Technical Appendix to Accompany

"Implementing High-Powered Contracts to Motivate Intertemporal Effort Supply"

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Recall from section 2 in the text that the principal's problem $[P]$ is to:

$$\underset{r(\cdot),\, e_1(\cdot),\, \cdots,\, e_n(\cdot)}{Maximize} \quad \int \left[\beta + \sum_{j=1}^{n} e_j(\cdot) - r(\beta + \sum_{j=1}^{n} e_j(\cdot)) \right] dF((\beta_i)_{i \le n}) \tag{1}$$

$$\text{subject to:} \quad r(\cdot) \ge 0\,; \;\text{ and} \tag{2}$$

$$e_j\,((\beta_i)_{i \le j} \,|\, e_i(\cdot)_{i<j}) \in$$

$$\underset{e}{argmax} \; \{e \mid \int U_{j+1}(\sum_{i=1}^{j-1} e_i(\cdot) + e \;|(\beta_i)_{i \le j+1}) dF(\beta_{j+1}|(\beta_i)_{i \le j}) - C_j(e)\}. \tag{3}$$

Also recall from section 3 in the text that the principal's objective can be restated as:

$$\begin{aligned} & \left[\frac{1}{1-t_n}\right] \int \left[e_n((\beta_i)_{i \le j})) - C(e_n((\beta_i)_{i \le n}))\right] dF((\beta_i)_{i \le n}) \\ - & \left[\frac{1}{1-t_n}\right] \int \left(C'(e_n((\beta_i)_{i \le n})) \left[\frac{1 - F(\beta_n|(\beta_i)_{i<n})}{f(\beta_n|(\beta_i)_{i<n})}\right] \right) dF((\beta_i)_{i \le n}) \\ - & \int u(\underline{\beta}|(\beta_i)_{i<n}) dF((\beta_i)_{i<n}). \end{aligned} \tag{4}$$

In addition, when there are two time periods, $[0, t)$ and $[t, 1]$, and when the agent has no information advantage during the first period but learns the realization of β at the start of the second period (i.e., at time t), the principal's problem $[P]$ is:

$$\underset{r(\cdot),\, e_1,\, e_2(\cdot)}{Maximize} \quad \int_{\underline{\beta}}^{\overline{\beta}} [\beta + e_1 + e_2(\beta; e_1) - r(\beta + e_1 + e_2(\beta; e_1))] dF(\beta) \tag{5}$$

$$\text{subject to:} \qquad r(\cdot) \ge 0\;; \tag{6}$$

$$e_1 \;\in\; \underset{e}{argmax} \; \{e \mid \int_{\underline{\beta}}^{\overline{\beta}} U_2(e|\beta) dF(\beta) - C_1(e)\}; \;\text{ and} \tag{7}$$

$$e_2(\beta; e_1) \;\in\; \underset{e}{argmax} \; \{e \mid r(\beta + e_1 + e) - C_2(e)\}. \tag{8}$$

The corresponding restatement of the principal's objective is:

$$\left[\frac{1}{1-t}\right] \int_{\underline{\beta}}^{\overline{\beta}} \left(e_2(\beta) - C(e_2(\beta)) - C'(e_2(\beta)) \left[\frac{1 - F(\beta)}{f(\beta)}\right] \right) dF(\beta) - u(\underline{\beta}). \tag{9}$$

Finally, recall the following conclusions from section 3 in the text:

$$u(\beta_n|(\beta_i)_{i<n}) \;=\; U_n(\sum_{j=1}^{n-1} e_k((\beta_i)_{i \le j})|(\beta_i)_{i \le n}) \tag{10}$$

$$U_n'(\sum_{j=1}^{n-1} e_k((\beta_i)_{i\leq j})|(\beta_i)_{i\leq n}) = u'(\beta_n|(\beta_i)_{i<n}) = C_n'(e_n((\beta_i)_{i\leq n})). \tag{11}$$

Lemma 1. *The minimum cost of delivering effort e during time interval* $[t_j, t_{j+1}]$ *is* $\frac{e^2}{4k[t_{j+1}-t_j]}$, *since the agent's effort cost is minimized when he delivers effort at the constant rate* $\frac{e}{t_{j+1}-t_j}$.

Proof of Lemma 1.

Because the agent's instantaneous cost of delivering effort rate $\lambda(\tau)$ is $\frac{1}{4k}\lambda(\tau)^2$, his cost of supplying effort at rate $\lambda(\tau)$ on the interval $[t_0, t_1]$ is $\int_{t_0}^{t_1} \frac{1}{4k}\lambda(\tau)^2 d\tau$. The agent will minimize his personal cost of delivering any chosen effort level e, and so will choose effort rate $\lambda(\tau)$ over time period $[t_0, t_1]$ to:

$$\underset{\lambda(\tau)}{Minimize} \int_{t_0}^{t_1} \frac{1}{4k}\lambda(\tau)^2 d\tau \quad \text{subject to} \quad \int_{t_0}^{t_1} \lambda(\tau) d\tau = e \ . \tag{12}$$

Letting γ denote the Lagrange multiplier associated with the constraint in (12), the relevant Lagrangian function is:

$$\int_{t_0}^{t_1} \left[\frac{1}{4k}\lambda(\tau)^2 - \gamma\lambda(\tau) \right] d\tau + \gamma e \ . \tag{13}$$

Maximizing (13) with respect to $\lambda(\tau)$ reveals that the agent's cost-minimizing level of effort supply is given by:

$$\lambda(\tau) = 2k\gamma. \tag{14}$$

Integrating (14) over the $[t_0, t_1]$ interval and using the equality in (12) provides:

$$\gamma = \frac{e}{2k[t_1 - t_0]} \ . \tag{15}$$

(14) and (15) imply that the agent's minimum cost of supplying effort e over the time interval $[t_0, t_1]$ is:

$$\int_{t_0}^{t_1} \frac{1}{4k}\lambda(\tau)^2 d\tau \ = \ \frac{\gamma^2}{4k}\int_{t_0}^{t_1} [2k]^2 d\tau \ = \ \frac{e^2}{4k[t_1 - t_0]}. \ \blacksquare \tag{16}$$

Lemma 2. *At the solution to* $[P]$, *for each* $j = 1, \cdots, n-1$:

$$\frac{e_j}{e_j^*} = \frac{e_j}{2k[t_{j+1} - t_j]} = \frac{\int e_h((\beta_i)_{i\leq h}) dF((\beta_i)_{j<i\leq h}|(\beta_i)_{i\leq j})}{e_h^*} = \frac{E_j\{e_h(\beta)\}}{2k[t_{h+1} - t_h]}, \tag{17}$$

for $h = j+1, ..., n$.

Proof of Lemma 2.

Because the agent's effort in period h is a perfect substitute for his effort in earlier period j,

$$C_j'(e_j((\beta_i)_{i\leq j})) = \int C_h'(e_h((\beta_i)_{i\leq h})) dF((\beta_i)_{j<i\leq h}|(\beta_i)_{i\leq j}) \tag{18}$$

for $j = 1, ..., n-1$ and $h = j+1, ..., n$.

(16) implies:

$$C_j'(e_j) = \frac{e_j}{2k[t_{j+1} - t_j]} = \frac{e_j}{e_j^*} \quad \text{and} \quad C_h'(e_h) = \frac{e_h}{2k[t_{h+1} - t_h]} = \frac{e_h}{e_h^*}\ . \tag{19}$$

Therefore, (18) can be rewritten as:

$$\frac{e_j}{e_j^*} = \frac{e_j}{2k[t_{j+1} - t_j]} = \frac{\int e_h((\beta_i)_{i\leq h})dF((\beta_i)_{j<i\leq h}|(\beta_i)_{i\leq j})}{e_h^*} = \frac{E_j\{e_h(\beta)\}}{2k[t_{h+1} - t_h]}. \ \blacksquare \tag{20}$$

Lemma 3. *Suppose $t = 0$. Then $e_2(\beta) = \max\ \{0,\ 2k + \beta - \overline{\beta}\}$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$ at the solution to $[P]$.*

Proof of Lemma 3.

The proof is provided in Laffont and Tirole (1986). $\blacksquare$

Lemma 4. *Suppose $t = 1$. Then the principal can ensure the first-best solution with the contract* $r(x) = \begin{cases} 0 & \text{for } x < \underline{x} \\ [\max\{1, \frac{\Delta}{2k}\}][x - \underline{x}] & \text{for } x \geq \underline{x}. \end{cases}$

Proof of Lemma 4.

The proof for the case where $\Delta \leq 2k$ follows from the proof of Proposition 1. The proof for the case where $\Delta > 2k$ is analogous. $\blacksquare$

Proposition 1. *Suppose $\Delta \leq 2k$ and $t \in [\frac{\Delta}{2k}, 1]$. Then $e_1 = e_1^*$ and $E\{e_2(\beta)\} = e_2^*$ at the solution to $[P]$, and the first-best solution is feasible.*

Proof of Proposition 1.

When the agent supplies effort $2k$, output is at least $\underline{\beta} + 2k \geq \frac{1}{2}[\underline{\beta} + \overline{\beta} + 2k]$. This inequality holds because:

$$\underline{\beta} + 2k \geq \frac{1}{2}[\underline{\beta} + \overline{\beta} + 2k] \ \Leftrightarrow\ 2\underline{\beta} + 4k \geq \underline{\beta} + \overline{\beta} + 2k \ \Leftrightarrow\ \Delta \leq 2k. \tag{21}$$

Therefore, when the agent supplies effort $2k$, the payment to the agent is always non-negative, the principal's sure return is $\frac{1}{2}[\underline{\beta} + \overline{\beta} + 2k]$, and the agent's expected utility is $\frac{1}{2}[\underline{\beta} + \overline{\beta}] + 2k - \frac{1}{2}[\underline{\beta} + \overline{\beta} + 2k] - \frac{1}{4k}[2k]^2 = k - k = 0$. Thus, the contract specified in Lemma 4 will secure the first-best outcome if it induces the agent to supply the first-best effort.

To show that this contract will induce the agent to supply the first-best effort when $\Delta \leq 2k$, notice that the agent will secure a payoff of 0 if he supplies 0 effort under the contract (since $0 + \overline{\beta} \leq \frac{1}{2}[\underline{\beta} + \overline{\beta} + 2k] \Leftrightarrow 2\overline{\beta} \leq \underline{\beta} + \overline{\beta} + 2k \Leftrightarrow 2k \geq \Delta$). If the agent delivers sufficient second-period effort to ensure output of at least $\frac{1}{2}[\underline{\beta} + \overline{\beta} + 2k]$, he retains the entire incremental surplus generated by his effort (since $r'(x) = 1$). Therefore, he will supply the efficient second-period effort $2k[1 - t]$. From (16), this effort entails personal cost:

$$\frac{1}{4k[1 - t]}[2k(1 - t)]^2 = k[1 - t]. \tag{22}$$

When $\beta + e_1 = \frac{1}{2}[\underline{\beta} + \overline{\beta}] + kt$, second-period effort $2k[1-t]$ generates reward:

$$\frac{1}{2}[\underline{\beta} + \overline{\beta}] + kt + 2k[1-t] - \frac{1}{2}[\underline{\beta} + \overline{\beta} + 2k] = k[1-t]. \tag{23}$$

(22) and (23) imply that the agent will secure an expected utility of 0 by delivering second-period effort $2k[1-t]$ when $\beta + e_1 = \frac{1}{2}[\underline{\beta} + \overline{\beta}] + kt$.

When $\beta + e_1 > \frac{1}{2}[\underline{\beta} + \overline{\beta}] + kt$, the agent can secure a strictly positive payoff by supplying first-best effort $2k[1-t]$ in the second period. In this case, output will exceed $\beta + e_1 + 2k[1-t] > \frac{1}{2}[\underline{\beta} + \overline{\beta}] + 2k[1-t] + kt \geq \frac{1}{2}[\underline{\beta} + \overline{\beta} + 2k]$. Therefore, the only relevant portion of the contract is that is which $r'(x) = 1$. Consequently, under this contract:

$$U_2(e_1|\beta) = \begin{cases} 0 & \text{for } \beta + e_1 < \frac{1}{2}[\underline{\beta} + \overline{\beta}] + kt \\ \beta + e_1 - \frac{1}{2}[\underline{\beta} + \overline{\beta}] - kt & \text{for } \beta + e_1 \geq \frac{1}{2}[\underline{\beta} + \overline{\beta}] + kt. \end{cases} \tag{24}$$

The last line in (24) reflects the fact that when $e = 2k[1-t]$, $\beta + e_1 + e - C(e) - \frac{1}{2}[\underline{\beta} + \overline{\beta} + 2k] = \beta + e_1 + 2k[1-t] - k[1-t] - \frac{1}{2}[\underline{\beta} + \overline{\beta} + 2k] = \beta + e_1 - \frac{1}{2}[\underline{\beta} + \overline{\beta}] - kt$.

The agent chooses first-period effort e_1 to maximize $\int_{\underline{\beta}}^{\overline{\beta}} U_2(e_1|\beta) dF(\beta) - C_1(e_1)$. The marginal cost of first-period effort is $\frac{e_1}{2kt}$. By (10) and (11), the marginal expected payoff from first-period effort is:

$$\int_{\underline{\beta}}^{\overline{\beta}} U_2'(e_1|\beta) dF(\beta) = \int_{\underline{\beta}}^{\overline{\beta}} u'(\beta) dF(\beta) = \frac{1}{\Delta}[U_2(e_1|\overline{\beta}) - U_2(e_1|\underline{\beta})]. \tag{25}$$

Notice that $U_2(0|\overline{\beta}) = 0$ because $\overline{\beta} + 0 \leq \frac{\underline{\beta}+\overline{\beta}}{2} + kt \Leftrightarrow 2\overline{\beta} \leq \underline{\beta} + \overline{\beta} + 2kt \Leftrightarrow \Delta \leq 2kt$.

To find the effort e_1 that maximizes $\int_{\underline{\beta}}^{\overline{\beta}} U_2(e_1|\beta) dF(\beta)$, compare the marginal benefit and cost of e_1 as e_1 varies. When e_1 increases from 0 to infinity, the marginal cost $\frac{e_1}{2kt}$ increases linearly, while the marginal benefit varies through three phases: (i) For small e_1, $U_2(e_1|\overline{\beta}) = 0 = U_2(e_1|\underline{\beta})$. Consequently, from (25), the marginal gain from increased effort is zero, and so the marginal gain is less than the marginal cost. (ii) For intermediate e_1, $U_2(e_1|\underline{\beta}) = 0$, while $U_2(e_1|\overline{\beta})$ increases at rate 1. Consequently, the marginal benefit increases linearly at the rate $\frac{1}{\Delta}$, which exceeds $\frac{1}{2kt}$, the rate at which marginal cost increases with e_1. The marginal benefit is first smaller than the marginal cost, and then greater than the marginal cost. At the point where the marginal benefit equals the marginal cost, the agent's payoff is minimized. (iii) For large e_1, both $U_2(e_1|\overline{\beta})$ and $U_2(e_1|\underline{\beta})$ increase with e_1 at the rate 1, and so the marginal benefit is the constant $[\overline{\beta} - \underline{\beta}]/\Delta = 1$. The marginal cost $\frac{e_1}{2kt}$ is first smaller than 1, and then greater than 1. The marginal benefit equals the marginal cost at the first-best effort level, $e_1 = 2kt$.

To summarize, the agent's payoff is maximized at either $e_1 = 0$ or $e_1 = 2kt$. Both effort levels provide an expected payoff of zero. Therefore, the agent is willing to supply first-best effort $2kt$ and the principal can secure the first-best outcome with the contract specified in Lemma 4 when $t \geq \frac{\Delta}{2k}$. ■

Proposition 3. *Suppose* $\Delta \leq 2k$ *and* $t \in (0, \frac{\Delta}{\Delta+2k})$. *Then at the solution to* $[P]$: $e_2(\beta) = 2k + \beta - \overline{\beta}$ *for all* $\beta \in [\underline{\beta}, \overline{\beta}]$ *and* $e_1 = \left[\frac{t}{1-t}\right]\left[\frac{4k-\Delta}{2}\right]$. *Consequently: (i)* $e_1 < e_1^*$ *and* $E\{e_2(\beta)\} < e_2^*$ *for* $t \in (0, \frac{\Delta}{4k})$; *(ii)* $e_1 = e_1^*$ *and* $E\{e_2(\beta)\} = e_2^*$ *when* $t = \frac{\Delta}{4k}$; *and (iii)* $e_1 > e_1^*$ *and* $E\{e_2(\beta)\} > e_2^*$ *for* $t \in (\frac{\Delta}{4k}, \frac{\Delta}{\Delta+2k})$.

Proposition 5. *Suppose* $\Delta > 2k$ *and* $t \in (0, \frac{\Delta}{\Delta+2k})$*. Then at the solution to* $[P]$*:* $e_2(\beta) = \max\{0, 2k + \beta - \overline{\beta}\}$ *for all* $\beta \in [\underline{\beta}, \overline{\beta}]$*, and so* $E\{e_2(\beta)\} < e_2^*$*. Furthermore,* $e_1 = \left[\frac{t}{1-t}\right] \frac{2k^2}{\Delta} < e_1^*$*.*

Proof of Proposition 3 and 5.

(9) and Lemma 3 imply that as long as the second order conditions are satisfied at the identified solution, the solution to $[P]$ entails $u(\underline{\beta}) = 0$[1] and:

$$e_2(\beta) = \begin{cases} 2k + \beta - \overline{\beta} & \text{for } \beta > \overline{\beta} - 2k \\ 0 & \text{for } \beta \leq \overline{\beta} - 2k. \end{cases} \tag{26}$$

From (20), the optimal value of e_1 is:

$$\begin{aligned} e_1 &= \left[\frac{t}{1-t}\right] E[e_2(\beta)] \quad = \quad \left[\frac{t}{1-t}\right] \int_{\overline{\beta}-2k}^{\overline{\beta}} [2k + \beta - \overline{\beta}] dF(\beta) \\ &= \left[\frac{t}{1-t}\right] \frac{1}{\Delta} \int_0^{2k} x dx \quad = \quad \left[\frac{t}{1-t}\right] \frac{1}{\Delta} \left[\frac{1}{2}(2k)^2\right] \\ &= \left[\frac{t}{1-t}\right] \frac{2k^2}{\Delta} \quad \text{for } \Delta \geq 2k; \quad \text{and} \end{aligned} \tag{27}$$

$$\begin{aligned} e_1 &= \left[\frac{t}{1-t}\right] E[e_2(\beta)] \quad = \quad \left[\frac{t}{1-t}\right] \int_{\overline{\beta}-\Delta}^{\overline{\beta}} [2k + \beta - \overline{\beta}] dF(\beta) \\ &= \left[\frac{t}{1-t}\right] \frac{1}{\Delta} \int_{2k-\Delta}^{2k} x dx = \left[\frac{t}{1-t}\right] \frac{1}{\Delta} \frac{1}{2} [(2k)^2 - (2k - \Delta)^2] \\ &= \left[\frac{t}{1-t}\right] \left[\frac{4k - \Delta}{2}\right] \quad \text{for } \Delta \leq 2k. \end{aligned} \tag{28}$$

The agent's second-period objective function in (8) is a concave function of e for the reasons specified in LT. It remains to verify that the first-period objective function in (7) is a concave function of e for $e \in [0, \infty)$ when $t \leq \frac{\Delta}{\Delta+2k}$.

Using (10), the second partial derivative of the agent's objective function in (7) is:

$$\begin{aligned} & \int_{\underline{\beta}}^{\overline{\beta}} u''(\beta + e - e_1) dF(\beta) - C_1''(e_1) \\ = & \int_{\underline{\beta}}^{\overline{\beta}} C_2''(e_2(\beta + e - e_1)) e_2'(\beta + e - e_1) dF(\beta) - C_1''(e_1) \\ = & \left[\frac{1}{1-t}\right] \frac{1}{2k} \int_{\underline{\beta}}^{\overline{\beta}} e_2'(\beta + e - e_1) dF(\beta) - \left[\frac{1}{t}\right] \frac{1}{2k} \\ = & \left[\frac{1}{1-t}\right] \frac{1}{2k} \left[\frac{1}{\Delta}\right] \left[e_2(\overline{\beta} + e - e_1) - e_2(\underline{\beta} + e - e_1)\right] - \left[\frac{1}{t}\right] \frac{1}{2k}. \end{aligned} \tag{29}$$

The first two equalities in (29) follow from (11) and (16), respectively.

[1] To define the complete reward contract for any output, we can set $r(x) = r(\overline{\beta} + e_1 + e_2(\overline{\beta}))$ for $x > \overline{\beta} + e_1 + e_2(\overline{\beta})$. This implies that $e_2(\beta) = \max\{0, e_2(\overline{\beta}) + \overline{\beta} - \beta\}$ for $\beta > \overline{\beta}$.

Because $e_2(\overline{\beta}+e-e_1) \leq 2k$ from (26) and $e_2(\underline{\beta}+e-e_1)$ is non-negative, the expression in (29) is at most

$$\left[\frac{1}{1-t}\right]\frac{1}{\Delta} - \left[\frac{1}{t}\right]\frac{1}{2k} = \frac{2kt - \Delta[1-t]}{2k\Delta t[1-t]} = \frac{[2k+\Delta]t - \Delta}{2k\Delta t[1-t]}. \tag{30}$$

This expression in (30) is non-positive when $t \leq \frac{\Delta}{\Delta+2k}$. Therefore, $\int_{\underline{\beta}}^{\overline{\beta}}[U_2(e|\beta) - C_1(e)]dF(\beta)$ is a concave function of e when $t \leq \frac{\Delta}{\Delta+2k}$.

From (28), when $\Delta \leq 2k$:

$$\begin{aligned} e_1 \gtreqless e_1^* \;&\Leftrightarrow\; \left[\frac{t}{1-t}\right]\left[\frac{4k-\Delta}{2}\right] \gtreqless 2kt = e_1^* \\ &\Leftrightarrow\; 4k - \Delta \gtreqless 4k(1-t) \;\Leftrightarrow\; 4kt \gtreqless \Delta. \end{aligned}$$

From (27), when $\Delta > 2k$:

$$e_1 < e_1^* \;\Leftrightarrow\; \left[\frac{t}{1-t}\right]\left[\frac{2k^2}{\Delta}\right] < 2kt = e_1^* \;\Leftrightarrow\; \frac{k}{1-t} < \Delta. \tag{31}$$

If $t \leq \frac{1}{2}$, then $1-t \geq \frac{1}{2}$ and so $\frac{k}{1-t} \leq 2k < \Delta$. If $\frac{1}{2} < t < 1$, then $\frac{k}{1-t} < \frac{2kt}{1-t} \leq \Delta$ because $[1-t]\Delta \geq 2kt \Leftrightarrow t \leq \frac{\Delta}{\Delta+2k}$. Therefore, the last inequality in (31) holds for all $t \leq \frac{\Delta}{\Delta+2k}$. ■

Proposition 2. *Suppose* $\Delta \leq 2k$ *and* $t \in (\frac{\Delta}{\Delta+2k}, \frac{\Delta}{2k})$. *Then at the solution to* $[P]$*:*

$$e_2(\beta) = \begin{cases} 2k+\beta-\overline{\beta} & \text{for } \beta \in (\underline{\beta}, \overline{\beta}-2k+\left[\frac{1-t}{t}\right]\Delta) \\ \left[\frac{1-t}{t}\right]\Delta & \text{for } \beta \in [\overline{\beta}-2k+\left[\frac{1-t}{t}\right]\Delta, \overline{\beta}], \end{cases}$$

so $E\{e_2(\beta)\} > e_2^*$. *Also,* $e_1 = 2k - \frac{1}{2}\left[\frac{1-t}{t}\right]\Delta - \frac{1}{2}\left[\frac{t}{1-t}\right]\frac{1}{\Delta}[2k-\Delta]^2 > e_1^*$.

Proposition 4. *Suppose* $\Delta > 2k$ *and* $t \in [\frac{\Delta}{\Delta+2k}, 1)$. *Then at the solution to* $[P]$*:*

$$e_2(\beta) = \begin{cases} 0 & \text{for } \beta \leq \overline{\beta}-2k \\ 2k+\beta-\overline{\beta} & \text{for } \beta \in (\overline{\beta}-2k, \overline{\beta}-2k+\left[\frac{1-t}{t}\right]\Delta) \\ \left[\frac{1-t}{t}\right]\Delta & \text{for } \beta \in [\overline{\beta}-2k+\left[\frac{1-t}{t}\right]\Delta, \overline{\beta}]. \end{cases}$$

Also, $e_1 = 2k - \frac{1}{2}\left[\frac{1-t}{t}\right]\Delta$. *Consequently: (i)* $e_1 < e_1^*$ *and* $E\{e_2(\beta)\} < e_2^*$ *when* $t \in [\frac{\Delta}{\Delta+2k}, \frac{\Delta}{4k})$; *(ii)* $e_1 = e_1^*$ *and* $E\{e_2(\beta)\} = e_2^*$ *when* $t = \frac{\Delta}{4k}$; *and (iii)* $e_1 > e_1^*$ *and* $E\{e_2(\beta)\} > e_2^*$ *when* $t \in (\frac{\Delta}{4k}, 1)$.

Proof of Propositions 2 and 4.

When $\frac{\Delta}{\Delta+2k} < t$, (26) does not constitute the solution to $[P]$ because the expression in (29) is positive, and so the second order condition is violated at this candidate solution. We will show that the solution to $[P]$ when $\frac{\Delta}{\Delta+2k} < t < \frac{\Delta}{2k}$ entails:

$$e_2(\beta) = \begin{cases} 0 & \text{for } \beta \leq \overline{\beta}-2k \\ 2k+\beta-\overline{\beta} & \text{for } \beta \in (\overline{\beta}-2k, \overline{\beta}-2k+\left[\frac{1-t}{t}\right]\Delta) \\ \left[\frac{1-t}{t}\right]\Delta & \text{for } \beta \in [\overline{\beta}-2k+\left[\frac{1-t}{t}\right]\Delta, \overline{\beta}] \end{cases} \tag{32}$$

$$\text{and} \quad u(\underline{\beta}) = 0.^2 \tag{33}$$

[2] $u(\underline{\beta}) = 0$ implies that $e_2(\beta) = 0$ for $\beta < \underline{\beta}$.

Given (32), (20) implies that for $\Delta \geq 2k$:

$$\begin{aligned}
e_1 &= \left[\frac{t}{1-t}\right] E[e_2(\beta)] \\
&= \left[\frac{t}{1-t}\right] \left[\int_{\overline{\beta}-2k}^{\overline{\beta}-2k+\left[\frac{1-t}{t}\right]\Delta} [2k+\beta-\overline{\beta}]dF(\beta) + \int_{\overline{\beta}-2k+\left[\frac{1-t}{t}\right]\Delta}^{\overline{\beta}} \left(\frac{1-t}{t}\right) \Delta dF(\beta)\right] \\
&= \left[\frac{t}{1-t}\right] \frac{1}{\Delta} \left[\int_0^{\left[\frac{1-t}{t}\right]\Delta} x dx + \left[\frac{1-t}{t}\right] \Delta \left(2k - \left[\frac{1-t}{t}\right] \Delta\right)\right] \\
&= \left[\frac{t}{1-t}\right] \frac{1}{\Delta} \frac{1}{2} \left[\frac{1-t}{t}\right]^2 \Delta^2 + 2k - \left[\frac{1-t}{t}\right] \Delta \\
&= \frac{1}{2}\left[\frac{1-t}{t}\right] \Delta + 2k - \left[\frac{1-t}{t}\right] \Delta \; = \; 2k - \frac{1}{2}\left[\frac{1-t}{t}\right] \Delta. \qquad (34)
\end{aligned}$$

Similarly, for $\Delta \leq 2k$:

$$\begin{aligned}
e_1 &= \left[\frac{t}{1-t}\right] E[e_2(\beta)] \\
&= \left[\frac{t}{1-t}\right] \left[\int_{\overline{\beta}-\Delta}^{\overline{\beta}-2k+\left[\frac{1-t}{t}\right]\Delta} [2k+\beta-\overline{\beta}]dF(\beta) + \int_{\overline{\beta}-2k+\left[\frac{1-t}{t}\right]\Delta}^{\overline{\beta}} \left(\frac{1-t}{t}\right) \Delta dF(\beta)\right] \\
&= \left[\frac{t}{1-t}\right] \frac{1}{\Delta} \left[\int_{2k-\Delta}^{\left[\frac{1-t}{t}\right]\Delta} x dx + \left[\frac{1-t}{t}\right] \Delta \left(2k - \left[\frac{1-t}{t}\right] \Delta\right)\right] \\
&= \frac{1}{2}\left[\frac{1-t}{t}\right] \Delta - \frac{1}{2}\left[\frac{t}{1-t}\right] \frac{1}{\Delta}[2k-\Delta]^2 + 2k - \left[\frac{1-t}{t}\right] \Delta \\
&= 2k - \frac{1}{2}\left[\frac{1-t}{t}\right] \Delta - \frac{1}{2}\left[\frac{t}{1-t}\right] \frac{1}{\Delta}[2k-\Delta]^2. \qquad (35)
\end{aligned}$$

The solution identified in (32) and (33) is feasible because no output pooling is induced and the second partial derivative of the objective function in (7) with respect to e is, from (29):

$$\begin{aligned}
&\left[\frac{1}{1-t}\right] \frac{1}{2k} \left[\frac{1}{2k}\right] \left[e_2(\overline{\beta}+e-e_1) - e_2(\underline{\beta}+e-e_1)\right] - \frac{1}{t}\left[\frac{1}{2k}\right] \\
&\qquad \leq \left[\frac{1}{1-t}\right] \frac{1}{2k} \left[\frac{1}{\Delta}\right] \left[\frac{1-t}{t}\right] \Delta - \frac{1}{t}\left[\frac{1}{2k}\right] = 0. \qquad (36)
\end{aligned}$$

The inequality in (36) holds because, from (32), $e_2(\overline{\beta}+e-e_1) - e_2(\underline{\beta}+e-e_1) \leq \left[\frac{1-t}{t}\right]\Delta$. (36) implies that $\int_{\underline{\beta}}^{\overline{\beta}} [U_2(e|\beta) - C_1(e)]\, dF(\beta)$ is a concave function of e and so the relevant second order condition is satisfied.

To determine the principal's expected payoff under the identified contract, recall from (9) that the principal seeks to maximize:

$$\begin{aligned}
&\left[\frac{1}{1-t}\right] \int_{\underline{\beta}}^{\overline{\beta}} \left(e_2(\beta) - \frac{1}{4k}(e_2(\beta))^2 - \frac{2e_2(\beta)}{4k}\left[\overline{\beta}-\beta\right]\right) dF(\beta) - u(\underline{\beta}) \\
= \; & -\left[\frac{1}{4k(1-t)}\right] \int_{\underline{\beta}}^{\overline{\beta}} \{[e_2(\beta)]^2 - 2e_2(\beta)\left[2k+\beta-\overline{\beta}\right]\} dF(\beta) - u(\underline{\beta})
\end{aligned}$$

$$
\begin{aligned}
&= -\left[\frac{1}{4k(1-t)}\right]\int_{\underline{\beta}}^{\overline{\beta}}\{[e_2(\beta)-(2k+\beta-\overline{\beta})]^2-[2k+\beta-\overline{\beta}]^2\}dF(\beta)-u(\underline{\beta})\\
&= -\left[\frac{1}{4k(1-t)}\right]\int_{\underline{\beta}}^{\overline{\beta}}\left[e_2(\beta)-(2k+\beta-\overline{\beta})\right]^2 dF(\beta)\\
&\qquad +\left[\frac{1}{4k(1-t)}\right]\int_{\underline{\beta}}^{\overline{\beta}}[2k+\beta-\overline{\beta}]^2dF(\beta)-u(\underline{\beta}).
\end{aligned}
\tag{37}
$$

(37) implies that under the feasible solution identified in (32) and (33), the principal's expected gain (i.e., her expected payoff net of $\frac{1}{2}[\underline{\beta}+\overline{\beta}]$) when $\Delta \geq 2k$ is:

$$
\begin{aligned}
&\left[\frac{1}{4k(1-t)}\right]\left[\int_{\overline{\beta}-2k}^{\overline{\beta}}[2k+\beta-\overline{\beta}]^2dF(\beta)-\int_{\overline{\beta}-2k}^{\overline{\beta}}\left[e_2(\beta)-(2k+\beta-\overline{\beta})\right]^2dF(\beta)\right]\\
&= \left[\frac{1}{4k(1-t)}\right]\frac{1}{\Delta}\left[\int_{\overline{\beta}-2k}^{\overline{\beta}}[2k+\beta-\overline{\beta}]^2d\beta-\int_{\overline{\beta}-2k}^{\overline{\beta}-2k+\Delta\left[\frac{1-t}{t}\right]}[0]\,d\beta\right.\\
&\qquad\left.-\int_{\overline{\beta}-2k+\Delta\left[\frac{1-t}{t}\right]}^{\overline{\beta}}\left[\left(\frac{1-t}{t}\right)\Delta-(2k+\beta-\overline{\beta})\right]^2d\beta\right]\\
&= \left[\frac{1}{4k(1-t)}\right]\frac{1}{\Delta}\left[\int_0^{2k}x^2dx-\int_0^{2k-\left[\frac{1-t}{t}\right]\Delta}x^2dx\right]\\
&= \left[\frac{1}{4k(1-t)}\right]\frac{1}{3\Delta}\left[(2k)^3-\left(2k-\left[\frac{1-t}{t}\right]\Delta\right)^3\right]\\
&= \left[\frac{1}{4k(1-t)}\right]\frac{1}{3\Delta}\left[\frac{1-t}{t}\right]\Delta\left\{4k^2+4k^2-2k\left[\frac{1-t}{t}\right]\Delta\right.\\
&\qquad\left.+4k^2-4k\left[\frac{1-t}{t}\right]\Delta+\left(\left[\frac{1-t}{t}\right]\Delta\right)^2\right\}\\
&= \frac{1}{12kt}\left[12k^2-6k\left[\frac{1-t}{t}\right]\Delta+\left(\left[\frac{1-t}{t}\right]\Delta\right)^2\right].
\end{aligned}
\tag{38}
$$

The principal's corresponding expected gain when $\Delta \leq 2k$ is:

$$
\begin{aligned}
&\left[\frac{1}{4k[1-t]}\right]\left[\int_{\overline{\beta}-\Delta}^{\overline{\beta}}[2k+\beta-\overline{\beta}]^2dF(\beta)-\int_{\overline{\beta}-\Delta}^{\overline{\beta}}\left[e_2(\beta)-(2k+\beta-\overline{\beta})\right]^2dF(\beta)\right]\\
&= \left[\frac{1}{4k[1-t]}\right]\frac{1}{\Delta}\left[\int_{2k-\Delta}^{2k}x^2dx-\int_0^{2k-\left[\frac{1-t}{t}\right]\Delta}x^2dx\right]\\
&= \left[\frac{1}{4k[1-t]}\right]\frac{1}{3\Delta}\left[(2k)^3-[2k-\Delta]^3-\left(2k-\left[\frac{1-t}{t}\right]\Delta\right)^3\right]\\
&= \frac{1}{12kt}\left[12k^2-6k\left[\frac{1-t}{t}\right]\Delta+\left(\left[\frac{1-t}{t}\right]\Delta\right)^2\right]-\frac{[2k-\Delta]^3}{12k[1-t]\Delta}.
\end{aligned}
\tag{39}
$$

We next prove that the identified solution is optimal. The proof proceeds by establishing the following conclusions: (i) $e_1 < \Delta$; (ii) $u(\underline{\beta}) = 0$; (iii) $e_2(\beta) = \left[\frac{1-t}{t}\right]\Delta$ for the (high innate output) region where the second order condition is binding; (iv) the final output $\beta + e_1 + e_2(\beta)$ is a non-decreasing function of β; (v) $e_2(\beta)$ is non-decreasing on $[\underline{\beta}, \overline{\beta}]$; and so (vi) the identified solution is indeed optimal. To facilitate the proofs, we let $V(\beta + e) \equiv U(e|\beta)$. It is readily verified that $V(\beta + e)$ is well defined because innate output and first-period effort are perfect substitutes in increasing output.

Lemma A1. *$e_1 < \Delta$ at the solution to $[P]$ when $\frac{\Delta}{\Delta+2k} < t < \frac{\Delta}{2k}$.*

Proof: From (20), $E\{e_2(\beta)\} = \left[\frac{1-t}{t}\right] e_1$. Therefore, total expected effort throughout the $[0, 1]$ time period is $e_1 + \left[\frac{1-t}{t}\right] e_1 = \frac{1}{t} e_1$. Due to the convexity of the agent's effort cost function, the total expected gain (i.e., the sum of the principal's expected payoff and the agent's expected payoff net of $\frac{1}{2}[\underline{\beta} + \overline{\beta}]$) for the chosen effort is at most the total expected gain for the expected effort, which is:

$$\frac{1}{t}e_1 - C\left(\frac{1}{t}e_1\right) = \frac{1}{t}e_1 - \left[\frac{1}{4kt^2}\right] e_1^2. \tag{40}$$

The expression in (40) is maximized at $e_1 = 2kt$. When $e_1 \geq \Delta > 2kt$, the total expected gain cannot exceed the expected gain when $e_1 = \Delta$ (since (40) is maximized at $e_1 = 2kt \leq \Delta$). From (40), the total expected gain when $e_1 = \Delta$ is $\frac{1}{t}\Delta - \left[\frac{1}{4kt^2}\right] \Delta^2$. The principal's expected gain cannot exceed the total expected gain because $r(\cdot) \geq 0$.

From (38) and (39), the principal's expected gain under the identified solution is:

$$\begin{cases} \frac{1}{12kt}\left[12k^2 - 6k\left[\frac{1-t}{t}\right]\Delta + \left(\left[\frac{1-t}{t}\right]\Delta\right)^2\right] & \text{for } \Delta \geq 2k \\ \frac{1}{12kt}\left[12k^2 - 6k\left[\frac{1-t}{t}\right]\Delta + \left(\left[\frac{1-t}{t}\right]\Delta\right)^2\right] - \frac{[2k-\Delta]^3}{12k[1-t]\Delta} & \text{for } \Delta \leq 2k. \end{cases} \tag{41}$$

We will demonstrate that $e_1 \leq \Delta$ by proving that the principal's expected gain under the identified solution is at least $\frac{1}{t}\Delta - \left[\frac{1}{4kt^2}\right] \Delta^2$.

When $\Delta \geq 2k$, it suffices to show that:

$$G(\Delta) \equiv \frac{1}{12kt}\left[12k^2 - 6k\left[\frac{1-t}{t}\right]\Delta + \left(\left[\frac{1-t}{t}\right]\Delta\right)^2\right] - \left[\frac{1}{t}\Delta - \frac{\Delta^2}{4kt^2}\right] \geq 0. \tag{42}$$

Notice that:

$$\begin{aligned} G(\Delta) &= \left[\frac{(1-t)^2}{12kt^3} + \frac{3t}{12kt^3}\right]\Delta^2 - \left[\frac{1-t}{2t^2} + \frac{2t}{2t^2}\right]\Delta + \frac{k}{t} \\ &= \left[\frac{1+t+t^2}{12kt^3}\right]\Delta^2 - \left[\frac{1+t}{2t^2}\right]\Delta + \frac{k}{t} \qquad (43) \\ &= \left[\frac{1+t+t^2}{12kt^3}\right]\left[\Delta - \left(\frac{12kt^3}{1+t+t^2}\right)\left(\frac{1+t}{4t^2}\right)\right]^2 + \frac{k}{t} - \left[\frac{12kt^3}{1+t+t^2}\right]\left(\frac{1+t}{4t^2}\right)^2 \\ &= \left[\frac{1+t+t^2}{12kt^3}\right]\left[\Delta - \frac{3kt(1+t)}{1+t+t^2}\right]^2 + \frac{k}{t} - \left[\frac{3k}{1+t+t^2}\right]\frac{(1+t)^2}{4t} \\ &= \left[\frac{1+t+t^2}{12kt^3}\right]\left[\Delta - \frac{3kt(1+t)}{1+t+t^2}\right]^2 + \frac{k}{4t(1+t+t^2)}\left[4+4t+4t^2-3(1+t)^2\right] \\ &= \left[\frac{1+t+t^2}{12kt^3}\right]\left[\Delta - \frac{3kt(1+t)}{1+t+t^2}\right]^2 + \frac{k}{4t(1+t+t^2)}[1-t]^2 \;\geq\; 0. \qquad (44) \end{aligned}$$

The inequality in (44) ensures that $e_1 \leq \Delta$ at the solution to $[P]$ when $\frac{\Delta}{\Delta+2k} < t < \frac{\Delta}{2k}$ and $\Delta \geq 2k$.

When $\Delta \leq 2k$, (41) implies that $e_1 \leq \Delta$ if $G(\Delta) - \frac{(2k-\Delta)^3}{12k(1-t)\Delta} \geq 0$. By (43), this inequality holds if:

$$\left[\frac{1+t+t^2}{12kt^3}\right]\Delta^2 - \left[\frac{1+t}{2t^2}\right]\Delta + \frac{k}{t} - \frac{[2k-\Delta]^3}{12k[1-t]\Delta} \geq 0$$

$$\Leftrightarrow \quad H(\Delta) \equiv \left[\frac{1+t+t^2}{12kt^3}\right]\Delta^3 - \left[\frac{1+t}{2t^2}\right]\Delta^2 + \frac{k}{t}\Delta - \frac{[2k-\Delta]^3}{12k[1-t]} \geq 0.$$

Notice that when $t < \frac{\Delta}{2k}$, $\Delta > 2kt$. Also, at $\Delta = 2kt$:

$$\begin{aligned} H(\Delta) &= \left[\frac{2+2t+2t^2}{3}\right]k^2 - (2+2t)k^2 + 2k^2 - \frac{2[1-t]^2}{3}k^2 \\ &= \frac{6t}{3}k^2 - 2tk^2 = 0. \end{aligned} \tag{45}$$

(45) implies that $e_1 \leq \Delta$ if $H'(\Delta) \geq 0$ for $\Delta \geq 2kt$. This inequality holds because:

$$\begin{aligned} H'(\Delta) &= \left[\frac{1+t+t^2}{4kt^3}\right]\Delta^2 - \left[\frac{1+t}{t^2}\right]\Delta + \frac{k}{t} + \frac{(2k-\Delta)^2}{4k(1-t)} \\ &= \left[\frac{1-t^3}{4kt^3(1-t)} + \frac{t^3}{4kt^3(1-t)}\right]\Delta^2 - \left[\frac{1-t^2}{t^2(1-t)} + \frac{t^2}{t^2(1-t)}\right]\Delta \\ &\qquad + \left[\frac{k[1-t]}{t[1-t]} + \frac{kt}{t[1-t]}\right] \\ &= \frac{\Delta^2}{4kt^3[1-t]} - \frac{\Delta}{t^2[1-t]} + \frac{k}{t[1-t]} \\ &= \frac{1}{4kt^3[1-t]}[\Delta^2 - 4kt\Delta + 4k^2t^2] \\ &= \frac{1}{4kt^3[1-t]}[\Delta - 2kt]^2 \geq 0. \end{aligned} \tag{46}$$

(45) and (46) imply that e_1 must be less than Δ at the solution to $[P]$ when $\frac{\Delta}{\Delta+2k} < t < \frac{\Delta}{2k}$ and $\Delta \leq 2k$. ■

Lemma A2. $u(\underline{\beta}) = 0$ *at the solution to* $[P]$.

Proof: It is readily verified that $V(\beta)$ is a non-decreasing continuous function, as shown in Figure A1. (11), (18), and (19) imply that:

$$\begin{aligned} \int_{\underline{\beta}}^{\overline{\beta}} V'(\beta+e_1)\frac{1}{\Delta}d\beta &= C_1'(e_1) = \frac{e_1}{2kt} \\ \Rightarrow \quad \int_{\underline{\beta}}^{\overline{\beta}} V'(\beta+e_1)d\beta &= V(\overline{\beta}+e_1) - V(\underline{\beta}+e_1) = \frac{\Delta}{2kt}e_1. \end{aligned} \tag{47}$$

(47) implies that the two thick vertical segments in Figure A1 are of the same length.

If $u(\underline{\beta}) = U_2(e_1|\underline{\beta}) = V(\underline{\beta}+e_1) > 0$, another contract can be found that secures a larger expected payoff for the principal. Denote by $V^*(\beta+e_1)$ the agent's expected utility under this alternative contract. $V^*(\cdot)$ is constructed by first systematically reducing $V(\cdot)$ by $\min\{V(\beta), V(\underline{\beta}+e_1)\}$. This modification increases the principal's objective function by reducing the agent's utility, provided the agent's effort supply is not changed. To ensure that the agent delivers the same first-period effort e_1, define:

$$e_0 = \min \ \{e \mid V(\overline{\beta}+e) - V(\underline{\beta}+e_1) = \frac{\Delta}{2kt}e, \ e \geq 0\}. \tag{48}$$

Notice that $e_0 \leq e_1$ because $V(\overline{\beta}+e_1) - V(\underline{\beta}+e_1) = \frac{\Delta}{2kt}e_1$, from (47). Furthermore, because $V(\cdot)$ is continuous, $V(\overline{\beta}+e) - V(\underline{\beta}+e_1) - \frac{\Delta}{2kt}e$ is either strictly positive or strictly negative for $0 \leq e < e_0$. Because $\overline{\beta} > \underline{\beta} + e_1$ by Lemma A1 and because $V(\cdot)$ is monotonic:

$$V(\overline{\beta}+e) - V(\underline{\beta}+e_1) > \frac{\Delta}{2kt}e \qquad \text{for} \ \ 0 \leq e < e_0.$$

Now define $V^*(\beta)$ such that:

$$V^*(\beta) = \begin{cases} 0 & \text{for} \ \ \beta \leq \underline{\beta}+e_1 \\ V(\beta) - V(\underline{\beta}+e_1) & \text{for} \ \ \beta \in (\underline{\beta}+e_1, \overline{\beta}+e_0] \\ \frac{\Delta}{2kt}[\beta - \overline{\beta}] & \text{for} \ \ \beta > \overline{\beta}+e_0 \ . \end{cases} \tag{49}$$

Under the reward structure identified by (49), the agent chooses his first-period effort to maximize:

$$\int_{\underline{\beta}}^{\overline{\beta}} V^*(\beta+e)\frac{1}{\Delta}d\beta - C_1(e) \ . \tag{50}$$

The first-order condition corresponding to (50) is:

$$\int_{\underline{\beta}}^{\overline{\beta}} V^{*\prime}(\beta+e)d\beta = V^*(\overline{\beta}+e) - V^*(\underline{\beta}+e) = \frac{\Delta}{2kt}e. \tag{51}$$

It follows that for $e \in [0, e_0)$:

$$V^*(\overline{\beta}+e) - V^*(\underline{\beta}+e) = V^*(\overline{\beta}+e) = V(\overline{\beta}+e) - V(\underline{\beta}+e_1) > \frac{\Delta}{2kt}e \ , \tag{52}$$

while for $e > e_1$:

$$V^*(\overline{\beta}+e) - V^*(\underline{\beta}+e) \leq V^*(\overline{\beta}+e) = \frac{\Delta}{2kt}e \ . \tag{53}$$

For $e \in [e_0, e_1]$:

$$V^*(\overline{\beta}+e) - V^*(\underline{\beta}+e) = V^*(\overline{\beta}+e) = \frac{\Delta}{2kt}e. \tag{54}$$

(52) - (54) imply that the agent is indifferent among effort levels between e_0 and e_1 under $V^*(\cdot)$. Therefore, the agent can be assumed to deliver first-period effort e_1 under $V^*(\cdot)$, and the agent's utility will be as depicted in Figure A2.

It remains to determine the second-period effort, $e_2(\beta)$, that will arise under $V(\cdot)$ and $V^*(\cdot)$, and to compare the expected surpluses under the two structures. From (5), the principal can be viewed as seeking to maximize:

$$\int_{\underline{\beta}}^{\overline{\beta}} [e_1 + e_2(\beta) - C_2(e_2(\beta)) - V(\beta+e_1)]dF(\beta)$$

$$
\begin{aligned}
&= e_1 + \int_{\underline{\beta}}^{\overline{\beta}} e_2(\beta)dF(\beta) - \int_{\underline{\beta}}^{\overline{\beta}+e_0-e_1} V(\beta+e_1)dF(\beta) - \int_{\overline{\beta}+e_0-e_1}^{\overline{\beta}} V(\beta+e_1)dF(\beta) \\
&\quad - \int_{\underline{\beta}}^{\overline{\beta}+e_0-e_1} C_2(e_2(\beta))dF(\beta) - \int_{\overline{\beta}+e_0-e_1}^{\overline{\beta}} C_2(e_2(\beta))dF(\beta). \qquad (55)
\end{aligned}
$$

Now compare the expressions in (55) term by term.

$e_1 + \int_{\underline{\beta}}^{\overline{\beta}} e_2(\beta)dF(\beta)$

Both reward structures induce the same e_1. Therefore, (20) implies that both structures induce the same $\int_{\underline{\beta}}^{\overline{\beta}} e_2(\beta)dF(\beta)$.

$\int_{\underline{\beta}}^{\overline{\beta}+e_0-e_1} V(\beta+e_1)dF(\beta)$

(49) implies that:

$$V^*(\beta+e_1) = V(\beta+e_1) - V(\underline{\beta}+e_1) \quad \text{for} \ \ \beta \in [\underline{\beta}, \overline{\beta}+e_0-e_1]. \qquad (56)$$

Therefore:

$$\int_{\underline{\beta}}^{\overline{\beta}+e_0-e_1} V^*(\beta+e_1)dF(\beta) < \int_{\underline{\beta}}^{\overline{\beta}+e_0-e_1} V(\beta+e_1)dF(\beta). \qquad (57)$$

(57) follows from (56) because $e_1 < \Delta$ and $e_0 \geq 0$. (57) implies that $V^*(\cdot)$ increases the principal's expected payoff relative to $V(\cdot)$ for the term $\int_{\underline{\beta}}^{\overline{\beta}+e_0-e_1} V(\beta+e_1)dF(\beta)$.

$\int_{\overline{\beta}+e_0-e_1}^{\overline{\beta}} V(\beta+e_1)dF(\beta)$

Because e_1 provides at least as great a return for the agent as e_0 under the contract that provides utility $V(\cdot)$:

$$
\begin{aligned}
\int_{\underline{\beta}}^{\overline{\beta}} V(\beta+e_1)dF(\beta) - C_1(e_1) &\geq \int_{\underline{\beta}}^{\overline{\beta}} V(\beta+e_0)dF(\beta) - C_1(e_0) \\
\Leftrightarrow \int_{\overline{\beta}+e_0-e_1}^{\overline{\beta}} V(\beta+e_1)dF(\beta) &\geq C_1(e_1) - C_1(e_0) \\
&\quad + \int_{\underline{\beta}}^{\overline{\beta}} V(\beta+e_0)dF(\beta) - \int_{\underline{\beta}}^{\overline{\beta}+e_0-e_1} V(\beta+e_1)dF(\beta). \qquad (58)
\end{aligned}
$$

The last integral in (58) can be written as:

$$
\begin{aligned}
&\int_{\underline{\beta}}^{\underline{\beta}+e_0-e_1} V(\beta+e_1)dF(\beta) + \int_{\underline{\beta}+e_0-e_1}^{\overline{\beta}+e_0-e_1} V(\beta+e_1)dF(\beta) \\
&= \int_{\underline{\beta}+e_0-e_1}^{\overline{\beta}+e_0-e_1} V(\beta+e_1)dF(\beta) - \int_{\underline{\beta}+e_0-e_1}^{\underline{\beta}} V(\beta+e_1)dF(\beta) \\
&= \int_{\underline{\beta}}^{\overline{\beta}} V(\beta+e_0)dF(\beta) - \int_{\underline{\beta}+e_0-e_1}^{\underline{\beta}} V(\beta+e_1)dF(\beta)\ . \qquad (59)
\end{aligned}
$$

(58) and (59) imply:

$$\begin{aligned}\int_{\overline{\beta}+e_0-e_1}^{\overline{\beta}} V(\beta+e_1)dF(\beta) &\geq C_1(e_1)-C_1(e_0)+\int_{\underline{\beta}+e_0-e_1}^{\underline{\beta}} V(\beta+e_1)dF(\beta)\\ &\geq C_1(e_1)-C_1(e_0)\\ &= \int_{\underline{\beta}}^{\overline{\beta}} V^*(\beta+e_1)dF(\beta)-\int_{\underline{\beta}}^{\overline{\beta}} V^*(\beta+e_0)dF(\beta)\\ &= \int_{\overline{\beta}+e_0-e_1}^{\overline{\beta}} V^*(\beta+e_1)dF(\beta). \end{aligned} \tag{60}$$

The second inequality in (60) holds because $V(\beta+e)\geq 0$. The first equality in (60) holds because the agent is indifferent between e_1 and e_0 under $V^*(\cdot)$. The last equality in (60) holds because $V^*(\beta)=0$ for $\beta\leq\underline{\beta}+e_1$. Therefore, $V^*(\cdot)$ may increase the principal's expected payoff relative to $V(\cdot)$ for the term $\int_{\overline{\beta}+e_0-e_1}^{\overline{\beta}} V(\beta+e_1)dF(\beta)$.

$\underline{\int_{\underline{\beta}}^{\overline{\beta}+e_0-e_1} C_2(e_2(\beta))dF(\beta)}$

(11) implies that $V'(\beta+e_1)=u'(\beta)=C'(e_2(\beta))$. Therefore, both reward structures induce the same $e_2(\beta)$ and thus the same expected payoff for the principal with regard to this term for $\beta\in[\underline{\beta},\overline{\beta}+e_0-e_1]$.

$\underline{\int_{\overline{\beta}+e_0-e_1}^{\overline{\beta}} C_2(e_2(\beta))dF(\beta)}$

For $\beta\in[\overline{\beta}+e_0-e_1,\overline{\beta}]$, the e_2 induced under both reward structures satisfies:

$$\begin{aligned}\int_{\overline{\beta}+e_0-e_1}^{\overline{\beta}} e_2(\beta)d\beta &= [1-t]2k\int_{\overline{\beta}+e_0-e_1}^{\overline{\beta}} C_2'(e_2(\beta))d\beta\\ &= [1-t]2k\int_{\overline{\beta}+e_0-e_1}^{\overline{\beta}} u'(\beta)d\beta = [1-t]2k\left[u(\overline{\beta})-u(\overline{\beta}+e_0-e_1)\right]\\ &= [1-t]2k\left[V(\overline{\beta}+e_1)-V(\overline{\beta}+e_0)\right] = \left[\frac{1-t}{t}\right]\Delta[e_1-e_0]. \end{aligned} \tag{61}$$

The four equalities in (61) follow from (19), (11), (10), and (48), respectively. From (49), $V^{*\prime}(\beta+e_1)=\frac{\Delta}{2kt}$ for $\beta\in[\overline{\beta}+e_0-e_1,\overline{\beta}]$. (11) implies that the e_2 induced by $V^*(\cdot)$ is equal to:

$$[1-t]2kC_2'(e_2(\beta)) = [1-t]2kV^{*\prime}(\beta+e_1) = \left[\frac{1-t}{t}\right]\Delta \quad \text{on } [\overline{\beta}+e_0-e_1,\overline{\beta}].$$

Because $C_2(e)$ is a convex function and $e_2(\beta)$ is a constant on the interval $[\overline{\beta}+e_0-e_1,\overline{\beta}]$ under $V^*(\cdot)$, $\int_{\overline{\beta}+e_0-e_1}^{\overline{\beta}} C_2(e_2(\beta))dF(\beta)$ under $V^*(\beta)$ does not exceed $\int_{\overline{\beta}+e_0-e_1}^{\overline{\beta}} C_2(e_2(\beta))dF(\beta)$ under $V(\beta)$. Therefore, if $V(\cdot)\neq V^*(\cdot)$, $V^*(\cdot)$ increases the principal's expected payoff relative to $V(\cdot)$ for the term $\int_{\overline{\beta}+e_0-e_1}^{\overline{\beta}} C_2(e_2(\beta))dF(\beta)$.

Because $V^*(\beta)$ secures a larger expected payoff for the principal than $V(\beta)$ if $u(\underline{\beta})>0$, it must be the case that $u(\underline{\beta})=V(\underline{\beta}+e_1)=0$ at the solution to [P]. ■

Corollary A1. $e_2(\beta) = \left[\frac{1-t}{t}\right]\Delta$ *on* $[\overline{\beta} + e_0 - e_1, \overline{\beta}]$*, where* e_0 *is defined in (48).*

Proof: The proof parallels the last part of the proof of Lemma A2, as the principal can secure a higher expected payoff by implementing $V^*(\cdot)$ such that:

$$V^*(\beta) = \begin{cases} 0 & \text{for } \beta \leq \underline{\beta} + e_1 \\ V(\beta) - V(\underline{\beta} + e_1) = V(\beta) & \text{for } \beta \in (\underline{\beta} + e_1, \overline{\beta} + e_0] \\ \frac{\Delta}{2kt}[\beta - \overline{\beta}] & \text{for } \beta > \overline{\beta} + e_0 \end{cases}$$

(11) implies that the e_2 that arises under $V^*(\cdot)$ is a constant $\left[\frac{1-t}{t}\right]\Delta$ on $[\overline{\beta} + e_0 - e_1, \overline{\beta}]$. ■

Lemma A3. *Final output* $\beta + e_1 + e_2(\beta)$ *is a non-decreasing function of* β*.*

Proof: Consider β_1, β_2 for which $\underline{\beta} \leq \beta_1 < \beta_2 \leq \overline{\beta}$. We need to prove that

$$\beta_1 + e_2(\beta_1) \leq \beta_2 + e_2(\beta_2).$$

Incentive compatibility requires $u(\beta_1|\beta_1) \geq u(\beta_2|\beta_1)$ and $u(\beta_2|\beta_2) \geq u(\beta_1|\beta_2)$. Summing these two inequalities provides:

$$\begin{aligned} & C_2(e_2(\beta_1)) + C_2(e_2(\beta_2)) \leq C_2(e_2(\beta_1) + \beta_1 - \beta_2) + C_2(e_2(\beta_2) + \beta_2 - \beta_1) \\ \Leftrightarrow\ & C_2(e_2(\beta_2) + \beta_2 - \beta_1) - C_2(e_2(\beta_2)) \geq C_2(e_2(\beta_1)) - C_2(e_2(\beta_1) + \beta_1 - \beta_2) \\ \Leftrightarrow\ & \int_0^{\beta_2 - \beta_1} C_2'(e_2(\beta_2) + t)dt \ \geq \ \int_0^{\beta_2 - \beta_1} C_2'(e_2(\beta_1) + \beta_1 - \beta_2 + t)dt. \end{aligned} \tag{62}$$

Because $C_2(\cdot)$ is a convex function, $C_2'(\cdot)$ is weakly monotone increasing in general and strictly monotone increasing for a positive domain. Therefore, (62) implies that $e_2(\beta_2) \geq e_2(\beta_1) + \beta_1 - \beta_2$ or $\beta_1 + e_2(\beta_1) \leq \beta_2 + e_2(\beta_2)$. ■

Lemma A4. $e_2(\beta)$ *is non-decreasing on* $[\underline{\beta}, \overline{\beta}]$*.*

Proof: From (37), the optimal $e_2(\beta)$ maximizes:

$$-\frac{1}{4k[1-t]} \int_{\underline{\beta}}^{\overline{\beta}} [e_2(\beta) - (2k + \beta - \overline{\beta})]^2 dF(\beta). \tag{63}$$

Because the second order condition is not binding on $[\underline{\beta}, \overline{\beta} + e_0 - e_1]$, the solution $e_2(\beta)$ on $[\underline{\beta}, \overline{\beta} + e_0 - e_1]$ must maximize the expression in (63) for given $u(\underline{\beta}) = 0$ and $u(\overline{\beta} + e_0 - e_1)$.

Consider a disturbance $\lambda\epsilon(\beta)$ around the optimal effort $e_2(\beta)$ on $[\underline{\beta}, \overline{\beta} + e_0 - e_1]$. To ensure $e_2(\beta) \pm \lambda\epsilon(\beta) \geq 0$ for all λ, we consider $\epsilon(\beta)$ for which $\epsilon(\beta) = 0$ if $e_2(\beta) = 0$. To maintain the fixed $u(\overline{\beta} + e_0 - e_1)$ while ensuring $u(\underline{\beta}) = 0$, we must have:

$$u(\overline{\beta} + e_0 - e_1) = \int_{\underline{\beta}}^{\overline{\beta} + e_0 - e_1} u'(\beta)d\beta \ = \ \left[\frac{1}{1-t}\right] \frac{1}{2k} \int_{\underline{\beta}}^{\overline{\beta} + e_0 - e_1} e_2(\beta)d\beta. \tag{64}$$

The second equality in (64) follows from (11) and (19), since $e_2^* = 2k[1-t]$. To ensure that (64) holds, we further restrict $\epsilon(\beta)$ to satisfy:

$$\int_{\underline{\beta}}^{\overline{\beta}+e_0-e_1} \epsilon(\beta)d\beta \;=\; 0. \tag{65}$$

To ensure that $e_2(\beta)$ is optimal, the partial derivative of the expression in (63) with respect to λ must be 0 at $\lambda = 0$ when the agent's effort is $e_2(\beta) + \lambda\epsilon(\beta)$. Setting this partial derivative equal to 0 provides:

$$\int_{\underline{\beta}}^{\overline{\beta}+e_0-e_1} [e_2(\beta) - (2k + \beta - \overline{\beta})]\epsilon(\beta)d\beta = 0. \tag{66}$$

Because $\epsilon(\beta)$ can be any continuous function satisfying (65), (66) implies that $[e_2(\beta)-(2k+\beta-\overline{\beta})]$ is a constant for $e_2(\beta) > 0$ on $[\underline{\beta}, \overline{\beta}+e_0-e_1]$. Therefore, $e_2(\beta)$ is non-decreasing on $[\underline{\beta}, \overline{\beta}+e_0-e_1]$ when $e_2(\beta)$ is positive. Moreover, Lemma A3 implies that $e_2(\beta)$ does not decrease discontinuously at any $\beta \in [\underline{\beta}, \overline{\beta}]$. In addition, Corollary A1 implies that $e_2(\beta) = \left[\frac{1-t}{t}\right]\Delta$, a constant on $[\overline{\beta} + e_0 - e_1, \overline{\beta}]$. Therefore, $e_2(\beta)$ is non-decreasing on $[\underline{\beta}, \overline{\beta}]$. ■

Lemma A4 implies that $e_2(\beta) \leq \left[\frac{1-t}{t}\right]\Delta$ for $\beta \in [\underline{\beta}, \overline{\beta}]$. Consider the following relaxed version of $[P]$:

$$\underset{e_2(\cdot)}{Maximize} \left[\frac{1}{1-t}\right] \int_{\underline{\beta}}^{\overline{\beta}} \left(e_2(\beta) - C(e_2(\beta)) - C'(e_2(\beta))\left[\frac{1-F(\beta)}{f(\beta)}\right]\right) dF(\beta)$$

$$\text{subject to:} \quad e_2(\beta) \leq \left[\frac{1-t}{t}\right]\Delta\ .$$

The solution to this problem is $\min\{e^{LT}, \left[\frac{1-t}{t}\right]\Delta\}$, where $e^{LT} = \max\{0, 2k + \beta - \overline{\beta}\}$ from Lemma 3. We have shown that this proposed solution is a feasible solution to $[P]$, and so is optimal.

From (35), when $\Delta \leq 2k$ and $t \in (\frac{\Delta}{\Delta+2k}, \frac{\Delta}{2k})$:

$$\begin{aligned}
e_1 > e_1^* \;&\Leftrightarrow\; 2k - \frac{1}{2}\left[\frac{1-t}{t}\right]\Delta - \frac{1}{2}\left[\frac{t}{1-t}\right]\frac{1}{\Delta}[2k-\Delta]^2 > 2kt = e_1^* \\
&\Leftrightarrow\; 2k[1-t]\Delta > \frac{1}{2}\left[\frac{1-t}{t}\right]\Delta^2 + \frac{1}{2}\left[\frac{t}{1-t}\right][2k-\Delta]^2 \\
&\Leftrightarrow\; 4k[1-t]^2 t\Delta > [1-t]^2\Delta^2 + t^2[2k-\Delta]^2 \\
&\Leftrightarrow\; 4kt[1-2t+t^2]\Delta > [1-2t+2t^2]\Delta^2 - 4kt^2\Delta + 4k^2t^2 \\
&\Leftrightarrow\; [1-2t+2t^2]\Delta^2 - 4kt[1-t+t^2]\Delta + 4k^2t^2 < 0 \\
&\Leftrightarrow\; [\Delta - 2kt]\{[1-2t+2t^2]\Delta - 2kt\} < 0 \\
&\Leftrightarrow\; [1-2t+2t^2]\Delta < 2kt.
\end{aligned} \tag{67}$$

First suppose $t \leq \frac{1}{2}$. In this case $1 - t \geq 1 - 2t + 2t^2$. Therefore:

$$\frac{2kt}{1-t} \leq \frac{2kt}{1-2t+2t^2}. \tag{68}$$

Because $t > \frac{\Delta}{\Delta+2k}$ in this case, $2kt > [1-t]\Delta$. Therefore, from (68):

$$\Delta < \frac{2kt}{1-t} \leq \frac{2kt}{1-2t+2t^2}. \tag{69}$$

(67) and (69) imply that $e_1 > e_1^*$.

Now suppose $t > \frac{1}{2}$. $1-2t<0$ in this case. Also, $t < \frac{\Delta}{2k} \leq 1$ in this case. Therefore:

$$1-3t+2t^2 = [1-t][1-2t] < 0 \quad \Leftrightarrow \quad 1-2t+2t^2 < t. \tag{70}$$

Notice that $1-2t+2t^2 = [1-t]^2 + t^2 > 0$. Therefore:

$$[1-2t+2t^2]\Delta < t[2k] = 2kt \quad \Leftrightarrow \quad e_1 > e_1^*. \tag{71}$$

The first inequality in (71) follows from (70), since $\Delta < 2k$ in this case. The equivalence in (71) follows from (67).

From (34), when $\Delta > 2k$ and $t \in (\frac{\Delta}{\Delta+2k}, \frac{\Delta}{2k})$:

$$\begin{aligned} e_1 \gtreqless e_1^* \quad &\Leftrightarrow \quad 2k - \frac{1}{2}\left[\frac{1-t}{t}\right]\Delta \ \gtreqless \ 2kt = e_1^* \\ &\Leftrightarrow \quad 2k[1-t] \gtreqless \frac{1}{2}\left[\frac{1-t}{t}\right]\Delta \quad \Leftrightarrow \quad 4kt \gtreqless \Delta. \quad \blacksquare \end{aligned}$$

Corollary 1. *Suppose $\Delta > 2k$. Then at the solution to [P], $e_1 \to e_1^*$, $E\{e_2(\beta)\} \to e_2^*$, and the principal's expected net return approaches the entire expected surplus from efficient production as $t \to 1$.*

Proof of Corollary 1.

From (34), when $t \in (\frac{\Delta}{\Delta+2k}, \frac{\Delta}{2k})$:

$$e_1 = 2k - \frac{1}{2}\left[\frac{1-t}{t}\right]\Delta. \tag{72}$$

It is apparent from (72) that as $t \uparrow 1$, e_1 approaches $2k$, as does $e_1^* = 2kt$.

From (38), the principal's expected gain in this case is:

$$\frac{1}{12kt}\left[12k^2 - 6k\left[\frac{1-t}{t}\right]\Delta + \left(\left[\frac{1-t}{t}\right]\Delta\right)^2\right]. \tag{73}$$

As $t \uparrow 1$, the expression in (73) approaches k, which is the expected gain from efficient production. $\blacksquare$

Proposition 6. *At the solution to [P], $\max_{\Delta}\{\frac{e_1}{e_1^*}\} = \max_{\Delta}\{\frac{E\{e_2(\beta)\}}{e_2^*}\}$ is non-decreasing in t for $t \in (0, \frac{1}{2})$ and non-increasing in t for $t \in (\frac{1}{2}, 1)$. At $t = \frac{1}{2}$, $\max\{\frac{e_1}{e_1^*}\}$ is non-decreasing in Δ for $\Delta \in (0, \sqrt{2}k)$ and non-increasing in Δ for $\Delta > \sqrt{2}k$. The maximum value of $\frac{e_1}{e_1^*} = \frac{E\{e_2(\beta)\}}{e_2^*}$ is $4 - 2\sqrt{2} \approx 1.17$.*

Proof of Proposition 6.

From Propositions 2 - 4, $e_1 > e_1^*$ when $t \in (\frac{\Delta}{4k}, \min\{\frac{\Delta}{2k}, 1\})$.

When $\Delta \geq 2k$, $\frac{\Delta}{\Delta+2k} \leq \frac{\Delta}{4k}$. From (34), when $\Delta \geq 2k$ and $t \in (\frac{\Delta}{\Delta+2k}, 1)$:

$$e_1 = 2k - \frac{1}{2}\left[\frac{1-t}{t}\right]\Delta \quad \text{and} \quad \frac{e_1}{e_1^*} = \frac{1}{t} - \frac{\Delta}{4k}\left[\frac{1-t}{t^2}\right]. \tag{74}$$

Since e_1 and $\frac{e_1}{e_1^*}$ are declining with Δ, e_1 and $\frac{e_1}{e_1^*}$ attain their highest values in this region at $\Delta = 2k$.

When $\Delta \leq 2k$, $\frac{\Delta}{\Delta+2k} \geq \frac{\Delta}{4k}$. From Proposition 3, when $t \leq \frac{\Delta}{\Delta+2k}$:

$$e_1 = \left[\frac{t}{1-t}\right]\left[\frac{4k-\Delta}{2}\right] \quad \text{and} \quad \frac{e_1}{e_1^*} = \left[\frac{1}{1-t}\right]\left[\frac{4k-\Delta}{4k}\right]. \tag{75}$$

Since e_1 and $\frac{e_1}{e_1^*}$ are increasing in t, e_1 and $\frac{e_1}{e_1^*}$ attain their highest values in this region at $t = \frac{\Delta}{\Delta+2k}$.

Therefore, $\frac{e_1}{e_1^*}$ is maximized when $\Delta \leq 2k$ and $t \in [\frac{\Delta}{\Delta+2k}, \frac{\Delta}{2k}]$. From Proposition 2, when $\Delta \leq 2k$ and $t \in [\frac{\Delta}{\Delta+2k}, \frac{\Delta}{2k}]$:

$$e_1 = 2k - \frac{1}{2}\left[\frac{1-t}{t}\right]\Delta - \frac{1}{2}\left[\frac{t}{1-t}\right]\frac{1}{\Delta}[2k-\Delta]^2. \tag{76}$$

To find the value of Δ at which $\frac{e_1}{e_1^*}$ is maximized, it suffices to find the value of Δ at which e_1 is maximized because $e_1^* = 2kt$ does not vary with Δ. Therefore, the value of Δ for which $\frac{e_1}{e_1^*}$ is maximized can be identified by determining the value of Δ at which the partial derivative of the expression for e_1 in (76) is zero:

$$-\frac{1}{2}\left[\frac{1-t}{t}\right] + \frac{1}{2}\left[\frac{t}{1-t}\right]\frac{1}{\Delta^2}[2k-\Delta]^2 + \left[\frac{t}{1-t}\right]\frac{1}{\Delta}[2k-\Delta] = 0$$

$$\begin{aligned}
&\Leftrightarrow \quad -\left[\frac{1-t}{t}\right]^2 + \frac{1}{\Delta^2}[2k-\Delta]^2 + 2\frac{1}{\Delta}[2k-\Delta] = 0 \\
&\Leftrightarrow \quad \left(\frac{1}{\Delta}[2k-\Delta]+1\right)^2 = \left[\frac{1-t}{t}\right]^2 + 1 \\
&\Leftrightarrow \quad \left(\frac{2k}{\Delta}\right)^2 = \frac{(1-t)^2+t^2}{t^2} \quad \Leftrightarrow \quad \Delta = 2k\sqrt{\frac{t^2}{(1-t)^2+t^2}}\,.
\end{aligned} \tag{77}$$

Substituting for Δ from (77) into (76) and dividing by e_1^* provides:

$$\begin{aligned}
\frac{e_1}{e_1^*} &= \frac{2k - \frac{1}{2}\left[\frac{1-t}{t}\right]\Delta - \frac{1}{2}\left[\frac{t}{1-t}\right]\frac{1}{\Delta}[2k-\Delta]^2}{2kt} \\
&= \frac{1 - \frac{1}{2}\left[\frac{1-t}{t}\right]\frac{\Delta}{2k} - \frac{1}{2}\left[\frac{t}{1-t}\right]\frac{2k}{\Delta} + \left[\frac{t}{1-t}\right] - \frac{1}{2}\left[\frac{t}{1-t}\right]\frac{\Delta}{2k}}{t} \\
&= \frac{1 + \left[\frac{t}{1-t}\right] - \frac{1}{2}\left(\left[\frac{1-t}{t}\right] + \left[\frac{t}{1-t}\right]\right)\frac{\Delta}{2k} - \frac{1}{2}\left[\frac{t}{1-t}\right]\frac{2k}{\Delta}}{t} \\
&= \frac{\left[\frac{1}{1-t}\right] - \frac{1}{2}\left[\frac{(1-t)^2+t^2}{t[1-t]}\right]\frac{\Delta}{2k} - \frac{1}{2}\left[\frac{t}{1-t}\right]\frac{2k}{\Delta}}{t} \\
&= \frac{1 - \frac{1}{2}\left[\frac{(1-t)^2+t^2}{t}\right]\frac{\Delta}{2k} - \frac{1}{2}t\frac{2k}{\Delta}}{t[1-t]}\,.
\end{aligned} \tag{78}$$

Substituting from (77) into (78) provides:

$$\begin{aligned}\frac{e_1}{e_1^*} &= \frac{1-\frac{t}{2}\left[\frac{(1-t)^2+t^2}{t^2}\right]\sqrt{\frac{t^2}{(1-t)^2+t^2}}-\frac{1}{2}t\frac{2k}{\Delta}}{t[1-t]} \\ &= \frac{1-\frac{t}{2}\sqrt{\frac{(1-t)^2+t^2}{t^2}}-\frac{1}{2}t\frac{2k}{\Delta}}{t[1-t]} = \frac{1-\frac{t}{2}\frac{2k}{\Delta}-\frac{1}{2}\frac{2kt}{\Delta}}{t[1-t]} = \frac{1-\frac{2kt}{\Delta}}{t[1-t]}\,. \end{aligned} \tag{79}$$

Substituting again for $\frac{2kt}{\Delta}$ from (77) into (79) provides:

$$\begin{aligned}\frac{e_1}{e_1^*} &= \frac{1-\sqrt{[1-t]^2+t^2}}{t[1-t]} = \frac{1-t^2-[1-t]^2}{t[1-t]\left[1+\sqrt{(1-t)^2+t^2}\right]} \\ &= \frac{2}{1+\sqrt{[1-t]^2+t^2}} = \frac{2}{1+\sqrt{2[t-\frac{1}{2}]^2+\frac{1}{2}}}\,. \end{aligned} \tag{80}$$

The last term in (80) is maximized at $t=\frac{1}{2}$. When $t=\frac{1}{2}$, (77) implies:

$$\Delta = 2k\sqrt{\frac{1/4}{1/4+1/4}} = 2k\sqrt{\frac{1}{2}} = \sqrt{2}k. \tag{81}$$

Also, from (80), the maximum value of $\frac{e_1}{e_1^*}$ is:

$$\frac{2}{1+\sqrt{1/2}} = 4-2\sqrt{2} = 1.172. \quad \blacksquare \tag{82}$$

Proposition 7. *(i)* $\overline{m} \equiv \frac{e_2(\overline{\beta})}{e_2^*} \geq 1$*, with strict inequality unless* $\Delta < 2k$ *and* $t \in (\frac{\Delta}{2k}, 1]$*; (ii)* $\overline{m}$ *is a non-decreasing function of* Δ *for all* $t \in [0,1]$*; and (iii) for any given* $\Delta > 0$*,* $\overline{m}$ *is increasing in* t *for* $t \in [0, \frac{\Delta}{\Delta+2k})$ *and non-increasing in* t *for* $t \in (\frac{\Delta}{\Delta+2k}, 1]$*.*

Proof of Proposition 7.

Proposition 1 implies that $e_2(\overline{\beta}) = e_2^*$ if $t \in [\frac{\Delta}{2k}, 1]$. From Propositions 3 and 5, when $t \leq \frac{\Delta}{\Delta+2k}$, $e_2(\overline{\beta}) = 2k > 2k[1-t] = e_2^*$. From Propositions 2 and 4, when $t \in (\frac{\Delta}{\Delta+2k}, \frac{\Delta}{2k})$, $e_2(\overline{\beta}) = \left[\frac{1-t}{t}\right]\Delta > 2k[1-t] = e_2^* \;\Leftrightarrow\; \Delta > 2kt \;\Leftrightarrow\; t < \frac{\Delta}{2k}$. Therefore, $\overline{m} \equiv \frac{e_2(\overline{\beta})}{e_2^*} \geq 1$, with strict inequality unless $\Delta < 2k$ and $t \in (\frac{\Delta}{2k}, 1]$.

We now show that for every $t \in [0,1]$, $\overline{m} \equiv \frac{e_2(\overline{\beta})}{e_2^*}$ is a non-decreasing function of Δ. From Proposition 1, when $\Delta \leq 2kt$, $e_2(\beta) = e_2^* = 2k[1-t]$. Therefore:

$$\overline{m} = \frac{2k[1-t]}{2k[1-t]} = 1, \tag{83}$$

which does not vary with Δ.

From Propositions 2 and 4, when $\Delta \in (2kt, \frac{2kt}{1-t})$, $e_2(\overline{\beta}) = \left[\frac{1-t}{t}\right]\Delta$. Therefore:

$$\overline{m} = \frac{\left[\frac{1-t}{t}\right]\Delta}{2k[1-t]} = \frac{\Delta}{2kt}, \tag{84}$$

which is an increasing function of Δ.

From Propositions 3 and 5, when $\Delta \geq \frac{2kt}{1-t}$, $e_2(\overline{\beta}) = 2k$, and so

$$\overline{m} = \frac{2k}{2k[1-t]} = \frac{1}{1-t}, \tag{85}$$

which does not vary with Δ.

(83), (84), and (85) imply that for any fixed duration of the information asymmetry (t), the maximum slope of the optimal contract is non-decreasing in Δ.

We now show that for given Δ, $\overline{m}$ is increasing with t on $t \in [0, \frac{\Delta}{\Delta+2k}]$ and is non-increasing with t on $t \in [\frac{\Delta}{\Delta+2k}, 1]$. From Propositions 3 and 5, when $t \in [0, \frac{\Delta}{\Delta+2k}]$, $e_2(\overline{\beta}) = 2k$, and so $\overline{m} = \frac{1}{1-t}$, which is an increasing function of t.

From Propositions 2 and 4, when $t \in [\frac{\Delta}{\Delta+2k}, \min\{\frac{\Delta}{2k}, 1\}]$, $e_2(\overline{\beta}) = \left[\frac{1-t}{t}\right]\Delta$, and so $\overline{m} = \frac{\Delta}{2kt}$, which is a decreasing function of t.

From Proposition 1, when $\Delta \leq 2k$ and $t \in [\frac{\Delta}{2k}, 1]$, $e_2(\beta) = e_2^* = 2k[1-t]$. Therefore, $\overline{m} = 1$.

Thus, $\overline{m}$ attains its maximum value at $t = \frac{\Delta}{\Delta+2k}$. $\overline{m} = \frac{\Delta+2k}{2k}$ when $t = \frac{\Delta}{\Delta+2k}$. ■

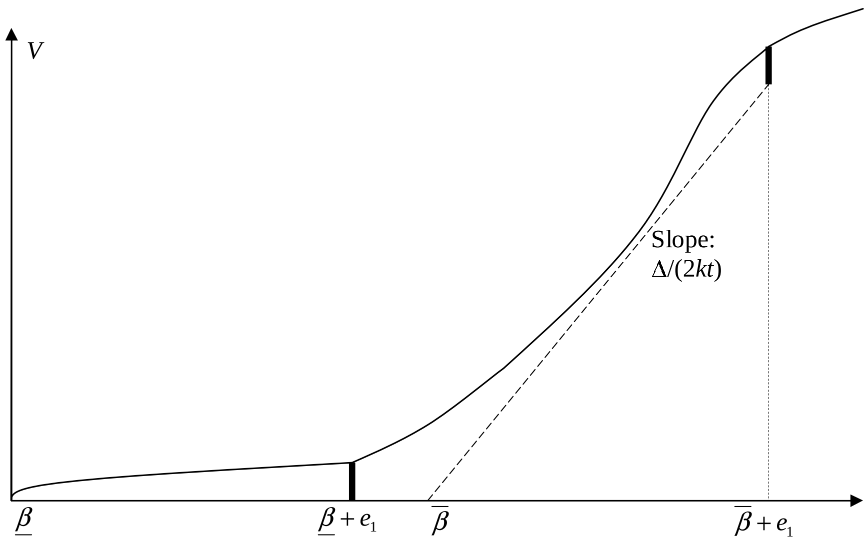

Figure A1

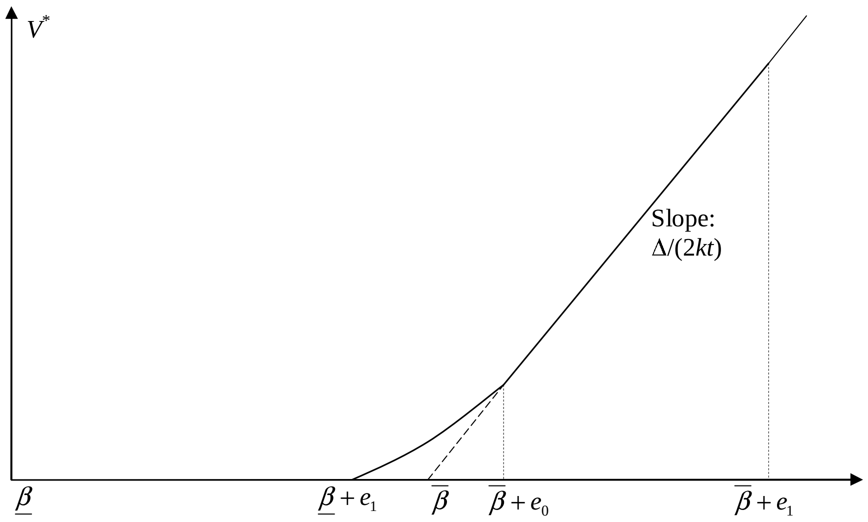

Figure A2