Technical Appendix to Accompany

"Employing Lenders' Deep Pockets to Resolve Judgement-Proof Problems"

by

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Proof of Proposition 1.

Expression (3) implies that:

$$P(D) = \begin{cases} D & \text{for } D \in [0,\, W + V + L] \\ W + V + L & \text{for } D \in [W + V + L,\, \overline{D}]\,. \end{cases} \tag{A1}$$

(A1) and (5) imply that:

$$R(D) = 0 \text{ for all } D \in [\, W + V + L,\, \overline{D}\,]\,. \tag{A2}$$

Since (A1) and (A2) imply that the producer will forfeit all of his assets and make no payments to the lender when social damages exceed $W + V + L$, the producer's voluntary participation constraint, (2), can be written as:

$$\int_0^{W+V+L} [\, W + V + L - P(D) - R(D)\,]\, dF(D|\, c) - K(c) \geq W\,. \tag{A3}$$

Let $\lambda,\ \mu,\ \xi(D)$, and λ^L denote the Langrange multipliers associated with constraints (A3), (4), (5), and (6), respectively. Also let γ denote the Lagrange multiplier associated with the constraint that $\overline{L} \geq L$. Then the Lagrangean function associated with [P] can be written as:

$$\begin{aligned} \mathfrak{L} = {} & V - \int_0^{\overline{D}} D\, dF(D|\, c) - K(c) \\ & + \lambda \Big\{ \int_0^{W+V+L} [\, W + V + L - D - R(D)]\, dF(D|\, c) - K(c) - W \Big\} \\ & + \mu \Big\{ \int_0^{W+V+L} [\, W + V + L - D - R(D)]\, f_c(D\,|\, c)\, dD - K'(c) \Big\} \\ & + \int_0^{W+V+L} \xi(D)\, [\, W + V + L - D - R(D)]\, f(D|\, c)\, dD \\ & + \lambda^L \Big[\int_0^{W+V+L} R(D)\, dF(D|\, c) - L \Big] + \gamma\, [\overline{L} - L]\,. \end{aligned} \tag{A4}$$

Differentiating (A4) with respect to $R(D)$ provides the following necessary condition for a solution to [P]:

$$[\lambda^L - \lambda - \xi(D)] f(D|c) - \mu f_c(D|c) \leq 0 \text{ ; and}$$

$$R(D)[\cdot] = 0 \text{ for all } D \in [0, W+V+L] . \tag{A5}$$

<u>Result 1.</u> Suppose $L > 0$. Then there exists a $D^o(L) \in (0, W+V+L)$ such that:

$$Z(D^o) \equiv \lambda^L - \lambda - \mu \frac{f_c(D^o|c)}{f(D^o|c)} = 0 . \tag{A6}$$

Proof. Here and throughout, let $A \equiv W + V + L$. Suppose $Z(D) > 0$ for all $D \in [0, A]$. Then $\xi(D) > 0$ for all $D \in [0, A]$ from (A5). Consequently, $R(D) = A - D$ for all $D \in [0, A]$, from (5). But then $\pi(c) = 0$ for all c, so $c = 0$, which is a contradiction.

Similarly, if $Z(D) < 0$ for all $D \in [0, A]$, then (A5) implies that $R(D) = 0$ for all $D \in [0, \overline{D}]$. Hence, $L = 0$ from (6). Therefore, since $L > 0$ by hypothesis, there exists a $D^o \in (0, A)$ such that $Z(D^o) = 0$. ■

<u>Result 2.</u> $\mu > 0$.

Proof. Suppose $\mu < 0$, and consider the case where $L > 0$. (The proof for the case where $L = 0$ is similar, and so is omitted.) Since $f_c(D|c)/f(D|c)$ is decreasing in D, it follows from (A5) and (A6) that $Z(D) > 0$ and hence $\xi(D) > 0$ for all $D \in [0, D^o)$. Therefore, from (5), $R(D) = W + V + L - D$ for all $D \in [0, D^o)$. Also, since $Z(D) < 0$ for all $D > D^o$, $R(D) = 0$ for all $D > D^o$. Therefore:

$$\pi(c) = \int_{D^o}^{A} [A - D]\, dF(D|c) - K(c) . \tag{A7}$$

But (A7) implies that $c = 0$, which is a contradiction.

An analogous proof by contradiction reveals $\mu \neq 0$ whenever the unobservability of c is constraining. Therefore, $\mu > 0$, which implies that total expected surplus would increase if c were increased above its level at the solution to [P]. ■

Result 3. If $\xi(D') > 0$ for some $D' \in [0, A)$, then $\xi(D'') > 0$ for all $D'' \in [D', A]$.

Proof. From (5), if $\xi(D') > 0$, then $R(D') = A - D' > 0$. Therefore, (A5) implies that:

$$0 < \xi(D') = \lambda^L - \lambda - \mu \frac{f_c(D' | c)}{f(D' | c)} . \tag{A8}$$

Since $\mu > 0$ and $f_c(\cdot)/f(\cdot)$ is decreasing in D, the expression to the right of the equality in (A8) increases with D. Consequently, $\xi(D'') > \xi(D') > 0$ for all $D'' \in (D', A]$. ■

Result 4. There exists a $\hat{D}(L) \in [0, W + V + L]$ such that $\xi(D) = 0$ for all $D \in [0, \hat{D}(L)]$ and $\xi(D) > 0$ for all $D \in (\hat{D}(L), W + V + L]$.

Proof. Suppose $\xi(D) > 0$ for all $D \in [0, A]$. Then $R(D) = A - D$ for all $D \in [0, A]$, and so $\pi(c) = 0$ for all c. Hence, $c = 0$, which is a contradiction. Therefore, $\xi(D) = 0$ for some $D \in [0, A]$, and so Result 4 follows from Result 3. ■

Result 5. There exists a $\hat{D}(L) \in [0, W + V + L]$ such that:

$$R(D) = \begin{cases} 0 & \text{for all } D \in [0, \hat{D}(L)] \\ W + V + L - D & \text{for all } D \in (\hat{D}(L), W + V + L] . \end{cases}$$

Proof. The $\hat{D}(L)$ identified in Result 5 is the same $\hat{D}(L)$ identified in Result 4. Suppose $R(D) > 0$ for some $D' \in [0, \hat{D}(L))$. Then the arguments employed in the proof of Result 3 imply that $\xi(D'') > 0$ for all $D'' \in [D', \hat{D}(L))$, which contradicts Result 4. Therefore, $R(D) = 0$ for all $D \in [0, \hat{D}(L)]$. Furthermore, (5) and Result 4 imply that $R(D) = W + V + L - D$ for all $D \in (\hat{D}(L), W + V + L]$. ■

Result 4 implies that the solution to $[P]$ is characterized in part by the following equalities:

$$H^1 \equiv \int_0^{\hat{D}(L)} [W + V + L - D] f_c(D | c)\, dD - K'(c) = 0 ; \quad \text{and} \tag{A9}$$

$$H^2 \equiv \int_{\hat{D}(L)}^{W+V+L} [W + V + L - D] \, dF(D|c) - L = 0 \quad . \tag{A10}$$

(A9) reflects the producer's profit-maximizing choice of c, given the repayment policy summarized in Result 5. (A10) reflects constraint (6) in $[P]$.

Letting H_j^i denote the partial derivative of H^i with respect to variable $j \in \{c, L, \hat{D}\}$, it follows from (A9), (A10), and Cramer's Rule that:

$$\frac{dc}{dL} = \frac{\begin{vmatrix} -H_L^1 & H_{\hat{D}}^1 \\ -H_L^2 & H_{\hat{D}}^2 \end{vmatrix}}{\begin{vmatrix} H_c^1 & H_{\hat{D}}^1 \\ H_c^2 & H_{\hat{D}}^2 \end{vmatrix}} . \tag{A11}$$

For future reference, notice that:

$$H_c^1 = \int_0^{\hat{D}} [W + V + L - D] \, f_{cc}(D|c) \, dD - K''(c) < 0 \; ; \tag{A12}$$

$$H_{\hat{D}}^1 = [W + V + L - \hat{D}] \, f_c(\hat{D}|c) \; ; \tag{A13}$$

$$H_L^1 = F_c(\hat{D}|c) \; ; \tag{A14}$$

$$H_c^2 = \int_{\hat{D}}^{W+V+L} [W + V + L - D] \, f_c(D|c) \, dD \tag{A15}$$

$$H_{\hat{D}}^2 = -[W + V + L - \hat{D}] \, f(\hat{D}|c) \; \text{; and} \tag{A16}$$

$$H_L^2 = F(W + V + L|c) - F(\hat{D}|c) - 1 \, . \tag{A17}$$

Now define $\tilde{D}$ to be the damage realization for which $f_c(\tilde{D}|c) = 0$ at the solution to the problem corresponding to [P] where L is restricted to be zero. If $W + V \geq \tilde{D}$, then, by raising L above zero, it is possible to: (1) induce the producer to increase c without altering his expected profit; and (2) ensure the lender's voluntary participation. These two outcomes, which secure an increase in surplus, are effected by reducing the producer's payoff for some realizations of D for which $f_c(D|\,c) < 0$ and increasing the producer's payoff for some realizations of D for which $f_c(D|\,c) > 0$. Consequently, $\left.\frac{dc}{dL}\right|_{L=0} > 0$ and $L > 0$ at the solution to $[P]$ if $V + W \geq \bar{D}$.

Now suppose $V + W < \bar{D}$. Notice that when $L > 0$ at the solution to $[P]$, Result 5 implies that constraint (6) in $[P]$ will only be satisfied if:

$$\hat{D}(L) < W + V + L \,. \tag{A18}$$

(A12), (A13), (A15) and (A16) imply that the denominator of the fraction in (A11) is:

$$\begin{aligned} H_c^1 H_{\hat{D}}^2 - H_{\hat{D}}^1 H_c^2 = {} & [W + V + L - \hat{D}]\Big\{ - f(\hat{D}|\,c)\, H_c^1 \\ & - f_c(\hat{D}|\,c) \int_{\hat{D}(L)}^{W+V+L} [W + V + L - D]\, f_c(D|\,c)\; dD \Big\} \,. \end{aligned} \tag{A19}$$

As $L \to 0$, $\hat{D}(L) \to W + V + L$. Therefore, (A12), (A18), and (A19) imply that:

$$H_c^1 H_{\hat{D}}^2 - H_{\hat{D}}^1 H_c^2 \stackrel{s}{=} - f(\hat{D}|\,c)\, H_c^1 > 0 \,. \tag{A20}$$

(A20) and (A11) imply that:

$$\left.\frac{dc}{dL}\right|_{L=0} \stackrel{s}{=} H_L^2 H_{\hat{D}}^1 - H_L^1 H_{\hat{D}}^2 \,. \tag{A21}$$

From (A13), (A14), (A16), and (A17):

$$\begin{aligned} H_L^2 H_{\hat{D}}^1 - H_L^1 H_{\hat{D}}^2 = {} & [W + V + L - \hat{D}]\Big\{ f(\hat{D}|\,c)\, F_c(\hat{D}|\,c) \\ & + f_c(\hat{D}|\,c)\, [F(W + V + L|\,c) - F(\hat{D}|\,c) - 1] \Big\} \,. \end{aligned} \tag{A22}$$

(A21) and (A22) imply that as $L \to 0$:

$$\left.\frac{dc}{dL}\right|_{L=0} \stackrel{s}{=} F_c(V + W|\,c) - \frac{f_c(V + W|\,c)}{f(V + W|\,c)} \,. \tag{A23}$$

Notice that $F_c(0|\,c) = 0$, and that $f_c(0|\,c) > 0$ under the maintained assumptions. Also notice that as $V + W \to \bar{D}$, $\frac{f_c(V + W|\,c)}{f(V + W|\,c)} \to 0$ while $F_c(V + W|\,c)$ is bounded strictly above zero. Therefore, since the maintained assumptions imply that $F_c(D|\,c)$ is strictly increasing in D and

$\dfrac{f_c(D \mid c)}{f(D \mid c)}$ is strictly decreasing in D, it follows from (A23) that there exists a $\bar{D}' < \bar{D}$ such that

$$\left.\frac{dc}{dL}\right|_{L=0} \gtreqless 0 \quad \text{as} \quad V + W \gtreqless \bar{D}' . \tag{A24}$$

(A24) implies that $L > 0$ at the solution to $[P]$ whenever $V + W > \bar{D}'$. ■