$$= \; -\,[\,\overline{\alpha} - \underline{\alpha}\,]\,\frac{1}{9\,b}\,[\,c_H - c_L\,]\,[\,2\,a + 2\,u + c_L + c_H - 4\,c_v\,]$$

$$+ \; [\,2\,r - 1\,]\,D_R + \left[\,\overline{d} - \underline{d}\,\right] D_C \,. \tag{29}$$

From (24) and (27):

$$\begin{aligned} A_1 - B_1 \;&=\; \frac{1}{9\,b}\,[\,5\,a + 5\,u - c_v - 4\,\underline{c}\,] - \frac{1}{9\,b}\,[\,5\,a + 5\,u - c_v - 4\,\overline{c}\,] \\ &=\; \frac{4}{9\,b}\,[\,\overline{c} - \underline{c}\,] \;=\; \frac{4}{9\,b}\,[\,\overline{\alpha}\;c_H + (\,1 - \overline{\alpha}\,)\,c_L - \underline{\alpha}\;c_H - (\,1 - \underline{\alpha}\,)\,c_L] \\ &=\; \frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\,. \end{aligned} \tag{30}$$

(28), (29), and (30) provide:

$$\begin{aligned} \underline{\pi}_v - \overline{\pi}_v \;&=\; A_0 - B_0 + [\,A_1 - B_1\,]\,w \\ &=\; -\,\frac{1}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\,[\,2\,a + 2\,u + c_L + c_H - 4\,c_v - 4\,w\,] \\ &\qquad + \; [\,2\,r - 1\,]\,D_R + \left[\,\overline{d} - \underline{d}\,\right] D_C \,. \end{aligned} \tag{31}$$

The conclusion follows from (18), (20), and (31). ■

(1) is the profitability constraint (PC) and (2) is the behavioral constraint (BC). We now identify conditions under which the PC and the BC both bind at the solution to [RP] and the solution is unique. To do so, it is useful to fix D_R at an exogenous level, and define the BC curve as the set of (w, r) for which (2) holds as an equality, given the specified D_R. The BC is not satisfied for (w, r) points to the left of the BC curve in (w, r) space, which is defined for $w \geq 0$ and $r \in \left[\frac{1}{2}, 1\right]$. The BC is satisfied, but does not bind, for points to the right of the BC curve in (w, r) space.

To further characterize the PC, define for a fixed D_R:

$$g\,(w, r) \;\; = \;\; \frac{w}{3\,b}\,[\,a + 3\,u + c_v - 2\,\underline{c}\,] - \frac{2\,w^2}{3\,b} - \phi\,. \tag{32}$$

(4) and (32) imply that $g\,(0, r) < 0$, $g\,(\infty, r) < 0$, and $g\,(w, r)$ is a concave function of w, given r. Because $g(w, r)$ is quadratic in w, given r, the equation $g\,(w, r) = 0$ has two real solutions if the PC holds, which are given by:

$$\widetilde{w}_1 \;=\; \frac{1}{4}\left[\,a + 3\,u + c_v - 2\,\underline{c} - \sqrt{\,[\,a + 3\,u + c_v - 2\,\underline{c}\,]^2 - 24\,b\,\phi}\;\right]\,, \text{ and} \tag{33}$$

$$\widetilde{w}_2 \;=\; \frac{1}{4}\left[a + 3\,u + c_v - 2\,\underline{c} + \sqrt{\,[\,a + 3\,u + c_v - 2\,\underline{c}\,]^2 - 24\,b\,\phi\,}\;\right] . \tag{34}$$

Define the $PC1$ curve as the set of (w, r) for which (33) holds, and define the $PC2$ curve as the set of (w, r) for which (34) holds. Because $\widetilde{w}_2 > \widetilde{w}_1$, the $PC2$ curve lies to the right of the $PC1$ curve in (w, r) space. The set of (w, r) that satisfy the profitability constraint consists of the values of (w, r) bounded to the left in (w, r) space by the $PC1$ curve and to the right by the $PC2$ curve.

From (4), (33), and (34), the slopes of the PC1 and PC2 curves in (w, r) space are:

$$\frac{\partial r}{\partial \widetilde{w}_1} \;\overset{s}{=}\; -\,\frac{1}{3\,b\,D_R}\sqrt{\,(\,a + 3\,u + c_v - 2\,\underline{c}\,)^2 - 24\,b\,\phi\,} \;<\; 0\,; \;\text{ and} \tag{35}$$

$$\frac{\partial r}{\partial \widetilde{w}_2} \;\overset{s}{=}\; \frac{1}{3\,b\,D_R}\sqrt{\,(\,a + 3\,u + c_v - 2\,\underline{c}\,)^2 - 24\,b\,\phi\,} \;>\; 0\,. \tag{36}$$

From (2), the slope of the BC in (w, r) space is:

$$\frac{dr}{dw} \;=\; -\,\frac{2\,[\,\overline{\alpha} \,-\, \underline{\alpha}\,]\,[\,c_H \,-\, c_L\,]}{9\,b\,D_R} \;<\; 0\,. \tag{37}$$

From (3), the slope of an iso-W curve in (w, r) space is:

$$\frac{dr}{dw} \;=\; -\,\frac{2\,a \,-\, w \,-\, u \,-\, \underline{c} \,-\, c_v}{18\,b\,k\,\left[\,r \,-\, \frac{1}{2}\,\right] \,+ 9\,b\,D_R\,[\,1 \,-\, f_R\,]}\,. \tag{38}$$

Let w^*, r^*, and W^*, respectively, denote the values of w, r, and W at the solution to [RP]. Also let $w^*(D_R)$ and $r^*(D_R)$, respectively, denote the values of w and r at the solution to [RP] when the optimal regulatory penalty is $D_R \in [\,0, \overline{D}_R\,]$.

Before proceeding, we restate Assumptions 1 and A1 from the text, along with Assumptions 2 and 3. The latter two assumptions refer to $w\,(\widehat{r})|_j$, which is the value of w on the j curve when $r = \widehat{r}$, for $j \in \{PC1,\; PC2,\; BC\}$.

Assumption A1. $a \,>\, \max\{\, a_1,\, a_2,\, a_3,\, a_4,\, a_5,\, 7\,u + 2\,c_v,\, 2\,[\,u + c_v\,] - c_L\,\}$,

where:

$$a_1 \;\equiv\; \frac{1}{2}\,[\,u + c_v + \underline{c}\,] + \frac{4\,\underline{d}\left[\,f_C + \frac{k}{D_R} - f_R\,\right][\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]}{\overline{d} + \underline{d} - 3\left[\,\frac{k}{D_R} + 1 - f_R\,\right][\,\overline{d} - \underline{d}\,] - 6\,\underline{d}\,[\,1 - f_C\,]}\,;$$

$$a_2 \;\equiv\; u + \frac{5\,c_v}{7} + \frac{2\,\underline{c}}{7} + \frac{16}{7}\left[\,\frac{k}{\overline{D}_R} + f_C \,-\, f_R\,\right][\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\,;$$

$$a_3 \;\equiv\; \frac{1}{2\,f_C - f_R - \frac{1}{3}}\;\left\{\,3\left[\,f_R - 2\,f_C + \frac{10}{9}\,\right]u + \left[\,f_R - 2\,f_C + \frac{4}{3}\,\right]c_v\right.$$

$$- 2 \left[f_R - 2 f_C + \frac{5}{6} \right] \underline{c} - \frac{4}{3} [\overline{\alpha} - \underline{\alpha}] [c_H - c_L] \left[f_R - f_C - \frac{k}{\overline{D}_R} \right] \Bigg\} ;$$

$$a_4 \equiv \frac{5u}{3} + c_v + \frac{2}{3} [k + \overline{D}_R (1 - f_R)] \sqrt{\frac{9b}{2k}} ;$$

$$a_5 \equiv u - c_v + 2 \underline{c} + \sqrt{24 b \left[\frac{\overline{D}_R}{2} + \underline{d}\, D_C + F_u \right] + \frac{16}{9} [\overline{\alpha} - \underline{\alpha}]^2 [c_H - c_L]^2} \; .$$

Assumption 1.

$$[a - u + c_v - 2 \underline{c}]^2 > \left[\frac{2a - 2u - \underline{c} - c_v}{3(1 - f_R)} \right]^2 + 24 b \left[\frac{\overline{D}_R}{2} + \underline{d}\, D_C + F_u \right] .$$

Assumption 2. $w\left(\frac{1}{2}\right)\big|_{PC1} < w\left(\frac{1}{2}\right)\big|_{BC}$ and $w(1)|_{PC1} > w(1)|_{BC}$ when $D_R = \overline{D}_R$.

Assumption 3. $w\left(\frac{1}{2}\right)\big|_{PC2} > w\left(\frac{1}{2}\right)\big|_{BC}$ when $D_R = \overline{D}_R$.

It can be shown that Assumption 1 ensures:[2]

$$\frac{a + 3u + c_v - 2\underline{c} - 4 w^*(\overline{D}_R)}{3 b \overline{D}_R} > \frac{\frac{1}{9b} \left[2a - w^*(\overline{D}_R) - u - \underline{c} - c_v \right]}{2k \left[r^*(\overline{D}_R) - \frac{1}{2} \right] + \overline{D}_R [1 - f_R]}$$

$$> \frac{2}{9b} \frac{[\overline{\alpha} - \underline{\alpha}] [c_H - c_L]}{\overline{D}_R} . \qquad (39)$$

Therefore, from (35), (37), and (38), Assumption 1 ensures that when $D_R = \overline{D}_R$, the slope of the $PC1$ curve at $(w^*(\overline{D}_R), r^*(\overline{D}_R))$ in (w, r) space exceeds the slope of the iso-W curves, which in turn exceeds the slope of the BC curve. Assumptions 2 and 3 state that if $D_R = \overline{D}_R$, then: (i) when $r = \frac{1}{2}$, the value of w on the BC curve exceeds the value of w on the $PC1$ curve, and the value of w on the $PC2$ curve exceeds the value of w on the BC curve; and (ii) when $r = 1$, the value of w on the $PC1$ curve exceeds the value of w on the BC curve.

Conclusion 2. *If Assumptions 1 – 3 hold, then the solution to [RP] is unique. At this solution,* $D_R = \overline{D}_R$ *,* $r \in (\frac{1}{2}, 1)$ *, and the PC and the BC both bind.*

<u>Proof</u>. The proof proceeds by first characterizing the welfare-maximizing values of w and r for a fixed $D_R \in [0, \overline{D}_R]$. Let [RP-$D_R$] denote problem [RP] where $D_R \in [0, \overline{D}_R]$ is specified exogenously. The proof consists of the following fourteen Claims.

[2] See Conclusion 4 below.

Claim 1. *The BC curve ($h(w,r) = 0$) is quasi-concave (so the set of (w,r) for which (2) holds is convex).*

Proof. From (2), the equation of the BC curve is:

$$h\,(w,r) \;\equiv\; -\,\frac{1}{9\,b}\,[\,\overline{\alpha}-\underline{\alpha}\,]\,[\,c_H-c_L\,]\,[\,2\,a+2\,u+c_L+c_H-4\,c_v-4\,w\,] \\ +\,[\,2\,r-1\,]\,D_R+\left[\,\overline{d}-\underline{d}\,\right]\,D_C \;=\; 0\,.$$

It is apparent that $h\,(w,r)$ is linear in w and r. Hence, it is quasi-concave in w and r. ■

Claim 2. *The PC curve ($g(w,r) = 0$) is quasi-concave (so the set of (w,r) for which (1) holds is convex).*

Proof. From (1), the equation of the PC curve is:

$$g\,(w,r) \;\equiv\; \frac{w}{3\,b}\,[\,a+3\,u+c_v-2\,\underline{c}\,]-\frac{2\,w^2}{3\,b}-\phi \;=\; 0\,. \tag{40}$$

Differentiating (40), letting subscripts denote partial derivatives, provides:

$$g_w \;=\; \frac{1}{3\,b}\,[\,a+3\,u+c_v-2\,\underline{c}\,]-\frac{4\,w}{3\,b}\,;\;\; g_{ww} \;=\; -\,\frac{4}{3\,b}\,;$$

$$g_r \;=\; D_R;\;\; \text{and}\;\; g_{rr} \;=\; g_{wr} \;=0\,. \tag{41}$$

(41) implies that $g(\cdot)$ is quasi-concave if:[3]

$$\begin{vmatrix} 0 & g_w & g_r \\ g_w & g_{ww} & g_{wr} \\ g_r & g_{rw} & g_{rr} \end{vmatrix} \;\geq\; 0 \;\;\Leftrightarrow\;\; 2\,g_w\,g_r\,g_{wr}-g_{ww}\,(g_r)^2-g_{rr}\,(g_w)^2 \;\geq\; 0$$

$$\Leftrightarrow\;\; -\,g_{ww}\,(g_r)^2 \;\geq\; 0 \;\;\Leftrightarrow\;\; \frac{4}{3\,b}\,(D_R)^2 \;\geq\; 0\,. \tag{42}$$

(42) implies that $g\,(w,r)$ is quasi-concave. ■

Claim 3. *The iso-W curves are strictly quasi-concave for all relevant values of w, r and $D_R \in [0,\,\overline{D}_R]$.*

Proof. From (3), the equation of an iso-W curve is:

$$W\,(w,r) \;\equiv\; \frac{\alpha}{18\,b}\,[\,2\,a-w-u-c_H-c_v\,]^2+\left[\frac{1-\alpha}{18\,b}\right][\,2\,a-w-u-c_L-c_v\,]^2 \\ -\,k\left[r-\frac{1}{2}\right]^2+[\,1-r\,]\,D_R\,[\,1-f_R\,]+[\,1-f_C\,]\,\underline{d}\;D_C \;=\; \overline{W}. \tag{43}$$

[3] Chiang and Wainwright (2005, pp. 368-370).

Differentiating (43), letting subscripts denote partial derivatives, provides:

$$\begin{aligned} W_w &= -\frac{\alpha}{9\,b}\,[\,2\,a - w - u - c_H - c_v\,] - \left[\frac{1-\alpha}{9\,b}\right][\,2\,a - w - u - c_L - c_v\,] \\ &= -\frac{1}{9\,b}\,[\,2\,a - w - u - \underline{c} - c_v\,] \quad \Rightarrow \quad W_{ww} = \frac{1}{9\,b}\,; \\ W_r &= -2\,k\left[r - \frac{1}{2}\right] - D_R\,[\,1 - f_R\,] \quad \Rightarrow \quad W_{rr} = -2\,k \ \text{ and } \ W_{wr} = 0\,. \end{aligned} \tag{44}$$

As in (42), $W(\cdot)$ is strictly quasi-concave if:

$$2\,W_w\,W_r\,W_{wr} - W_{ww}\,(W_r)^2 - W_{rr}\,(W_w)^2 \ > \ 0. \tag{45}$$

(44) and (45) imply that $W(\cdot)$ is strictly quasi-concave if:

$$\begin{aligned} & -\frac{1}{9\,b}\left[2\,k\left(r - \frac{1}{2}\right) + D_R\,(1 - f_R)\right]^2 + 2\,k\left[\frac{1}{9\,b}\,(\,2\,a - w - u - \underline{c} - c_v\,)\right]^2 \ > \ 0 \\ \Leftrightarrow \quad & 2\,k\left[\frac{1}{9\,b}\,(\,2\,a - w - u - \underline{c} - c_v\,)\right]^2 \ > \ \frac{1}{9\,b}\left[2\,k\left(r - \frac{1}{2}\right) + D_R\,(1 - f_R)\right]^2 \\ \Leftrightarrow \quad & [\,2\,a - w - u - \underline{c} - c_v\,]^2 \ > \ \frac{9\,b}{2\,k}\left[2\,k\left(r - \frac{1}{2}\right) + D_R\,(1 - f_R)\right]^2 \\ \Leftrightarrow \quad & \frac{2\,a - w - u - \underline{c} - c_v}{2\,k\,[\,r - \frac{1}{2}\,] + D_R\,[\,1 - f_R\,]} \ > \ \sqrt{\frac{9\,b}{2\,k}}\,. \end{aligned} \tag{46}$$

(34) implies:

$$w \ \leq \ \widetilde{w}_2 \ \text{ if } \ w \ \leq \ \widehat{w}_2 \ \equiv \ \frac{1}{2}\,[\,a + 3\,u + c_v - 2\,\underline{c}\,]\,. \tag{47}$$

Therefore, since $D_R \leq \overline{D}_R$, it must be the case that for all $w \leq \widehat{w}_2$:

$$\begin{aligned} & \frac{2\,a - w - u - \underline{c} - c_v}{2\,k\,[\,r - \frac{1}{2}\,] + D_R\,[\,1 - f_R\,]} \ \geq \ \frac{2\,a - w - u - \underline{c} - c_v}{2\,k\,[\,1 - \frac{1}{2}\,] + \overline{D}_R\,[\,1 - f_R\,]} \ = \ \frac{2\,a - w - u - \underline{c} - c_v}{k + \overline{D}_R\,[\,1 - f_R\,]} \\ & \geq \ \frac{2\,a - \frac{1}{2}\,[\,a + 3\,u + c_v - 2\,\underline{c}\,] - u - \underline{c} - c_v}{k + \overline{D}_R\,[\,1 - f_R\,]} \ = \ \frac{3\,a - 5\,u - 3\,c_v}{2\,[\,k + \overline{D}_R\,(\,1 - f_R\,)\,]}\,. \end{aligned} \tag{48}$$

(48) implies that (46) holds when, as is assumed to be the case, a is sufficiently large.[4] ■

Claim 4. *For any $D_R \in [0, \overline{D}_R]$, there is a unique (w, r) that solves [RP-D_R].*

[4] Specifically, (46) holds when Assumption A1 holds.

Proof. The objective function in [RP-D_R] is strictly quasi-concave and the constraint set is convex. Therefore, the problem has a unique solution. ■

Claim 5. *Suppose Assumption 2 holds. Then when $D_R = \overline{D}_R$, the $PC1$ curve and the BC curve intersect exactly once. $r \in (\frac{1}{2}, 1)$ at the point of intersection.*

Proof. (35) and (37) imply that the $PC1$ curve and the BC curve both have negative slopes. Therefore, Assumption 2 ensures that when $D_R = \overline{D}_R$, the two curves intersect at least once and they do not intersect where $r = \frac{1}{2}$ or where $r = 1$. (35) implies that $\frac{\partial}{\partial r}(\frac{\partial r}{\partial \widetilde{w}_1}) < 0$. Therefore, the $PC1$ curve is convex to the origin in (w, r) space. (37) implies that the BC curve is linear. Therefore, the two curves intersect exactly once at a point where $r \in (\frac{1}{2}, 1)$. ■

Claim 6. *Suppose Assumption 3 holds. Then for any $D_R \in \left[0, \overline{D}_R\right]$, the solution to [RP-$D_R$] does not lie on the $PC2$ curve.*

Proof. $\frac{\partial}{\partial D_R}\left\{ w\left(\frac{1}{2}\right)\Big|_{PC2}\right\} < 0$ from (34). Therefore, $w\left(\frac{1}{2}\right)\Big|_{PC2}$ increases as D_R decreases. Also, from (2), $w\left(\frac{1}{2}\right)\Big|_{BC}$ does not change as D_R changes. Consequently, $w\left(\frac{1}{2}\right)\Big|_{PC2} > w\left(\frac{1}{2}\right)\Big|_{BC}$ for all $D_R \in \left[0, \overline{D}_R\right]$ if Assumption 3 holds.

(36) implies that PC2 has a positive slope in (w, r) space for all $D_R \in \left[0, \overline{D}_R\right]$. (37) implies that the BC curve has a negative slope. Therefore, when Assumption 3 holds, $w(r)|_{PC2} > w(r)|_{BC}$ for all $r \in \left[\frac{1}{2}, 1\right]$, so the $PC2$ curve lies strictly to the right of the BC curve in (w, r) space for all $D_R \in \left[0, \overline{D}_R\right]$.

Suppose (w^*, r^*), a candidate solution to [RP-D_R], lies on the $PC2$ curve. Because the $PC2$ curve lies the right of the BC curve in (w, r) space, the BC does not bind. Therefore, there exist values of $w \in [0, w^*)$ for which (w, r^*) satisfy both the PC and the BC. (3) implies that W is higher at all such values of (w, r^*) than at (w^*, r^*). Consequently, (w^*, r^*) is not a solution to [RP-D_R]. ■

Let $\widehat{w}(D_R)$ and $\widehat{r}(D_R)$ denote the values of w and r that solve the following two equations, given D_R:[5]

$$\begin{aligned} \widehat{w}(D_R) \;=\; & \frac{1}{4}\left[a + 3u + c_v - 2\underline{c}\right] - \frac{1}{4}\Big\{\left[a + 3u + c_v - 2\underline{c}\right]^2 \\ & - 24b\left[\frac{u}{3b}(a + u + c_v - 2\underline{c}) + (1 - \widehat{r}(D_R))D_R + \underline{d}\,D_C + F_u\right]\Big\}^{\frac{1}{2}}. \end{aligned} \qquad (49)$$

$$\begin{aligned} & -\frac{1}{9b}\left[\overline{\alpha} - \underline{\alpha}\right]\left[c_H - c_L\right]\left[2a + 2u + c_L + c_H - 4c_v - 4\widehat{w}(D_R)\right] \\ & \qquad = \left[2\widehat{r}(D_R) - 1\right]D_R + \left[\overline{d} - \underline{d}\right]D_C. \end{aligned} \qquad (50)$$

[5]Observe that (49) reflects (33), and (50) reflects (2).

Claim 7. *Suppose Assumptions 1 – 3 hold. Then* $\widehat{w}(\overline{D}_R)$ *and* $\widehat{r}(\overline{D}_R)$ *solve [RP-*$\overline{D}_R$*]. In addition,* $\widehat{r}(\overline{D}_R) \in \left(\frac{1}{2}, 1\right)$*, the profitability constraint binds, and the behavioral constraint binds at the solution to [RP-*$\overline{D}_R$*].*

Proof. Claim 5 implies that when $D_R = \overline{D}_R$, the $PC1$ curve and the BC curve intersect exactly once at $(\widehat{w}(\overline{D}_R), \widehat{r}(\overline{D}_R))$, where $\widehat{r}(\overline{D}_R) \in \left(\frac{1}{2}, 1\right)$. Claim 6 implies that the feasible solutions to [RP-$\overline{D}_R$] consist of the (w, r) pairs that lie to the right of the $PC1$ curve for $r \in [\,\widehat{r}(\overline{D}_R), 1\,]$ and to the right of the BC curve for $r \in [\,\frac{1}{2}, \widehat{r}(\overline{D}_R)\,]$ in (w, r) space. Assumption 1 ensures that the iso-W curves are more steeply sloped than the BC curve and less steeply sloped than the $PC1$ curve in the neighborhood of $(\widehat{w}(\overline{D}_R), \widehat{r}(\overline{D}_R))$. Furthermore, iso-$W$ curves further to the left in (w, r) space correspond to higher levels of W. Therefore, the highest feasible level of W in the neighborhood of $(\widehat{w}(\overline{D}_R), \widehat{r}(\overline{D}_R))$ is uniquely achieved at $(\widehat{w}(\overline{D}_R), \widehat{r}(\overline{D}_R))$. Furthermore, the constraint set is convex and Claim 4 implies that the objective function in [RP-$\overline{D}_R$] is strictly quasi-concave. Therefore, $(\widehat{w}(\overline{D}_R), \widehat{r}(\overline{D}_R))$ is the unique solution to [RP-$\overline{D}_R$]. ■

Claim 8. $w\left(\frac{1}{2}\right)\big|_{PC1}$ *increases as* D_R *increases, whereas* $w\left(\frac{1}{2}\right)\big|_{BC}$ *is independent of* D_R*.*

Proof. From (4) and (33):

$$\frac{\partial}{\partial D_R}\left\{ w(\frac{1}{2})\bigg|_{PC1}\right\} \;=\; -\,\frac{1}{8}\left[\,(\,a + 3\,u + c_v - 2\,\underline{c}\,)^2 - 24\,b\,\phi\,\right]^{-\frac{3}{2}}[\,-\,24\,b\,]\,\frac{1}{2} \;>\; 0\,.$$

From (2):
$$\frac{\partial}{\partial D_R}\left\{ w(\frac{1}{2})\bigg|_{BC}\right\} \;=\; [\,2\,r - 1\,]\,\big|_{r=\frac{1}{2}} \;=\; 0\,. \quad ■$$

Claim 9. *The* $PC1$ *curve and the* BC *curve are vertical straight lines when* $D_R = 0$*.*

Proof. Follows immediately from (1), (2), (4) and (33). ■

Claim 10. *Suppose Assumptions 1 – 3 hold. Then there exists a* $\widetilde{D}_R \in \left(0, \overline{D}_R\right)$ *such that: (i) the* $PC1$ *curve lies everywhere to the left of the* BC *curve in* (w, r) *space if* $D_R \in [\,0, \widetilde{D}_R\,)$*; (ii) the two curves intersect exactly once at* $r = 1$ *if* $D_R = \widetilde{D}_R$*; and (iii) the two curves intersect for some* $w > 0$ *and* $r \in (\,\frac{1}{2}, 1\,)$ *if* $D_R \in (\,\widetilde{D}_R, \overline{D}_R\,]$*.*

Proof. From Assumption 2 and Claim 8: (i) $w\left(\frac{1}{2}\right)\big|_{PC1} < w\left(\frac{1}{2}\right)\big|_{BC}$ at $D_R = \overline{D}_R$; (ii) $w\left(\frac{1}{2}\right)\big|_{PC1}$ declines as D_R declines; and (iii) $w\left(\frac{1}{2}\right)\big|_{BC}$ is independent of D_R. Therefore, $w\left(\frac{1}{2}\right)\big|_{PC1} < w\left(\frac{1}{2}\right)\big|_{BC}$ for all $D_R \in \left[\,0, \overline{D}_R\,\right]$. Furthermore, as demonstrated in the proof of Claim 5, for a fixed D_R: (i) the $PC1$ curve is convex to the origin in (w, r) space; (ii) the BC curve is a straight line; and (iii) both curves have a negative slope. Therefore, the curves intersect at most once.

Claim 9 implies that when $D_R = 0$, the $PC1$ curve and the BC curve are vertical lines in (w, r) space. Furthermore, because $w\left(\frac{1}{2}\right)\big|_{PC1} < w\left(\frac{1}{2}\right)\big|_{BC}$ for all $D_R \in \left[0, \overline{D}_R\right]$, the $PC1$ curve lies to the left of the BC curve. Consequently, the two curves do not intersect.

Claim 5 implies that the $PC1$ curve and the BC curve intersect when $D_R = \overline{D}_R$.

From (2) and (33), $w\left(\frac{1}{2}\right)\big|_{PC1}$ increases and $w\left(\frac{1}{2}\right)\big|_{BC}$ does not change as D_R increases from 0 to $\overline{D}_R$. In addition, from (35) and (37), the $PC1$ curve and the BC curve both become flatter in (w, r) space as D_R increases. The claim then follows from the established fact that for a fixed D_R: (i) the $PC1$ curve is convex to the origin in (w, r) space; (ii) the BC curve is a straight line; and (iii) both curves have a negative slope. ■

Claim 11. *Suppose Assumptions 1 – 3 hold. Then it is not the case that only BC binds at a solution to [RP].*

<u>Proof</u>. If $D_R = \overline{D}_R$, then the PC and the BC both bind at a solution to [RP], from Claim 7. Consider a candidate solution to [RP] at which $D_R < \overline{D}_R$ and the BC is the only binding constraint. Suppose D_R is increased by an arbitrarily small amount. This increase in D_R is feasible because $D_R < \overline{D}_R$. Following this increase: (i) the PC continues to hold because the constraint is not binding; (ii) the BC continues to hold because the expression to the left of the inequality in (2) is increasing in D_R; and (iii) W increases because it is increasing in D_R, from (3). Therefore, the candidate solution cannot be a solution to [RP]. ■

Claim 12. *Suppose Assumptions 1 – 3 hold. Then it is not the case that only the PC binds at a solution to [RP].*

<u>Proof</u>. First suppose $D_R = 0$ at a solution to [RP]. Then from Claim 9, the $PC1$ curve and the BC curve are both vertical lines in (w, r) space. Furthermore, Assumption 2 implies that the $PC1$ curve lies everywhere to the left of the BC curve in (w, r) space. Therefore, Claim 6 implies that if the PC binds at a solution to [RP], then the BC is violated. Consequently, it cannot be the case that only the PC binds when $D_R = 0$ at a solution to [RP].

Now suppose $r = \frac{1}{2}$ and only the PC binds at a solution to [RP]. $w\left(\frac{1}{2}\right)\big|_{PC} < w\left(\frac{1}{2}\right)\big|_{BC}$ for all $D_R \in \left[0, \overline{D}_R\right]$ from Claim 8 and Assumption 2. Therefore, if the PC binds at a solution to [RP], then the BC is violated. Consequently, it cannot be the case that $r = \frac{1}{2}$ and only the PC binds at a solution to [RP].

Now suppose $r > \frac{1}{2}$, $D_R > 0$, and only the PC binds at a solution to [RP]. Since $r > \frac{1}{2}$ and $D_R > 0$, it is possible to find $\underline{r} \in (\frac{1}{2}, r)$ and $\underline{D}_R \in (0, D_R)$ such that $[\,1 - r\,]\, D_R = [\,1 - \underline{r}\,]\, \underline{D}_R$. Observe from (1) and (4) that the PC continues to bind at $(w, \underline{r}, \underline{D}_R)$. Also, if $D_R - \underline{D}_R$ is sufficiently small, then the inequality in (2) will continue to hold because, by assumption, it holds strictly when the regulatory penalty is D_R. W is higher at $(w, \underline{r}, \underline{D}_R)$ than at (w, r, D_R) because, from (3), $\frac{\partial W}{\partial r}\big|_{[\,1-r\,]\, D_R = \text{constant}} = -\,2\,k\left[\,r - \frac{1}{2}\,\right] < 0$. Therefore, it cannot be the case that $D_R = 0$, $r = \frac{1}{2}$, and only the PC binds at a solution to [RP]. ■

Claim 13. *Suppose Assumptions 1 – 3 hold. Then at a solution to [RP]: (i)* $D_R \in [\,\widetilde{D}_R, \overline{D}_R\,]$*; and (ii)* $w^* = \widehat{w}(D_R)$ *and* $r^* = \widehat{r}(D_R)$*, as specified in (49) and (50).*

Proof. From Claim 10, the $PC1$ curve lies everywhere to the left of the BC curve in (w, r) space if $D_R \in [\,0, \widetilde{D}_R\,)$. Therefore, because the PC and the BC both bind at the solution to [RP] from Claims 11 and 12, it must be the case that $D_R \in [\,\widetilde{D}_R, \overline{D}_R\,]$.

The remainder of the proof follows from (2), (4), and (33) because the PC and the BC both bind at the solution to [RP]. ■

Claim 14. *Suppose Assumptions 1 – 3 hold. Then* $\frac{dW^*}{dD_R} > 0$ *for all* $D_R \in (\,\widetilde{D}_R, \overline{D}_R)$*.*

Proof. From (1), differentiating $g(w, r) = 0$ with respect to D_R, using (4), provides:

$$D_R \left[\frac{\partial r}{\partial D_R} \right] + \left[\frac{a + 3\,u + c_v - 2\,\underline{c} - 4\,w}{3\,b} \right] \frac{\partial w}{\partial D_R} \; = \; 1 - r\,. \tag{51}$$

From (2), differentiating $h(w, r) = 0$ with respect to D_R provides:

$$\frac{4}{9\,b} [\,\overline{\alpha} - \underline{\alpha}\,] [\,c_H - c_L\,] \frac{\partial w}{\partial D_R} + 2\,r - 1 + 2\,D_R \left[\frac{\partial r}{\partial D_R} \right] \; = \; 0$$

$$\Rightarrow \quad 2\,D_R \left[\frac{\partial r}{\partial D_R} \right] + \frac{4}{9\,b} [\,\overline{\alpha} - \underline{\alpha}\,] [\,c_H - c_L\,] \frac{\partial w}{\partial D_R} \; = \; 1 - 2\,r\,. \tag{52}$$

(51) and (52) can be written as:

$$\Lambda \begin{bmatrix} \frac{\partial r}{\partial D_R} \\ \\ \frac{\partial w}{\partial D_R} \end{bmatrix} = \begin{bmatrix} 1 - r \\ \\ 1 - 2\,r \end{bmatrix} \quad \text{where} \;\; \Lambda \; \equiv \begin{bmatrix} D_R & \frac{a + 3\,u + c_v - 2\,\underline{c} - 4\,w}{3\,b} \\ \\ 2\,D_R & \frac{4}{9\,b} [\,\overline{\alpha} - \underline{\alpha}\,] [\,c_H - c_L\,] \end{bmatrix} \tag{53}$$

$$\Rightarrow \quad |\,\Lambda\,| \; = \; \frac{D_R}{3\,b} \left[\frac{4}{3} (\,\overline{\alpha} - \underline{\alpha}\,) (\,c_H - c_L\,) - 2\,(a \, + \, 3\,u \, + \, c_v \, - \, 2\,\underline{c} \, - \, 4\,w) \right] \; < \; 0. \tag{54}$$

The inequality in (54) holds because:

$$a + 3\,u + c_v - 2\,\underline{c} - 4\,w$$

$$\geq \; \sqrt{[\,a + 3\,u + c_v - 2\,\underline{c}\,]^2 - 24\,b \left[\frac{u}{3\,b} (\,a + u + c_v - 2\,\underline{c}\,) + \frac{\overline{D}_R}{2} + \underline{d}\; D_C + F_u \right]}. \tag{55}$$

(55) follows from (33) and Claim 6, since the PC binds at the solution to [RP]. (55) implies:

$$a + 3\,u + c_v - 2\,\underline{c} - 4\,w - \frac{2}{3} [\,\overline{\alpha} - \underline{\alpha}\,] [\,c_H - c_L\,]$$

$$\geq\; a + 3u + c_v - 2\underline{c} \;-\frac{2}{3}[\overline{\alpha} - \underline{\alpha}][c_H - c_L] - [a + 3u + c_v - 2\underline{c}]$$

$$+\;\sqrt{[a + 3u + c_v - 2\underline{c}]^2 - 24b\left[\frac{u}{3b}(a + u + c_v - 2\underline{c}) + \frac{\overline{D}_R}{2} + \underline{d}\; D_C + F_u\right]}$$

$$=\; -\frac{2}{3}[\overline{\alpha} - \underline{\alpha}][c_H - c_L]$$

$$+\;\sqrt{[a + 3u + c_v - 2\underline{c}]^2 - 24b\left[\frac{u}{3b}(a + u + c_v - 2\underline{c}) + \frac{\overline{D}_R}{2} + \underline{d}\; D_C + F_u\right]}$$

$$>\; -\frac{2}{3}[\overline{\alpha} - \underline{\alpha}][c_H - c_L] + \sqrt{\frac{4}{9}[\overline{\alpha} - \underline{\alpha}]^2[c_H - c_L]^2} \;=\; 0\,. \tag{56}$$

The inequality in (56) holds because

$$[a - u + c_v - 2\underline{c}]^2 - 24b\left[\frac{\overline{D}_R}{2} + \underline{d}\; D_C + F_u\right] \;>\; \frac{4}{9}[\overline{\alpha} - \underline{\alpha}]^2[c_H - c_L]^2 \tag{57}$$

from Assumption A1 and because:

$$[a + 3u + c_v - 2\underline{c}]^2 - 24b\,\frac{u}{3b}[a + u + c_v - 2\underline{c}]$$

$$=\; [a + u + c_v - 2\underline{c}]^2 + 4u[a + u + c_v - 2\underline{c}] + 4u^2 - \;8u[a + u + c_v - 2\underline{c}]$$

$$=\; [a + u + c_v - 2\underline{c}]^2 - 4u[a + u + c_v - 2\underline{c}] + 4u^2$$

$$=\; [a + u + c_v - 2\underline{c} - 2u]^2 \;=\; [a - u + c_v - 2\underline{c}]^2\,. \tag{58}$$

(54) follows from (56).

Because $|\Lambda| < 0$, (53) implies:

$$\frac{\partial w}{\partial D_R} \;=\; \frac{|\Omega_2|}{|\Lambda|} \;\stackrel{s}{=}\; -\,|\Omega_2| \quad \text{where}\;\; \Omega_2 \;\equiv\; \begin{bmatrix} D_R & 1 - r \\ 2\,D_R & 1 - 2r \end{bmatrix} \tag{59}$$

$$\Rightarrow \quad |\Omega_2| \;=\; [1 - 2r]\,D_R - 2[1 - r]\,D_R \;=\; -\,D_R\,. \tag{60}$$

(59) and (60) imply $\frac{\partial w}{\partial D_R} \stackrel{s}{=} -\,|\Omega_2| = D_R > 0$.

(52) implies $\frac{\partial r}{\partial D_R} < 0$, since $\frac{\partial w}{\partial D_R} > 0$ and $r > \frac{1}{2}$.

$$\text{Define}\;\; \alpha_1 \;\equiv\; \frac{2}{3}[\overline{\alpha} - \underline{\alpha}][c_H - c_L] \;\;\text{and}\;\; \alpha_2 \;\equiv\; a + 3u + c_v - 2\underline{c} - 4w\,. \tag{61}$$

(54) and (61) imply:

$$|\Lambda| \;=\; \frac{2\,D_R}{3b}[\alpha_1 - \alpha_2] \;<\; 0 \quad\;\Rightarrow\quad \alpha_2 \;>\; \alpha_1\,. \tag{62}$$

(59), (60), and (62) provide:

$$\frac{\partial w}{\partial D_R} = \frac{|\,\Omega_2\,|}{|\,\Lambda\,|} = \frac{3\,b}{2\,[\,\alpha_2 - \alpha_1\,]} > 0\,. \tag{63}$$

From (53):

$$\frac{\partial r}{\partial D_R} = \frac{|\,\Omega_3\,|}{|\,\Lambda\,|} \quad \text{where} \quad \Omega_3 \equiv \begin{bmatrix} 1 - r & \frac{a + 3\,u + c_v - 2\,\underline{c} - 4\,w}{3\,b} \\ 1 - 2\,r & \frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,] \end{bmatrix} \tag{64}$$

Because $\frac{\partial r}{\partial D_R} < 0$ and $|\,\Lambda\,| < 0$:

$$|\,\Omega_3\,| = [\,1 - r\,]\,\frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]$$
$$- [\,1 - 2\,r\,]\left[\frac{a + 3\,u + c_v - 2\,\underline{c} - 4\,w}{3\,b}\right] > 0. \tag{65}$$

(61) and (65) imply:

$$|\,\Omega_3\,| = \frac{2\,[\,1 - r\,]\,\alpha_1}{3\,b} - \frac{[\,1 - 2\,r\,]\,\alpha_2}{3\,b} > 0 \tag{66}$$

$$\Rightarrow \; 2\,[\,1 - r\,]\,\alpha_1 - [\,1 - 2\,r\,]\,\alpha_2 = 2\,[\,1 - r\,]\,[\,\alpha_1 - \alpha_2\,] + \alpha_2 > 0\,. \tag{67}$$

(63), (64), and (66) provide:

$$\frac{\partial r}{\partial D_R} = \frac{\frac{2\,[1-r]\,\alpha_1}{3\,b} - \frac{[1-2\,r]\,\alpha_2}{3\,b}}{\frac{2\,D_R}{3\,b}\,[\,\alpha_1 - \alpha_2\,]} = \frac{2\,[\,1 - r\,]\,\alpha_1 - [\,1 - 2\,r\,]\,\alpha_2}{2\,D_R\,[\,\alpha_1 - \alpha_2\,]}\,. \tag{68}$$

From (3):

$$\frac{dW^*}{dD_R} = -\frac{\underline{\alpha}}{9\,b}\,[\,2\,a - w - u - c_H - c_v\,]\,\frac{\partial w}{\partial D_R}$$
$$- \left[\frac{1 - \underline{\alpha}}{9\,b}\right][\,2\,a - w - u - c_L - c_v\,]\,\frac{\partial w}{\partial D_R}$$
$$- 2\,k\left[r - \frac{1}{2}\right]\frac{\partial r}{\partial D_R} - D_R\,[\,1 - f_R\,]\,\frac{\partial r}{\partial D_R} + [\,1 - r\,]\,[\,1 - f_R\,] \tag{69}$$

$$= -\frac{1}{9\,b}\,[\,2\,a - w - u - \underline{c} - c_v\,]\,\frac{\partial w}{\partial D_R}$$
$$- [\,k\,(\,2r - 1\,) + D_R\,(\,1 - f_R\,)]\,\frac{\partial r}{\partial D_R} + [\,1 - r\,]\,[\,1 - f_R\,]\,. \tag{70}$$

(62), (68), and (70) provide:

$$\frac{dW^*}{dD_R} = \frac{1}{9\,b}\left[\,2\,a - w - u - \underline{c} - c_v\,\right] \frac{3\,b}{2\left[\,\alpha_1 - \alpha_2\,\right]}$$

$$- \left[\,k\,(\,2\,r - 1\,) + D_R\,(\,1 - f_R\,)\,\right] \frac{2\left[\,1 - r\,\right]\alpha_1 - \left[\,1 - 2\,r\,\right]\alpha_2}{2\,D_R\left[\,\alpha_1 - \alpha_2\,\right]} + \left[\,1 - r\,\right]\left[\,1 - f_R\,\right]$$

$$= \frac{1}{6}\left[\frac{2\,a - w - u - \underline{c} - c_v}{\alpha_1 - \alpha_2}\right] - k\left[\,2\,r - 1\,\right] \frac{2\left[\,1 - r\,\right]\left[\,\alpha_1 - \alpha_2\,\right] + \alpha_2}{2\,D_R\left[\,\alpha_1 - \alpha_2\,\right]}$$

$$- \; D_R\left[\,1 - f_R\,\right] \frac{2\left[\,1 - r\,\right]\left[\,\alpha_1 - \alpha_2\,\right] + \alpha_2}{2\,D_R\left[\,\alpha_1 - \alpha_2\,\right]} + \left[\,1 - r\,\right]\left[\,1 - f_R\,\right]$$

$$= \frac{1}{6}\left[\frac{2\,a - w - u - \underline{c} - c_v}{\alpha_1 - \alpha_2}\right] - k\left[\,2\,r - 1\,\right] \frac{2\left[\,1 - r\,\right]\left[\,\alpha_1 - \alpha_2\,\right] + \alpha_2}{2\,D_R\left[\,\alpha_1 - \alpha_2\,\right]}$$

$$- \left[\,1 - f_R\,\right]\left[\,1 - r\,\right] - \; D_R\left[\,1 - f_R\,\right] \frac{\alpha_2}{2\,D_R\left[\,\alpha_1 - \alpha_2\,\right]} + \left[\,1 - r\,\right]\left[\,1 - f_R\,\right]$$

$$= \frac{1}{6}\left[\frac{2\,a - w - u - \underline{c} - c_v}{\alpha_1 - \alpha_2}\right] - k\left[\,2\,r - 1\,\right] \frac{2\left[\,1 - r\,\right]\left[\,\alpha_1 - \alpha_2\,\right] + \alpha_2}{2\,D_R\left[\,\alpha_1 - \alpha_2\,\right]}$$

$$- \left[\,1 - f_R\,\right] \frac{\alpha_2}{2\left[\,\alpha_1 - \alpha_2\,\right]}$$

$$\Rightarrow \quad 2\left[\,\alpha_1 - \alpha_2\,\right] \frac{dW^*}{dD_R} = \frac{1}{3}\left[\,2\,a - w - u - \underline{c} - c_v\,\right]$$

$$- \frac{k\left[\,2\,r - 1\,\right]}{D_R}\left[\,2\,(\,1 - r\,)\,(\,\alpha_1 - \alpha_2\,) + \alpha_2\,\right] - \left[\,1 - f_R\,\right]\alpha_2\,. \tag{71}$$

Since $\alpha_1 < \alpha_2$, (71) implies:

$$\frac{dW^*}{dD_R} > 0 \;\text{ if }\; \frac{1}{3}\left[\,2\,a - w^* - u - \underline{c} - c_v\,\right] - \left[\,1 - f_R\,\right]\alpha_2^*$$

$$< \frac{k\left[\,2\,r^* - 1\,\right]}{D_R}\left[\,2\,(\,1 - r^*\,)\,(\,\alpha_1 - \alpha_2^*\,) + \alpha_2^*\,\right] \tag{72}$$

where $\alpha_1 \equiv \frac{2}{3}\left[\,\overline{\alpha} - \underline{\alpha}\,\right]\left[\,c_H - c_L\,\right]$ and $\alpha_2^* \equiv a + 3\,u + c_v - 2\,\underline{c} - 4\,w^*$.

Since $2\left[\,1 - r^*\right]\left[\,\alpha_1 - \alpha_2^*\,\right] + \alpha_2^* \geq 0$ from (67), (72) holds if:

$$\frac{1}{3}\left[\,2\,a - w^* - u - \underline{c} - c_v\,\right] - \left[\,1 - f_R\,\right]\alpha_2^* < 0. \tag{73}$$

Observe that (73) holds when:

$$3\left[\,1 - f_R\,\right]\left[\,a + 3\,u + c_v - 2\,\underline{c} - 4\,w^*(\,D_R)\,\right] > 2\,a - w^*(D_R) - u - \underline{c} - c_v\,. \tag{74}$$

To complete the proof, we will show that (74) holds when Assumption 1 holds.

The PC binds at a solution to [RP], from Claims 11 and 12. Therefore, from (33):

$$a + 3\,u + c_v - 2\,\underline{c} - 4\;w^*\,(D_R) \;\;=\;\; \sqrt{[\,a + 3\,u + c_v - 2\,\underline{c}\,]^2 - 24\,b\,\phi}\,.$$

Consequently, (74) holds if and only if:

$$3\,[\,1 - f_R\,]\,\sqrt{[\,a + 3\,u + c_v - 2\,\underline{c}\,]^2 - 24\,b\,\phi} \;>\; 2\,a - w^*(\,D_R) - u - \underline{c} - c_v\,. \tag{75}$$

(4) and (58) imply:

$$[\,a + 3\,u + c_v - 2\,\underline{c}\,]^2 - 24\,b\,[\,\phi - (\,1 - r\,)\,D_R - \underline{d}\;D_C - F_u\,] \;=\; [\,a - \;u + c_v - 2\,\underline{c}]^2$$

$$\begin{aligned}\Rightarrow\quad & [\,a + 3\,u + c_v - 2\,\underline{c}\,]^2 - 24\,b\,\phi \\ & = \; [\,a - \;u + c_v - 2\,\underline{c}]^2 - 24\,b\;[\,(\,1 - r\,)\,D_R + \underline{d}\;D_C + F_u\,] \\ & \geq \; [\,a - \;u + c_v - 2\,\underline{c}]^2 - 24\,b\,\left[\frac{D_R}{2} + \underline{d}\;D_C + F_u\right].\end{aligned} \tag{76}$$

Also, because $w^* > u$ to ensure the PC is satisfied:

$$2\,a - w^*(\,D_R) - u - \underline{c} - c_v \;<\; 2\,a - 2\,u - \underline{c} - c_v\,. \tag{77}$$

(76) and (77) imply that (75) holds if:

$$\begin{aligned} & 3\,[\,1 - f_R\,]\sqrt{[\,a - \;u + c_v - 2\,\underline{c}]^2 - 24\,b\,\left[\frac{D_R}{2} + \underline{d}\;D_C + F_u\right]} \;>\; 2\,a - 2\,u - \underline{c} - c_v \\ \Leftrightarrow\quad & [\,1 - f_R\,]^2\left[(\,a - \;u + c_v - 2\,\underline{c}\,)^2 - 24\,b\,\left(\frac{D_R}{2} + \underline{d}\;D_C + F_u\right)\right] \\ & \qquad > \left[\frac{2\,a - 2\,u - \underline{c} - c_v}{3}\right]^2.\end{aligned} \tag{78}$$

Assumption 1 ensures that the inequality in (78) holds. ■

Conclusion 3. *E will produce strictly positive output in equilibrium if* $c_v > c_H$. *V will produce strictly positive output in equilibrium if Assumption A1 holds (so* $a > 2\left[\,u + c_v - \frac{c_L}{2}\,\right]$*).*

Proof. Because the BC binds at the solution to [RP], (2) implies that when $D_R > 0$ and/or $D_C > 0$:

$$4\,w \;<\; 2\,a + 2\,u + c_L + c_H - 4\,c_v \quad\Rightarrow\quad 2\,w \;<\; a + u + \frac{c_L + c_H}{2} - 2\,c_v\,. \tag{79}$$

Therefore:

$$a+u+c_v-2\,w-2\,c_H \;>\; a+u+c_v-a-u-\frac{c_L+c_H}{2}+2\,c_v-2\,c_H$$

$$= \; 3\,c_v-\frac{c_L+c_H}{2}-2\,c_H \;>\; 3\,[\,c_v-c_H\,] \;>\; 0 \text{ if } c_v > c_H\,.$$

Consequently, (11) implies that $x_e > 0$ if $c_v > c_H$.

Since $w \geq 0$ and $c_L < c_H$, (10) implies that if $a > 2\left[\,u+c_v-\frac{c_L}{2}\,\right]$, then:

$$x_v \;=\; \frac{1}{3\,b}\,[\,a+w+c_i-2\,u-2\,c_v\,] \;>\; \frac{1}{3\,b}\,[\,a+c_L-2\,u-2\,c_v\,] \;>\; 0\,. \quad \blacksquare$$

Proof of Observation 1.

The proof follows directly from the proof of Conclusion 2. ■

Proof of Observation 2.

From (1), differentiating $g(w,r) = 0$ with respect to $\underline{\alpha}$, using (4), provides:

$$\frac{1}{3\,b}\,[\,a+3\,u+c_v-2\,\underline{c}\,]\,\frac{\partial w}{\partial \underline{\alpha}}-\frac{4\,w}{3\,b}\left[\frac{\partial w}{\partial \underline{\alpha}}\right]+\overline{D}_R\left[\frac{\partial r}{\partial \underline{\alpha}}\right]$$

$$-\;\frac{2\,w}{3\,b}\,[\,c_H-c_L\,]+\frac{2\,u}{3\,b}\,[\,c_H-c_L\,] \;=\; 0$$

$$\Rightarrow \quad \overline{D}_R\left[\frac{\partial r}{\partial \underline{\alpha}}\right]+\left[\frac{a+3\,u+c_v-2\,\underline{c}-4\,w}{3\,b}\right]\frac{\partial w}{\partial \underline{\alpha}} \;=\; \frac{2}{3\,b}\,[\,w-u\,]\,[\,c_H-c_L\,]\,. \tag{80}$$

From (2), differentiating $h(w,r) = 0$ with respect to $\underline{\alpha}$ provides:

$$\frac{4}{9\,b}\,[\,\overline{\alpha}-\underline{\alpha}\,]\,[\,c_H-c_L\,]\,\frac{\partial w}{\partial \underline{\alpha}}+2\,\overline{D}_R\left[\frac{\partial r}{\partial \underline{\alpha}}\right]$$

$$+\;\frac{1}{9\,b}\,[\,c_H-c_L\,]\,[\,2\,a+2\,u+c_L+c_H-4\,c_v-4\,w\,] \;=\; 0$$

$$\Rightarrow \quad 2\,\overline{D}_R\left[\frac{\partial r}{\partial \underline{\alpha}}\right]+\frac{4}{9\,b}\,[\,\overline{\alpha}-\underline{\alpha}\,]\,[\,c_H-c_L\,]\,\frac{\partial w}{\partial \underline{\alpha}}$$

$$= \; -\,\frac{1}{9\,b}\,[\,c_H-c_L\,]\,[\,2\,a+2\,u+c_L+c_H-4\,c_v-4\,w\,]\,. \tag{81}$$

(80) and (81) can be written as:

$$[\,\Lambda\,]\begin{bmatrix}\frac{\partial r}{\partial \underline{\alpha}}\\[2ex] \frac{\partial w}{\partial \underline{\alpha}}\end{bmatrix} \;=\; \begin{bmatrix}\frac{2}{3\,b}\,[\,w-u\,]\,[\,c_H-c_L\,]\\[2ex] -\,\frac{1}{9\,b}\,[\,c_H-c_L\,]\,[\,2\,a+2\,u+c_L+c_H-4\,c_v-4\,w\,]\end{bmatrix}, \tag{82}$$

where Λ is defined in (53) with $D_R = \overline{D}_R$. Since $|\Lambda| < 0$, (82) implies:

$$\frac{\partial r}{\partial \underline{\alpha}} \;=\; \frac{|\Upsilon_{r\underline{\alpha}}|}{|\Lambda|} \;\overset{s}{=}\; -\,|\Upsilon_{r\underline{\alpha}}|$$

where

$$\Upsilon_{r\underline{\alpha}} \;\equiv\; \begin{bmatrix} \frac{2}{3b}[w-u][c_H-c_L] & \frac{a+3u+c_v-2\underline{c}-4w}{3b} \\ -\frac{1}{9b}[c_H-c_L][2a+2u+c_L+c_H-4c_v-4w] & \frac{4}{9b}[\overline{\alpha}-\underline{\alpha}][c_H-c_L] \end{bmatrix}$$

$$\Rightarrow\; |\Upsilon_{r\underline{\alpha}}| \;=\; \frac{1}{27\,b^2}[c_H-c_L][8(\overline{\alpha}-\underline{\alpha})(w-u)(c_H-c_L)$$
$$+\,(2a+2u+c_L+c_H-4c_v-4w)(a+3u+c_v-2\underline{c}-4w)] \;>\; 0\,. \qquad (83)$$

The inequality in (83) reflects (33) and (79). The inequality implies $\frac{\partial r}{\partial \underline{\alpha}} < 0$.

Similarly, since $|\Lambda| < 0$, (82) implies:

$$\frac{\partial w}{\partial \underline{\alpha}} \;=\; \frac{|\Upsilon_{w\underline{\alpha}}|}{|\Lambda|} \;\overset{s}{=}\; -\,|\Upsilon_{w\underline{\alpha}}|$$

where

$$\Upsilon_{w\underline{\alpha}} \;\equiv\; \begin{bmatrix} \overline{D}_R & \frac{2}{3b}[w-u][c_H-c_L] \\ 2\,\overline{D}_R & -\frac{1}{9b}[c_H-c_L][2a+2u+c_L+c_H-4c_v-4w] \end{bmatrix}$$

$$\Rightarrow\; |\Upsilon_{w\underline{\alpha}}| \;=\; -\frac{\overline{D}_R}{9\,b}[c_H-c_L][2a+2u+c_L+c_H-4c_v-4w+12(w-u)]$$
$$=\; -\frac{\overline{D}_R}{9\,b}[c_H-c_L][2a+c_L+c_H-4c_v+8w-10u]. \qquad (84)$$

From (79):

$$2a+c_L+c_H-4c_v+8w-10u \;\geq\; 4w-2u+8w-10u \;=\; 12[w-u] \;>\; 0\,. \qquad (85)$$

(84) and (85) imply $|\Upsilon_{w\underline{\alpha}}| < 0$ and so $\frac{\partial w}{\partial \underline{\alpha}} > 0$.

From (1), differentiating $g(w,r) = 0$ with respect to $\overline{\alpha}$, using (4), provides:

$$\frac{1}{3\,b}[a+3u+c_v-2\underline{c}]\frac{\partial w}{\partial \overline{\alpha}} - \frac{4\,w}{3\,b}\left[\frac{\partial w}{\partial \overline{\alpha}}\right] + \overline{D}_R\left[\frac{\partial r}{\partial \overline{\alpha}}\right] \;=\; 0$$

$$\Rightarrow\; \overline{D}_R\left[\frac{\partial r}{\partial \overline{\alpha}}\right] + \left[\frac{a+3u+c_v-2\underline{c}-4w}{3\,b}\right]\frac{\partial w}{\partial \overline{\alpha}} \;=\; 0\,. \qquad (86)$$

From (2), differentiating $h(w,r) = 0$ with respect to $\overline{\alpha}$ provides:

$$\frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\,\frac{\partial w}{\partial \overline{\alpha}} + 2\,\overline{D}_R\left[\frac{\partial r}{\partial \overline{\alpha}}\right]$$

$$-\;\frac{1}{9\,b}\,[\,c_H - c_L\,]\,[\,2\,a + 2\,u + c_L + c_H - 4\,c_v - 4\,w\,] \;=\; 0$$

$$\Rightarrow\quad 2\,\overline{D}_R\left[\frac{\partial r}{\partial \overline{\alpha}}\right] + \frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\,\frac{\partial w}{\partial \overline{\alpha}}$$

$$=\;\frac{1}{9\,b}\,[\,c_H - c_L\,]\,[\,2\,a + 2\,u + c_L + c_H - 4\,c_v - 4\,w\,]\,. \tag{87}$$

(86) and (87) can be written as:

$$[\,\Lambda\,]\begin{bmatrix}\frac{\partial r}{\partial \overline{\alpha}}\\[2ex] \frac{\partial w}{\partial \overline{\alpha}}\end{bmatrix} \;=\; \begin{bmatrix} 0 \\[2ex] \frac{1}{9\,b}\,[\,c_H - c_L\,]\,[\,2\,a + 2\,u + c_L + c_H - 4\,c_v - 4\,w\,]\end{bmatrix}, \tag{88}$$

where Λ is defined in (53). Since $|\,\Lambda\,| < 0$, (88) implies:

$$\frac{\partial r}{\partial \overline{\alpha}} \;=\; \frac{|\,\Upsilon_{r\underline{\alpha}}\,|}{|\,\Lambda\,|} \;\stackrel{s}{=}\; -\,|\,\Upsilon_{r\underline{\alpha}}\,|$$

where

$$\Upsilon_{r\overline{\alpha}} \;\equiv\; \begin{bmatrix} 0 & \frac{a + 3\,u + c_v - 2\,\underline{c} - 4\,w}{3\,b} \\[2ex] \frac{1}{9\,b}\,[\,c_H - c_L\,]\,[\,2\,a + 2\,u + c_L + c_H - 4\,c_v - 4\,w\,] & \frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\end{bmatrix}$$

$$\Rightarrow\quad |\,\Upsilon_{r\overline{\alpha}}\,| \;=\; -\,\frac{1}{27\,b^2}\,[\,c_H - c_L\,]\,[\,2\,a + 2\,u + c_L + c_H - 4\,c_v - 4\,w\,]$$

$$\cdot\,[\,a + 3\,u + c_v - 2\,\underline{c}\, - 4\,w\,] \;<\; 0\,. \tag{89}$$

The inequality in (89) reflects (33) and (79). The inequality implies $\frac{\partial r}{\partial \overline{\alpha}} > 0$.

Similarly, since $|\,\Lambda\,| < 0$, (88) implies:

$$\frac{\partial w}{\partial \overline{\alpha}} \;=\; \frac{|\,\Upsilon_{w\overline{\alpha}}\,|}{|\,\Lambda\,|} \;\stackrel{s}{=}\; -\,|\,\Upsilon_{w\overline{\alpha}}\,|$$

where

$$\Upsilon_{w\overline{\alpha}} \;\equiv\; \begin{bmatrix} \overline{D}_R & 0 \\[2ex] 2\,\overline{D}_R & \frac{1}{9\,b}\,[\,c_H - c_L\,]\,[\,2\,a + 2\,u + c_L + c_H - 4\,c_v - 4\,w\,]\end{bmatrix}$$

$$\Rightarrow \quad |\,\Upsilon_{w\overline{\alpha}}\,| \;=\; \frac{\overline{D}_R}{9\,b}\,[\,c_H - c_L\,]\,[\,2\,a + 2\,u + c_L + c_H - 4\,c_v - 4\,w\,] \;>\; 0. \tag{90}$$

The inequality in (90) reflects (79). The inequality implies $\frac{\partial w}{\partial \overline{\alpha}} < 0$. ■

Proof of Observation 3.

From (1), differentiating $g(w,r) = 0$ with respect to k, using (4), provides:

$$\frac{1}{3\,b}\,[\,a + 3\,u + c_v - 2\,\underline{c}\,]\,\frac{\partial w}{\partial k} - \frac{4\,w}{3\,b}\left[\frac{\partial w}{\partial k}\right] + \overline{D}_R\left[\frac{\partial r}{\partial k}\right] \;=\; 0$$

$$\Rightarrow \quad \overline{D}_R\left[\frac{\partial r}{\partial k}\right] + \left[\frac{a + 3\,u + c_v - 2\,\underline{c} - 4\,w}{3\,b}\right]\frac{\partial w}{\partial k} \;=\; 0\,. \tag{91}$$

From (2), differentiating $h(w,r) = 0$ with respect to k provides:

$$\frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\,\frac{\partial w}{\partial k} + 2\,\overline{D}_R\left[\frac{\partial r}{\partial k}\right] \;=\; 0$$

$$\Rightarrow \quad 2\,\overline{D}_R\left[\frac{\partial r}{\partial k}\right] + \frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\,\frac{\partial w}{\partial k} \;=\; 0\,. \tag{92}$$

(91) and (92) can be written as:

$$[\,\Lambda\,]\begin{bmatrix} \frac{\partial r}{\partial k} \\ \\ \frac{\partial w}{\partial k} \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}, \tag{93}$$

where Λ is defined in (53). Since $|\,\Lambda\,| < 0$, (93) implies:

$$\frac{\partial r}{\partial k} \;=\; \frac{|\,\Upsilon_{rk}\,|}{|\,\Lambda\,|} \;\stackrel{s}{=}\; -\,|\,\Upsilon_{rk}\,| \quad \text{where} \quad \Upsilon_{rk} \;\equiv\; \begin{bmatrix} 0 & \frac{a + 3\,u + c_v - 2\,\underline{c} - 4\,w}{3\,b} \\ \\ 0 & \frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,] \end{bmatrix}$$

$$\Rightarrow \quad |\,\Upsilon_{rk}\,| \;=\; 0 \quad \Rightarrow \quad \frac{\partial r}{\partial k} \;=\; 0\,.$$

(93) also implies:

$$\frac{\partial w}{\partial k} \;=\; \frac{|\,\Upsilon_{wk}\,|}{|\,\Lambda\,|} \;\stackrel{s}{=}\; -\,|\,\Upsilon_{wk}\,| \quad \text{where} \quad \Upsilon_{wk} \;\equiv\; \begin{bmatrix} \overline{D}_R & 0 \\ \\ 2\,\overline{D}_R & 0 \end{bmatrix}$$

$$\Rightarrow \quad |\,\Upsilon_{wk}\,| \;=\; 0 \quad \Rightarrow \quad \frac{\partial w}{\partial k} \;=\; 0\,. \qquad \blacksquare$$

Proof of Proposition 1.

From (1), differentiating $g(w,r) = 0$ with respect to D_C, using (4), provides:

$$\frac{\partial r}{\partial D_C}\,\overline{D}_R + \left[\frac{a + 3\,u + c_v - 2\,\underline{c} - 4\,w}{3\,b}\right]\frac{\partial w}{\partial D_C} \;=\; \underline{d}\,. \tag{94}$$

From (2), differentiating $h(w,r) = 0$ with respect to D_C provides:

$$\frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\,\frac{\partial w}{\partial D_C} + 2\,\overline{D}_R\,\frac{\partial r}{\partial D_C} + \overline{d} - \underline{d} \;=\; 0$$

$$\Rightarrow \quad 2\,\overline{D}_R\,\frac{\partial r}{\partial D_C} + \frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\,\frac{\partial w}{\partial D_C} \;=\; -\,\left[\,\overline{d} - \underline{d}\,\right]. \tag{95}$$

(94) and (95) can be written as:

$$\Lambda \begin{bmatrix} \frac{\partial r}{\partial D_C} \\ \\ \frac{\partial w}{\partial D_C} \end{bmatrix} \;=\; \begin{bmatrix} \underline{d} \\ \\ -\,\left[\,\overline{d} - \underline{d}\,\right] \end{bmatrix} \tag{96}$$

where Λ is defined in (53). Since $|\,\Lambda\,| < 0$, (96) implies:

$$\frac{\partial r}{\partial D_C} \;=\; \frac{|\,\Lambda_1\,|}{|\,\Lambda\,|} \;\overset{s}{=}\; -\,|\,\Lambda_1\,| \quad \text{where} \quad \Lambda_1 \;\equiv\; \begin{bmatrix} \underline{d} & \frac{a + 3\,u + c_v - 2\,\underline{c} - 4\,w}{3\,b} \\ \\ -\left[\overline{d} - \underline{d}\right] & \frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,] \end{bmatrix} \tag{97}$$

$$\Rightarrow \quad |\,\Lambda_1\,| \;=\; \frac{4\,\underline{d}}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,] + \left[\,\overline{d} - \underline{d}\,\right]\left[\frac{a + 3\,u + c_v - 2\,\underline{c} - 4\,w}{3\,b}\right] \;>\; 0. \tag{98}$$

The inequality in (98) follows from (33). (97) and (98) imply $\frac{\partial r}{\partial D_C} < 0$.

From (1), differentiating $g(w,r) = 0$ with respect to $\overline{d}$, using (4), provides:

$$\frac{1}{3\,b}\,[\,a + 3\,u + c_v - 2\,\underline{c}\,]\,\frac{\partial w}{\partial \overline{d}} - \frac{4\,w}{3\,b}\left[\frac{\partial w}{\partial \overline{d}}\right] + \overline{D}_R\left[\frac{\partial r}{\partial \overline{d}}\right] \;=\; 0$$

$$\Rightarrow \quad \overline{D}_R\left[\frac{\partial r}{\partial \overline{d}}\right] + \left[\frac{a + 3\,u + c_v - 2\,\underline{c} - 4\,w}{3\,b}\right]\frac{\partial w}{\partial \overline{d}} \;=\; 0\,. \tag{99}$$

From (2), differentiating $h(w,r) = 0$ with respect to $\overline{d}$ provides:

$$\frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\,\frac{\partial w}{\partial \overline{d}} + 2\,\overline{D}_R\left[\frac{\partial r}{\partial \overline{d}}\right] + D_C \;=\; 0$$

$$\Rightarrow \quad 2\,\overline{D}_R\left[\frac{\partial r}{\partial \overline{d}}\right] + \frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\,\frac{\partial w}{\partial \overline{d}} \;=\; -\,D_C\,. \tag{100}$$

(99) and (100) can be written as:

$$[\,\Lambda\,]\begin{bmatrix} \frac{\partial r}{\partial \overline{d}} \\ \\ \frac{\partial w}{\partial \overline{d}} \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ \\ -\,D_C \end{bmatrix}, \tag{101}$$

where Λ is defined in (53). Since $|\,\Lambda\,| < 0$, (101) implies:

$$\frac{\partial r}{\partial \overline{d}} \;=\; \frac{|\,\Upsilon_1\,|}{|\,\Lambda\,|} \;\stackrel{s}{=}\; -\,|\,\Upsilon_1\,| \quad \text{where} \;\; \Upsilon_1 \;\equiv\; \begin{bmatrix} 0 & \frac{a+3\,u+c_v-2\,\underline{c}-4\,w}{3\,b} \\ \\ -\,D_C & \frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,] \end{bmatrix} \tag{102}$$

$$\Rightarrow \quad |\,\Upsilon_1\,| \;=\; D_C\left[\frac{a+3\,u+c_v-2\,\underline{c}-4\,w}{3\,b}\right] \;>\; 0 \quad \Rightarrow \quad \frac{\partial r}{\partial \overline{d}} \;<\; 0\,.$$

From (1), differentiating $g(w,r) = 0$ with respect to $\underline{d}$, using (4), provides:

$$\frac{1}{3\,b}\,[\,a+3\,u+c_v-2\,\underline{c}\,]\,\frac{\partial w}{\partial \underline{d}} - \frac{4\,w}{3\,b}\left[\frac{\partial w}{\partial \underline{d}}\right] + \overline{D}_R\left[\frac{\partial r}{\partial \underline{d}}\right] - D_C \;=\; 0$$

$$\Rightarrow \quad \overline{D}_R\left[\frac{\partial r}{\partial \underline{d}}\right] + \left[\frac{a+3\,u+c_v-2\,\underline{c}-4\,w}{3\,b}\right]\frac{\partial w}{\partial \underline{d}} \;=\; D_C\,. \tag{103}$$

From (2), differentiating $h(w,r) = 0$ with respect to $\underline{d}$ provides:

$$\frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\,\frac{\partial w}{\partial \underline{d}} + 2\,\overline{D}_R\left[\frac{\partial r}{\partial \underline{d}}\right] - D_C \;=\; 0$$

$$\Rightarrow \quad 2\,\overline{D}_R\left[\frac{\partial r}{\partial \underline{d}}\right] + \frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\,\frac{\partial w}{\partial \underline{d}} \;=\; D_C\,. \tag{104}$$

(103) and (104) can be written as:

$$\begin{bmatrix} \overline{D}_R & \frac{a+3\,u+c_v-2\,\underline{c}-4\,w}{3\,b} \\ \\ 2\,\overline{D}_R & \frac{4}{9\,b}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,] \end{bmatrix}\begin{bmatrix} \frac{\partial r}{\partial \underline{d}} \\ \\ \frac{\partial w}{\partial \underline{d}} \end{bmatrix} \;=\; \begin{bmatrix} D_C \\ \\ D_C \end{bmatrix}$$

$$\Leftrightarrow \quad [\,\Lambda\,] \begin{bmatrix} \frac{\partial r}{\partial \underline{d}} \\ \\ \frac{\partial w}{\partial \underline{d}} \end{bmatrix} \;=\; \begin{bmatrix} D_C \\ \\ D_C \end{bmatrix}, \tag{105}$$

where Λ is defined in (53). Since $|\,\Lambda\,| < 0$, (105) implies:

$$\frac{\partial r}{\partial \underline{d}} \;=\; \frac{|\,\Sigma_1\,|}{|\,\Lambda\,|} \;\overset{s}{=}\; -\,|\,\Sigma_1\,| \quad \text{where} \;\; \Sigma_1 \;\equiv\; \begin{bmatrix} D_C & \frac{a+3\,u+c_v-2\,\underline{c}-4\,w}{3\,b} \\ \\ D_C & \frac{4}{9\,b}\,[\,\overline{\alpha}-\underline{\alpha}\,]\,[\,c_H-c_L\,] \end{bmatrix} \tag{106}$$

$$\Rightarrow \quad |\,\Sigma_1\,| \;=\; D_C \left[\frac{4}{9\,b}\,(\,\overline{\alpha}-\underline{\alpha}\,)\,(\,c_H-c_L\,) - \frac{a+3\,u+c_v-2\,\underline{c}-4\,w}{3\,b} \right]. \tag{107}$$

From (55):

$$a+3\,u+c_v-2\,\underline{c}-4\,w-\frac{4}{3}\,[\,\overline{\alpha}-\underline{\alpha}\,]\,[\,c_H-c_L\,]$$

$$\geq \sqrt{[\,a+3\,u+c_v-2\,\underline{c}\,]^2 - 24\,b\left[\frac{u}{3\,b}\,(\,a+u+c_v-2\,\underline{c}\,)+\frac{\overline{D}_R}{2}+\underline{d}\;D_C+F_u\right]}$$

$$-\,\frac{4}{3}\,[\,\overline{\alpha}-\underline{\alpha}\,]\,[\,c_H-c_L\,]. \tag{108}$$

Also, (58) implies:

$$[\,a+3\,u+c_v-2\,\underline{c}\,]^2 - 24\,b\left[\frac{u}{3\,b}\,(\,a+u+c_v-2\,\underline{c}\,)+\frac{\overline{D}_R}{2}+\underline{d}\;D_C+F_u\right]$$

$$=\;[\,a-u+c_v-2\,\underline{c}\,]^2 - 24\,b\left[\frac{\overline{D}_R}{2}+\underline{d}\;D_C+F_u\right]. \tag{109}$$

(108), (109), and Assumption A1 imply:

$$a+3\,u+c_v-2\,\underline{c}-4\,w-\frac{4}{3}\,[\,\overline{\alpha}-\underline{\alpha}\,]\,[\,c_H-c_L\,]$$

$$\geq \sqrt{[\,a-u+c_v-2\,\underline{c}\,]^2 - 24\,b\left[\frac{\overline{D}_R}{2}+\underline{d}\;D_C+F_u\right]} - \frac{4}{3}\,[\,\overline{\alpha}-\underline{\alpha}\,]\,[\,c_H-c_L\,]$$

$$> \sqrt{\frac{16}{9}\,[\,\overline{\alpha}-\underline{\alpha}\,]^2\,[\,c_H-c_L\,]^2} - \frac{4}{3}\,[\,\overline{\alpha}-\underline{\alpha}\,]\,[\,c_H-c_L\,] \;=\; 0. \tag{110}$$

(107) and (110) imply:

$$|\,\Sigma_1\,| \;=\; \frac{D_C}{3\,b}\left[\frac{4}{3}\,(\,\overline{\alpha}-\underline{\alpha}\,)\,(\,c_H-c_L\,)-(\,a+3\,u+c_v-2\,\underline{c}-4\,w\,)\right] \;<\; 0. \tag{111}$$

(106) and (111) imply $\frac{\partial r}{\partial \underline{d}} > 0$. ■

Proof of Proposition 2.

Since $|\Lambda| < 0$, (53) and (96) imply:

$$\frac{\partial w}{\partial D_C} = \frac{|\Lambda_2|}{|\Lambda|} \overset{s}{=} -|\Lambda_2| \tag{112}$$

where

$$\Lambda_2 = \begin{bmatrix} \overline{D}_R & \underline{d} \\ 2\,\overline{D}_R & -\left[\overline{d} - \underline{d}\right] \end{bmatrix}$$

$$\Rightarrow \quad |\Lambda_2| = -\overline{D}_R\left[\overline{d} - \underline{d}\right] - 2\,\overline{D}_R\,\underline{d} = -\overline{D}_R\left[\overline{d} + \underline{d}\right] < 0\,. \tag{113}$$

(112) and (113) imply $\frac{\partial w}{\partial D_C} > 0$.

Similarly, (101) implies:

$$\frac{\partial w}{\partial \overline{d}} = \frac{|\Upsilon_2|}{|\Lambda|} \overset{s}{=} -|\Upsilon_2| \quad \text{where} \quad \Upsilon_2 \equiv \begin{bmatrix} \overline{D}_R & 0 \\ 2\,\overline{D}_R & -D_C \end{bmatrix} \tag{114}$$

$$\Rightarrow \quad |\Upsilon_2| = -\overline{D}_R\,D_C < 0 \quad \Rightarrow \quad \frac{\partial w}{\partial \overline{d}} > 0\,.$$

In addition, (105) implies:

$$\frac{\partial w}{\partial \underline{d}} = \frac{|\Sigma_2|}{|\Lambda|} \overset{s}{=} -|\Sigma_2| \quad \text{where} \quad \Sigma_2 \equiv \begin{bmatrix} \overline{D}_R & D_C \\ 2\,\overline{D}_R & D_C \end{bmatrix}$$

$$\Rightarrow \quad |\Sigma_2| = -\overline{D}_R\,D_C < 0 \quad \Rightarrow \quad \frac{\partial w}{\partial \underline{d}} > 0\,. \quad ■$$

Proof of Proposition 3.

From (1), (2), and (3), the Lagrangian function associated with [RP] is:

$$\begin{aligned} \pounds \;=\; & \frac{\alpha}{18\,b}\left[2\,a - w - u - c_H - c_v\right]^2 + \left[\frac{1-\alpha}{18\,b}\right]\left[2\,a - w - u - c_L - c_v\right]^2 \\ & - k\left[r - \frac{1}{2}\right]^2 + \left[1 - r\right]\overline{D}_R\left[1 - f_R\right] + \left[1 - f_C\right]\underline{d}\,D_C \\ & + \lambda\left\{\frac{w}{3\,b}\left[a + 3\,u + c_v - 2\,\underline{c}\right] - \frac{2\,w^2}{3\,b} - \phi\right\} \end{aligned}$$

$$+ \ \mu \{ - \frac{[\,\overline{\alpha} - \underline{\alpha}\,]}{9\,b} [\, c_H - c_L \,] [\, 2\,a + 2\,u + c_L + c_H - 4\,c_v - 4\,w \,] + [\, 2\,r - 1 \,] \overline{D}_R + \left[\, \overline{d} - \underline{d} \,\right] D_C \} \tag{115}$$

where ϕ is defined in (4).

Differentiating (115), using (4), provides:

$$\frac{\partial \pounds}{\partial r} \ = \ -2\,k \left[r - \frac{1}{2} \right] - \overline{D}_R [\, 1 - f_R \,] + \lambda\, \overline{D}_R \, + \, 2\,\mu\, \overline{D}_R \ = \ 0 \tag{116}$$

$$\Leftrightarrow \ \ \lambda\, \overline{D}_R \, + \, 2\,\mu\, \overline{D}_R \ = \ 2\,k \left[r - \frac{1}{2} \right] + \overline{D}_R [\, 1 - f_R \,] \, ; \quad \text{and} \tag{117}$$

$$\frac{\partial \pounds}{\partial w} \ = \ - \frac{\underline{\alpha}}{9\,b} [\, 2\,a - w - u - c_H - c_v \,] \ - \ \left[\frac{1 - \underline{\alpha}}{9\,b} \right] [\, 2\,a - w - u - c_L - c_v \,]$$

$$+ \ \lambda \left[\frac{1}{3\,b} (\, a + 3\,u + c_v - 2\,\underline{c} \,) - \frac{4\,w}{3\,b} \right] + \mu \ \frac{4\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\, c_H - c_L \,]}{9\,b} \ = \ 0$$

$$\Leftrightarrow \ \ \frac{\lambda}{3\,b} [\, a + 3\,u + c_v - 2\,\underline{c} \, - 4\,w] + \frac{4\,\mu \ [\,\overline{\alpha} - \underline{\alpha}\,]\,[\, c_H - c_L \,]}{9\,b} = \ \frac{1}{9\,b} [\, 2\,a - w - u - \underline{c} - c_v \,] \, . \tag{118}$$

(117) and (118) can be written as:

$$[\, M \,] \begin{bmatrix} \lambda \\ \\ \mu \end{bmatrix} \ = \ \begin{bmatrix} \frac{1}{9\,b} [\, 2\,a - w - u - \underline{c} - c_v \,] \\ \\ 2\,k \left[r - \frac{1}{2} \right] + \overline{D}_R [\, 1 - f_R \,] \end{bmatrix}$$

$$\text{where} \quad M \ \equiv \ \begin{bmatrix} \frac{1}{3\,b} [\, a + 3\,u + c_v - 2\,\underline{c} \, - 4\,w \,] & \frac{4\,[\,\overline{\alpha} - \underline{\alpha}\,][\, c_H - c_L \,]}{9\,b} \\ \\ \overline{D}_R & 2\,\overline{D}_R \end{bmatrix} \tag{119}$$

$$\Rightarrow \ \ |\, M \,| \ = \ - \overline{D}_R \left\{ \frac{4\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\, c_H - c_L \,]}{9\,b} - \frac{2}{3\,b} [\, a + 3\,u + c_v - 2\,\underline{c} \, - 4\,w \,] \right\}$$

$$= \ - \,|\, \Lambda \,| \ > \ 0 \, . \tag{120}$$

From (119):

$$\mu \ = \ \frac{|\, M_\mu \,|}{|\, M \,|}, \quad \text{where} \tag{121}$$

$$M_\mu \ \equiv \ \begin{bmatrix} \frac{1}{3\,b} [\, a + 3\,u + c_v - 2\,\underline{c} \, - 4\,w \,] & \frac{1}{9\,b} [\, 2\,a - w - u - \underline{c} - c_v \,] \\ \\ \overline{D}_R & 2\,k \left[r - \frac{1}{2} \right] + \overline{D}_R [\, 1 - f_R \,] \end{bmatrix} . \tag{122}$$

(116) implies that at a solution to [RP]:

$$- 2\,k \left[r - \frac{1}{2} \right] - \overline{D}_R \left[1 - f_R \right] + \lambda\, \overline{D}_R + 2\,\mu\, \overline{D}_R \;=\; 0$$

$$\Rightarrow \quad \lambda + 2\,\mu \;=\; \frac{2\,k}{\overline{D}_R} \left[r - \frac{1}{2} \right] + 1 - f_R\,. \tag{123}$$

From the envelope theorem, (4), and (115):

$$\begin{aligned}
\frac{dW^*}{dD_C} \;=\; \frac{\partial \pounds}{\partial D_C} \;&=\; [\,1 - f_C\,]\;\underline{d} - \lambda\;\underline{d} + \mu\,[\;\overline{d} - \underline{d}\,] \\
&=\; [\,1 - f_C\,]\;\underline{d} - \lambda\;\underline{d} + \mu\;\overline{d} - 2\,\mu\;\underline{d} + \mu\;\underline{d} \\
&=\; [\,1 - f_C\,]\;\underline{d} - [\,\lambda + 2\,\mu\,]\,\underline{d} + \mu\,[\;\overline{d} + \underline{d}\,]\,.
\end{aligned} \tag{124}$$

(123) and (124) provide:

$$\frac{dW^*}{dD_C} \;=\; [\,1 - f_C\,]\;\underline{d} - \left[\frac{2\,k}{\overline{D}_R} \left(r - \frac{1}{2} \right) + 1 - f_R \right] \underline{d} \;+\; \mu\,[\;\overline{d} + \underline{d}\,]$$

$$=\; \underline{d}\; \left[1 - f_C - \frac{2\,k}{\overline{D}_R} \left(r - \frac{1}{2} \right) - (\,1 - f_R\,) \right] + \;\mu\,[\;\overline{d} + \underline{d}\,]$$

$$=\; \underline{d}\; \left[f_R - f_C - \frac{2\,k}{\overline{D}_R} \left(r - \frac{1}{2} \right) \right] + \frac{|\,M_\mu\,|}{|\,M\,|}\,[\;\overline{d} + \underline{d}\,] \tag{125}$$

$$\stackrel{s}{=}\; |\,M\,|\;\underline{d} \left[f_R - f_C - \frac{2\,k}{\overline{D}_R} \left(r - \frac{1}{2} \right) \right] + |\,M_\mu\,|\;[\;\overline{d} + \underline{d}\,]\,. \tag{126}$$

The equality in (125) reflects (121). (126) holds because $|\,M\,| > 0$, from (120).

Recall $\alpha_1 \equiv \frac{2}{3}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]$ and $\alpha_2 \equiv a + 3\,u + c_v - 2\,\underline{c} - 4\,w$ from (61). Define:

$$\alpha_3 \;\equiv\; 2\,a - w - u - \underline{c} - c_v\,. \tag{127}$$

Then, from (122):

$$\begin{aligned}
|\,M_\mu\,| \;&=\; \frac{1}{3\,b}\,[\,a + 3\,u + c_v - 2\,\underline{c}\, - 4\,w\,] \left[2\,k \left(r - \frac{1}{2} \right) + \overline{D}_R\,(\,1 - f_R\,) \right] \\
&\quad - \frac{1}{9\,b}\,[\,2\,a - w - u - \underline{c} - c_v\,]\,\overline{D}_R \\
&=\; \frac{1}{3\,b} \left\{ \alpha_2 \left[2\,k \left(r - \frac{1}{2} \right) + \overline{D}_R\,(\,1 - f_R\,) \right] - \frac{1}{3}\,\alpha_3\,\overline{D}_R \right\}
\end{aligned}$$

$$= \frac{\overline{D}_R}{3\,b}\left\{\alpha_2\left[\frac{2\,k}{\overline{D}_R}\left(r-\frac{1}{2}\right)+(1-f_R)\right]-\frac{1}{3}\,\alpha_3\right\}. \tag{128}$$

(120) implies:

$$|\,M\,| \;=\; \frac{2\,\overline{D}_R}{3\,b}\,[\,\alpha_2-\alpha_1\,]\,. \tag{129}$$

(126), (128), and (129) provide:

$$\begin{aligned}
\frac{dW^*}{dD_C} \;&\overset{s}{=}\; |\,M\,|\,\underline{d}\left[f_R-f_C-\frac{2\,k}{\overline{D}_R}\left(r-\frac{1}{2}\right)\right]+|\,M_\mu\,|\,[\,\overline{d}+\underline{d}\,] \\
&= \frac{2\,\overline{D}_R}{3\,b}\,[\,\alpha_2-\alpha_1\,]\,\underline{d}\left[f_R-f_C-\frac{2\,k}{\overline{D}_R}\left(r-\frac{1}{2}\right)\right] \\
&\qquad + \frac{\overline{D}_R}{3\,b}\left\{\alpha_2\left[\frac{2\,k}{\overline{D}_R}\left(r-\frac{1}{2}\right)+(1-f_R)\right]-\frac{1}{3}\,\alpha_3\right\}[\,\overline{d}+\underline{d}\,] \\
&= \frac{2}{3\,b}\,[\,\alpha_2-\alpha_1\,]\,\underline{d}\left[\overline{D}_R\,(f_R-f_C)-2\,k\left(r-\frac{1}{2}\right)\right] \\
&\qquad + \frac{1}{3\,b}\left\{\alpha_2\left[2\,k\left(r-\frac{1}{2}\right)+\overline{D}_R\,(1-f_R)\right]-\frac{\overline{D}_R}{3}\,\alpha_3\right\}[\,\overline{d}+\underline{d}\,]
\end{aligned} \tag{130}$$

$$\begin{aligned}
&= -\,\frac{4}{3\,b}\,[\,\alpha_2-\alpha_1\,]\,\underline{d}\,k\left[r-\frac{1}{2}\right]+\frac{2\,k}{3\,b}\left[r-\frac{1}{2}\right]\alpha_2\,[\,\overline{d}+\underline{d}\,] \\
&\qquad + \frac{2}{3\,b}\,[\,\alpha_2-\alpha_1\,]\,\underline{d}\,\overline{D}_R\,[\,f_R-f_C\,]+\frac{\overline{D}_R}{3\,b}\left[\alpha_2\,(1-f_R)-\frac{\alpha_3}{3}\right][\,\overline{d}+\underline{d}\,]
\end{aligned} \tag{131}$$

$$\begin{aligned}
&= \frac{4\,\alpha_1}{3\,b}\,\underline{d}\,k\left[r-\frac{1}{2}\right]-\frac{4}{3\,b}\,[\,\alpha_2\,]\,\underline{d}\,k\left[r-\frac{1}{2}\right]+\frac{2\,k}{3\,b}\left[r-\frac{1}{2}\right]\alpha_2\,[\,\overline{d}+\underline{d}\,] \\
&\qquad + \frac{2}{3\,b}\,[\,\alpha_2-\alpha_1\,]\,\underline{d}\,\overline{D}_R\,[\,f_R-f_C\,]+\frac{\overline{D}_R}{3\,b}\left[\alpha_2\,(1-f_R)-\frac{\alpha_3}{3}\right][\,\overline{d}+\underline{d}\,] \\
&= \frac{4\,\alpha_1}{3\,b}\,\underline{d}\,k\left[r-\frac{1}{2}\right]+\frac{2\,k}{3\,b}\left[r-\frac{1}{2}\right]\alpha_2\,\left[-2\underline{d}+\left(\overline{d}+\underline{d}\,\right)\right] \\
&\qquad + \frac{2}{3\,b}\,[\,\alpha_2-\alpha_1\,]\,\underline{d}\,\overline{D}_R\,[\,f_R-f_C\,]+\frac{\overline{D}_R}{3\,b}\left[\alpha_2\,(1-f_R)-\frac{\alpha_3}{3}\right][\,\overline{d}+\underline{d}\,] \\
&= \frac{4\,\alpha_1}{3\,b}\,\underline{d}\,k\left[r-\frac{1}{2}\right]+\frac{2\,k}{3\,b}\left[r-\frac{1}{2}\right]\alpha_2\,[\,\overline{d}-\underline{d}\,] \\
&\qquad + \frac{2}{3\,b}\,[\,\alpha_2-\alpha_1\,]\,\underline{d}\,\overline{D}_R\,[\,f_R-f_C\,]+\frac{\overline{D}_R}{3\,b}\left[\alpha_2\,(1-f_R)-\frac{\alpha_3}{3}\right][\,\overline{d}+\underline{d}\,]\,.
\end{aligned} \tag{132}$$

$\alpha_2 > \alpha_1$ from (62). Furthermore, Assumption 1 ensures $\alpha_2\,[\,1-f_R\,] \geq \frac{\alpha_3}{3}$. (See the proof of Conclusion 4 below.) Therefore, (132) implies that $\frac{dW^*}{dD_C} > 0$ if $f_R \geq f_C$.

α_1 is independent of $\overline{D}_R$ and α_2 and α_3 are bounded above. Consequently, $[\,\alpha_2 - \alpha_1\,]\,\overline{D}_R \to 0$ and $\left[\,\alpha_2\,(\,1 - f_R\,) - \frac{\alpha_3}{3}\,\right]\overline{D}_R \to 0$ as $\overline{D}_R \to 0$. Therefore, as $\overline{D}_R \to 0$:

$$\frac{2}{3\,b}\,[\,\alpha_2 - \alpha_1\,]\,\underline{d}\;\overline{D}_R\,[\,f_R - f_C\,] + \frac{\overline{D}_R}{3\,b}\left[\,\alpha_2\,(\,1 - f_R\,) - \frac{\alpha_3}{3}\,\right]\left[\,\overline{d} + \underline{d}\,\right] \;\to\; 0\,. \tag{133}$$

Recall that $\frac{\partial w}{\partial \overline{D}_R} > 0$ and $\frac{\partial r}{\partial \overline{D}_R} < 0$. Therefore, $\frac{\partial \alpha_1}{\partial \overline{D}_R} = 0$ and $\frac{\partial \alpha_2}{\partial \overline{D}_R} < 0$. Consequently, when $\overline{D}_R$ is sufficiently close to 0, the first two terms in (131) are strictly positive, so (131) and (133) imply $\frac{dW^*}{dD_C} > 0$. ■

Proof of Proposition 4.

From (3), using (97) and (112):

$$\begin{aligned}
\frac{dW^*}{dD_C} \;&=\; -\,\frac{1}{9\,b}\,[\,2\,a - w - u - \underline{c} - c_v\,]\,\frac{\partial w}{\partial D_C} \\
&\qquad - \left[\,2\,k\,(\,r - \frac{1}{2}\,) + \overline{D}_R\,(\,1 - f_R\,)\,\right]\frac{\partial r}{\partial D_C} + [\,1 - f_C\,]\,\underline{d} \\
&=\; -\,\frac{1}{9\,b}\,[\,2\,a - w - u - \underline{c} - c_v\,]\,\frac{|\,\Lambda_2\,|}{|\,\Lambda\,|} \\
&\qquad - \left[\,2\,k\,(\,r - \frac{1}{2}\,) + \overline{D}_R\,(\,1 - f_R\,)\,\right]\frac{|\,\Lambda_1\,|}{|\,\Lambda\,|} + [\,1 - f_C\,]\,\underline{d} \;\lesseqgtr\; 0
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow\quad H \;&\equiv\; -\,\frac{1}{9\,b}\,[\,2\,a - w - u - \underline{c} - c_v\,]\,|\,\Lambda_2\,| \\
&\qquad - \left[\,2\,k\,(\,r - \frac{1}{2}\,) + \overline{D}_R\,(\,1 - f_R\,)\,\right]|\,\Lambda_1\,| + [\,1 - f_C\,]\;\underline{d}\;|\,\Lambda\,| \;\gtreqless\; 0\,.
\end{aligned} \tag{134}$$

The inequality in (134) holds because $|\,\Lambda\,| < 0$.

Recall from (61) and (127) that:

$$\begin{aligned}
&\alpha_1 \;\equiv\; \frac{2}{3}\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]\,, \quad \alpha_2 \;\equiv\; a + 3\,u + c_v - 2\,\underline{c} - 4\,w\,, \quad \text{and} \\
&\alpha_3 \;\equiv\; 2\,a - w - u - \underline{c} - c_v\,.
\end{aligned} \tag{135}$$

Since $|\,\Lambda_1\,| > 0$ from (98):

$$\begin{aligned}
H \;\geq\; &-\,\frac{1}{9\,b}\,[\,2\,a - w - u - \underline{c} - c_v\,]\,|\,\Lambda_2\,| \\
&\qquad - \left[\,2\,k\,(\,r - \frac{1}{2}\,) + \overline{D}_R\,(\,1 - f_R\,)\,\right]|\,\Lambda_1\,| + [\,1 - f_C\,]\;\underline{d}\;|\,\Lambda\,|
\end{aligned}$$

$$= -\frac{1}{9\,b}\,[\,2\,a - w - u - \underline{c} - c_v\,]\,|\,\Lambda_2\,| - \,\big[\,k + \overline{D}_R\,(\,1 - f_R\,)\big]\,|\,\Lambda_1\,|$$

$$+ \,[\,1 - f_C\,]\;\underline{d}\;|\,\Lambda\,| \;=\; G\,(w) \tag{136}$$

where, from (98), (113), (120), (135), and (136):

$$G\,(w) \;\equiv\; \frac{\alpha_3}{9\,b}\,\overline{D}_R\,[\,\overline{d} + \underline{d}\,] - \big[\,k + \overline{D}_R\,(\,1 - f_R\,)\big]\left[\frac{4\,\underline{d}}{9\,b}\,(\,\overline{\alpha} - \underline{\alpha}\,)\,(\,c_H - c_L\,) + \big(\,\overline{d} - \underline{d}\,\big)\,\frac{\alpha_2}{3\,b}\right]$$

$$-\,[\,1 - f_C\,]\,\underline{d}\left[\frac{2\,\overline{D}_R}{3\,b}\right]\left[\alpha_2 - \frac{2}{3}\,(\,\overline{\alpha} - \underline{\alpha}\,)\,(\,c_H - c_L\,)\right]. \tag{137}$$

(127), (135), and (137) imply that $G(\cdot)$ is linear in w. Therefore, $G\,(w^*) > 0$ if: (i) $G\,(0) > 0$; (ii) $G\,(\widetilde{w}) > 0$; and (iii) $w^* \in [\,0, \widetilde{w}\,]$, where $\widetilde{w} \,=\, \frac{1}{4}\,[\,a + 3\,u + c_v - 2\,\underline{c}\,]$.

(108) implies that $w^* \le \widetilde{w}$. To determine when $G\,(0) > 0$, note that (135) and (137) imply:

$$G\,(w) \;=\; \frac{\alpha_3}{9\,b}\,\overline{D}_R\,[\,\overline{d} + \underline{d}\,] - \big[\,k + \overline{D}_R\,(\,1 - f_R\,)\big]\left[\frac{2\,\underline{d}}{3\,b}\,\alpha_1 + \big(\,\overline{d} - \underline{d}\,\big)\,\frac{\alpha_2}{3\,b}\right] \tag{138}$$

$$-\,[\,1 - f_C\,]\,\underline{d}\left[\frac{2\,\overline{D}_R}{3\,b}\right][\,\alpha_2 - \,\alpha_1\,]$$

$$= \frac{\alpha_3}{9\,b}\,\overline{D}_R\,[\,\overline{d} + \underline{d}\,] - \frac{\overline{D}_R}{3\,b}\left[\frac{k}{\overline{D}_R} + 1 - f_R\right][\,2\,\underline{d}\,\alpha_1 + \big(\,\overline{d} - \underline{d}\,\big)\,\alpha_2\,]$$

$$-\,[\,1 - f_C\,]\,\underline{d}\left[\frac{2\,\overline{D}_R}{3\,b}\right][\,\alpha_2 - \,\alpha_1\,] \;=\; \frac{\overline{D}_R}{3\,b}\,\widetilde{G}\,(w)\;, \tag{139}$$

where $\quad \widetilde{G}\,(w) \;\equiv\; \frac{\alpha_3}{3}\,[\,\overline{d} + \underline{d}\,] - \left[\frac{k}{\overline{D}_R} + 1 - f_R\right][\,2\,\underline{d}\,\alpha_1 + \big(\,\overline{d} - \underline{d}\,\big)\,\alpha_2\,]$

$$-\;2\,\underline{d}\,[\,1 - f_C\,]\;[\,\alpha_2 - \,\alpha_1\,]\,. \tag{140}$$

(135) and (140) imply:

$$\widetilde{G}\,(0) \;=\; \frac{1}{3}\,[\,\overline{d} + \underline{d}\,]\,[\,2\,a - u - c_v - \underline{c}\,] - \left[\frac{k}{\overline{D}_R} + 1 - f_R\right][\,\overline{d} - \underline{d}\,]\,[\,a + 3\,u + c_v - \underline{c}\,]$$

$$-\,2\,\underline{d}\,[\,1 - f_C\,]\,[\,a + 6\,u + c_v - \underline{c}\,] - 2\,\underline{d}\left[\,f_C + \frac{k}{\overline{D}_R} - f_R\right]\alpha_1\,. \tag{141}$$

Since $a \,>\, 7\,u + 2\,c_v$ from Assumption A1:

$$2\,a - u - c_v - \underline{c} \;>\; a + 6\,u + c_v - \underline{c} \;>\; a + 3\,u + c_v - \underline{c}\,. \tag{142}$$

(141) and (142) imply:

$$\widetilde{G}\,(0) \;>\; \left\{\frac{1}{3}\,[\,\overline{d} + \underline{d}\,] - \left[\frac{k}{\overline{D}_R} + 1 - f_R\right][\,\overline{d} - \underline{d}\,] - 2\,\underline{d}\,[\,1 - f_C\,]\right\}[\,2\,a - u - c_v - \underline{c}\,]$$

$$- \, 2 \, \underline{d} \left[f_C + \frac{k}{\overline{D}_R} - f_R \right] \alpha_1 \, . \tag{143}$$

(135), (143), and Assumption A1 imply that $\widetilde{G}(0) > 0$ (and so $G(0) > 0$, from (139)) if:

$$\frac{1}{3} \left[\, \overline{d} + \underline{d} \, \right] - \left[\frac{k}{\overline{D}_R} + 1 - f_R \right] \left[\, \overline{d} - \underline{d} \, \right] - 2 \, \underline{d} \left[1 - f_C \, \right] \; > \; 0 \, . \tag{144}$$

It remains to demonstrate that $G(\widetilde{w}) \; > \; 0$. From (135) and (137):

$$\begin{aligned} G(\widetilde{w}) \;\; &\equiv \;\; \frac{1}{9\,b} \left[2\,a - u - \underline{c} - c_v - \frac{a + 3\,u + c_v - 2\,\underline{c}}{4} \right] \overline{D}_R \left[\, \overline{d} + \underline{d} \, \right] \\ &\quad - \left[k + \overline{D}_R \, (1 - f_R) \right] \frac{4\,\underline{d}}{9\,b} \left[\, \overline{\alpha} - \underline{\alpha} \, \right] \left[\, c_H - c_L \, \right] \\ &\quad + \left[1 - f_C \, \right] \underline{d} \left[\frac{2\,\overline{D}_R}{3\,b} \right] \frac{2}{3\,b} \left[\, \overline{\alpha} - \underline{\alpha} \, \right] \left[\, c_H - c_L \, \right] \\ &= \frac{1}{9\,b} \left[\frac{7\,a - 7\,u - 2\,\underline{c} - 5\,c_v}{4} \right] \overline{D}_R \left[\, \overline{d} + \underline{d} \, \right] - \left[k + \overline{D}_R \, (1 - f_R) \right] \frac{4\,\underline{d}}{9\,b} \left[\, \overline{\alpha} - \underline{\alpha} \, \right] \left[\, c_H - c_L \, \right] \\ &\quad + \left[1 - f_C \, \right] \underline{d} \left[\frac{2\,\overline{D}_R}{3\,b} \right] \frac{2}{3\,b} \left[\, \overline{\alpha} - \underline{\alpha} \, \right] \left[\, c_H - c_L \, \right] \\ &> \frac{1}{9\,b} \left[\frac{7\,a - 7\,u - 2\,\underline{c} - 5\,c_v}{4} \right] \overline{D}_R \left[2\,\underline{d} \, \right] - \left[k + \overline{D}_R \, (1 - f_R) \right] \frac{4\,\underline{d}}{9\,b} \left[\, \overline{\alpha} - \underline{\alpha} \, \right] \left[\, c_H - c_L \, \right] \\ &\quad + \left[1 - f_C \, \right] \underline{d} \left[\frac{2\,\overline{D}_R}{3\,b} \right] \frac{2}{3\,b} \left[\, \overline{\alpha} - \underline{\alpha} \, \right] \left[\, c_H - c_L \, \right] \\ &= \underline{d} \left\{ \frac{2}{9\,b} \left[\frac{7\,a - 7\,u - 2\,\underline{c} - 5\,c_v}{4} \right] \overline{D}_R - \left[k + \overline{D}_R \, (1 - f_R) \right] \frac{4}{9\,b} \left[\, \overline{\alpha} - \underline{\alpha} \, \right] \left[\, c_H - c_L \, \right] \right\} \\ &\quad + \left[1 - f_C \, \right] \underline{d} \left[\frac{2\,\overline{D}_R}{3\,b} \right] \frac{2}{3\,b} \left[\, \overline{\alpha} - \underline{\alpha} \, \right] \left[\, c_H - c_L \, \right] . \end{aligned} \tag{145}$$

Assumption A1 ensures:

$$\frac{2}{9\,b} \left[\frac{7\,a - 7\,u - 2\,\underline{c} - 5\,c_v}{4} \right] \overline{D}_R \;\; > \;\; \left[k + \overline{D}_R \, (1 - f_R) \right] \frac{4}{9\,b} \left[\, \overline{\alpha} - \underline{\alpha} \, \right] \left[\, c_H - c_L \, \right] . \tag{146}$$

(145) and (146) imply $G(\widetilde{w}) > 0$.

Finally, observe that since $\overline{d} + \underline{d} \; > \; 2\,\underline{d}$, (144) holds if:

$$\left[\frac{k}{\overline{D}_R} + f_C - f_R \right] \left[\, \overline{d} - \underline{d} \, \right] + 2\,\underline{d} \left[1 - f_C \, \right] \; < \; \frac{2}{3} \, \underline{d}$$

$$\Leftrightarrow \quad \left[\frac{k}{\overline{D}_R} + f_C - f_R \right] \left[\frac{\overline{d} - \underline{d}}{\underline{d}} \right] \; < \; 2 \left[f_C - \frac{2}{3} \right] . \quad \blacksquare$$

Proof of Proposition 5.

From (4) and (115): $$\frac{dW^*}{d\overline{d}} \; = \; \frac{\partial \pounds}{\partial \overline{d}} \; = \; \mu \, D_C \; > 0 \, . \quad \blacksquare$$

Proof of Proposition 6.

From (4) and (115):

$$\frac{dW^*}{d\underline{d}} \; = \; \frac{\partial \pounds}{\partial \underline{d}} \; = \; [\, 1 - f_C \,] \, D_C - \lambda \, D_C - \mu \, D_C \; = \; D_C \, [\, 1 - f_C - \lambda - \mu \,]$$

$$= \; D_C \, [\, 1 - f_C \; - (\, \lambda + 2 \, \mu \,) + \mu \,] \, . \tag{147}$$

Since $\lambda + 2\,\mu \; = \; \frac{2\,k}{\overline{D}_R} \left[\, r - \frac{1}{2} \, \right] + 1 - f_R$ from (123), (147) implies:

$$\frac{dW^*}{d\underline{d}} \; = \; D_C \left[1 - f_C - \left(\frac{2\,k}{\overline{D}_R} \left[r - \frac{1}{2} \right] + 1 - f_R \right) + \mu \right]$$

$$= \; D_C \left[f_R - f_C \; - \frac{2\,k}{\overline{D}_R} \left(r - \frac{1}{2} \right) + \mu \right] \; > \; D_C \left[f_R - f_C \; - \frac{k}{\overline{D}_R} + \mu \right] . \tag{148}$$

(148) implies:

$$\text{If } \; \frac{dW^*}{d\underline{d}} \; < \; 0 \, , \text{ then it must be the case that } \; f_C \; \geq \; f_R - \frac{k}{\overline{D}_R} \, . \tag{149}$$

From (117) and (118):

$$3\,\lambda\, [\, a + 3\,u + c_v - 2\,\underline{c} \; - 4\,w \,] + 4\,\mu\, [\, \overline{\alpha} - \underline{\alpha} \,] \, [\, c_H - c_L \,] \; = \; 2\,a - w - u - \underline{c} - c_v \, ,$$

and

$$\lambda \; = \; \frac{2\,k}{\overline{D}_R} \left[r - \frac{1}{2} \right] + 1 - f_R - 2\,\mu \, .$$

Therefore:

$$3 \left[\frac{2\,k}{\overline{D}_R} \left(r - \frac{1}{2} \right) + 1 - f_R - 2\,\mu \right] [\, a + 3\,u + c_v - 2\,\underline{c} \; - 4\,w \,]$$

$$+ \; 4\,\mu\, [\, \overline{\alpha} - \underline{\alpha} \,] \, [\, c_H - c_L \,] \; = \; 2\,a - w - u - \underline{c} - c_v$$

$$\Rightarrow \quad - \; 6\,\mu\, [\, a + 3\,u + c_v - 2\,\underline{c} \; - 4\,w \,]$$

$$+ \; 3 \left[\frac{2\,k}{\overline{D}_R} \left(r - \frac{1}{2} \right) + 1 - f_R \right] [\, a + 3\,u + c_v - 2\,\underline{c} \; - 4\,w \,]$$

$$+ 4\,\mu\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,] \;=\; 2\,a - w - u - \underline{c} - c_v$$

$$\Rightarrow \quad 6\,\mu\,[\,a + 3\,u + c_v - 2\,\underline{c} - 4\,w\,] - 4\,\mu\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]$$

$$= \; 3\left[\frac{2\,k}{\overline{D}_R}\left(r - \frac{1}{2}\right) + 1 - f_R\right][\,a + 3\,u + c_v - 2\,\underline{c} - 4\,w\,] - [\,2\,a - w - u - \underline{c} - c_v\,]$$

$$\Rightarrow \quad 6\,\mu\left[a + 3\,u + c_v - 2\,\underline{c} - 4\,w - \frac{2}{3}\,(\,\overline{\alpha} - \underline{\alpha}\,)\,(\,c_H - c_L\,)\right]$$

$$= \; 3\left[\frac{2\,k}{\overline{D}_R}\left(r - \frac{1}{2}\right) + 1 - f_R\right][\,a + 3\,u + c_v - 2\,\underline{c} - 4\,w\,] - [\,2\,a - w - u - \underline{c} - c_v\,]$$

$$\Rightarrow \quad \mu \;=\; \frac{3\left[\frac{2\,k}{\overline{D}_R}\,(\,r - \frac{1}{2}\,) + 1 - f_R\right][\,a + 3\,u + c_v - 2\,\underline{c} - 4\,w\,] - [\,2\,a - w - u - \underline{c} - c_v\,]}{6\,[\,a + 3\,u + c_v - 2\,\underline{c} - 4\,w - \frac{2}{3}\,(\,\overline{\alpha} - \underline{\alpha}\,)\,(\,c_H - c_L\,)\,]}\,. \tag{150}$$

Assumption 1 ensures the numerator in (150) is positive. Therefore, because $\mu > 0$, the denominator in (150) is also positive. Consequently, (148) and (150) imply:

$$\frac{dW^*}{d\underline{d}} \;<\; 0 \quad \Leftrightarrow \quad f_R - f_C - \frac{2\,k}{\overline{D}_R}\left[r - \frac{1}{2}\right] + \mu \;<\; 0 \quad \Leftrightarrow \quad \gamma\,(w) \;<\; 0 \tag{151}$$

where

$$\gamma\,(w) \;\equiv\; 6\left[f_R - f_C - \frac{2\,k}{\overline{D}_R}\left(r - \frac{1}{2}\right)\right]\left[a + 3\,u + c_v - 2\,\underline{c} - 4\,w - \frac{2}{3}\,(\,\overline{\alpha} - \underline{\alpha}\,)\,(\,c_H - c_L\,)\right]$$

$$+ \; 3\left[\frac{2\,k}{\overline{D}_R}\left(r - \frac{1}{2}\right) + 1 - f_R\right][\,a + 3\,u + c_v - 2\,\underline{c} - 4\,w\,]$$

$$- \;\; [\,2\,a - w - u - \underline{c} - c_v\,] \tag{152}$$

$$\Rightarrow \quad \gamma\,(0) \;=\; 6\left[f_R - f_C - \frac{2\,k}{\overline{D}_R}\left(r - \frac{1}{2}\right)\right]\left[a + 3\,u + c_v - 2\,\underline{c} - \frac{2}{3}\,(\,\overline{\alpha} - \underline{\alpha}\,)\,(\,c_H - c_L\,)\right]$$

$$+ \; 3\left[\frac{2\,k}{\overline{D}_R}\left(r - \frac{1}{2}\right) + 1 - f_R\right][\,a + 3\,u + c_v - 2\,\underline{c}\,] - [\,2\,a - u - \underline{c} - c_v\,] \tag{153}$$

$$= \; \left\{6\left[f_R - f_C - \frac{2\,k}{\overline{D}_R}\left(r - \frac{1}{2}\right)\right] + \; 3\left[\frac{2\,k}{\overline{D}_R}\left(r - \frac{1}{2}\right) + 1 - f_R\right] - 2\right\}a$$

$$+ \left\{18\left[f_R - f_C - \frac{2\,k}{\overline{D}_R}\left(r - \frac{1}{2}\right)\right] + 9\left[\frac{2\,k}{\overline{D}_R}\left(r - \frac{1}{2}\right) + 1 - f_R\right] + 1\right\}u$$

$$+ \left\{6\left[f_R - f_C - \frac{2\,k}{\overline{D}_R}\left(r - \frac{1}{2}\right)\right] + \; 3\left[\frac{2\,k}{\overline{D}_R}\left(r - \frac{1}{2}\right) + 1 - f_R\right] + 1\right\}c_v$$

$$- \left\{ 12 \left[f_R - f_C - \frac{2k}{\overline{D}_R} \left(r - \frac{1}{2} \right) \right] + 6 \left[\frac{2k}{\overline{D}_R} \left(r - \frac{1}{2} \right) + 1 - f_R \right] - 1 \right\} \underline{c}$$

$$- 6 \left[f_R - f_C - \frac{2k}{\overline{D}_R} \left(r - \frac{1}{2} \right) \right] \frac{2}{3} [\overline{\alpha} - \underline{\alpha}] [c_H - c_L] . \tag{154}$$

Define $$\varphi \equiv 2 \left[f_R - f_C - \frac{2k}{\overline{D}_R} \left(r - \frac{1}{2} \right) \right] + \left[\frac{2k}{\overline{D}_R} \left(r - \frac{1}{2} \right) + 1 - f_R \right] \tag{155}$$

$$= 2 f_R - 2 f_C - \frac{4k}{\overline{D}_R} \left[r - \frac{1}{2} \right] + \frac{2k}{\overline{D}_R} \left[r - \frac{1}{2} \right] + 1 - f_R$$

$$= 1 + f_R - 2 f_C - \frac{2k}{\overline{D}_R} \left[r - \frac{1}{2} \right] < 1 + f_R - 2 f_C . \tag{156}$$

(154) and (155) imply:

$$\gamma(0) = [3\varphi - 2] a + [9\varphi + 1] u + [3\varphi + 1] c_v - [6\varphi - 1] \underline{c}$$

$$- 6 \left[f_R - f_C - \frac{2k}{\overline{D}_R} \left(r - \frac{1}{2} \right) \right] \frac{2}{3} [\overline{\alpha} - \underline{\alpha}] [c_H - c_L]$$

$$= 3\varphi [a + 3u + c_v - 2\underline{c}] - [2a - u - c_v - \underline{c}]$$

$$- 6 \left[f_R - f_C - \frac{2k}{\overline{D}_R} \left(r - \frac{1}{2} \right) \right] \frac{2}{3} [\overline{\alpha} - \underline{\alpha}] [c_H - c_L] \tag{157}$$

$$\Rightarrow \gamma(0) < 0 \Leftrightarrow 3\varphi [a + 3u + c_v - 2\underline{c}] - [2a - u - c_v - \underline{c}]$$

$$< 6 \left[f_R - f_C - \frac{2k}{\overline{D}_R} \left(r - \frac{1}{2} \right) \right] \frac{2}{3} [\overline{\alpha} - \underline{\alpha}] [c_H - c_L] . \tag{158}$$

Since $r \in [\frac{1}{2}, 1]$, (158) implies:

$$\gamma(0) < 0 \text{ if } 3\varphi [a + 3u + c_v - 2\underline{c}] - [2a - u - c_v - \underline{c}]$$

$$< 6 \left[f_R - f_C - \frac{k}{\overline{D}_R} \right] \frac{2}{3} [\overline{\alpha} - \underline{\alpha}] [c_H - c_L] . \tag{159}$$

(156) and (159) imply:

$$\gamma(0) < 0 \text{ if } 3 [1 + f_R - 2 f_C] [a + 3u + c_v - 2\underline{c}] - [2a - u - c_v - \underline{c}]$$

$$< 6 \left[f_R - f_C - \frac{k}{\overline{D}_R} \right] \frac{2}{3} [\overline{\alpha} - \underline{\alpha}] [c_H - c_L] \tag{160}$$

$$\Leftrightarrow 3 \left[1 + f_R - 2 f_C - \frac{2}{3} \right] a < [-9 (1 + f_R - 2 f_C) - 1] u - [3 (1 + f_R - 2 f_C) + 1] c_v$$

$$+ \; [\, 6\, (\, 1 + f_R - 2\, f_C\,) - 1\,]\, \underline{c} \; + \; 6 \left[f_R - f_C \; - \; \frac{k}{\overline{D}_R} \right] \frac{2}{3} \, [\, \overline{\alpha} - \underline{\alpha}\,]\, [\, c_H - c_L\,]$$

$$\Leftrightarrow \quad \left[f_R - 2\, f_C + \frac{1}{3} \right] a \; < \; - \left[1 + f_R - 2\, f_C + \frac{1}{9} \right] 3\, u - \left[1 + f_R - 2\, f_C + \frac{1}{3} \right] c_v$$

$$+ \; \left[1 + f_R - 2\, f_C - \frac{1}{6} \right] 2\, \underline{c} \; + \left[f_R - f_C \; - \; \frac{k}{\overline{D}_R} \right] \frac{4}{3} \, [\, \overline{\alpha} - \underline{\alpha}\,]\, [\, c_H - c_L\,]$$

$$\Leftrightarrow \quad \left[f_R - 2\, f_C + \frac{1}{3} \right] a \; < \; - \left[f_R - 2\, f_C + \frac{10}{9} \right] 3\, u - \left[f_R - 2\, f_C + \frac{4}{3} \right] c_v$$

$$+ \; \left[f_R - 2\, f_C + \frac{5}{6} \right] 2\, \underline{c} \; + \; \frac{4}{3} \, [\, \overline{\alpha} - \underline{\alpha}\,]\, [\, c_H - c_L\,] \left[f_R - f_C \; - \; \frac{k}{\overline{D}_R} \right]$$

$$\Rightarrow \quad \gamma\, (0) \;\; < \;\; 0 \;\; \text{if} \;\; f_R - 2\, f_C + \frac{1}{3} \;\; < \;\; 0 \;\; \text{and Assumption A1 holds.} \qquad (161)$$

From (33), $w \; < \; \widetilde{w} \; = \; \frac{1}{4}\, [\, a + 3\, u + c_v - 2\, \underline{c}\,]$ at the solution to [RP]. From (152):

$$\gamma\, (\widetilde{w}) \;\; = \; -\, 6 \left[f_R - f_C \; - \; \frac{2\, k}{\overline{D}_R} \left(r - \frac{1}{2} \right) \right] \frac{2}{3} \, [\, \overline{\alpha} - \underline{\alpha}\,]\, [\, c_H - c_L\,]$$

$$- \; \left[2\, a - \frac{a + 3\, u + c_v - 2\, \underline{c}}{4} - u - \underline{c} - c_v \right]$$

$$= \; -\, 6 \left[f_R - f_C \; - \; \frac{2\, k}{\overline{D}_R} \left(r - \frac{1}{2} \right) \right] \frac{2}{3} \, [\, \overline{\alpha} - \underline{\alpha}\,]\, [\, c_H - c_L\,]$$

$$- \; \frac{1}{4}\, [\, 7\, a - 3\, u - c_v + 2\, \underline{c} - 4\, u - 4\, \underline{c} - 4\, c_v\,]$$

$$= \; -\, 4 \left[f_R - f_C \; - \; \frac{2\, k}{\overline{D}_R} \left(r - \frac{1}{2} \right) \right] \, [\, \overline{\alpha} - \underline{\alpha}\,]\, [\, c_H - c_L\,] - \; \frac{1}{4}\, [\, 7\, a - 7\, u - 5\, c_v - 2\, \underline{c}\,]$$

$$< \; -\, 4 \left[f_R - f_C \; - \; \frac{k}{\overline{D}_R} \right] [\, \overline{\alpha} - \underline{\alpha}\,]\, [\, c_H - c_L\,] - \frac{1}{4}\, [\, 7\, a - 7\, u - 5\, c_v - 2\, \underline{c}\,] \; < \; 0\, . \qquad (162)$$

The last inequality in (162) reflects Assumption A1.

Observe from (152) that $\gamma\, (w)$ is linear in w. Therefore, since $w < \widetilde{w}$ at the solution to [RP], $\gamma\, (w) \, < \, 0$ for all relevant w if $\gamma\, (0) \; < \; 0$ and $\gamma\, (\widetilde{w}) \; < \; 0$. Consequently, the proposition follows from (149), (161), and (162). ■

Proof of Proposition 7.

(148) implies:

$$\frac{dW^*}{d\underline{d}} = D_C \left[1 - f_C - \left(\frac{2\,k}{\overline{D}_R} \left[r - \frac{1}{2} \right] + 1 - f_R \right) + \mu \right]$$

$$= D_C \left[f_R - f_C - \frac{2\,k}{\overline{D}_R} \left(r - \frac{1}{2} \right) + \mu \right]$$

$$> D_C \left[f_R - f_C - \frac{k}{\overline{D}_R} + \mu \right] > 0 \text{ when } f_R - f_C - \frac{k}{\overline{D}_R} > 0 . \quad \blacksquare$$

Conclusion 4. (39) holds if Assumption 1 holds.

Proof. Assumption A1 ensures:

$$7\,a - 7\,u - 5\,c_v - 2\,\underline{c} > 16 \left[\frac{k}{\overline{D}_R} + 1 - f_R \right] [\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,] . \tag{163}$$

Observe that:

$$\frac{\frac{1}{9\,b}\left[\,2\,a - w^*(\overline{D}_R) - u - \underline{c} - c_v\,\right]}{2\,k\left[\,r^*(\overline{D}_R) - \frac{1}{2}\,\right] + \overline{D}_R\,[\,1 - f_R\,]} > \frac{2\,[\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,]}{9\,b\,\overline{D}_R}$$

$$\Leftrightarrow \quad 2\,a - w^*(\overline{D}_R) - u - \underline{c} - c_v$$

$$> 2 \left[\frac{2\,k}{\overline{D}_R} \left(r^*(\overline{D}_R) - \frac{1}{2} \right) + 1 - f_R \right] [\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,] . \tag{164}$$

Since $r^*(\overline{D}_R) \in [\,\frac{1}{2}, 1\,]$, (164) holds if:

$$2\,a - w^*(\overline{D}_R) - u - \underline{c} - c_v > 2 \left[\frac{k}{\overline{D}_R} + 1 - f_R \right] [\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,] . \tag{165}$$

Because $w^*(\overline{D}_R) \leq \frac{1}{4}\,[\,a + 3\,u + c_v - 2\,\underline{c}\,]$:

$$2\,a - w^*(\overline{D}_R) - u - \underline{c} - c_v \geq 2\,a - \frac{1}{4}\,[\,a + 3\,u + c_v - 2\,\underline{c}\,] - u - \underline{c} - c_v$$

$$= \frac{1}{4}\,[\,8\,a - a - 3\,u - c_v + 2\,\underline{c} - 4\,u - 4\,\underline{c} - 4\,c_v\,] = \frac{1}{4}\,[\,7\,a - 7\,u - 5\,c_v - 2\,\underline{c}\,] .$$

Therefore, (164) holds if

$$\frac{1}{4}\,[\,7\,a - 7\,u - 5\,c_v - 2\,\underline{c}\,] > 2 \left[\frac{k}{\overline{D}_R} + 1 - f_R \right] [\,\overline{\alpha} - \underline{\alpha}\,]\,[\,c_H - c_L\,] . \tag{166}$$

(163) implies that (166) holds.

It remains to show that when Assumption 1 holds:

$$\frac{a + 3\,u + c_v - 2\,\underline{c} - 4\,w^*(\overline{D}_R)}{3\,b\,\overline{D}_R} \;>\; \frac{\frac{1}{9\,b}\left[\,2\,a - w^*(\overline{D}_R) - u - \underline{c} - c_v\,\right]}{2\,k\left[\,r^*(\overline{D}_R) - \frac{1}{2}\,\right] + \overline{D}_R\,[\,1 - f_R\,]}\,. \tag{167}$$

Assumption 1 holds if and only if:

$$3\,[\,1 - f_R\,]\sqrt{(\,a - \;u + c_v - 2\,\underline{c}\,)^2 - 24\,b\left(\frac{\overline{D}_R}{2} + \,\underline{d}\;D_C + F_u\right)} \;>\; 2\,a - 2\,u - \underline{c} - c_v\,. \tag{168}$$

(167) holds if and only if:

$$a + 3\,u + c_v - 2\,\underline{c} - 4\,w^*(\overline{D}_R) \;>\; \frac{1}{3}\left[\frac{2\,a - w^*(\overline{D}_R) - u - \underline{c} - c_v}{\frac{2\,k}{\overline{D}_R}\left(r^*(\overline{D}_R) - \frac{1}{2}\right) + 1 - f_R}\right]. \tag{169}$$

Since $r^*(\overline{D}_R) \geq \frac{1}{2}$:

$$\frac{2\,a - w^*(\overline{D}_R) - u - \underline{c} - c_v}{\frac{2\,k}{\overline{D}_R}\left[\,r^*(\overline{D}_R) - \frac{1}{2}\,\right] + 1 - f_R} \;\leq\; \frac{2\,a - w^*(\overline{D}_R) - u - \underline{c} - c_v}{1 - f_R}\,.$$

Therefore, (169) holds (and so (167) holds) if:

$$a + 3\,u + c_v - 2\,\underline{c} - 4\,w^*(\overline{D}_R) \;>\; \frac{1}{3}\left[\frac{2\,a - w^*(\overline{D}_R) - u - \underline{c} - c_v}{1 - f_R}\right]$$

$$\Leftrightarrow \quad 3\,[\,1 - f_R\,]\left[\,a + 3\,u + c_v - 2\,\underline{c} - 4\,w^*(\overline{D}_R)\right] \;>\; 2\,a - w^*(\overline{D}_R) - u - \underline{c} - c_v\,. \tag{170}$$

Since $w^*(\overline{D}_R) \geq u$ to ensure non-negative upstream profit for V:

$$2\,a - w^*(\overline{D}_R) - u - \underline{c} - c_v \;\leq\; 2\,a - 2\,u - \underline{c} - c_v\,.$$

Therefore, (170) holds (and so (167) holds) if:

$$3\,[\,1 - f_R\,]\left[\,a + 3\,u + c_v - 2\,\underline{c} - 4\,w^*(\overline{D}_R)\right] \;>\; 2\,a - 2\,u - \underline{c} - c_v\,. \tag{171}$$

From (4):

$$\begin{aligned}
&[\,a + 3\,u + c_v - 2\,\underline{c}\,]^2 - 24\,b\,\phi \\
&= [\,a + 3\,u + c_v - 2\,\underline{c}\,]^2 - 24\,b\left[\frac{u}{3\,b}\,(\,a + u + c_v - 2\,\underline{c}\,) + \left(1 - r^*(\overline{D}_R)\right)\overline{D}_R + \,\underline{d}\;D_C + F_u\right] \\
&= [\,a + 3\,u + c_v - 2\,\underline{c}\,]^2 - 8\,u\,[\,a + u + c_v - 2\,\underline{c}\,] - 24\,b\left[\left(1 - r^*(\overline{D}_R)\right)\overline{D}_R + \,\underline{d}\;D_C + F_u\,\right] \\
&= [\,a + 3\,u + c_v - 2\,\underline{c}\,]^2 - 8\,u\,[\,a + 3\,u + c_v - 2\,\underline{c} - 2\,u\,] \\
&\qquad - \;24\,b\left[\left(1 - r^*(\overline{D}_R)\right)\overline{D}_R + \,\underline{d}\;D_C + F_u\,\right]
\end{aligned}$$

$$
\begin{aligned}
&= \; [\, a + 3\,u + c_v - 2\,\underline{c}\,]^2 - 8\,u\,[\,a + 3\,u + c_v - 2\,\underline{c}\,] + 16\,u^2 \\
&\qquad\qquad - \; 24\,b\left[\,\left(1 - r^*(\overline{D}_R)\right)\overline{D}_R + \,\underline{d}\; D_C + F_u\,\right] \\
&= \; [\, a + 3\,u + c_v - 2\,\underline{c}\, - 4\,u\,]^2 - 24\,b\left[\,\left(1 - r^*(\overline{D}_R)\right)\overline{D}_R + \,\underline{d}\; D_C + F_u\,\right] \\
&= \; [\, a - \; u + c_v - 2\,\underline{c}\,]^2 - 24\,b\left[\,\left(1 - r^*(\overline{D}_R)\right)\overline{D}_R + \,\underline{d}\; D_C + F_u\,\right] \\
&\geq \; [\, a - \; u + c_v - 2\,\underline{c}\,]^2 - 24\,b\left[\,\frac{1}{2}\;\overline{D}_R + \,\underline{d}\; D_C + F_u\,\right]. \qquad (172)
\end{aligned}
$$

The inequality in (172) holds because $r^*(\overline{D}_R) \in (\,\frac{1}{2}, 1\,)$.

(33) and (172) imply:

$$
a + 3\,u + c_v - 2\,\underline{c} - 4\;w^*(\overline{D}_R) \;\geq\; \sqrt{[\, a - \; u + c_v - 2\,\underline{c}\,]^2 - 24\,b\left[\,\frac{1}{2}\;\overline{D}_R + \,\underline{d}\; D_C + F_u\,\right]}
$$

$$
\begin{aligned}
&\Rightarrow \;\; 3\,[\,1 - f_R\,]\left[\,a + 3\,u + c_v - 2\,\underline{c} - 4\;w^*(\overline{D}_R)\,\right] \\
&\qquad\qquad \geq \; 3\,[\,1 - f_R\,]\sqrt{[\, a - \; u + c_v - 2\,\underline{c}\,]^2 - 24\,b\left[\,\frac{1}{2}\;\overline{D}_R + \,\underline{d}\; D_C + F_u\,\right]}\,. \qquad (173)
\end{aligned}
$$

(168) and (173) ensure that (171) holds (and so (167) holds) when Assumption 1 holds. ■

Technical Appendix B

This appendix identifies conditions under which the behavioral constraint (BC) does not bind at the solution to [RP] and characterizes the optimal regulatory policy in this case.

Observation B1. *V's equilibrium expected profit increases as E's output increases if and only if V's upstream profit margin $(w-u)$ exceeds its equilibrium downstream profit margin $(P-u-c_v)$.*

<u>Proof</u>. $\frac{\partial \widehat{\pi}_v}{\partial x_e} = w - u + P'(\cdot)\, x_v$ from (6). Furthermore, given x_e, V's profit-maximizing choice of x_v is determined by $\frac{\partial \widehat{\pi}_v}{\partial x_v} = P(\cdot) - u - c_v + P'(\cdot)\, x_v = 0$. Therefore:

$$\frac{\partial \widehat{\pi}_v}{\partial x_e} = w - u - (\, P(\cdot) - u - c_v\,) \gtreqless 0 \;\Leftrightarrow\; w - u \gtreqless P(\cdot) - u - c_v\,. \quad \blacksquare$$

Observation B1 implies that V will not wish to raise E's cost when V's upstream profit margin exceeds its equilibrium downstream profit margin.[6] Lemmas B1 and B2 help to identify exogenous conditions under which V will have no incentive to raise E's cost in equilibrium.

Lemma B1. *When $D_R > 0$, the PC curve and the BC curve both have a negative slope and the PC curve is more steeply sloped than the BC curve in (w, r) space. When $D_R = 0$, both the PC curve and the BC curve are vertical straight lines in (w, r) space.*

<u>Proof</u>. The first conclusion reflects (35) and (37). The second conclusion reflects Claim 8 in the proof of Conclusion 2 in Appendix A. $\blacksquare$

From (38):
$$\left.\frac{\partial r}{\partial w}\right|_{dW=0} = -\,\frac{\frac{1}{9b}\left[\,2a - w\left(\tfrac{1}{2}\right)\big|_{PC} - u - \underline{c} - c_v\,\right]}{D_R\,[\,1 - f_R\,]} \quad \text{at } r = \frac{1}{2}, \text{ and}$$

$$\left.\frac{\partial r}{\partial w}\right|_{dW=0} = -\,\frac{\frac{1}{9b}\left[\,2a - w\,(1)|_{PC} - u - \underline{c} - c_v\,\right]}{k + D_R\,[\,1 - f_R\,]} \quad \text{at } r = 1, \tag{174}$$

where:
$$\left. w\left(\frac{1}{2}\right)\right|_{PC1} = \frac{1}{4}\left[\,a + 3u + c_v - 2\,\underline{c}\,\right] - \frac{1}{4}\left(\left[\,a - u + c_v - 2\,\underline{c}\,\right]^2 - 24\,b\left[\frac{D_R}{2} + \underline{d}\,D_C + F_u\right]\right)^{\frac{1}{2}}, \text{ and} \tag{175}$$

$$w\,(1)|_{PC1} = \frac{1}{4}\left[\,a + 3u + c_v - 2\,\underline{c}\,\right]$$

[6] Observe that $P(\cdot) - u - c_v$ can be viewed as V's marginal opportunity cost of providing access to E (Baumol, Ordover, and Willig, 1997).

$$- \frac{1}{4} \left([\, a - \; u + c_v - 2\, \underline{c} \,]^2 - 24\, b\, [\, \underline{d} \; D_C + F_u \,] \right)^{\frac{1}{2}} . \tag{176}$$

(175) and (176) follow from (1) and (4) because:

$$\begin{aligned} & [\, a + 3\, u + c_v - 2\, \underline{c} \,]^2 - \; 24\, b \, \frac{u}{3\, b} \, [\, a + u + c_v - 2\, \underline{c} \,] \\ & \qquad = \; [\, a + 3\, u + c_v - 2\, \underline{c} \,]^2 - 8\, u\, [\, a + u + c_v - 2\, \underline{c} \,] \\ & \qquad = \; [\, a + 3\, u + c_v - 2\, \underline{c} \,]^2 - 8\, u\, [\, a + 3\, u + c_v - 2\, \underline{c} \,] + 16\, u^2 \\ & \qquad = \; [\, a + 3\, u + c_v - 2\, \underline{c} \; - 4\, u]^2 \; = \; [\, a - \; u + c_v - 2\, \underline{c} \,]^2 . \end{aligned} \tag{177}$$

Also, from (2):

$$w(\frac{1}{2})\Big|_{BC} \; = \; - \frac{\left[\, \overline{d} - \underline{d} \,\right] D_C}{\frac{4}{9\, b} \left[\, \overline{q} - \underline{q} \,\right] [\, c_H - c_L \,]} + \frac{1}{4} [\, 2\, a + 2\, u + c_L + c_H - 4\, c_v \,] , \; \text{and} \tag{178}$$

$$w\, (1)|_{BC} \; = \; - \frac{D_R + \left[\, \overline{d} - \underline{d} \,\right] D_C}{\frac{4}{9\, b} \left[\, \overline{q} - \underline{q} \,\right] [\, c_H - c_L \,]} + \frac{1}{4} [\, 2\, a + 2\, u + c_L + c_H - 4\, c_v \,] . \tag{179}$$

Proposition B1. *Only the PC binds at the solution to [RP] if:*

$$a - \; u + c_L + c_H - 5\, c_v + 2\, \underline{c} \; < \; \frac{\left[\, \overline{d} - \underline{d} \,\right] D_C}{\frac{1}{9\, b} \left[\, \overline{q} - \underline{q} \,\right] [\, c_H - c_L \,]} - \left([\, a - u + c_v - 2\, \underline{c} \,]^2 - 24\, b\, [\, \underline{d} \; D_C + F_u \,] \right)^{\frac{1}{2}} . \tag{180}$$

Proof. The proof proceeds by demonstrating that: (i) the $PC1$ curve, the BC curve, and the iso-W constraints are all downward sloping in (w, r) space; (ii) the $PC1$ curve is more steeply sloped than the BC curve; (iii) the $PC1$ curve lies to the right of the BC curve at $r = \frac{1}{2}$.

Lemma B1 establishes that both the $PC1$ curve and the BC curve have a negative slope in (w, r) space, and the $PC1$ curve is everywhere more steeply sloped than the BC curve for $r \in \left[\frac{1}{2}, 1\right]$. (175) and (178) imply that $w\left(\frac{1}{2}\right)\big|_{BC} < \; w\left(\frac{1}{2}\right)\big|_{PC1}$ when (180) holds, so the $PC1$ curve and the BC curve do not intersect and the BC curve lies to the left of the $PC1$ curve (and the $PC2$ curve), and so is not constraining (since an iso-W curve that is closer to the origin in (w, r) space represents a higher level of W, from (3)).

Finally, note that $\frac{\partial}{\partial D_R} \left([\, a - u + c_v - 2\, \underline{c} \,]^2 - 24\, b \left[\, \frac{D_R}{2} + \underline{d} \; D_C + F_u \,\right] \right) < \; 0$. Therefore, the inequality in (180) will hold for all $D_R \geq 0$ if it holds for $D_R = 0$. ■

Proposition B2. *If only the PC binds at the solution to [RP], then* $r^* \; = \; \frac{1}{2}$.

Proof. When only the PC binds, [RP] is as specified in Conclusion 1, except that the BC in (2) is omitted. Let $\mathcal{L}_B$ denote the relevant Lagrangian function in this case. Then:

$$\frac{\partial \mathcal{L}_B}{\partial r} \;=\; -2\,k\,[\,r-\frac{1}{2}\,]-D_R\,[\,1-f_R\,-\lambda\,]\,; \tag{181}$$

$$\frac{\partial \mathcal{L}_B}{\partial w} \;=\; -\frac{1}{9\,b}\,[\,2\,a-w-u-\underline{c}-c_v\,]+\frac{\lambda}{3\,b}\,[\,a+3\,u+c_v-2\,\underline{c}\,-4\,w\,]\,;\text{ and} \tag{182}$$

$$\frac{\partial \mathcal{L}_B}{\partial D_R} \;=\; [\,1-r\,]\,[\,1-f_R\,-\lambda\,]\,. \tag{183}$$

The remainder of the proof consists primarily of the following findings.

Finding 1. $r \notin \left(\frac{1}{2},1\right)$ *at the solution to [RP].*

Proof. If $r \in \left(\frac{1}{2},1\right)$ at the solution to [RP], then (181) implies:

$$D_R\,[\,1-f_R\,-\lambda\,] \;=\; -2\,k\,[\,r-\frac{1}{2}\,]. \tag{184}$$

Continuing to let $\mathcal{L}$ denote the Lagrangian function associated with [RP], (183) and (184) imply:

$$\frac{\partial \mathcal{L}}{\partial D_R} \;=\; [\,1-r\,]\,[\,1-f_R\,-\lambda\,] \;=\; -\frac{2\,k}{D_R}\,[\,r-\frac{1}{2}\,]\,[\,1-r\,]\,. \tag{185}$$

(185) implies $\frac{\partial \mathcal{L}}{\partial D_R} < 0$ for all $D_R > 0$ because $r \in \left(\frac{1}{2},1\right)$ by assumption. Therefore, $D_R = 0$. Consequently, from (181), for all $r > \frac{1}{2}$:

$$\frac{\partial \mathcal{L}}{\partial r} \;=\; -2\,k\,[\,r-\frac{1}{2}\,]-D_R\,[\,1-f_R\,-\lambda\,] \;=\; -2\,k\,[\,r-\frac{1}{2}\,] \;<\; 0\,. \tag{186}$$

(186) implies $r \notin \left(\frac{1}{2},1\right)$ at the solution to [RP].

Finding 2. *It is not the case that* $r = 1$ *and* $D_R = 0$ *at the solution to [RP].*

Proof. From (181), $\frac{\partial \mathcal{L}}{\partial r}\Big|_{D_R=0} \;=\; -2\,k\left[\,r-\frac{1}{2}\,\right] \;<\; 0$ for all $r > \frac{1}{2}$. Therefore, $r < 1$ if $D_R = 0$ at a solution to [RP]. ■

Finding 3. *Suppose*

$$2\,a-w^*-u-\underline{c}-c_v \;>\; 3\,[\,1-f_R\,]\,[\,a+3\,u+c_v-2\,\underline{c}-4\,w^*\,]\,, \tag{187}$$

$$\text{where}\quad r^* \;=\; \frac{1}{2}\,;\;\; D_R \;=\; 0\,;\;\; \lambda^* \;=\; \frac{2a-w^*-u-\underline{c}-c_v}{3\,[\,a+3\,u+c_v-2\,\underline{c}-4\,w^*]}\,;\;\; \text{and} \tag{188}$$

$$w^* \;=\; \frac{1}{4}\,[\,a+3\,u+c_v-2\,\underline{c}\,]-\frac{1}{4}\left(\,[\,a-u+c_v-2\,\underline{c}\,]^2-24\,b\,[\,\underline{d}\;D_C+F_u\,]\,\right)^{\frac{1}{2}}. \tag{189}$$

Then these values of r^*, w^*, D_R, *and* λ^* *satisfy the necessary conditions for a solution to [RP].*

Proof. (182) implies that λ^* must be as specified in (188) to ensure $\frac{\partial \mathcal{L}}{\partial w} = 0$. Because $\lambda^* > 0$, the PC must hold as an equality. (175) implies that w^* must be as specified in (189) to ensure the PC holds as an equality when $D_R = 0$ and $r = \frac{1}{2}$.

(183), (187), and (188) imply that if $r < 1$, then for all $D_R \in \left[0, \overline{D}_R\right]$:

$$\frac{\partial \mathcal{L}}{\partial D_R} = [1 - r][1 - f_R - \lambda^*] \overset{s}{=} 1 - f_R - \frac{2a - w^* - u - \underline{c} - c_v}{3[a + 3u + c_v - 2\underline{c} - 4w^*]} < 0. \tag{190}$$

(190) implies that when (187) holds, D_R must be 0 to satisfy the relevant necessary condition for a solution to [RP] when $r < 1$.

From (181), $\left.\frac{\partial \mathcal{L}}{\partial r}\right|_{D_R=0} = -2k\left[r - \frac{1}{2}\right] < 0$ for all $r > \frac{1}{2}$. Therefore, r must be $\frac{1}{2}$ to satisfy the relevant necessary condition for a solution to [RP] when $D_R = 0$. ■

Finding 4. *Suppose λ^* and w^* are as specified in (188) and (189), respectively, and:*

$$2a - w^* - u - \underline{c} - c_v > 3\left[\frac{k}{D_R} + 1 - f_R\right][a + 3u + c_v - 2\underline{c} - 4w^*], \tag{191}$$

where $r^* = 1$ and $D_R \in \left(0, \overline{D}_R\right]$. Then these values of r^*, w^*, D_R, and λ^* satisfy the necessary conditions for a solution to [RP].

Proof. (182) implies that λ^* must be as specified in (188) to ensure $\frac{\partial \mathcal{L}}{\partial w} = 0$. Because $\lambda^* > 0$, the PC must hold as an equality. (176) implies that w^* must be as specified in (189) to ensure the PC holds as an equality when $r = 1$.

Since $D_R > 0$, (181) and (188) imply:

$$\left.\frac{\partial \mathcal{L}}{\partial r}\right|_{r=1} = -2k\left[r - \frac{1}{2}\right] - D_R[1 - f_R - \lambda^*] = -k - D_R[1 - f_R - \lambda^*] \geq 0$$

$$\Leftrightarrow \lambda^* = \frac{2a - w^* - u - \underline{c} - c_v}{3[a + 3u + c_v - 2\underline{c} - 4w^*]} \geq \frac{k}{D_R} + 1 - f_R. \tag{192}$$

(192) holds when (191) holds. Therefore, when (191) holds, $r = 1$ satisfies the relevant necessary condition for a solution to [RP].

From (183), $\left.\frac{\partial \mathcal{L}}{\partial D_R}\right|_{r=1} = [1 - r][1 - f_R - \lambda] = 0$ for all $D_R \in \left(0, \overline{D}_R\right]$. Therefore, when $r = 1$, any $D_R \in \left(0, \overline{D}_R\right]$ satisfies the relevant necessary condition for a solution to [RP]. ■

Finding 5. *Suppose λ^* is as specified in (188) and*

$$2a - w^* - u - \underline{c} - c_v < 3[1 - f_R][a + 3u + c_v - 2\underline{c} - 4w^*], \tag{193}$$

$$\textit{where} \quad r^* = \frac{1}{2}, \quad D_R = \overline{D}_R, \quad \textit{and} \tag{194}$$

$$w^* = \frac{1}{4}[a + 3u + c_v - 2\underline{c}]$$

$$- \frac{1}{4} \left([\, a - u + c_v - 2\,\underline{c}\,]^2 - 24\, b \left[\frac{D_R}{2} + \underline{d}\; D_C + F_u \right] \right)^{\frac{1}{2}} . \tag{195}$$

Then these values of r^*, w^*, D_R, and λ^* satisfy the necessary conditions for a solution to [RP].

Proof. (182) implies that λ^* must be as specified in (188) to ensure $\frac{\partial \mathcal{L}}{\partial w} = 0$. Because $\lambda^* > 0$, the PC must hold as an equality. (175) implies that w^* must be as specified in (195) to ensure the PC holds as an equality when $r = \frac{1}{2}$.

If $D_R > 0$, then (181) and (188) imply:

$$\frac{\partial \mathcal{L}}{\partial r} \;=\; -\,2\,k \left[r - \frac{1}{2} \right] - D_R\,[\,1 - f_R - \lambda^*\,] \;\le\; -\,D_R\,[\,1 - f_R - \lambda^*\,]$$

$$< \; 0 \quad \Leftrightarrow \quad \lambda^* \;=\; \frac{2\,a - w^* - u - \underline{c} - c_v}{3\,[\,a + 3\,u + c_v - 2\,\underline{c} - 4\,w^*\,]} \;<\; 1 - f_R\,. \tag{196}$$

(196) implies that r must be $\frac{1}{2}$ to satisfy the relevant necessary condition for a solution to [RP] when $D_R > 0$ and (193) holds.

From (183) and (193), if $r < 1$, then $\frac{\partial \mathcal{L}}{\partial D_R} = [\,1 - r\,]\,[\,1 - f_R - \lambda^*\,] > 0$ for all $D_R \in \left[\,0, \overline{D}_R\,\right]$, so $D_R = \overline{D}_R$ satisfies the relevant necessary condition for a solution to [RP]. ■

Finding 6. *Suppose* $2\,a - w^* - u - \underline{c} - c_v \;=\; 3\,[\,1 - f_R\,]\,[\,a + 3\,u + c_v - 2\,\underline{c} - 4\,w^*\,]\,,$ (197)

where $r^* = \frac{1}{2}$*;* $\lambda^* = 1 - f_R$*, and* w^* *and* D_R *solve:*

$$\frac{w^*}{3\,b}\,[\,a + 3\,u + c_v - 2\,\underline{c}\,] - \frac{2\,(w^*)^2}{3\,b} \;=\; \frac{u}{3\,b}\,[\,a + u + c_v - 2\,\underline{c}\,] + \frac{D_R}{2} + \underline{d}\; D_C + F_u\,. \tag{198}$$

Then if $D_R \in [\,0, \overline{D}_R\,]$, these values of r^*, w^*, D_R, and λ^* satisfy the necessary conditions for a solution to [RP].

Proof. (182) implies that $\lambda^* = \frac{2\,a - w^* - u - \underline{c} - c_v}{3\,[\,a + 3\,u + c_v - 2\,\underline{c} - 4\,w^*\,]}$ at a solution to [RP]. Therefore, $\lambda^* = 1 - f_R$ when (198) holds. Consequently, (181) implies that when $r^* = \frac{1}{2}$:

$$\frac{\partial \mathcal{L}}{\partial D_R} \;=\; [\,1 - r\,]\,[\,1 - f_R - \lambda^*\,] \;=\; 0 \;\; \text{for all} \;\; D_R \in \left[\,0, \overline{D}_R\,\right].$$

(1) and (4) imply that when $r = \frac{1}{2}$ and (197) holds, the PC is as specified in (198). Therefore, under the stated conditions, the identified values of r^*, w^*, D_R, and λ^* satisfy the necessary conditions for a solution to [RP], provided $D_R \in [\,0, D_R\,]$. ■

Observe that the sets of values of r^*, w^*, D_R, and λ^* identified in Findings 3 – 6 are the only potential solutions to [RP]. This is the case because $r^* \notin \left(\frac{1}{2}, 1\right)$ at the solution to [RP]. Therefore, the only possible solutions to [RP] are of the form: (i) $r^* = \frac{1}{2}$, $D_R = 0$; (ii) $r^* = \frac{1}{2}$, $D_R = \overline{D}_R$; (iii) $r^* = \frac{1}{2}$, $D_R \in \left(0, \overline{D}_R\right)$; (iv) $r^* = 1$, $D_R = 0$; and (v) $r^* = 1$,

$D_R \in (0, \overline{D}_R\,]$. Finding 1 precludes possibility (iv). The other possibilities are accounted for in Findings 3, 4, 5, and 6.

Finding 7. *If (191) holds. Then* $W^*(\,r^* = \frac{1}{2},\, D_R = 0\,) \;=\; W^*(\,r^* = 1,\, D_R > 0\,) + \frac{k}{4}$.

Proof. (189) implies that if (187) and (191) hold, then $w^*(\,r^* = \frac{1}{2},\, D_R = 0\,) = w^*(\,r^* = 1,$ $D_R > 0\,)$. Therefore, (187) holds if (191) holds. The conclusion then follows from (3). ■

Finally, suppose that only the PC binds at the solution to [RP]. Then Findings 3 – 6 imply that if $r = 1$ at the solution to [RP], then (191) must hold. However, Finding 7 implies that if (191) holds, then $r \neq 1$. Therefore, $r = \frac{1}{2}$ because $r \notin (\frac{1}{2}, 1)$, from Finding 1. ■

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