Technical Appendix to Accompany

“Welfare-Enhancing Fraudulent Behavior”

by Haijin Lin and David Sappington

Condition (A1). $$\frac{(\theta_0)^2\, l_a}{c} \; < \; b \; < \frac{(\theta_0)^2\, l_a\, [\, 2\,(1-\lambda)\, l + l_a \,]}{c\left[\, l + l_a + \frac{b}{2}\left(\overline{\Pi} - \; \overline{\Pi}_f\,\right)\right]}\,,$$

where l, l_a, b, θ_0, and c are all strictly positive parameters.

Definition. In the Extended Linear-Quadratic (ELQ) Example: (i) Condition (A1) holds; (ii) $\theta\,(E) = \theta_0\, E$; (iii) $C\,(E) = \frac{c}{2}\, E^2$; (iv) $\overline{\Pi}_f$ and $\overline{\Pi}$ do not vary with M; (v) $L\,(M) = l\, M$; (vi) $L_a\,(M) = l_a\, M$; (vii) $K(M) = k\, M$; and (viii) $\phi\,(M) = b\, M$ for M in the relevant range.[1]

Proposition 1. *In the setting of the ELQ Example, there exists a unique* $\widehat{M} \in (0, \frac{1}{b})$ *such that* $R'(M) \lesseqqgtr 0$ *as* $M \lesseqqgtr \widehat{M}$ *, where* $\widehat{M} > 0$ *is defined by:*[2]

$$\frac{2\, b^2\, l_a\, (\theta_0)^2}{c}\, [\, 2\,(1-\lambda)\, l + l_a \,]\; \widehat{M}^{\,3} - 2\, b\, [\, l + \, l_a \,]\; \widehat{M} \; - b\left[\, \overline{\Pi} - \; \overline{\Pi}_f \,\right] \; = \; 0\,. \tag{1}$$

Proof. Under the maintained assumptions:

$$\begin{aligned}\frac{\partial \pi(M,E)}{\partial M} \;&=\; \phi\,(M)\, [\, -\, (\, 1 - \theta(E)\,)\, l\,] + \; \phi'(M)\; \left[\, \overline{\Pi}_f - \overline{\Pi} - (\, 1 - \theta(E)\,)\, l\, M \,\right] \\ &=\; -\, b\, l\, M\, [\, 1 - \theta_0\, E\,] \, + \, b\left[\, \overline{\Pi}_f - \overline{\Pi}\,\right] - b\, l\, M\, [\, 1 - \theta_0\, E\,]\,. \end{aligned} \tag{2}$$

Furthermore:

$$\begin{aligned} F'(M) \;&=\; \lambda\, \{\, \phi(M)\, \theta'(E(M))\, l\, M\, E'(M) + \theta(E(M))\, [\, \phi(M)\, l + l\, M\, \phi'(M)\,]\, \} \\ &\quad + \; [\, 1 - \theta(E(M))\,]\, [\, l_a\; \phi(M) + l_a\, M\, \phi'(M)\,] \\ &=\; \lambda\, \left[\, b\, l\, \theta_0\, M^2\, E'(M) + 2\, b\, M\, l\, \theta_0\, E(M) \right] + 2\, b\, l_a\, M\, [\, 1 - \theta_0\, E(M)\,]\,. \end{aligned} \tag{3}$$

[1] The relevant range for M is the range in which probabilities $\phi(M)$ and $\theta(E(M))$ are well-defined.

[2] Observe that $\widehat{M} = 0$ is a root of equation (1) when $\overline{\Pi} = \overline{\Pi}_f$. As the proof of Proposition 1 reveals, equation (1) also has a strictly positive root. This larger root is the value of $\widehat{M}$ identified in Proposition 1 and discussed further below.

From (2) in the text, the optimal $E(M)$ is determined by:

$$b\,M\;l_a\;M\;\theta_0 \;=\; c\,E(M) \quad \Rightarrow \quad E\,(M) \;=\; \frac{b\,\theta_0\,l_a}{c}\,M^2$$

$$\Rightarrow \quad E'(M) \;=\; \frac{2\,b\,\theta_0\,l_a}{c}\,M \quad \text{and} \quad \theta\,(E\,(M)) \;=\; \frac{b\,(\theta_0)^2\,l_a}{c}\,M^2\,. \tag{4}$$

(2), (3), and (4) imply:

$$\begin{aligned}
R'(M) \;&=\; \frac{\partial\pi(M,E)}{\partial M} - F'(M) + b\,l\,\theta_0\,M^2\,E'(M) \\
&=\; -\,b\,l\,M\left[1 - \frac{b\,(\theta_0)^2\,l_a}{c}\,M^2\right] + \,b\,[\,\overline{\Pi}_f - \,\overline{\Pi}\,] - b\,l\,M\left[1 - \frac{b\,(\theta_0)^2\,l_a}{c}\,M^2\right] \\
&\qquad -\,\lambda\left[b\,l\,\theta_0\,M^2\,\frac{2\,b\,\theta_0\,l_a}{c}\,M + 2\,b\,M\,l\,\theta_0\,\frac{b\,\theta_0\,l_a}{c}\,M^2\right] \\
&\qquad -\,2\,b\,M\,l_a\left[1 - \frac{b\,(\theta_0)^2\,l_a}{c}\,M^2\right] + b\,l\,\theta_0\,M^2\,\frac{2\,b\,\theta_0\,l_a}{c}\,M \\
&=\; -\left[1 - \frac{b\,(\theta_0)^2\,l_a}{c}\,M^2\right][\,2\,b\,l\,M + 2\,b\,l_a\,M\,] - b\,[\,\overline{\Pi} - \,\overline{\Pi}_f\,] \\
&\qquad +\,[\,1-\lambda\,]\,\frac{2\,b^2\,(\theta_0)^2\,l\,l_a}{c}\,M^3 - \,2\,\lambda\,\frac{b^2\,(\theta_0)^2\,l\,l_a}{c}\,M^3 \\
&=\; [\,1-\lambda\,]\,\frac{2\,b^2\,(\theta_0)^2\,l\,l_a}{c}\,M^3 - b\,[\,\overline{\Pi} - \,\overline{\Pi}_f\,] \\
&\qquad -\,2\,b\,M\left[1 - \frac{b\,(\theta_0)^2\,l_a}{c}\,M^2\right][\,l + l_a\,] - 2\,\lambda\,\frac{b^2\,(\theta_0)^2\,l_a\,l}{c}\,M^3 \\
&=\; \frac{2\,b^2\,(\theta_0)^2\,l\,l_a}{c}\,M^3 - b\,[\,\overline{\Pi} - \,\overline{\Pi}_f] - 2\,b\,M\,[\,l + l_a\,] \\
&\qquad +\,\frac{2\,b^2\,(\theta_0)^2\,l_a}{c}\,M^3\,[\,l + l_a\,] - \,\frac{4\,\lambda\,b^2\,(\theta_0)^2\,l\,l_a}{c}\,M^3 \\
&=\; \frac{2\,b^2\,(\theta_0)^2}{c}\,[\,l\,l_a + l_a\,(l + l_a) - 2\,\lambda\,l\,l_a\,]\,M^3 - 2\,b\,[\,l + \,l_a\,]\,M - b\,[\,\overline{\Pi} - \,\overline{\Pi}_f\,] \\
&=\; \frac{2\,b^2 l_a\,(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,M^3 - 2\,b\,[\,l + \,l_a\,]\,M - b\,[\,\overline{\Pi} - \,\overline{\Pi}_f\,]\,.
\end{aligned} \tag{5}$$

From (5):

$$R''(M) \;=\; \frac{6\,b^2\,l_a\,(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,M^2 - 2\,b\,[\,l + \,l_a\,] \;=\; 0 \tag{6}$$

$$\Leftrightarrow \;\; M \;=\; M^* \;\equiv\; \sqrt{\frac{2\,b\,[\,l + \,l_a\,]}{\frac{6\,b^2 l_a (\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]}} \;=\; \sqrt{\frac{c\,[\,l + \,l_a\,]}{3\,b\,l_a\,(\theta_0)^2\,[\,2\,(1-\lambda)\,l + l_a\,]}}\,. \tag{7}$$

Condition (A1) ensures:

$$3\,l_a\,(\theta_0)^2\,[\,2\,(1-\lambda)\,l + l_a\,] \;>\; 3b\,c\left[\,l + \,l_a + \frac{b}{2}\left(\overline{\Pi} - \,\overline{\Pi}_f\,\right)\right] > \;b\,c\,[\,l + \,l_a\,]\,,$$

and so, from (7), $b\,M^* \,<\, 1 \;\;\Rightarrow\;\; M^* \,<\, \frac{1}{b}$. From (4), $\theta(E(M)) \leq 1 \;\;\Leftrightarrow\;\; M \leq \sqrt{\frac{c}{b(\theta_0)^2 l_a}}$. Therefore, Condition (A1) ensures $M^* < \frac{1}{b} < \sqrt{\frac{c}{b(\theta_0)^2 l_a}}$.

From (5):

$$R'(M)|_{M\,=\,0} \;=\; -\,b\left[\,\overline{\Pi} - \,\overline{\Pi}_f\,\right] \;\leq\; 0\,.$$

(6) implies that $R''(M) \leq 0$ for $M \in [\,0,\, M^*\,]$, whereas $R''(M) > 0$ for $M \in (M^*,\, \frac{1}{b}\,]$. (5) and Condition (A1) imply that $R'(M)|_{M\,=\,\frac{1}{b}} > 0$. Consequently, there exists a unique $\widehat{M} \in \left(M^*,\, \frac{1}{b}\right)$ such that $R'(M)|_{M\,=\,\widehat{M}} \;=\; 0$. ■

Lemma A1. If $\overline{\Pi} \,=\, \overline{\Pi}_f$, then $\widehat{M} \,=\, \sqrt{\frac{c\,[\,l + l_a\,]}{b(\theta_0)^2 l_a [\,2(1-\lambda)l + l_a\,]}}$ in the setting of the ELQ Example.

<u>Proof</u>. From (1), when $\overline{\Pi}_f \,=\, \overline{\Pi}$:

$$\frac{2\,b^2\,l_a\,(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,3} \;=\; 2\,b\,[\,l + \,l_a\,]\,\widehat{M}$$

$$\Rightarrow \;\; \widehat{M}^{\,2} \;=\; \frac{2\,b\,[\,l + \,l_a\,]}{\frac{2\,b^2\,l_a(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]} \;=\; \frac{c\,[\,l + \,l_a\,]}{b\,(\theta_0)^2\;\; l_a\,[\,2\,(1-\lambda)\,l + l_a\,]}\,. \quad ■$$

Proposition 2. *In the ELQ Example,* $\widehat{M}$*: (i) increases as* c *or* λ *increases; (ii) decreases as* b*,* θ_0*, or* l_a *increases; and (iii) decreases as* l *increases if* $\lambda \leq \frac{1}{2}$*.*

<u>Proof</u>. Differentiating (1) with respect to $\widehat{M}$ and b provides:

$$d\,\widehat{M}\;\left\{\frac{6}{c}\,b^2\,l_a\,(\theta_0)^2\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,2} - 2\,b\,[\,l + \,l_a\,]\right\}$$

$$+\;\; d\,b\left\{\frac{4}{c}\,b\,l_a\,(\theta_0)^2\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,3} - 2\,[\,l + \,l_a\,]\,\widehat{M} - \left[\,\overline{\Pi} - \,\overline{\Pi}_f\right]\right\} \;=\; 0$$

$$\Rightarrow \quad \frac{d\widehat{M}}{db} \;=\; -\; \frac{\frac{4\,b\,l_a(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,3} - 2\,[\,l + \,l_a\,]\,\widehat{M} - \left[\,\overline{\Pi} - \,\overline{\Pi}_f\,\right]}{\frac{6\,b^2\,l_a(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,2} - 2\,b\,[\,l + \,l_a\,]}\,. \tag{8}$$

(1) and (8) provide:

$$\frac{d\widehat{M}}{db} \;=\; -\; \frac{\frac{4\,b\,l_a(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,3} - \frac{2\,b\,l_a(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{3}}{\frac{6\,b^2 l_a(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,2} - 2\,b\,[\,l + \,l_a\,]}$$

$$=\; -\; \frac{\frac{2\,b\,l_a(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,3}}{\frac{6\,b^2\,l_a(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,2} - 2\,b\,[\,l + \,l_a\,]} \;<\; 0\,. \tag{9}$$

The inequality in (9) holds because $\widehat{M} > M^{*}$, and so, from (6):

$$\frac{6\,b^2\,l_a\,(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,2} - 2\,b\,[\,l + \,l_a\,]$$

$$>\; \frac{6\,b^2\,l_a\,(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,M^{*2} - 2\,b\,[\,l + \,l_a\,] \;=\; 0\,. \tag{10}$$

Differentiating (1) with respect to $\widehat{M}$ and θ_0 provides:

$$d\widehat{M}\;\left\{\frac{6}{c}\,b^2\,l_a\,(\theta_0)^2\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,2} - 2\,b\,[\,l + \,l_a\,]\right\}$$

$$+\; d\theta_0\left\{\frac{4\,b^2\,l_a\,\theta_0}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,3}\right\} \;=\; 0$$

$$\Rightarrow \quad \frac{d\widehat{M}}{d\theta_0} \;=\; -\; \frac{\frac{4\,b^2\,l_a\,\theta_0}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,3}}{\frac{6\,b^2\,l_a(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,2} - 2\,b\,[\,l + \,l_a\,]} \;<\; 0\,. \tag{11}$$

The inequality in (11) follows from (10).

Differentiating (1) with respect to $\widehat{M}$ and c provides:

$$d\widehat{M}\;\left\{\frac{6}{c}\,b^2\,l_a\,(\theta_0)^2\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,2} - 2\,b\,[\,l + \,l_a\,]\right\}$$

$$+\; dc\left\{-\,\frac{2\,b^2\,l_a\,(\theta_0)^2}{c^2}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,3}\right\} = 0$$

$$\Rightarrow \quad \frac{d\widehat{M}}{dc} \;=\; \frac{\frac{2\,b^2\,l_a(\theta_0)^2}{c^2}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{3}}{\frac{6\,b^2\,l_a(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,2} - 2\,b\,[\,l + \,l_a\,]} \;>\; 0\,. \tag{12}$$

The inequality in (12) follows from (10).

Differentiating (1) with respect to $\widehat{M}$ and λ provides:

$$d\widehat{M}\ \left\{\frac{6}{c}\,b^2\,l_a\,(\theta_0)^2\,[\,2\,(1-\lambda)\,l+l_a\ ]\,\widehat{M}^{\,2}-2\,b\,[\,l+\ l_a\,]\right\}+d\lambda\left\{-\ \frac{4\,b^2\,l\,l_a\,(\theta_0)^2}{c}\ \widehat{M}^{\,3}\right\}\ =\ 0$$

$$\Rightarrow\quad \frac{d\widehat{M}}{d\lambda}\ =\ \frac{\frac{4\,b^2\ l\ l_a(\theta_0)^2}{c}\ \widehat{M}^{\,3}}{\frac{6\,b^2\,l_a(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l+l_a\ ]\,\widehat{M}^{\,2}-2\,b\,[\,l+\ l_a\,]}\ >\ 0. \qquad (13)$$

The inequality in (13) follows from (10).

Differentiating (1) with respect to $\widehat{M}$ and l_a provides:

$$d\widehat{M}\ \left\{\frac{6}{c}\,b^2\,l_a\,(\theta_0)^2\,[\,2\,(1-\lambda)\,l+l_a\ ]\,\widehat{M}^{\,2}-2\,b\,[\,l+\ l_a\,]\right\}$$
$$+\ d\,l_a\left\{\frac{2\,b^2\,(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l+l_a\ ]\,\widehat{M}^{\,3}+\frac{2\,b^2 l_a\,(\theta_0)^2}{c}\,\widehat{M}^{\,3}-2\,b\,\widehat{M}\right\}\ =\ 0$$

$$\Rightarrow\quad \frac{d\widehat{M}}{d\,l_a}\ =\ -\ \frac{2\,b\,\widehat{M}\left[\frac{2\,b(\theta_0)^2}{c}\,[\,(1-\lambda)\,l+\ l_a\,]\,\widehat{M}^{\,2}-1\right]}{\frac{6\,b^2\,l_a(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l+l_a\ ]\,\widehat{M}^{\,2}-2\,b\,[\,l+\ l_a\,]}\ <\ 0\,. \qquad (14)$$

The inequality in (14) follows from (10) and from the fact that, using (1):

$$2\,b\,\widehat{M}\left[\frac{2\,b\,(\theta_0)^2}{c}\,[\,(1-\lambda)\,l+\ l_a\,]\,\widehat{M}^{\,2}-1\right]\ =\ \frac{2\,b^2\,(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l+2\,l_a\,]\,\widehat{M}^{\,3}-2\,b\,\widehat{M}$$
$$=\ \frac{2\,b^2\,(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l+l_a\,]\,\widehat{M}^{\,3}+\frac{2\,b^2\,l_a\,(\theta_0)^2}{c}\,\widehat{M}^{\,3}-2\,b\,\widehat{M}$$
$$=\ \frac{1}{l_a}\,\frac{2\,b^2\,l_a\,(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l+l_a\,]\,\widehat{M}^{\,3}+\frac{2\,b^2\,l_a\,(\theta_0)^2}{c}\,\widehat{M}^{\,3}-2\,b\,\widehat{M}$$
$$=\ \frac{1}{l_a}\left[2\,b\,(l+l_a)\,\widehat{M}+b\left(\overline{\Pi}-\ \overline{\Pi}_f\,\right)\right]+\frac{2\,b^2\,l_a\,(\theta_0)^2}{c}\,\widehat{M}^{\,3}-2\,b\,\widehat{M}$$
$$=\ \frac{2\,b^2\,l_a\,(\theta_0)^2}{c}\,\widehat{M}^{\,3}+\frac{2\,b\,l\,\widehat{M}+b\left[\,\overline{\Pi}-\ \overline{\Pi}_f\right]}{l_a}\ >\ 0\,. \qquad (15)$$

Differentiating (1) with respect to $\widehat{M}$ and l provides:

$$d\widehat{M}\ \left\{\frac{6}{c}\,b^2\,l_a\,(\theta_0)^2\,[\,2\,(1-\lambda)\,l+l_a\ ]\,\widehat{M}^{\,2}-2\,b\,[\,l+\ l_a\,]\right\}$$
$$+\ dl\left\{\frac{4}{c}\,[\,1-\lambda\,]\ b^2\,l_a\,(\theta_0)^2\ \widehat{M}^{\,3}-2\,b\,\widehat{M}\right\}\ =\ 0$$

$$\Rightarrow \quad \frac{d\widehat{M}}{dl} \;=\; - \, \frac{2\,b\,\widehat{M}\left[\,\frac{2}{c}\,(1-\lambda)\,b\,l_a\,(\theta_0)^2\,\widehat{M}^{\,2} - 1\right]}{\frac{6\,b^2\,l_a(\theta_0)^2}{c}\,[\,2\,(1-\lambda)\,l + l_a\,]\,\widehat{M}^{\,2} - 2\,b\,[\,l + \,l_a\,]}\,. \tag{16}$$

From (10) and (16):

$$\frac{d\widehat{M}}{dl} \;\overset{s}{=}\; 1 - \frac{2\,b\,(\theta_0)^2}{c}\,[\,1-\lambda\,]\,l_a\,\widehat{M}^{\,2} \;<\; 0 \quad \Leftrightarrow \quad \widehat{M}^{\,2} \;>\; \frac{c}{2\,[\,1-\lambda\,]\,b\,l_a\,(\theta_0)^2}\,. \tag{17}$$

The inequality in (17) holds when $\lambda \leq \frac{1}{2}$ because:

$$\widehat{M}^{\,2} \;\geq\; \frac{c\,[\,l + \,l_a\,]}{b\,(\theta_0)^2\;l_a\,[\,2\,(1-\lambda)\,l + l_a\,]} \;\geq\; \frac{c}{2\,[\,1-\lambda\,]\,b\,l_a\,(\theta_0)^2}\,. \tag{18}$$

The first inequality in (18) holds because: (i) $\widehat{M}^{\,2} = \frac{c[\,l + \,l_a\,]}{b(\theta_0)^2\;l_a[\,2(1-\lambda)l + l_a\,]}$ when $\overline{\Pi} - \overline{\Pi}_f \;=\; 0$, from Lemma A1; (ii) $R''(M) > 0$ for $M \in (M^*,\, \frac{1}{b}\,]$; and (iii) $R'(M)$ declines as $\overline{\Pi} - \overline{\Pi}_f$ increases, from (5). Therefore, $\widehat{M}$, the value of M at which $R'(M) = 0$, increases as $\overline{\Pi} - \overline{\Pi}_f$ increases.

The second inequality in (18) holds because:

$$\frac{c\,[\,l + \,l_a\,]}{b\,(\theta_0)^2\;\;l_a\,[\,2\,(1-\lambda)\,l + l_a\,]} \;\;\geq\;\; \frac{c}{2\;(1-\lambda)\,b\,l_a\,(\theta_0)^2}$$

$$\Leftrightarrow \quad \frac{l + \,l_a}{2\,[\,1-\lambda\,]\,l + l_a} \;\;\geq\;\; \frac{1}{2\,[\,1-\lambda\,]} \quad \Leftrightarrow \quad 2\,[\,1-\lambda\,] \;\;\geq\;\; 1 \quad \Leftrightarrow \quad \lambda \;\;\leq\;\; \frac{1}{2}\,. \quad \blacksquare$$

Proposition 3. *In the setting of the ELQ Example, the client's expected net return and the auditor's expected utility both increase as the probability of fraud within the client's organization increases whenever* $M > \widehat{M}$.

<u>Proof</u>. The proposition follows directly from Proposition 1 and from Lemma 3 in the paper. ■

Proposition 4 refers to the following conditions.

(*A2*) $\quad c\,b\,b_m \;<\; 3\,[\,2\,(\,b_m + l_m\,) - b\,B\,]\,l_a\,(\theta_0)^2\,.$

(*A3*) $\quad M^{**} \;>\; \dfrac{12\,[k\,+ b\,B]\,[\,b_m + l_m\,] - B\,b\,b_m}{10\,b\,b_m\,[\,b_m + l_m\,] + \frac{3}{c}\,B^2\,b^2\,(\theta_0)^2\;l_a}$, where M^{**} is defined in (24) below.

(*A4*) $\quad c\,b\,b_m \;<\; 2\,(\theta_0)^2\;l_a\,[\,b_m + l_m\,] - \dfrac{3}{2}\,b\,B\,(\theta_0)^2\;l_a + \dfrac{b\,c}{2}\,[\,b\,B + k\,]\,.$

$(A5)$ $\quad \widehat{M}^* \; > \; \dfrac{\alpha}{\beta}\,$, where $\widehat{M}^*$ is defined in (47) below, and where

$$\alpha \;\equiv\; \frac{8\,c\,[\,b_m + l_m\,]\left[\,\overline{\Pi} - \;\overline{\Pi}_f\,\right][\,b_m\,(\,2\,\lambda\,l + l_a\,) + 2\;l_m\,(\,l + l_a\,)\,]}{9\,B^2\,b\,(\theta_0)^2\,l_a\,[\,2\,(1-\lambda)\,l + l_a\,]^2}$$

$$+\;\frac{4\,c\,[\,b\,B + k\,]\,[\,b_m\,(\,2\,\lambda\,l + l_a\,) + 2\,l_m\,(\,l + l_a\,)\,]}{9\,B^2\,b^2\,(\theta_0)^2\,l_a\,[\,2\,(1-\lambda)\,l + l_a\,]} - \frac{\overline{\Pi} - \;\overline{\Pi}_f}{2\,[\,1-\lambda\,]\,l + l_a}\,, \tag{19}$$

and

$$\beta \;\equiv\; -\,\frac{8\,b\,c\,[\,b_m\,(\,2\,\lambda\,l + l_a\,) + 2\,l_m\,(\,l + l_a\,)\,]^2}{9\,B^2\,b^2\,(\theta_0)^2\,l_a\,[\,2\,(1-\lambda)\,l + l_a\,]^2} - \;\frac{2\,[\,b\,B + k\,]}{3\,B\,b}$$

$$-\;\frac{4\,[\,b_m + l_m\,]\left[\,\overline{\Pi} - \;\overline{\Pi}_f\,\right]}{3\,B\,[\,2\,(1-\lambda)\,l + l_a\,]} + \frac{2\,[\,l + \;l_a\,]}{2\,[\,1-\lambda\,]\,l + l_a}\,. \tag{20}$$

Proposition 4. *Suppose conditions (A2) – (A5) hold. Then $\widetilde{M}$, the manager's preferred M, increases as k declines in the ELQ Example. Furthermore, $\widetilde{M} > \widehat{M}$, so the expected utility of the client and the auditor both increase as k decreases (and so $\widetilde{M}$ increases).*

Proof. In the ELQ Example:

$$U_m(M,w) \;=\; w + \phi(M)B_m(M) - \phi(M)\,\theta(E(M))\,[\,B_m(M) + L_m(M)\,] - K(M)$$

$$=\; w + b\,M\,[\,b_m\,M - B\,] - b\,M\,\theta(E(M))\,[\,b_m M - B + l_m\,M\,] - K(M)$$

$$=\; w + b\;b_m\;M^2 - b\;B\;M - b\;M\,\frac{b\,(\theta_0)^2\;l_a}{c}\,M^2\;[\,b_m + l_m\,]\,M + b\,B\,M\,\frac{b\,(\theta_0)^2\;l_a}{c}\,M^2 - k\,M$$

$$=\; w + b\;b_m\;M^2 + B\,\frac{b^2\,(\theta_0)^2\;l_a}{c}\,M^3 - \frac{b^2\,(\theta_0)^2\;l_a}{c}\,[\,b_m + l_m\,]\,M^4 - [\,k + b\,B\,]\,M\,. \tag{21}$$

(21) implies:

$$\frac{\partial U_m\,(\cdot)}{\partial M} \;=\; 2\,b\,b_m\;M + \frac{3}{c}\,B\,b^2\,(\theta_0)^2\;l_a\,M^2 - \frac{4}{c}\,[\,b_m + l_m\,]\,b^2\,(\theta_0)^2\;l_a\,M^3 - k - b\,B\,. \tag{22}$$

Differentiating (22) provides:

$$\frac{\partial^2 U_m\,(M,w)}{\partial M^2} \;=\; 2\,b\;b_m - \frac{1}{c}\,[\,b_m + l_m\,]\,12\,b^2\,(\theta_0)^2\;l_a\,M^2 + \frac{6\,B}{c}\,b^2\,(\theta_0)^2\,l_a\,M \;=\; 0 \tag{23}$$

$$\Leftrightarrow\quad [\,b_m + l_m\,]\,12\,b^2\,(\theta_0)^2\;l_a\,M^2 - 6\,B\,b^2\,(\theta_0)^2\,l_a\;M - 2\,b\;b_m\,c \;=\; 0$$

$$\Leftrightarrow\quad M \;=\; \frac{6\,B\,b^2\,(\theta_0)^2\,l_a + \sqrt{36\,B^2\,b^4\,(\theta_0)^4\,(l_a)^2 + 8\,b\;b_m\,c\,[\,b_m + l_m\,]\,12\,b^2\,(\theta_0)^2\;l_a}}{24\,[\,b_m + l_m\,]\,b^2\,(\theta_0)^2\,l_a}$$

$$= \frac{6\,B\,b^2\,(\theta_0)^2\,l_a + 2\,b\,\theta_0\,\sqrt{9\,B^2\,b^2\,(\theta_0)^2\,(l_a)^2 + 24\,b\;b_m\,c\,l_a\,[\,b_m + l_m\,]}}{24\,[\,b_m + l_m\,]\,b^2\,(\theta_0)^2\,l_a}$$

$$= \frac{3\,B\,b\,\theta_0\,l_a + \sqrt{9\,B^2\,b^2\,(\theta_0)^2\,(l_a)^2 + 24\,b\;b_m\,c\,l_a\,[\,b_m + l_m\,]}}{12\,b\,\theta_0\,l_a\,[\,b_m + l_m\,]}$$

$$= \frac{3\,B + \sqrt{9\,B^2 + \frac{24\,b_m\,c[\,b_m + l_m\,]}{b\,l_a(\theta_0)^2}}}{12\,[\,b_m + l_m\,]} \;\equiv\; M^{**}\,. \tag{24}$$

(24) implies that $\phi\,(M^{**}) \;=\; b\,M^{**} <\; 1$ when condition (A2) holds because:

$$3\,b\,B + b\,\sqrt{9\,B^2 + \frac{24\,c}{b\,l_a\,(\theta_0)^2}\,[\,b_m + l_m\,]\,b_m} \;<\; 12\,[\,b_m + l_m\,]$$

$$\Leftrightarrow\quad b^2\left[9\,B^2 + \frac{24\,c\,b_m\,(\,b_m + l_m\,)}{b\,l_a\,(\theta_0)^2}\right] \;<\; [\,12\,(\,b_m + l_m\,) - 3\,b\,B\,]^2$$

$$\Leftrightarrow\quad b^2\left[\,9\,B^2\,b\,l_a\,(\theta_0)^2 + 24\,c\,b_m\,(\,b_m + l_m\,)\,\right]$$

$$<\; \left[\,144\,(\,b_m + l_m\,)^2 - 72\,b\,B\,(\,b_m + l_m\,) + 9\,b^2\,B^2\,\right]b\,l_a\,(\theta_0)^2$$

$$\Leftrightarrow\quad 8\,c\,b^2\,b_m\,[\,b_m + l_m\,] \;<\; [\,b_m + l_m\,]\,[\,48\,(\,b_m + l_m\,) - 24\,b\,B\,]\,b\,l_a\,(\theta_0)^2$$

$$\Leftrightarrow\quad c\,b\,b_m \;<\; 3\,[\,2\,(\,b_m + l_m\,) - b\,B\,]\,l_a\,(\theta_0)^2\;.$$

From (23):

$$\left.\frac{\partial^2 U_m\,(M,w)}{\partial M^2}\right|_{M\,=\,0} \;=\; 2\,b\,b_m \;>\; 0\,. \tag{25}$$

(23), (24), and (25) imply that $\frac{\partial^2 U_m(M,w)}{\partial M^2} \geq 0$ for $M \in [0, M^{**}]$ and $\frac{\partial^2 U_m(M,w)}{\partial M^2} < 0$ for $M \in (\,M^{**}, \frac{1}{b}\,]$. Therefore, $U_m\,(\cdot)$ is convex in M for $M \in [0, M^{**}]$ and concave in M for $M \in (\,M^{**}, \frac{1}{b}\,]$. Consequently, $\widetilde{M}$, defined as the value of M at which the expression in (22) is zero, will exceed M^{**} if $U_m\,(M, w)$ is increasing in M at M^{**} and $U_m(M^{**}, w) \;>\; U_m\,(0, w)$.

(21), (23), and (24) provide:

$$U_m(M^{**}, w) \;=\; w + b\;b_m\,(M^{**})^2 + \frac{B\,b^2\,(\theta_0)^2\,l_a}{c}\;(M^{**})^3$$

$$-\;\frac{1}{c}\,b^2\,(\theta_0)^2\;l_a\,(M^{**})^4\;[\,b_m + l_m\,] - [\,k\; + b\,B\,]\;M^{**}$$

$$=\; w + b\;b_m\,(M^{**})^2 + B\,\frac{b^2\,(\theta_0)^2\;l_a}{c}\;(M^{**})^3$$

$$- \left[\frac{1}{6} b\, b_m + \frac{B\, b^2 \left(\theta_0\right)^2\, l_a}{2\, c}\, M^{**} \right] (M^{**})^2 - [\, k\, + b\, B\,]\, M^{**}$$

$$= \; w + \frac{5}{6}\, b\, b_m\, (M^{**})^2 + \frac{B\, b^2 \left(\theta_0\right)^2\, l_a}{2\, c}\, (M^{**})^3 - [\, k\, + b\, B\,]\, M^{**}$$

$$= \; w + M^{**} \left[\frac{5}{6}\, b\, b_m\; M^{**} + \frac{B\, b^2 \left(\theta_0\right)^2\, l_a}{2\, c}\, (M^{**})^2 - k\, - b\; B \right] \tag{26}$$

$$= \; w + M^{**} \left[\frac{5}{6}\, b\, b_m\; M^{**} + \frac{B}{24\, [\, b_m + l_m\,]} \left(2\, b\, b_m + \frac{6\, B\, b^2 \left(\theta_0\right)^2\, l_a}{c}\, M^{**} \right) - k - b\, B \right]$$

$$= \; w + M^{**} \left[\frac{5}{6}\, b\, b_m\; M^{**} + \frac{B\, b\, b_m}{12\, [\, b_m + l_m\,]} + \frac{B^2\, b^2 \left(\theta_0\right)^2\, l_a}{4\, [\, b_m + l_m\,]\, c}\, M^{**} - k - b\, B \right]$$

$$= \; w + M^{**} \left[\left(\frac{5}{6}\, b\, b_m + \frac{B^2\, b^2 \left(\theta_0\right)^2\, l_a}{4\, [\, b_m + l_m\,]\, c} \right) M^{**} + \frac{B\, b\, b_m}{12\, [\, b_m + l_m\,]} - k - b\; B \right]. \tag{27}$$

The term in $(\cdot)$ brackets in (27) is positive if Condition (A3) holds because:

$$\left[\frac{5}{6}\, b\, b_m + \frac{B^2\, b^2 \left(\theta_0\right)^2\, l_a}{4\, [\, b_m + l_m\,]\, c} \right] M^{**} \; > \; k + b\; B - \frac{B\, b\, b_m}{12\, [\, b_m + l_m\,]}$$

$$\Leftrightarrow \quad M^{**} \; > \frac{k + b\; B - \frac{B\, b\, b_m}{12[\, b_m + l_m\,]}}{\frac{5}{6}\, b\, b_m + \frac{B^2\, b^2 (\theta_0)^2\, l_a}{4[\, b_m + l_m\,] c}} \; = \; \frac{12\, [\, k\, + b\; B\,]\, [\, b_m + l_m\,] - B\, b\, b_m}{10\, b\, b_m\, [\, b_m + l_m\,] + \frac{3}{c}\, B^2\, b^2 \left(\theta_0\right)^2\, l_a}\,. \tag{28}$$

(28) implies that when Condition (A3) holds:

$$U_m(M^{**}, w) \; > \; w \quad \Rightarrow \quad U_m(M^{**}, w) \; > \; U_m\, (0, w)\, .$$

From (22):

$$\left. \frac{\partial U_m\, (M, w)}{\partial M} \right|_{M\, =\, M^{**}} \; = \; 2\, b\, b_m\, M^{**} - \frac{1}{c}\, [\, b_m + l_m\,]\, 4\, b^2 \left(\theta_0\right)^2\, l_a\, (M^{**})^3$$

$$+ \; B \left[\frac{3\, b^2 \left(\theta_0\right)^2\, l_a}{c}\; (M^{**})^2 \right] - b\; B - k$$

$$= \; 2\, b\, b_m\, M^{**} - \left[\frac{2}{3}\, b\, b_m\, M^{**} + \frac{2\, B\, b^2 \left(\theta_0\right)^2\, l_a}{c}\, (M^{**})^2 \right]$$

$$+ \, B \left[\frac{3\, b^2 \,(\theta_0)^2 \; l_a}{c} \; (M^{**})^2 \right] - b \; B - k$$

$$= \; \frac{4}{3}\, b\, b_m \, M^{**} + \frac{B\, b^2 \,(\theta_0)^2 \; l_a}{c} \,(M^{**})^2 - b \; B - k$$

$$> \; \frac{5}{6}\, b \; b_m \; M^{**} + \frac{B\, b^2 \,(\theta_0)^2 \; l_a}{2\,c} \,(M^{**})^2 - k - b \; B \; > \; 0 \, . \qquad (29)$$

The first inequality in (29) holds because $M^{**} > 0$. (26) and (28) imply that the second inequality in (29) holds when Condition (A3) holds.

$\widetilde{M}$ will be interior if $U_m\,(M, w)$ is declining in M at $M = \frac{1}{b}$. (Recall $\phi(\frac{1}{b}) = 1$.) From (22):

$$\left.\frac{\partial U_m\,(M, w)}{\partial M}\right|_{M \,=\, \frac{1}{b}} \; = \; 2\, b_m \; - \frac{1}{b\, c}\,[\, b_m + l_m \,]\, 4\, (\theta_0)^2 \; l_a \; + \frac{3}{c}\, B\, (\theta_0)^2 \; l_a \; - b \; B - k \; < \; 0 \, . \quad (30)$$

The inequality in (30) holds when Condition (A4) holds.

Furthermore, from (22), $\widetilde{M}$ is determined by:

$$G \; \equiv \; 2\, b\, b_m \; \widetilde{M} + \frac{3}{c}\, B\, b^2 \,(\theta_0)^2 \, l_a \, \widetilde{M}^2 - \frac{4}{c}\, [\, b_m + l_m \,]\, b^2 \,(\theta_0)^2 \, l_a \, \widetilde{M}^3 - k - b\, B \; = \; 0 \, . \qquad (31)$$

(31) implies that $\frac{d\widetilde{M}}{dk} = - \frac{\partial G / \partial k}{\partial G / \partial M} \overset{s}{=} \frac{\partial G}{\partial k} = -1 < 0$ since $U_m\,(\cdot)$ is strictly concave in M at $\widetilde{M}$.

To prove that $\widetilde{M} > \widehat{M}$, it suffices to show $R'\,(M)\,|_{M \,=\, \widetilde{M}} > 0$. This is the case because if $R'(\widetilde{M}) > 0$, then the value of M at which $R(M)$ attains its minimum value (i.e., $\widehat{M}$) must be less than $\widetilde{M}$, given the shape of $R(M)$, as characterized in Proposition 1.

(31) can be written as:

$$\frac{1}{c}\,[\, b_m + l_m \,]\, 4\, b^2 \,(\theta_0)^2 \, l_a \, \widetilde{M}^3 \; = \; 2\, b\, b_m \; \widetilde{M} + \frac{3\, B\, b^2 \,(\theta_0)^2 \, l_a}{c} \; \widetilde{M}^2 - b \; B - k$$

$$\Leftrightarrow \quad \frac{2}{c}\, b^2 \,(\theta_0)^2 \, l_a \, \widetilde{M}^3 \; = \; \frac{1}{2\,[\, b_m + l_m \,]} \left[2\, b\, b_m \; \widetilde{M} + \frac{3\, B\, b^2 \,(\theta_0)^2 \, l_a}{c} \; \widetilde{M}^2 - b \; B - k \right] . \qquad (32)$$

(5) and (32) provide:

$$R'\,(M)|_{M \,=\, \widetilde{M}} \; = \; \frac{2\,[\, 1 - \lambda \,]\, l + l_a}{2\,[\, b_m + l_m \,]} \left[2\, b\, b_m \; \widetilde{M} + \frac{3\, B\, b^2 \,(\theta_0)^2 \, l_a}{c} \; \widetilde{M}^2 - b \; B - k \right]$$

$$- \; 2\, b\,[\, l + \; l_a \,] \; \widetilde{M} - b\,[\, \overline{\Pi} - \; \overline{\Pi}_f \,]$$

$$= \frac{3\,B\,b^2\,(\theta_0)^2\;l_a\,[\,2\,(1-\lambda)\,l + l_a\,]}{2\,[\,b_m + l_m\,]\,c}\,\widetilde{M}^2 + \left[\frac{b\,b_m\,[\,2\,(1-\lambda)\,l + l_a\,]}{b_m + l_m} - 2\,b\,(\,l + \,l_a\,)\right]\widetilde{M}$$

$$- \frac{[\,2\,(1-\lambda)\,l + l_a\,]\,[\,b\,B + k\,]}{2\,[\,b_m + l_m\,]} - b\,[\,\overline{\Pi} - \,\overline{\Pi}_f\,]\,. \tag{33}$$

Define $\widehat{M}^*$ as the value of $\widetilde{M}$ at which the expression in (33) is zero. Since $R'(M) \lesseqqgtr 0$ as $M \lesseqqgtr \widehat{M}$, $\widetilde{M}$ must exceed $\widehat{M}^*$ (the value of M at which $R'\,(M)|_{M\,=\,\widetilde{M}} = 0$) if $R'\,(M)|_{M\,=\,\widetilde{M}} > 0$. Therefore, provided $U_m(\cdot)$ is strictly concave in M:[3]

$$R'\,(M)|_{M\,=\,\widetilde{M}} \;>\; 0 \;\;\Leftrightarrow\;\; \widetilde{M} \;>\; \widehat{M}^* \;\;\Leftrightarrow\;\; \left.\frac{\partial U_m(\cdot)}{\partial M}\right|_{M\,=\,\widehat{M}^*} \;>\; 0\,.$$

To prove that $\left.\frac{\partial U_m(\cdot)}{\partial M}\right|_{M\,=\,\widehat{M}^*} > 0$, observe from (22) that:

$$\left.\frac{\partial U_m(\cdot)}{\partial M}\right|_{M\,=\,\widehat{M}^*} = 2\,b\,b_m\,\widehat{M}^* - \frac{1}{c}\,[\,b_m + l_m\,]\,4\,b^2\,(\theta_0)^2\;l_a\,(\widehat{M}^*)^3$$

$$+ \;B\left[\frac{3}{c}\,b^2\,(\theta_0)^2\,l_a\,(\widehat{M}^*)^2\right] - b\;B\; - k\,. \tag{34}$$

From (33) and the definition of $\widehat{M}^*$:

$$\frac{3\,B\,b^2\,(\theta_0)^2\;l_a\,[\,2\,(1-\lambda)\,l + l_a\,]}{2\,[\,b_m + l_m\,]\,c}\,(\widehat{M}^*)^2$$

$$= \;-\left[\frac{b\,b_m\,[\,2\,(1-\lambda)\,l + l_a\,]}{b_m + l_m} - 2\,b\,(\,l + \,l_a\,)\right]\widehat{M}^*$$

$$+ \;\frac{[\,2\,(1-\lambda)\,l + l_a\,]\,[\,b\,B + k\,]}{2\,[\,b_m + l_m\,]} + b\,[\,\overline{\Pi} - \,\overline{\Pi}_f\,]$$

$$\Rightarrow\;\; \frac{3\,B\,b^2\,(\theta_0)^2\;l_a}{c}\,(\widehat{M}^*)^2 \;=\; -\left[2\,b\,b_m - \frac{4\,b\,(\,l + \,l_a\,)\,(\,b_m + l_m\,)}{2\,(\,1-\lambda\,)\,l + l_a}\right]\widehat{M}^*$$

$$+ \;\;b\,B + k + \frac{2\,b\,[\,b_m + l_m\,]\,[\,\overline{\Pi} - \,\overline{\Pi}_f\,]}{2\,[\,1-\lambda\,]\,l + l_a}\,. \tag{35}$$

From (34) and (35):

$$\left.\frac{\partial U_m\,(M, w)}{\partial M}\right|_{M\,=\,\widehat{M}^*} = 2\,b\,b_m\,\widehat{M}^* - \frac{1}{c}\,[\,b_m + l_m\,]\,4\,b^2\,(\theta_0)^2\;l_a\,(\widehat{M}^*)^3$$

[3] Recall that $U_m(\cdot)$ is strictly concave in M for $M > M^{**}$.

$$- \left[2\, b\, b_m - \frac{4\, b\, (\, l + \, l_a\,)\, (\, b_m + l_m\,)}{2\, [\, 1 - \lambda\,]\, l + l_a} \right] \widehat{M}^* + b\, B + k$$

$$+ \; \frac{2\, b\, \left[\, \overline{\Pi} - \, \overline{\Pi}_f\, \right] [\, b_m + l_m\,]}{2\, [\, 1 - \lambda\,]\, l + l_a} - b\ B\ - k$$

$$= \; - \, \frac{1}{c}\, [\, b_m + l_m\,]\, 4\, b^2\, (\theta_0)^2\ l_a\, (\widehat{M}^*)^3 + \frac{4\, b\, [\, l + \, l_a\,]\, [\, b_m + l_m\,]}{2\, [\, 1 - \lambda\,]\, l + l_a}\, \widehat{M}^*$$

$$+ \; \frac{2\, b\, \left[\, \overline{\Pi} - \, \overline{\Pi}_f\, \right] [\, b_m + l_m\,]}{2\, [\, 1 - \lambda\,]\, l + l_a}$$

$$= \; 2\, b\, [\, b_m + l_m\,] \left[- \frac{2\, b}{c}\, (\theta_0)^2\ l_a\, (\widehat{M}^*)^3 + \frac{2\, (\, l + \, l_a\,)\ \widehat{M}^* + \overline{\Pi} - \, \overline{\Pi}_f}{2\, (\, 1 - \lambda\,) + l_a} \right]. \tag{36}$$

From (35):

$$- \frac{2\, b}{c}\, (\theta_0)^2\ l_a\, (\widehat{M}^*)^3 \; = \; \frac{2}{3\, B\, b} \left[2\, b\, b_m - \frac{4\, b\, (\, l + \, l_a\,)\, (\, b_m + l_m\,)}{2\, [\, 1 - \lambda\,]\, l + l_a} \right] (\widehat{M}^*)^2$$

$$- \; \frac{2}{3\, B\, b} \left[b\, B + k + \frac{2\, b\, (\, b_m + l_m\,)\, \left(\overline{\Pi} - \, \overline{\Pi}_f\, \right)}{2\, [\, 1 - \lambda\,]\, l + l_a} \right] \widehat{M}^*$$

$$= \; \left[\frac{4\, b\, b_m}{3\, B\, b} - \frac{8\, b\, (\, l + \, l_a\,)\, (\, b_m + l_m\,)}{3\, B\, b\, [\, 2\, (1 - \lambda)\, l + l_a\,]} \right] \left(\widehat{M}^* \right)^2$$

$$- \; \left[\frac{2\, (\, b\, B + k\,)}{3\, B\, b} + \frac{4\, b\, (\, b_m + l_m\,)\, \left(\overline{\Pi} - \, \overline{\Pi}_f\, \right)}{3\, B\, b\, [\, 2\, (1 - \lambda)\, l + l_a\,]} \right] \widehat{M}^*. \tag{37}$$

(36) and (37) imply:

$$\left. \frac{\partial U_m\, (M, w)}{\partial M} \right|_{M\, =\, \widehat{M}^*} \; = \; 2\, b\, [\, b_m + l_m\,] \left\{ \left[\frac{4\, b_m}{3\, B} - \frac{8\, (\, l + \, l_a\,)\, (\, b_m + l_m\,)}{3\, B\, [\, 2\, (1 - \lambda)\, l + l_a\,]} \right] (\widehat{M}^*)^2 \right.$$

$$- \; \left[\frac{2\, (\, b\, B + k\,)}{3\, B\, b} + \frac{4\, b\, (\, b_m + l_m\,)\, \left(\overline{\Pi} - \, \overline{\Pi}_f\, \right)}{3\, B\, b\, [\, 2\, (1 - \lambda)\, l + l_a\,]} \right] \widehat{M}^*$$

$$\left. + \; \frac{2\, [\, l + \, l_a\,]}{2\, [\, 1 - \lambda\,]\, l + l_a}\, \widehat{M}^* + \; \frac{\overline{\Pi} - \, \overline{\Pi}_f}{2\, [\, 1 - \lambda\,]\, l + l_a} \right\}$$

$$= \; 2\, b\, [\, b_m + l_m\,] \left\{ \left[\frac{4\, b_m}{3\, B} - \frac{8\, (\, l + \, l_a\,)\, (\, b_m + l_m\,)}{3\, B\, (\, 2\, [\, 1 - \lambda\,]\, l + l_a\,)} \right] (\widehat{M}^*)^2 + \frac{\overline{\Pi} - \, \overline{\Pi}_f}{2\, [\, 1 - \lambda\,]\, l + l_a} \right.$$

$$\left. - \; \left[\frac{2\, (\, b\, B + k\,)}{3\, B\, b} + \frac{4\, b\, (\, b_m + l_m\,)\, \left(\overline{\Pi} - \, \overline{\Pi}_f\, \right)}{3\, B\, b\, [\, 2\, (1 - \lambda)\, l + l_a\,]} - \frac{2\, (\, l + \, l_a\,)}{2\, (\, 1 - \lambda\,)\, l + l_a} \right] \widehat{M}^* \right\}. \tag{38}$$

From (33):

$$(\widehat{M}^{*})^2 \;=\; \frac{2\,[\,b_m + l_m\,]\,c}{3\,B\,b^2\,(\theta_0)^2\,l_a\,[\,2\,(1-\lambda)\,l + l_a\,]}\left\{ - \left[\frac{b\,b_m\,[\,2\,(1-\lambda)\,l + l_a\,]}{b_m + l_m} - 2\,b\,(\,l + \,l_a\,) \right] \widehat{M}^{*} \right.$$
$$\left. + \frac{[\,2\,(1-\lambda)\,l + l_a\,]\,[\,b\,B + k\,]}{2\,[\,b_m + l_m\,]} + b\,[\,\overline{\Pi} - \,\overline{\Pi}_f\,] \right\}$$

$$= \; - \frac{2\,[\,b_m + l_m\,]\,c}{3\,B\,b^2\,(\theta_0)^2\;l_a\,[\,2\,(1-\lambda)\,l + l_a\,]} \left[\frac{b\,b_m\,[\,2\,(1-\lambda)\,l + l_a\,]}{b_m + l_m} - 2\,b\,(\,l + \,l_a\,) \right] \widehat{M}^{*}$$
$$+ \; \frac{c\,[\,b\,B + k]}{3\,B\,b^2\,(\theta_0)^2\;l_a} + \frac{2\,b\,c\,[\,b_m + l_m\,]\,[\,\overline{\Pi} - \,\overline{\Pi}_f\,]}{3\,B\,b^2\,(\theta_0)^2\,l_a\,[\,2\,(1-\lambda)\,l + l_a\,]}$$

$$= \left[- \frac{2\,b\,b_m\,c}{3\,B\,b^2\,(\theta_0)^2\,l_a} + \frac{4\,b\,c\,(\,b_m + l_m\,)\,(\,l + \,l_a\,)}{3\,B\,b^2\,(\theta_0)^2\,l_a\,(\,2\,[\,1-\lambda\,]\,l + l_a\,)} \right] \widehat{M}^{*}$$
$$+ \; \frac{c\,[\,b\,B + k\,]}{3\,B\,b^2\,(\theta_0)^2\,l_a} + \frac{2\,b\,c\,[\,b_m + l_m\,]\,[\,\overline{\Pi} - \,\overline{\Pi}_f\,]}{3\,B\,b^2\,(\theta_0)^2\,l_a\,[\,2\,(1-\lambda)\,l + l_a\,]}\,. \tag{39}$$

(38) and (39) provide:

$$\left. \frac{\partial U_m\,(M, w)}{\partial M} \right|_{M \,=\, \widehat{M}^{*}} \;=\; 2\,b\,[\,b_m + l_m\,]$$
$$\cdot \left\{ \left[\frac{4\,b_m}{3\,B} - \frac{8\,(\,l + \,l_a\,)\,(\,b_m + l_m\,)}{3\,B\,(\,2\,[\,1-\lambda\,]\,l + l_a\,)} \right] \left[- \frac{2\,b\,b_m\,c}{3\,B\,b^2\,(\theta_0)^2\,l_a} + \frac{4\,b\,c\,(\,b_m + l_m\,)\,(\,l + \,l_a\,)}{3\,B\,b^2\,(\theta_0)^2\,l_a\,(\,2\,[\,1-\lambda\,]\,l + l_a\,)} \right] \widehat{M}^{*} \right.$$
$$+ \; \frac{c\,[\,b\,B + k\,]}{3\,B\,b^2\,(\theta_0)^2\,l_a} \left[\frac{4\,b_m}{3\,B} - \frac{8\,(\,l + \,l_a\,)\,(\,b_m + l_m\,)}{3\,B\,(\,2\,[\,1-\lambda\,]\,l + l_a\,)} \right]$$
$$+ \; \frac{2\,b\,c\,[\,b_m + l_m\,]\,[\,\overline{\Pi} - \,\overline{\Pi}_f\,]}{3\,B\,b^2\,(\theta_0)^2\,l_a\,[\,2\,(1-\lambda)\,l + l_a\,]} \left[\frac{4\,b_m}{3\,B} - \frac{8\,(\,l + \,l_a\,)\,(\,b_m + l_m\,)}{3\,B\,(\,2\,[\,1-\lambda\,]\,l + l_a\,)} \right]$$
$$- \; \left[\frac{2\,(\,b\,B + k\,)}{3\,B\,b} + \frac{4\,b\,(\,b_m + l_m\,)\left(\overline{\Pi} - \,\overline{\Pi}_f\right)}{3\,B\,(\,2\,[\,1-\lambda\,]\,l + l_a\,)} - \frac{2\,(\,l + \,l_a\,)}{2\,(\,1-\lambda\,)\,l + l_a} \right] \widehat{M}^{*}$$
$$\left. + \; \frac{\overline{\Pi} - \,\overline{\Pi}_f}{2\,[\,1-\lambda\,]\,l + l_a} \right\}. \tag{40}$$

(40) implies:

$$\left. \frac{\partial U_m(\cdot)}{\partial M} \right|_{M \,=\, \widehat{M}^{*}} \;>\; 0 \;\; \text{if} \;\; \widehat{M}^{*} \;>\; \frac{\alpha}{\beta}\,, \;\; \text{where } \alpha \text{ and } \beta \text{ are defined in (19) and (20).}$$

The expressions for α and β follow from (40). In particular:

$$\alpha \;=\; -\frac{2\,b\,c\,[\,b_m + l_m\,]\left[\,\overline{\Pi} - \overline{\Pi}_f\,\right]}{3\,B\,b^2\,(\theta_0)^2\,l_a\,[\,2\,(1-\lambda)\,l + l_a\,]}\left[\frac{4\,b_m}{3\,B} - \frac{8\,(\,l + l_a\,)\,(\,b_m + l_m\,)}{3\,B\,(\,2\,[\,1-\lambda\,]\,l + l_a\,)}\right]$$

$$- \frac{c\,[\,b\,B + k\,]}{3\,B\,b^2\,(\theta_0)^2\,l_a}\left[\frac{4\,b_m}{3\,B} - \frac{8\,(\,l + l_a\,)\,(\,b_m + l_m\,)}{3\,B\,(\,2\,[\,1-\lambda\,]\,l + l_a\,)}\right] - \frac{\overline{\Pi} - \overline{\Pi}_f}{2\,[\,1-\lambda\,]\,l + l_a}$$

$$= \frac{8\,c\,[\,b_m + l_m\,]\left[\,\overline{\Pi} - \overline{\Pi}_f\,\right][\,b_m\,(\,2\,\lambda\,l + l_a\,) + 2\;l_m\,(\,l + l_a\,)\,]}{9\,B^2\,b\,(\theta_0)^2\,l_a\,[\,2\,(1-\lambda)\,l + l_a\,]^2}$$

$$+ \frac{4\,c\,[\,b\,B + k\,]\,[\,b_m\,(\,2\,\lambda\,l + l_a\,) + 2\,l_m\,(\,l + l_a\,)\,]}{9\,B^2\,b^2\,(\theta_0)^2\,l_a\,[\,2\,(1-\lambda)\,l + l_a\,]} - \frac{\overline{\Pi} - \overline{\Pi}_f}{2\,[\,1-\lambda\,]\,l + l_a}. \qquad (41)$$

The last equality in (41) holds because:

$$\frac{4\,b_m}{3\,B} - \frac{8\,[\,l + l_a\,]\,[\,b_m + l_m\,]}{3\,B\,[\,2\,(\,1-\lambda\,)\,l + l_a\,]} \;=\; \frac{4\,b_m\,[\,2\,(\,1-\lambda\,)\,l + l_a\,] - 8\,[\,l + l_a\,]\,[\,b_m + l_m\,]}{3\,B\,[\,2\,(\,1-\lambda\,)\,l + l_a\,]}$$

$$= \frac{4\,b_m\,[\,2\,(\,1-\lambda\,)\,l + l_a - 2\,l - 2\,l_a\,] - 8\,l_m\,[\,l + l_a\,]}{3\,B\,[\,2\,(\,1-\lambda\,)\,l + l_a\,]}$$

$$= -\frac{4\,b_m\,[\,2\,\lambda\,l + l_a\,] + 8\,l_m\,[\,l + l_a\,]}{3\,B\,[\,2\,(\,1-\lambda\,)\,l + l_a\,]}. \qquad (42)$$

Also, from (40) and (42):

$$\beta \;=\; \left[\frac{4\,b_m}{3\,B} - \frac{8\,(\,l + l_a\,)\,(\,b_m + l_m\,)}{3\,B\,(\,2\,[\,1-\lambda\,]\,l + l_a\,)}\right]\left[-\frac{2\,b\,b_m\,c}{3\,B\,b^2\,(\theta_0)^2\,l_a} + \frac{4\,b\,c\,(\,b_m + l_m\,)\,(\,l + l_a\,)}{3\,B\,b^2\,(\theta_0)^2\,l_a\,[\,2\,(1-\lambda)\,l + l_a\,]}\right]$$

$$- \frac{2\,[\,b\,B + k\,]}{3\,B\,b} - \frac{4\,b\,[\,b_m + l_m\,]\left[\,\overline{\Pi} - \overline{\Pi}_f\,\right]}{3\,B\,b\,[\,2\,(1-\lambda)\,l + l_a\,]} + \frac{2\,[\,l + l_a\,]}{2\,[\,1-\lambda\,]\,l + l_a}$$

$$= -\frac{4}{3\,B}\left[\frac{b_m\,(\,2\,\lambda\,l + l_a\,) + 2\;l_m\,(\,l + l_a\,)}{2\,[\,1-\lambda\,]\,l + l_a}\right]\left[-\frac{2\,b\,b_m\,c}{3\,B\,b^2\,(\theta_0)^2\,l_a} + \frac{4\,b\,c\,(\,b_m + l_m\,)\,(\,l + l_a\,)}{3\,B\,b^2\,(\theta_0)^2\,l_a\,[\,2\,(1-\lambda)\,l + l_a\,]}\right]$$

$$- \frac{2\,[\,b\,B + k\,]}{3\,B\,b} - \frac{4\,b\,[\,b_m + l_m\,]\left[\,\overline{\Pi} - \overline{\Pi}_f\,\right]}{3\,B\,b\,[\,2\,(1-\lambda)\,l + l_a\,]} + \frac{2\,[\,l + l_a\,]}{2\,[\,1-\lambda\,]\,l + l_a}$$

$$= -\frac{4\,[\,b_m\,(\,2\,\lambda\,l + l_a\,) + 2\;l_m\,(\,l + l_a\,)\,]}{3\,B\,[\,2\,(1-\lambda)\,l + l_a\,]}\left[\frac{4\,b\,c\,(\,b_m + l_m\,)\,(\,l + l_a\,) - 2\,b\,b_m\,c\,(\,2\,[\,1-\lambda\,]\,l + l_a\,)}{3\,B\,b^2\,(\theta_0)^2\,l_a\,[\,2\,(1-\lambda)\,l + l_a\,]}\right]$$

$$- \frac{2\,[\,b\,B + k\,]}{3\,B\,b} - \frac{4\,b\,[\,b_m + l_m\,]\left[\,\overline{\Pi} - \overline{\Pi}_f\,\right]}{3\,B\,b\,[\,2\,(1-\lambda)\,l + l_a\,]} + \frac{2\,[\,l + l_a\,]}{2\,[\,1-\lambda\,]\,l + l_a}$$

$$
\begin{aligned}
= & -\frac{8\,b\,c\,[\,b_m\,(\,2\,\lambda\,l + l_a\,) + 2\;l_m\,(\,l + l_a\,)\,]^2}{9\,B^2\,b^2\,(\theta_0)^2\,l_a\,[\,2\,(1-\lambda)\,l + l_a\,]^2} - \frac{2\,[\,b\,B + k\,]}{3\,B\,b} \\
& - \frac{4\,[\,b_m + l_m\,]\,\left[\,\overline{\Pi} - \overline{\Pi}_f\,\right]}{3\,B\,[\,2\,(1-\lambda)\,l + l_a\,]} + \frac{2\,[\,l + l_a\,]}{2\,[\,1-\lambda\,]\,l + l_a}\,.
\end{aligned} \tag{43}
$$

The last equality in (43) reflects the fact that:

$$
\begin{aligned}
& 4\,b\,c\,[\,b_m + l_m\,]\,[\,l + l_a\,] - 2\,b\,b_m\,c\,[\,2\,(\,1-\lambda\,)\,l + l_a\,] \\
& \quad = \; 2\,b\,c\,b_m\,[\,2\,(\,l + l_a\,) - 2\,(1-\lambda)\,l - l_a\,] + 4\,b\,c\,l_m\,[\,l + l_a\,] \\
& \quad = \; 2\,b\,c\,[\,b_m\,(\,2\,\lambda\,l + l_a) + 2\,l_m\,(\,l + l_a)\,]\,.
\end{aligned}
$$

From (39):

$$
(\widehat{M}^*)^2 + h\,\widehat{M}^* + h_0 \;=\; 0\,, \tag{44}
$$

where

$$
\begin{aligned}
h \;\equiv\; & \frac{1}{D}\,\{\,2\,b\,b_m\,c\,[\,2\,(1-\lambda)\,l + l_a\,] - 4\,b\,c\,[\,b_m + l_m\,]\,[\,l + l_a\,]\,\} \\
= \; & -\frac{2\,b\,c}{D}\,[\,b_m\,(\,2\,\lambda\,l + l_a\,) + 2\,l_m\,(\,l + l_a\,)\,]\,, \qquad (45) \\
h_0 \;\equiv\; & -\frac{1}{D}\left\{\,2\,b\,c\,[\,b_m + l_m\,]\left[\,\overline{\Pi} - \overline{\Pi}_f\,\right] + c\,[\,b\,B + k\,]\,[\,2\,(1-\lambda)\,l + l_a\,]\right\}\,, \quad \text{and} \\
D \;\equiv\; & 3\,B\,b^2\,(\theta_0)^2\,l_a\,[\,2\,(1-\lambda)\,l + l_a\,]\,.
\end{aligned}
$$

The equality in (45) holds because:

$$
\begin{aligned}
& 2\,b\,b_m\,c\,[\,2\,(1-\lambda)\,l + l_a\,] - 4\,b\,c\,[\,b_m + l_m\,]\,[\,l + l_a\,] \\
& \quad = \; 2\,b\,c\,\{\,b_m\,[\,2\,(1-\lambda)\,l + l_a - 2\,(\,l + l_a\,)\,] - 2\,l_m\,(\,l + l_a\,)\,\} \\
& \quad = \; -2\,b\,c\,[\,b_m\,(2\,\lambda\,l + l_a) + 2\,l_m\,(l + l_a\,)\,]\,.
\end{aligned}
$$

(44) implies:

$$
\begin{aligned}
\widehat{M}^* \; = \; & \frac{-h + \sqrt{h^2 - 4\,h_0}}{2} \\
= \; & \frac{1}{2\,D}\,\{\,2\,b\,c\,[\,b_m\,(2\,\lambda\,l + l_a) + 2\,l_m\,(l + l_a\,)\,]\,\} \\
& + \; \frac{1}{2}\sqrt{\frac{4\,b^2\,c^2}{D^2}\,[\,b_m\,(\,2\,\lambda\,l + l_a\,) + 2\,l_m\,(\,l + l_a\,)\,]^2 - \frac{4\,h_0\,D^2}{D^2}}
\end{aligned} \tag{46}
$$

$$= \frac{1}{D} \left\{ b\,c\,[\,b_m\,(\,2\,\lambda\,l + l_a\,) + 2\,l_m\,(\,l + \,l_a\,\,)\,] + \sqrt{Y}\,\right\} \tag{47}$$

where

$$Y \;\equiv\; b^2\,c^2\,[\,b_m\,(\,2\,\lambda\,l + l_a\,\,) + 2\,l_m\,(\,l + \,l_a\,)\,]^2 - h_0\,D^2$$

$$= \; b^2\,c^2\,[\,b_m\,(\,2\,\lambda\,l + l_a\,) + 2\,l_m\,(\,l + \,l_a\,)\,]^2 + 2\,b\,c\,D\,[\,b_m + l_m\,]\,[\,\overline{\Pi} - \,\overline{\Pi}_f\,]$$

$$+ \; c\,D\,[\,b\,B + k\,]\,[\,2\,(1-\lambda)\,l + l_a\,\,]\,. \quad \blacksquare \tag{48}$$