Appendix to Accompany

"Welfare Effects of Limiting Bank Loans"

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Part I of this Appendix reviews the key elements of the analysis. Part II re-states and proves the formal conclusions in the text.

I. Key Elements of the Analysis.

The loan officer's formal problem, [PL], is:

$$\underset{G,\,B}{\text{Maximize}} \;\; U(G,B) \;\equiv\; r\,[\,g\,G + (\,1-b\,)\,B\,] - C(G,B)\,.$$

Assumption 1. $C(G,B) \,=\, c_g\,G^{k_g} + c_b\,B^{k_b}$, where $k_g,\, k_b \,>\, 1$ and $c_g, c_b > 0$.

The bank's problem in the setting with endogenous screening accuracies, [PB], is:

$$\underset{r,\,g,\,b}{\text{Maximize}} \;\; \Pi \;\equiv\; g\,[\,\pi_G - r\,]\,G + [\,1-b\,]\,[\,\pi_B - r\,]\,B - K(g,b)$$

$$\text{subject to: } \; g\,G + [\,1-b\,]\,B \;\leq\; L, \text{ where } (G,B) \in \arg\max \; U(\widetilde{G},\, \widetilde{B})\,. \tag{1}$$

In the quadratic cost setting: (i) $C(G,B) \;=\; \frac{1}{2}\,c_g\,G^2 + \frac{1}{2}\,c_b\,B^2$; and (ii) $K(g,b) \;=\; \frac{1}{2}\,\sigma_g\,[\,g-\frac{1}{2}\,]^2 + \frac{1}{2}\,\sigma_b\,[\,b-\frac{1}{2}\,]^2$.

II. Statements and Proofs of Formal Conclusions.

Observation 1. *At the solution to [PL]: (i)* $\frac{dG}{dr} > 0$ *if* $C_{BB} > \frac{1-b}{g}\,C_{GB}$*; (ii)* $\frac{dB}{dr} > 0$ *if* $C_{GG} > \frac{g}{1-b}\,C_{GB}$*; (iii)* $\frac{dG}{dg} > 0$*; (iv)* $\frac{dB}{dg} \overset{s}{=} -C_{GB}$*; (v)* $\frac{dB}{db} < 0$*; (vi)* $\frac{dG}{db} \overset{s}{=} C_{GB}$*.*

Proof. At the solution to [PL]:

$$r\,g \;=\; \frac{\partial C(G,B)}{\partial G} \quad \text{and} \quad r\,[\,1-b\,] \;=\; \frac{\partial C(G,B)}{\partial B} \tag{2}$$

$$\Rightarrow \;\; \begin{bmatrix} C_{GG} & C_{GB} \\ C_{GB} & C_{BB} \end{bmatrix} \begin{bmatrix} dG \\ dB \end{bmatrix} \;=\; \begin{bmatrix} g \\ 1-b \end{bmatrix} dr\,. \tag{3}$$

(3) and Cramer's Rule imply:

$$\frac{dG}{dr} \;=\; \left[\,C_{GG}\,C_{BB} - (C_{GB})^2\,\right]^{-1} \begin{vmatrix} g & C_{GB} \\ 1-b & C_{BB} \end{vmatrix}$$

$$\stackrel{s}{=} \; g\, C_{BB} - [\,1-b\,]\, C_{GB} \; > \; 0 \;\text{ if }\; C_{BB} \; > \; \left[\frac{1-b}{g} \right] C_{GB}\, ; \;\text{ and}$$

$$\frac{dB}{dr} \; = \; \left[C_{GG}\, C_{BB} - (C_{GB})^2 \right]^{-1} \begin{vmatrix} C_{GG} & g \\ C_{GB} & 1-b \end{vmatrix}$$

$$\stackrel{s}{=} \; [\,1-b\,]\, C_{GG} - g\, C_{GB} \; > \; 0 \;\text{ if }\; C_{GG} \; > \; \left[\frac{g}{1-b} \right] C_{GB}\, . \tag{4}$$

(2) also implies:

$$\begin{bmatrix} C_{GG} & C_{GB} \\ C_{GB} & C_{BB} \end{bmatrix} \begin{bmatrix} dG \\ dB \end{bmatrix} \; = \; \begin{bmatrix} r \\ 0 \end{bmatrix} dg\, . \tag{5}$$

(5) and Cramer's Rule imply:

$$\frac{dG}{dg} \; \stackrel{s}{=} \; \begin{vmatrix} r & C_{GB} \\ 0 & C_{BB} \end{vmatrix} \; = \; r\, C_{BB} \; > \; 0 \;\text{ and }\; \frac{dB}{dg} \; \stackrel{s}{=} \; \begin{vmatrix} C_{GG} & r \\ C_{GB} & 0 \end{vmatrix} \; = \; -\, r\, C_{GB}\, .$$

Finally, (2) implies:

$$\begin{bmatrix} C_{GG} & C_{GB} \\ C_{GB} & C_{BB} \end{bmatrix} \begin{bmatrix} dG \\ dB \end{bmatrix} \; = \; \begin{bmatrix} 0 \\ -\, r \end{bmatrix} db\, . \tag{6}$$

(6) and Cramer's Rule imply:

$$\frac{dG}{db} \; \stackrel{s}{=} \; \begin{vmatrix} 0 & C_{GB} \\ -\, r & C_{BB} \end{vmatrix} \; = \; r\, C_{GB} \;\text{ and }\; \frac{dB}{db} \; \stackrel{s}{=} \; \begin{vmatrix} C_{GG} & 0 \\ C_{GB} & -\, r \end{vmatrix} \; = \; -\, r\, C_{GG} \; < \; 0\, . \;\; \blacksquare$$

Observation 2. *Suppose Assumption 1 holds and the bank makes L loans in expectation. Then at the solution to [PL]: (i)* $\frac{dr}{dL} > 0$*; (ii)* $\frac{d}{dr}\left(\frac{G}{B}\right) \lesseqgtr 0 \;\Leftrightarrow\; k_g \gtreqless k_b$*; and (iii)* $\frac{d}{dL}\left(\frac{G}{B}\right) \lesseqgtr 0 \;\Leftrightarrow\; k_g \gtreqless k_b$*. If, in addition,* $w_B < 0$*, then:*

$$\frac{dW}{dL} \; \gtreqless \; 0 \;\; \Leftrightarrow \;\; \frac{w_G}{|\, w_B \,|} \; \gtreqless \; \frac{[\,1-b\,]\, [\,(1-b)\,/c_b\, k_b\,]^{\frac{1}{k_b - 1}}}{g\, [\,g/c_g\, k_g\,]^{\frac{1}{k_g - 1}}} \left[\frac{k_g - 1}{k_b - 1} \right] r^{\frac{k_g - k_b}{[\,k_g - 1\,][\,k_b - 1\,]}}\, . \tag{7}$$

Corollary 1. *If Assumption 1 holds,* $w_B < 0$*, and* $k_g = k_b = k$*, then:*

$$\frac{dW}{dL} \; \gtreqless \; 0 \;\; \Leftrightarrow \;\; \frac{w_G}{|\, w_B \,|} \; \gtreqless \; \left[\frac{1-b}{g} \right]^{\frac{k}{k-1}} \left[\frac{c_g}{c_b} \right]^{\frac{1}{k-1}}\, . \tag{8}$$

Proof. To prove conclusion (i) in the Observation, observe that when the bank makes L loans in expectation:

$$g\, G + [\,1-b\,]\, B \; = \; L \;\; \Rightarrow \;\; \left[g\, \frac{dG}{dr} + (\,1-b\,)\, \frac{dB}{dr} \right] dr \; = \; dL$$

$$\Rightarrow \quad \frac{dr}{dL} \;=\; \left[g\,\frac{dG}{dr} + (1-b)\,\frac{dB}{dr} \right]^{-1} \;>\; 0\,. \tag{9}$$

The inequality in (9) holds because, from (4):

$$g\,\frac{dG}{dr} + [1-b]\,\frac{dB}{dr} \;\stackrel{s}{=}\; g\,[\,g\,C_{BB} - (1-b)\,C_{GB}\,] + [1-b]\,[(1-b)\,C_{GG} - g\,C_{GB}\,]$$

$$=\; [1-b]^2\,C_{GG} + g^2\,C_{BB} - 2\,g\,[1-b]\,C_{GB} \;>\; 0\,.$$

The inequality here reflects Assumption 1.

We now prove conclusions (ii) and (iii) in the Observation. (2) implies that if Assumption 1 holds, then at the solution to [PL]:

$$r\,g \;=\; c_g\,k_g\,G^{k_g-1} \quad\Rightarrow\quad G \;=\; \left[\frac{r\,g}{c_g\,k_g}\right]^{\frac{1}{k_g-1}}, \text{ and}$$

$$r\,[1-b\,] \;=\; c_b\,k_b\,B^{k_b-1} \quad\Rightarrow\quad B \;=\; \left[\frac{r\,(1-b)}{c_b\,k_b}\right]^{\frac{1}{k_b-1}}. \tag{10}$$

(10) implies that at the solution to [PL]:

$$\frac{G}{B} \;=\; \frac{\left[\frac{g}{c_g\,k_g}\right]^{\frac{1}{k_g-1}}}{\left[\frac{1-b}{c_b\,k_b}\right]^{\frac{1}{k_b-1}}}\left[\,r^{\frac{1}{k_g-1}-\frac{1}{k_b-1}}\,\right] \;=\; \frac{\left[\frac{g}{c_g\,k_g}\right]^{\frac{1}{k_g-1}}}{\left[\frac{1-b}{c_b\,k_b}\right]^{\frac{1}{k_b-1}}}\left[\,r^{\frac{k_b-k_g}{[k_g-1][k_b-1]}}\,\right]. \tag{11}$$

(11) implies $\frac{d}{dr}\left(\frac{G}{B}\right) \lesseqgtr 0 \;\Leftrightarrow\; k_g \gtreqless k_b$. Consequently, conclusion (iii) in the Observation follows from (9) because $\frac{d}{dL}\left(\frac{G}{B}\right) = \frac{d}{dr}\left(\frac{G}{B}\right)\frac{dr}{dL} \stackrel{s}{=} \frac{d}{dr}\left(\frac{G}{B}\right)$.

To prove the last conclusion in the Observation and Corollary 1, observe that $\frac{dr}{dL} > 0$ from (9). Furthermore, (10) implies that expected welfare is:

$$W \;=\; w_G\,g\left[\frac{r\,g}{c_g\,k_g}\right]^{\frac{1}{k_g-1}} + w_B\,[1-b]\left[\frac{r\,(1-b)}{c_b\,k_b}\right]^{\frac{1}{k_b-1}}$$

$$\Rightarrow \quad \frac{dW}{dL} \;\stackrel{s}{=}\; \frac{dW}{dr} \;=\; w_G\,g\left[\frac{g}{c_g\,k_g}\right]^{\frac{1}{k_g-1}}\left[\frac{1}{k_g-1}\right] r^{\frac{2-k_g}{k_g-1}}$$

$$+\; w_B\,[1-b]\left[\frac{1-b}{c_b\,k_b}\right]^{\frac{1}{k_b-1}}\left[\frac{1}{k_b-1}\right] r^{\frac{2-k_b}{k_b-1}} \;\gtreqless\; 0$$

$$\Leftrightarrow \quad w_G\,g\left[\frac{g}{c_g\,k_g}\right]^{\frac{1}{k_g-1}}\left[\frac{1}{k_g-1}\right] r^{\frac{2-k_g}{k_g-1}} \;\gtreqless\; |\,w_B\,|\,[1-b]\left[\frac{1-b}{c_b\,k_b}\right]^{\frac{1}{k_b-1}}\left[\frac{1}{k_b-1}\right] r^{\frac{2-k_b}{k_b-1}}$$

$$\Leftrightarrow \quad \frac{w_G}{|\,w_B\,|} \;\gtreqless\; \frac{[1-b]\,[\,(1-b)\,/c_b\,k_b\,]^{\frac{1}{k_b-1}}}{g\,[\,g/c_g\,k_g\,]^{\frac{1}{k_g-1}}}\left[\frac{k_g-1}{k_b-1}\right] r^{\frac{k_g-k_b}{[k_g-1][k_b-1]}}\,. \tag{12}$$

(12) implies that when $k_g = k_b = k$:

$$\frac{dW}{dL} \gtreqless 0 \Leftrightarrow \frac{w_G}{|\, w_B \,|} \gtreqless \left[\frac{1-b}{g} \right]^{1+\frac{1}{k-1}} \left[\frac{c_g}{c_b} \right]^{\frac{1}{k-1}} = \left[\frac{1-b}{g} \right]^{\frac{k}{k-1}} \left[\frac{c_g}{c_b} \right]^{\frac{1}{k-1}} . \; \blacksquare$$

Observation 3. *Suppose $\widetilde{r} < \widehat{r} < r^*$ in the quadratic cost setting. Then: (i) $\widehat{g} < g^*$; whereas (ii) $\widehat{b}$ can either exceed b^* or be less than b^*.*

Proof. It is readily verified that, as in (10), at the solution to [PL] in the quadratic cost setting:

$$G = \frac{r\, g}{c_g} \text{ and } B = \frac{r\,[1-b]}{c_b} . \tag{13}$$

(13) implies that in the quadratic cost setting, the bank seeks to maximize:

$$\Pi \equiv g\,[\pi_G - r]\,\frac{r\,g}{c_g} + [1-b]\,[\pi_B - r]\,\frac{r\,[1-b]}{c_b} - \frac{\sigma_g}{2}\,[g - \frac{1}{2}]^2 - \frac{\sigma_b}{2}\,[b - \frac{1}{2}]^2. \tag{14}$$

Let $\widehat{\lambda}_L$ denote the value of the Lagrange multiplier associated with constraint (1) at the solution to [PB]. Then (14) implies that when constraint (1) binds in the quadratic cost setting, the necessary conditions for a solution to [PB] include:

$$[\pi_G - 2\,\widehat{r}\,]\,\frac{\widehat{g}^2}{c_g} + [\pi_B - 2\,\widehat{r}\,]\,\frac{\left[1-\widehat{b}\,\right]^2}{c_b} - \widehat{\lambda}_L \left[\frac{\widehat{g}^2}{c_g} + \frac{\left(1-\widehat{b}\right)^2}{c_b} \right] = 0 . \tag{15}$$

$$2\,[\pi_G - \widehat{r}\,]\,\frac{\widehat{r}\,\widehat{g}}{c_g} - \sigma_g \left[\widehat{g} - \frac{1}{2} \right] - 2\,\widehat{\lambda}_L\,\frac{\widehat{r}\,\widehat{g}}{c_g} = 0 . \tag{16}$$

$$-\,[\pi_B - \widehat{r}\,]\,\widehat{r}\,\frac{2\left[1-\widehat{b}\,\right]}{c_b} - \sigma_b \left[\widehat{b} - \frac{1}{2} \right] + \widehat{\lambda}_L\,\frac{2\,\widehat{r}\left[1-\widehat{b}\,\right]}{c_b} = 0 . \tag{17}$$

$$\widehat{r} \left[\frac{\widehat{g}^2}{c_g} + \frac{\left(1-\widehat{b}\right)^2}{c_b} \right] = L . \tag{18}$$

The corresponding necessary conditions for a solution to [PB] when constraint (1) does not bind are:

$$[\pi_G - 2\,r^*]\,\frac{(g^*)^2}{c_g} + [\pi_B - 2\,r^*]\,\frac{[1-b^*]^2}{c_b} = 0 . \tag{19}$$

$$2\,[\pi_G - r^*]\,\frac{r^*\,g^*}{c_g} - \sigma_g \left[g^* - \frac{1}{2} \right] = 0 . \tag{20}$$

$$- \left[\pi_B - r^* \right] r^* \, \frac{2 \left[1 - b^* \right]}{c_b} - \sigma_b \left[b^* - \frac{1}{2} \right] = 0 \, . \tag{21}$$

$$r^* \left[\frac{(g^*)^2}{c_g} + \frac{(1 - b^*)^2}{c_b} \right] = L + \delta \, , \text{ where } \delta > 0 \, . \tag{22}$$

Let $\widetilde{\lambda}_L$ denote the value of the Lagrange multiplier associated with constraint (1) at the solution to the corresponding problem in which: (i) the bank chooses r; and (ii) g and b are fixed at g^* and b^*, respectively. The necessary conditions for a solution to this problem include:

$$\left[\pi_G - 2 \, \widetilde{r} \, \right] \frac{(g^*)^2}{c_g} + \left[\pi_B - 2 \, \widetilde{r} \, \right] \frac{\left[1 - b^* \right]^2}{c_b} - \widetilde{\lambda}_L \left[\frac{(g^*)^2}{c_g} + \frac{(1 - b^*)^2}{c_b} \right] = 0 \, . \tag{23}$$

$$\widetilde{r} \left[\frac{(g^*)^2}{c_g} + \frac{(1 - b^*)^2}{c_b} \right] = L \, . \tag{24}$$

The proof of conclusion (i) in the Observation now follows from the following three Findings.

Finding 1. *If* $r^* > \widehat{r}$ *and* $\widehat{b} \geq b^*$, *then* $\widehat{g} < g^*$.

Proof. (16) implies:

$$2 \left[\pi_G - \widehat{r} - \widehat{\lambda}_L \right] \frac{\widehat{r} \, \widehat{g}}{c_g} = \sigma_g \left[\widehat{g} - \frac{1}{2} \right] \;\; \Rightarrow \;\; \pi_G - \widehat{r} - \widehat{\lambda}_L = \frac{\sigma_g \, c_g}{2} \left[\frac{\widehat{g} - \frac{1}{2}}{\widehat{g}} \right] \frac{1}{\widehat{r}} \, . \tag{25}$$

Similarly, (20) implies:

$$\pi_G - r^* = \frac{\sigma_g \, c_g}{2} \left[\frac{g^* - \frac{1}{2}}{g^*} \right] \frac{1}{r^*} \, . \tag{26}$$

Subtracting (26) from (25) provides:

$$r^* - \widehat{r} - \widehat{\lambda}_L = \frac{\sigma_g \, c_g}{2} \left[\left(\frac{\widehat{g} - \frac{1}{2}}{\widehat{g}} \right) \frac{1}{\widehat{r}} - \left(\frac{g^* - \frac{1}{2}}{g^*} \right) \frac{1}{r^*} \right] . \tag{27}$$

(17) implies:

$$\left[- \pi_B + \widehat{r} + \widehat{\lambda}_L \right] \, \widehat{r} \, \frac{2 \left[1 - \widehat{b} \, \right]}{c_b} = \sigma_b \left[\widehat{b} - \frac{1}{2} \right]$$

$$\Rightarrow \;\; - \pi_B + \widehat{r} + \widehat{\lambda}_L = \frac{\sigma_b \, c_b}{2} \left[\frac{\widehat{b} - \frac{1}{2}}{1 - \widehat{b}} \right] \frac{1}{\widehat{r}} \, . \tag{28}$$

Similarly, (21) implies:

$$- \pi_B + r^* = \frac{\sigma_b \, c_b}{2} \left[\frac{b^* - \frac{1}{2}}{1 - b^*} \right] \frac{1}{r^*} \, . \tag{29}$$

Subtracting (28) from (29) provides:

$$r^* - \widehat{r} - \widehat{\lambda}_L \;=\; \frac{\sigma_b\, c_b}{2}\left[\left(\frac{b^* - \frac{1}{2}}{1 - b^*}\right)\frac{1}{r^*} - \left(\frac{\widehat{b} - \frac{1}{2}}{1 - \widehat{b}}\right)\frac{1}{\widehat{r}}\right]. \tag{30}$$

(27) and (30) imply:

$$\frac{\sigma_g\, c_g}{2}\left[\left(\frac{\widehat{g} - \frac{1}{2}}{\widehat{g}}\right)\frac{1}{\widehat{r}} - \left(\frac{g^* - \frac{1}{2}}{g^*}\right)\frac{1}{r^*}\right] \;=\; \frac{\sigma_b\, c_b}{2}\left[\left(\frac{b^* - \frac{1}{2}}{1 - b^*}\right)\frac{1}{r^*} - \left(\frac{\widehat{b} - \frac{1}{2}}{1 - \widehat{b}}\right)\frac{1}{\widehat{r}}\right]. \tag{31}$$

Suppose $r^* > \widehat{r}$ and $\widehat{b} \geq b^*$. $\frac{1}{r^*} < \frac{1}{\widehat{r}}$ because $r^* > \widehat{r}$. Also, $\frac{b - \frac{1}{2}}{1 - b}$ is increasing in b. Therefore, the expression to the right of the equality in (31) is negative.

$\frac{g - \frac{1}{2}}{g}$ is increasing in g. Therefore, if $r^* > \widehat{r}$ and $\widehat{g} \geq g^*$, the expression to the left of the equality in (31) is non-negative, which is a contradiction. Consequently, $\widehat{g} < g^*$. $\square$

Finding 2. *If $r^* > \widehat{r}$ and $\widehat{g} \geq g^*$, then $\widehat{b} < b^*$.*

Proof. Finding 1 implies that if $r^* > \widehat{r}$ and $\widehat{b} \geq b^*$, then it cannot be the case that $\widehat{g} \geq g^*$. Therefore, if $r^* > \widehat{r}$ and $\widehat{g} \geq g^*$, it must be the case that $\widehat{b} < b^*$. $\square$

Finding 3. *If $\widehat{r} > \widetilde{r}$ and $\widehat{g} \geq g^*$, then $\widehat{b} > b^*$.*

Proof. From (18) and (24):

$$\widehat{r}\left[\frac{\widehat{g}^2}{c_g} + \frac{\left(1 - \widehat{b}\right)^2}{c_b}\right] \;=\; \widetilde{r}\left[\frac{(g^*)^2}{c_g} + \frac{(1 - b^*)^2}{c_b}\right]. \tag{32}$$

$\frac{(g)^2}{c_g} + \frac{(1 - b)^2}{c_b}$ is increasing in g and decreasing in $b \in [0, 1]$. Therefore, (32) implies that if $\widehat{r} > \widetilde{r}$ and $\widehat{g} \geq g^*$, then it must be the case that $\widehat{b} > b^*$. $\square$

Finally, suppose $\widetilde{r} < \widehat{r} < r^*$ and $\widehat{g} \geq g^*$. Then $\widehat{b} < b^*$ from Finding 2 and $\widehat{b} > b^*$ from Finding 3. This contradiction ensures $\widehat{g} < g^*$ when $\widetilde{r} < \widehat{r} < r^*$. $\square$

Conclusion (ii) in the Observation is proved by example in the text. ■