Technical Appendix to Accompany
"Hotelling Competition with Avoidable Horizontal Product Differentiation"

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Equations and Definitions from the Text

$$G_1 - p_1 - \min\{\, t\,x,\ t\,[\,1-x\,] + K_1 \,\} \;\geq\; G_2 - p_2 - \min\{\, t\,[\,1-x\,],\ t\,x + K_2 \,\}\,. \tag{1}$$

$$K_1 + t\,[\,1-x\,] \;<\; t\,x \quad \Leftrightarrow \quad x \;>\; \frac{1}{2} + \frac{K_1}{2\,t}\,. \tag{2}$$

$$K_2 + t\,x \;<\; t\,[\,1-x\,] \quad \Leftrightarrow \quad x \;<\; \frac{1}{2} - \frac{K_2}{2\,t}\,. \tag{3}$$

$$A \;\equiv\; \frac{1}{3}\,[\,G_1 - c_1 - (\,G_2 - c_2\,)\,] > 0. \tag{4}$$

Additional Lemmas

The following lemmas (Lemmas A1 – A17) are employed to prove the formal conclusions in the text.

Lemma A1. If a consumer buys Firm 2's product, she will change its default horizontal characteristic if and only if she is located in $[\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,)$.

Proof. If a consumer located at x buys the product from Firm 2, the consumer will change the default horizontal product characteristic if and only if:

$$G_2 - t\,x - K_2 \;>\; G_2 - t\,[\,1-x\,] \quad \Leftrightarrow \quad t\,[\,1-2\,x\,] \;>\; K_2 \quad \Leftrightarrow \quad 1 - 2\,x \;>\; \frac{K_2}{t}$$

$$\Leftrightarrow \quad 2\,x \;<\; 1 - \frac{K_2}{t} \quad \Leftrightarrow \quad x \;<\; \frac{1}{2} - \frac{K_2}{2\,t}. \quad \blacksquare$$

Lemma A2. If a consumer buys Firm 1's product, she will change its default horizontal characteristic if and only if she is located in $(\,\frac{1}{2} + \frac{K_1}{2\,t}, 1\,]$.

Proof. If a consumer located at x buys the product from Firm 1, the consumer will change the default horizontal product characteristic if:

$$G_1 - t\,[\,1-x\,] - K_1 \;>\; G_1 - t\,x \quad \Leftrightarrow \quad t\,[\,1-2\,x\,] \;<\; -\,K_1 \quad \Leftrightarrow \quad 1 - 2\,x \;<\; -\,\frac{K_1}{t}$$

$$\Leftrightarrow \quad 2\,x \;>\; 1 + \frac{K_1}{t} \quad \Leftrightarrow \quad x \;>\; \frac{1}{2} + \frac{K_1}{2\,t}. \quad \blacksquare$$

Lemma A3. A consumer located in $\left[\,\frac{1}{2}-\frac{K_2}{2\,t},\,\frac{1}{2}+\frac{K_1}{2\,t}\,\right]$ will not change the default horizontal characteristic of the product she purchases.

Proof. The proof follows directly from Lemmas A1 and A2. ■

Lemma A4. Suppose a consumer located at $x_0 \in [\,\frac{1}{2}-\frac{K_2}{2\,t},\,\frac{1}{2}+\frac{K_1}{2\,t}\,]$ is indifferent between buying Firm 1's product and buying Firm 2's product. Then: (i) all consumers located in $[\,0,\,x_0\,]$ will buy Firm 1's product; and (ii) all consumers located in $(\,x_0, 1\,]$ will buy Firm 2's product.

Proof. Lemma A3 implies that because the consumer at x_0 is indifferent between buying Firm 1's product and buying Firm 2's product:

$$G_1 - t\,x_0 - p_1 \;=\; G_2 - t\,[\,1-x_0\,] - p_2 \;\;\Leftrightarrow\;\; t\,[\,1-2\,x_0\,] \;=\; G_2 - G_1 + p_1 - p_2$$

$$\Leftrightarrow\;\; 2\,x_0 \;=\; 1 + \frac{1}{t}\,[\,G_1 - G_2 + p_2 - p_1\,] \;\;\Leftrightarrow\;\; x_0 \;=\; \frac{1}{2\,t}\,[\,t + G_1 - G_2 + p_2 - p_1\,]\,. \qquad (5)$$

(5) and Lemma A3 imply that a consumer located at $x \in [\,\frac{1}{2}-\frac{K_2}{2\,t}, x_0\,]$ will buy the Firm 1's product because:

$$G_1 - t\,x - p_1 \;\geq\; G_2 - t\,[\,1-x\,] - p_2 \;\;\Leftrightarrow\;\; t\,[\,1-2\,x\,] \;\geq\; G_2 - G_1 + p_1 - p_2$$

$$\Leftrightarrow\;\; 2\,x \;\leq\; 1 + \frac{1}{t}\,[\,G_1 - G_2 + p_2 - p_1\,] \;\;\Leftrightarrow\;\; x \;\leq\; \frac{1}{2\,t}\,[\,t + G_1 - G_2 + p_2 - p_1\,] \;=\; x_0\,.$$

(5) and Lemma A1 imply that a consumer located at $x \in [\,0, \frac{1}{2}-\frac{K_2}{2\,t}\,)$ will buy Firm 1's product because:

$$G_1 - t\,x - p_1 \;\geq\; G_2 - t\,x - p_2 - K_2$$

$$\Leftrightarrow\;\; G_1 - G_2 + p_2 - p_1 \;\geq\; -\,K_2 \;\;\Leftrightarrow\;\; \frac{1}{2\,t}\,[\,G_1 - G_2 + p_2 - p_1\,] \;\geq\; -\,\frac{K_2}{2\,t}$$

$$\Leftrightarrow\;\; \frac{1}{2} + \frac{1}{2\,t}\,[\,G_1 - G_2 + p_2 - p_1\,] \;\geq\; \frac{1}{2} - \frac{K_2}{2\,t} \;\;\Leftrightarrow\;\; x_0 \;\geq\; \frac{1}{2} - \frac{K_2}{2\,t}\,.$$

(5) and Lemma A3 imply that a consumer located at $x \in (\,x_0, \frac{1}{2}+\frac{K_1}{2\,t}\,]$ will buy Firm 2's product because:

$$G_2 - t\,[\,1-x\,] - p_2 \;>\; G_1 - t\,x - p_1 \;\;\Leftrightarrow\;\; t\,[\,1-2\,x_0\,] \;<\; G_2 - G_1 + p_1 - p_2$$

$$\Leftrightarrow\;\; 2\,x \;>\; 1 + \frac{1}{t}\,[\,G_1 - G_2 + p_2 - p_1\,] \;\;\Leftrightarrow\;\; x \;>\; \frac{1}{2\,t}\,[\,t + G_1 - G_2 + p_2 - p_1\,] \;=\; x_0\,.$$

(5) and Lemma A3 imply that a consumer located at $x \in (\,\frac{1}{2}+\frac{K_1}{2\,t}, 1\,]$ will buy Firm 2's product because:

$$G_2 - t\,[\,1-x\,] - p_2 \;>\; G_1 - t\,[\,1-x\,] - p_1 - K_1$$

$$\Leftrightarrow \quad G_1 - G_2 + p_2 - p_1 \;<\; K_1 \quad \Leftrightarrow \quad \frac{1}{2\,t}\,[\,G_1 - G_2 + p_2 - p_1\,] \;<\; \frac{K_1}{2\,t}$$

$$\Leftrightarrow \quad \frac{1}{2} + \frac{1}{2\,t}\,[\,G_1 - G_2 + p_2 - p_1\,] \;<\; \frac{1}{2} + \frac{K_1}{2\,t} \quad \Leftrightarrow \quad x_0 \;<\; \frac{1}{2} + \frac{K_1}{2\,t}\,. \quad \blacksquare$$

Lemma A5. Suppose a consumer located at $x_1 \in [\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,)$ is indifferent between buying Firm 1's product and buying Firm 2's product. Then: (i) all consumers located in $[\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,]$ are similarly indifferent; and (ii) all consumers located in $(\,\frac{1}{2} - \frac{K_2}{2\,t}, 1\,]$ will buy Firm 2's product.

<u>Proof</u>. Lemma A1 implies that when a consumer at $x \in [\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,)$ is indifferent between buying Firm 1's product and buying Firm 2's product:

$$G_1 - t\,x - p_1 \;=\; G_2 - t\,x - p_2 - K_2 \quad \Leftrightarrow \quad p_2 \;=\; p_1 + G_2 - G_1 - K_2\,. \tag{6}$$

(6) implies that when the consumer located at $x_1 \in [\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,)$ is indifferent between buying Firm 1's product and buying Firm 2's product, the same is true of all consumers located in $[\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,)$.

Lemma A3 implies that when (6) holds, the consumer at $\widetilde{x} = \frac{1}{2} - \frac{K_2}{2\,t}$ is indifferent between buying Firm 1's product and buying Firm 2's product because:

$$G_1 - t\,\widetilde{x} - p_1 \;=\; G_2 - t\,[\,1 - \widetilde{x}\,] - p_2$$

$$\Leftrightarrow \quad p_2 - (\,p_1 + G_2 - G_1\,) \;=\; -\,t\,[\,1 - 2\,\widetilde{x}\,] \quad \Leftrightarrow \quad -\,K_2 \;=\; -\,t\,[\,1 - 2\,\widetilde{x}\,]$$

$$\Leftrightarrow \quad \frac{K_2}{t} \;=\; 1 - 2\,\widetilde{x} \quad \Leftrightarrow \quad \widetilde{x} \;=\; \frac{1}{2} - \frac{K_2}{2\,t}\,.$$

Lemma A3 implies that when (6) holds, a consumer located at $x \in (\,\frac{1}{2} - \frac{K_2}{2\,t}, \frac{1}{2} + \frac{K_1}{2\,t}\,]$ will buy Firm 2's product. This is the case because when (6) holds:

$$G_2 - t\,[\,1 - x\,] - p_2 \;>\; G_1 - t\,x - p_1$$

$$\Leftrightarrow \quad p_2 - (\,p_1 + G_2 - G_1\,) \;<\; -\,t\,[\,1 - 2\,x\,] \quad \Leftrightarrow \quad -\,K_2 \;<\; -\,t\,[\,1 - 2\,x\,]$$

$$\Leftrightarrow \quad \frac{K_2}{t} \;>\; 1 - 2\,x \quad \Leftrightarrow \quad x \;>\; \frac{1}{2} - \frac{K_2}{2\,t}\,.$$

Lemma A2 implies that when (6) holds, a consumer located at $x \in (\,\frac{1}{2} + \frac{K_1}{2\,t}, 1\,]$ will buy Firm 2's product because:

$$G_2 - t\,[\,1 - x\,] - p_2 \;>\; G_1 - t\,[\,1 - x\,] - p_1 - K_1 \quad \Leftrightarrow \quad p_2 \;<\; p_1 + G_2 - G_1 + K_1\,. \tag{7}$$

(6) ensures that (7) holds. ■

Lemma A6. Suppose a consumer located at $x_2 \in (\frac{1}{2} + \frac{K_1}{2t}, 1]$ is indifferent between buying Firm 1's product and buying Firm 2's product. Then: (i) all consumers located in $[\frac{1}{2} + \frac{K_1}{2t}, 1]$ are similarly indifferent; and (ii) all consumers located in $[0, \frac{1}{2} + \frac{K_1}{2t}]$ will buy Firm 1's product.

Proof. Lemma A2 implies that when a consumer at $x \in (\frac{1}{2} + \frac{K_1}{2t}, 1]$ is indifferent between buying Firm 1's product and buying Firm 2's product:

$$G_1 - t[1-x] - p_1 - K_1 \;=\; G_2 - t[1-x] - p_2 \Leftrightarrow \;\; p_2 \;=\; p_1 + G_2 - G_1 + K_1 \,. \qquad (8)$$

(8) implies that when the consumer located at $x_2 \in (\frac{1}{2} + \frac{K_1}{2t}, 1]$ is indifferent between buying Firm 1's product and buying Firm 2's product, the same is true of all consumers located in $(\frac{1}{2} + \frac{K_1}{2t}, 1]$.

Lemma A3 implies that when (8) holds, the consumer at $\widehat{x} = \frac{1}{2} + \frac{K_1}{2t}$ is indifferent between buying Firm 1's product and buying Firm 2's product because:

$$\begin{aligned} & G_1 - t\,\widehat{x} - p_1 \;=\; G_2 - t[1-\widehat{x}] - p_2 \\ \Leftrightarrow\;\; & p_2 - (p_1 + G_2 - G_1) \;=\; -t[1-2\,\widehat{x}] \;\;\Leftrightarrow\;\; K_1 \;=\; -t[1-2\,\widehat{x}] \\ \Leftrightarrow\;\; & \frac{K_1}{t} \;=\; 2\,\widehat{x} - 1 \;\;\Leftrightarrow\;\; \widehat{x} \;=\; \frac{1}{2} + \frac{K_1}{2t}\,. \end{aligned}$$

Lemma A3 implies that when (8) holds, a consumer located at $x \in [\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t})$ will buy Firm 1's product. This is the case because when (8) holds:

$$\begin{aligned} & G_1 - t\,x - p_1 \;\geq\; G_2 - t[1-x] - p_2 \\ \Leftrightarrow\;\; & p_2 - (p_1 + G_2 - G_1) \;\geq\; -t[1-2\,x] \;\;\Leftrightarrow\;\; K_1 \;\geq\; -t[1-2\,x] \\ \Leftrightarrow\;\; & \frac{K_1}{t} \;\geq\; 2\,x - 1 \;\;\Leftrightarrow\;\; x \;\leq\; \frac{1}{2} + \frac{K_1}{2t}\,. \end{aligned}$$

Lemma A1 implies that when (8) holds, a consumer located at $x \in [0, \frac{1}{2} - \frac{K_2}{2t}]$ will buy Firm 1's product because:

$$G_1 - t\,x - p_1 \;\geq\; G_2 - t\,x - p_2 - K_2 \;\;\Leftrightarrow\;\; p_2 - (p_1 + G_2 - G_1) \;\geq\; -K_2\,. \qquad (9)$$

(8) ensures that (9) holds. ■

Lemma A7. If $p_1 \geq p_2 + G_1 - G_2 + K_2$, then all consumers located in $[0, \frac{1}{2} - \frac{K_2}{2t})$ (weakly) prefer to buy Firm 2's product than to buy Firm 1's product. The preference is strict if the inequality holds strictly.

Proof. Lemma A1 implies that a consumer located at $x \in [0, \frac{1}{2} - \frac{K_2}{2t})$ (weakly) prefers to buy the product from Firm 2 than from Firm 1 if:

$$G_1 - t\,x - p_1 \;\leq\; G_2 - t\,x - p_2 - K_2 \quad \Leftrightarrow \quad p_1 \;\geq\; p_2 + G_1 - G_2 + K_2\,. \tag{10}$$

It is apparent from (10) that the preference is strict if the inequality holds strictly. ■

Lemma A8. If $p_1 \geq p_2 + G_1 - G_2 + K_2$, then all consumers located in $[\,\frac{1}{2} - \frac{K_2}{2t}, 1\,]$ (weakly) prefer to buy Firm 2's product than to buy Firm 1's product. The preference is strict if the inequality holds strictly.

Proof. Lemma A3 implies that a consumer located at $x \in \left[\,\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}\,\right]$ (weakly) prefers to buy the product from Firm 2 than from Firm 1 if:

$$G_1 - t\,x - p_1 \;\leq\; G_2 - t\,[\,1 - x\,] - p_2 \quad \Leftrightarrow \quad p_1 \;\geq\; p_2 + G_1 - G_2 + t\,[\,1 - 2\,x\,]\,. \tag{11}$$

The maintained assumption ensures the inequality in (11) holds if:

$$K_2 \;\geq\; t\,[\,1 - 2\,x\,] \quad \Leftrightarrow \quad x \;\geq\; \frac{1}{2} - \frac{K_2}{2\,t}\,. \tag{12}$$

(12) holds for all consumers located in $\left[\,\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}\,\right]$. Furthermore, it is apparent from (11) and (12) that all consumers located in $\left[\,\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}\,\right]$ strictly prefer to buy Firm 2's product than to buy Firm 1's product if $p_1 > p_2 + G_1 - G_2 + K_2$.

Lemmas A1 and A2 imply that all consumers in $(\,\frac{1}{2} + \frac{K_1}{2t}, 1\,]$ strictly prefer to buy Firm 2's product than to buy Firm 1's product if:

$$G_1 - t\,[\,1 - x\,] - p_1 - K_1 \;<\; G_2 - t\,[\,1 - x\,] - p_2 \quad \Leftrightarrow \quad p_1 \;>\; p_2 + G_1 - G_2 - K_1\,. \tag{13}$$

The maintained assumption ensures that the inequality in (13) holds. ■

Lemma A9. If $p_2 \geq p_1 + G_2 - G_1 + K_1$, then all consumers located in $(\frac{1}{2} + \frac{K_1}{2t}, 1\,]$ (weakly) prefer to buy Firm 1's product than to buy Firm 2's product. The preference is strict if the inequality holds strictly.

Proof. Lemma A2 implies that a consumer located at $x \in (\frac{1}{2} + \frac{K_1}{2t}, 1\,]$ (weakly) prefers to buy Firm 1's product than to buy Firm 2's product if:

$$G_1 - t\,[\,1 - x\,] - p_1 - K_1 \;\geq\; G_2 - t\,[\,1 - x\,] - p_2 \quad \Leftrightarrow \quad p_2 \;\geq\; p_1 + G_2 - G_1 + K_1\,. \tag{14}$$

It is apparent from (14) that the preference is strict if the inequality holds strictly. ■

Lemma A10. If $p_2 \geq p_1 + G_2 - G_1 + K_1$, then all consumers located in $[\,0, \frac{1}{2} + \frac{K_1}{2t}\,]$ (weakly) prefer to buy Firm 1's product than to buy Firm 2's product. The preference is strict if the inequality holds strictly.

Proof. Lemma A3 implies that a consumer located at $x \in \left[\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}\right]$ (weakly) prefers to buy Firm 1's product than to buy Firm 2's product if:

$$G_1 - t\,x - p_1 \;\geq\; G_2 - t\,[\,1 - x\,] - p_2 \;\;\Leftrightarrow\;\; p_2 \;\geq\; p_1 + G_2 - G_1 - t\,[\,1 - 2\,x\,]\,. \tag{15}$$

The maintained assumption ensures the inequality in (15) holds if:

$$K_1 \;\geq\; -\,t\,[\,1 - 2\,x\,] \quad\Leftrightarrow\quad x \;\leq\; \frac{1}{2} + \frac{K_1}{2\,t}\,. \tag{16}$$

(16) holds for all consumers located in $\left[\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}\right]$. Furthermore, it is apparent from (15) and (16) that all consumers located in $\left[\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}\right]$ strictly prefer to buy Firm 1's product than to buy Firm 2's product if $p_2 > p_1 + G_2 - G_1 + K_1$.

Lemmas A1 and A2 imply that all consumers in $[\,0, \frac{1}{2} - \frac{K_2}{2t})$ strictly prefer to buy Firm 1's product than to buy Firm 2's product if:

$$G_1 - t\,x\, - p_1 \;>\; G_2 - t\,x\, - p_2 - K_2 \;\;\Leftrightarrow\;\; p_2 \;>\; p_1 + G_2 - G_1 - K_2\,. \tag{17}$$

The maintained assumption ensures the inequality in (17) holds. ■

Assumption 1. $K_1 \in [\,0,\,t\,)$, $K_2 \in [\,0,\,t\,)$, and $(K_1\,,\,K_2\,) \neq (0\,,\,0\,)$.

Lemma A11. Suppose Assumption 1 holds. Then in any equilibrium in which $p_1 - p_2 \in (G_1 - G_2 - K_1, G_1 - G_2 + K_2)$: (i) all consumers located in $[\,0, \frac{1}{2} - \frac{K_2}{2t}\,)$ strictly prefer to buy Firm 1's product than to buy Firm 2's product; (ii) all consumers located in $(\,\frac{1}{2} + \frac{K_1}{2t}, 1\,]$ strictly prefer to buy Firm 2's product than to buy Firm 1's product; and (iii) some consumer located in $[\,\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}\,]$ is indifferent between buying Firm 1's product and buying Firm 2's product.

Proof. Lemmas A1 and A2 imply that all consumers located in $[\,0, \frac{1}{2} - \frac{K_2}{2t}\,)$ strictly prefer to buy Firm 1's product than to buy Firm 2's product if:

$$\begin{aligned} & G_1 - t\,x\, - p_1 \;>\; G_2 - t\,x\, - p_2 - K_2 \\ \Leftrightarrow\;\; & p_1 \;<\; p_2 + G_1 - G_2 + K_2 \;\;\Leftrightarrow\;\; p_1 - p_2 \;<\; G_1 - G_2 + K_2\,. \end{aligned} \tag{18}$$

Lemmas A1 and A2 also imply that all consumers located in $(\,\frac{1}{2} + \frac{K_1}{2t}, 1\,]$ strictly prefer to buy Firm 2's product than to buy Firm 1's product if:

$$\begin{aligned} & G_1 - t\,[\,1 - x\,] - p_1 - K_1 \;<\; G_2 - t\,[\,1 - x\,] - p_2 \\ \Leftrightarrow\;\; & p_2 \;<\; p_1 + G_2 - G_1 + K_1 \;\;\Leftrightarrow\;\; p_1 - p_2 \;>\; G_1 - G_2 - K_1\,. \end{aligned} \tag{19}$$

(18) and (19) imply that conclusions (i) and (ii) in the lemma hold.

To prove conclusion (iii), first suppose all consumers located in $\left[\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}\right]$ strictly prefer to buy Firm 1's product than to buy Firm 2's product. Then the consumer located at

$\frac{1}{2} + \frac{K_1}{2t}$ weakly prefers to buy Firm 1's product than to buy Firm 2's product. Consequently, Lemma A3 implies:

$$G_1 - t\left[\frac{1}{2} + \frac{K_1}{2t}\right] - p_1 \;\geq\; G_2 - t\left[\frac{1}{2} - \frac{K_1}{2t}\right] - p_2$$

$$\Leftrightarrow \quad p_1 - p_2 \;\leq\; G_1 - G_2 - t\left[\frac{1}{2} + \frac{K_1}{2t} - \left(\frac{1}{2} - \frac{K_1}{2t}\right)\right]$$

$$\Leftrightarrow \quad p_1 - p_2 \;\leq\; G_1 - G_2 - K_1 \,.$$

(19) implies that this inequality cannot hold. Therefore, it is not the case that all consumers located in $[\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}]$ strictly prefer to buy Firm 1's product than to buy Firm 2's product when $p_1 - p_2 \in (G_1 - G_2 - K_1, G_1 - G_2 + K_2)$.

Now suppose all consumers located in $\left[\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}\right]$ strictly prefer to buy Firm 2's product than to buy Firm 1's product. Then the consumer located at $\frac{1}{2} - \frac{K_2}{2t}$ weakly prefers to buy Firm 2's product than to buy Firm 1's product. Consequently, Lemma A3 implies:

$$G_2 - t\left[\frac{1}{2} + \frac{K_2}{2t}\right] - p_2 \;\geq\; G_1 - t\left[\frac{1}{2} - \frac{K_2}{2t}\right] - p_1$$

$$\Leftrightarrow \quad p_1 - p_2 \;\geq\; G_1 - G_2 - t\left[\frac{1}{2} - \frac{K_2}{2t} - \left(\frac{1}{2} + \frac{K_2}{2t}\right)\right]$$

$$\Leftrightarrow \quad p_1 - p_2 \;\geq\; G_1 - G_2 + K_2 \,.$$

(18) implies that this inequality cannot hold. Therefore, it is not the case that all consumers located in $[\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}]$ strictly prefer to buy Firm 2's product than to buy Firm 1's product when $p_1 - p_2 \in (G_1 - G_2 - K_1, G_1 - G_2 + K_2)$. Consequently, it must be the case that some consumer located in $[\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}]$ is indifferent between buying Firm 1's product and buying Firm 2's product. ■

Lemma A12. When Assumption 1 holds: (i) c_1 is the lowest price that Firm 1 can profitably charge when all consumers buy its product; and (ii) c_2 is the lowest price that Firm 2 can profitably charge when all consumers buy its product.

Proof. Lemmas A1 – A3 imply that when Assumption 1 holds, Firm 1's profit when all consumers buy its product at price p_1 is:

$$\overline{\pi}_1 \;=\; [\,p_1 - c_1\,]\left[\frac{1}{2} + \frac{K_1}{2t}\right] + [\,p_1 - c_1\,]\left[\frac{1}{2} - \frac{K_1}{2t}\right]$$

$$=\; [\,p_1 - c_1\,]\left[\frac{1}{2} + \frac{K_1}{2t} + \frac{1}{2} - \frac{K_1}{2t}\right] \;=\; p_1 - c_1 \quad\Rightarrow\quad \overline{\pi}_1 \;\geq\; 0 \quad\Leftrightarrow\quad p_1 \;\geq\; c_1 \,.$$

Lemmas A1 – A3 also imply that when Assumption 1 holds, Firm 2's profit when all consumers buy its product at price p_2 is:

$$\overline{\pi}_2 \;=\; [\,p_2 - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] + [\,p_2 - c_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right]$$

$$=\; [\,p_2 - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t} + \frac{1}{2} - \frac{K_2}{2\,t}\right] \;=\; p_2 - c_2 \;\;\Rightarrow\;\; \overline{\pi}_2 \;\geq\; 0 \;\;\Leftrightarrow\;\; p_2 \;\geq\; c_2. \;\blacksquare$$

Lemma A13. Suppose Assumption 1 holds. Then an equilibrium does not exist in which $p_1 > p_2 + G_1 - G_2 + K_2$.

Proof. Setting $p_1 < c_1$ is a dominated strategy for Firm 1. Therefore, by assumption, Firm 1 never sets price $p_1 < c_1$ in equilibrium.

Now consider a putative equilibrium in which $p_1 > p_2 + G_1 - G_2 + K_2$ and $p_1 > c_1$.

First suppose that $p_1 > p_2 + G_1 - G_2 + K_2 > c_1$. Then Lemmas A7 and A8 imply that all consumers strictly prefer to buy Firm 2's product than to buy Firm 1's product. Firm 1 earns 0 profit. If Firm 1 reduces its price to $p_1 = p_2 + G_1 - G_2 + K_2$, then (6) and Lemma A5 imply that: (i) all consumers located in $[\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,]$ are indifferent between buying Firm 1's product buying Firm 2's product (and therefore, all these consumers buy Firm 1's product, by assumption); and (ii) all consumers located in $(\,\frac{1}{2} - \frac{K_2}{2\,t}, 1\,]$ buy Firm 2's product. Therefore, Firm 1's profit is:

$$\widetilde{\pi}_1 \;=\; [\,p_1 - c_1\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \;>\; 0 \quad \text{because } p_1 > c_1\,.$$

Because Firm 1 thereby strictly increases its profit, an equilibrium in which $p_1 > p_2 + G_1 - G_2 + K_2 > c_1$ does not exist.

Now suppose that $p_1 > c_1 \geq p_2 + G_1 - G_2 + K_2$. Lemmas A7 and A8 again imply that all consumers strictly prefer to buy Firm 2's product than to buy Firm 1's product. Firm 2 can increase its profit by increasing its price to ensure $p_2 + G_1 - G_2 + K_2 \in [\,c_1, p_1\,)$. Therefore, no equilibrium exists in which $p_1 > p_2 + G_1 - G_2 + K_2$ and $p_1 > c_1$.

Finally, consider a putative equilibrium in which $p_1 > p_2 + G_1 - G_2 + K_2$ and $p_1 = c_1$.

Lemmas A7 and A8 imply that in any putative equilibrium in which $p_1 > p_2 + G_1 - G_2 + K_2$, all consumers buy Firm 2's product. Observe that $p_2 < p_1 + G_2 - G_1 - K_2$ when $p_1 > p_2 + G_1 - G_2 + K_2$. Furthermore, Lemmas A7 and A8 imply that if $p_1 = c_1$, then among all values of $p_2 < p_1 + G_2 - G_1 - K_2$, the value of p_2 that is most profitable for Firm 2 is marginally below $c_1 + G_2 - G_1 - K_2$. Therefore, the maximum profit that Firm 2 can earn in any equilibrium in which $p_2 < c_1 + G_2 - G_1 - K_2$ is less than:

$$\pi_2^{\max} \;=\; c_1 + G_2 - G_1 - K_2 - c_2 \;=\; -\,G_1 + c_1 + G_2 - c_2 - K_2 \;=\; -\,3\,A - K_2 \;<\; 0. \quad (20)$$

The inequality in (20) follows from (4) and the assumption $A > 0$. Therefore, no equilibrium exists in which $p_2 < p_1 + G_2 - G_1 - K_2$ and $p_1 = c_1$. ■

Lemma A14. Suppose Assumption 1 holds. Then an equilibrium does not exist in which $p_1 = p_2 + G_1 - G_2 + K_2$.

Proof. Setting $p_1 < c_1$ and $p_2 < c_2$ are dominated strategies for Firms 1 and 2, respectively. Therefore, by assumption, Firms 1 and 2 never set price $p_2 < c_1$ and $p_1 < c_2$, respectively, in equilibrium.

Suppose that $p_1 = p_2 + G_1 - G_2 + K_2$, $p_1 > c_1$, and $p_2 > c_2$. Then (6) and Lemma A5 imply that: (i) all consumers located in $[\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,]$ are indifferent between buying Firm 1's product and buying Firm 2's product (and therefore, all these consumers buy the firm 1's product, by assumption); and (ii) all consumers located in $(\,\frac{1}{2} - \frac{K_2}{2\,t}, 1\,]$ buy Firm 2's product. Therefore, Firm 2's profit is:

$$\widetilde{\pi}_2 \;=\; [\,p_2 - c_2\,]\left[1 - \left(\frac{1}{2} - \frac{K_2}{2\,t}\right)\right] \;=\; [\,p_2 - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] \;>\; 0\,. \tag{21}$$

The inequality in (21) holds because $p_2 > c_2$. If Firm 2 were to reduce its price marginally to $p_2 - \varepsilon_1$ where $\varepsilon_1 > 0$, all consumers would purchase its product. Therefore, Firm 2's profit would be:

$$\begin{aligned}\pi_2 \;&=\; p_2 - \varepsilon_1 - c_2 \;=\; [\,p_2 - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] + [\,p_2 - c_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] - \varepsilon_1 \\ &=\; \widetilde{\pi}_2 + [\,p_2 - c_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] - \varepsilon_1 \;>\; \widetilde{\pi}_2 \;\text{ for } \varepsilon_1 \text{ sufficiently small.}\end{aligned}$$

The last inequality holds because $K_2 < t$ and $p_2 > c_2$. Because Firm 2 could increase its profit by reducing its price marginally, an equilibrium does not exist in which $p_1 = p_2 + G_1 - G_2 + K_2$, $p_1 > c_1$ and $p_2 > c_2$.

Now we consider a putative equilibrium in which $p_1 = p_2 + G_1 - G_2 + K_2$, $p_1 \geq c_1$ and $p_2 = c_2$. Again, Firm 2's profit is given by (21). However, $\widetilde{\pi}_2 = 0$ because $p_2 = c_2$. Firm 1's price is $p_1 = c_2 + G_1 - G_2 + K_2$ because $p_2 = c_2$. If Firm 2 increases its price to ensure that $p_2 > c_2$ and $p_1 \in (\,p_2 + G_1 - G_2 - K_1,\, p_2 + G_1 - G_2 + K_2\,)$, then (59) implies it will set:[1]

$$\begin{aligned}p_2 \;&=\; \frac{1}{2}\,[\,t + c_2 + G_2 - G_1 + p_1\,] \\ &=\; \frac{1}{2}\,[\,t + c_2 + G_2 - G_1 + c_2 + G_1 - G_2 + K_2\,] \;=\; \frac{1}{2}\,[\,t + 2\,c_2 + K_2\,]\,.\end{aligned} \tag{22}$$

[1]This response function for Firm 2 when $p_1 \in (\,p_2 + G_1 - G_2 - K_1,\, p_2 + G_1 - G_2 + K_2\,)$ is derived in the proof of Lemma 3. See (59) below.

Therefore, Firm 2's profit when $p_1 \in (p_2 + G_1 - G_2 - K_1, \, p_2 + G_1 - G_2 + K_2)$ is given by (57):[2]

$$\pi_2 = [p_2 - c_2] \frac{1}{2t} [t + G_2 - G_1 + p_1 - p_2]$$

$$= \frac{1}{2} [t + 2c_2 + K_2 - 2c_2]$$

$$\cdot \frac{1}{4t} [2t + 2G_2 - 2G_1 + 2c_2 + 2G_1 - 2G_2 + 2K_2 - t - 2c_2 - K_2]$$

$$= \frac{1}{8t} [t + K_2][t + K_2] = \frac{1}{8t} [t + K_2]^2 > 0. \tag{23}$$

The maintained assumptions imply that the inequality in (23) holds. Therefore, no equilibrium exists in which $p_1 = p_2 + G_1 - G_2 + K_2$, $p_1 \geq c_1$ and $p_2 = c_2$.

Finally, consider a putative equilibrium in which $p_1 = p_2 + G_1 - G_2 + K_2$, $p_1 = c_1$ and $p_2 > c_2$.

If $p_1 = p_2 + G_1 - G_2 + K_2$ and $p_1 = c_1$, then:

$$p_2 = p_1 + G_2 - G_1 - K_2 \;\; \Rightarrow \;\; p_2 = c_1 + G_2 - G_1 - K_2$$

$$\Rightarrow \;\; p_2 - c_2 = G_2 - G_1 + c_1 - c_2 - K_2. \tag{24}$$

Because $p_2 > c_2$ in the present case, the ensuing proof considers settings in which $p_2 - c_2 = G_2 - G_1 + c_1 - c_2 - K_2 > 0$.

(6) and Lemma A5 imply that when $p_2 = p_1 + G_2 - G_1 - K_2$: (i) all consumers located in $[0, \frac{1}{2} - \frac{K_2}{2t}]$ are indifferent between buying Firm 1's product and buying Firm 2's product (and therefore, all these consumers buy Firm 1's product, by assumption); and (ii) all consumers located in $(\frac{1}{2} - \frac{K_2}{2t}, 1]$ buy Firm 2's product. Therefore, (24) and Lemma A5 imply that Firm 2's profit is:

$$\widetilde{\pi}_2 = [p_2 - c_2] \left[1 - \left(\frac{1}{2} - \frac{K_2}{2t} \right) \right]$$

$$= [p_2 - c_2] \left[\frac{1}{2} + \frac{K_2}{2t} \right] = [G_2 - G_1 + c_1 - c_2 - K_2] \left[\frac{1}{2} + \frac{K_2}{2t} \right] > 0. \tag{25}$$

The assumption $p_2 - c_2 = G_2 - G_1 + c_1 - c_2 - K_2 > 0$ and Assumption 1 imply that the inequality in (25) holds. Lemmas A7 and A8 imply that if Firm 2 reduces p_2 below $c_1 + G_2 - G_1 - K_2$ by $\varepsilon_2 > 0$, it can induce all consumers to purchase its product. (25) implies that Firm 2's corresponding profit would be:

[2] Firm 2's profit function when $p_1 \in (p_2 + G_1 - G_2 - K_1, \, p_2 + G_1 - G_2 + K_2)$ is derived in the proof of Lemma 3. See (57) below.

$$
\begin{aligned}
\pi_2 \;=\; p_2 - c_2 - \varepsilon_2 \;&=\; [\,p_2 - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] + [\,p_2 - c_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] - \varepsilon_2 \\
&=\; \widetilde{\pi}_2 + [\,G_2 - G_1 - c_2 + c_1 - K_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] - \varepsilon_2 \\
&>\; \widetilde{\pi}_2 \;\; \text{for } \varepsilon_2 \text{ sufficiently small.}
\end{aligned} \tag{26}
$$

The inequality in (26) follows from (25) because $K_2 < t$ and $G_2 - G_1 + c_1 - c_2 - K_2 > 0$, by assumption. (26) implies that Firm 2 could increase its profit by reducing p_2 marginally below $c_1 + G_2 - G_1 - K_2$. Consequently, an equilibrium does not exist in which $p_1 = p_2 + G_1 - G_2 + K_2$, $p_1 = c_1$ and $G_2 - G_1 + c_1 - c_2 - K_2 > 0$. ■

Lemma A15. Suppose Assumption 1 holds. Then an equilibrium does not exist in which $p_2 \geq p_1 + G_2 - G_1 + K_1$ and $p_2 \neq c_2$.

Proof. $p_2 < c_2$ is a dominated strategy for Firm 2. Therefore, by assumption, Firm 2 never sets price $p_2 < c_2$.

Now consider a putative equilibrium in which $p_2 \geq p_1 + G_2 - G_1 + K_1$ and $p_2 > c_2$. Lemmas A9 and A10 imply that all consumers purchase Firm 1's product, so Firm 2 secures 0 profit in this putative equilibrium.

First consider a putative equilibrium in which $p_2 \geq p_1 + G_2 - G_1 + K_1 > c_2$. Suppose Firm 2 reduces its price to $p_2^{'}$ so that $p_2^{'} \in (p_1 + G_2 - G_1 - K_2, p_1 + G_2 - G_1 + K_1)$ and $p_2^{'} > c_2$. Lemmas A4 and A11 imply Firm 2's profit in this case is:

$$
\widehat{\pi}_2 \;=\; \left[p_2^{'} - c_2\right] [\,1 - x_0\,] \;>\; 0 \tag{27}
$$

where $1 - x_0 > 0$ is defined in (5). Therefore, Firm 2 strictly increases its profit by reducing p_2. Consequently, the putative equilibrium is not an equilibrium, so an equilibrium in which $p_2 \geq p_1 + G_2 - G_1 + K_1 > c_2$ does not exist.

Next consider a putative equilibrium in which $p_2 > c_2 \geq p_1 + G_2 - G_1 + K_1$. Lemmas A9 and A10 imply that Firm 1 can increase profit by increasing p_1 to ensure $p_2 = p_1 + G_2 - G_1 + K_1$. Therefore, the putative equilibrium is not an equilibrium, so no equilibrium exists in which $p_2 > c_2 \geq p_1 + G_2 - G_1 + K_1$. ■

Lemma A16. Suppose Assumption 1 holds. Then an equilibrium does not exist in which $p_2 > p_1 + G_2 - G_1 + K_1$ and $p_2 = c_2$.

Proof. Consider a putative equilibrium in which $p_2 > p_1 + G_2 - G_1 + K_1$ and $p_2 = c_2$. Lemmas A9 and A10 imply that all consumers buy Firm 1's product in this putative equilibrium. If Firm 1 increases p_1 to ensure that $p_2 = p_1 + G_2 - G_1 + K_1$ when $p_2 = c_2$, Lemmas A9 and A10 imply that Firm 1 can continue to attract all consumers while increasing its profit. Therefore, no equilibrium exists in which $p_2 > p_1 + G_2 - G_1 + K_1$ and $p_2 = c_2$. ■

Lemma A17. Suppose Assumption 1 holds. Then in any duopoly equilibrium, there is a consumer located in $(0,1)$ who is indifferent between buying Firm 1's product and buying Firm 2's product.

Proof. Lemmas A7 and A8 imply that if $p_1 > p_2 + G_1 - G_2 + K_2$, then all consumers buy Firm 2's product. Lemmas A9 and A10 imply that if $p_1 \leq p_2 + G_1 - G_2 - K_1$, then all consumers buy Firm 1's product. Therefore, it must be the case that $p_1 - p_2 \in (G_1 - G_2 - K_1, G_1 - G_2 + K_2]$ in any duopoly equilibrium.

The proof of Lemma A5 (see (6), in particular) implies that if $p_1 = p_2 + G_1 - G_2 + K_2$, then all consumers located in $[0, \frac{1}{2} - \frac{K_2}{2t})$ are indifferent between buying Firm 1's product and buying Firm 2's product.

Lemma A11 implies that if $p_1 - p_2 \in (G_1 - G_2 - K_1, G_1 - G_2 + K_2)$, then there is a consumer located in $[\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}]$ who is indifferent between buying Firm 1's product and buying Firm 2's product. ■

Formal Conclusions in the Text

Lemma 1. *Suppose $A < t$ and default product characteristics cannot be changed. Then the unique equilibrium is the duopoly equilibrium in which all consumers located in $[0, x_0]$ buy Firm 1's product, whereas all consumers located in $(x_0, 1]$ buy Firm 2's product, where $x_0 \equiv \frac{1}{2} + \frac{A}{2t} \in \left(\frac{1}{2}, 1\right)$. Furthermore, $p_1 = c_1 + t + A$, $p_2 = c_2 + t - A$, $\pi_1 = \frac{1}{2t}[t + A]^2$, and $\pi_2 = \frac{1}{2t}[t - A]^2$.*

Proof. The proof follows from the following lemmas (Lemmas A1.1 – A1.5).

Lemma A1.1. When horizontal product characteristics cannot be changed: (i) all consumers buy the product from Firm 1 if $p_2 - p_1 \geq G_2 - G_1 + t$; and (ii) all consumers buy the product from Firm 2 if $p_2 - p_1 < G_2 - G_1 - t$.

Proof. All consumers buy the product from Firm 1 if, for all $x \in [0,1]$:

$$G_1 - t\,x - p_1 \;\geq\; G_2 - t\,[1 - x] - p_2 \;\;\Leftrightarrow\;\; t\,[1 - 2\,x] \;\geq\; G_2 - G_1 - p_2 + p_1$$

$$\Leftrightarrow \;\; 1 - 2\,x \;\geq\; \frac{1}{t}\,[G_2 - G_1 - p_2 + p_1] \;\;\Leftrightarrow\;\; x \;\leq\; \frac{1}{2} + \frac{1}{2\,t}\,[G_1 - G_2 - p_1 + p_2]. \quad (28)$$

(28) holds for all $x \in [0,1]$ if:

$$1 \;\leq\; \frac{1}{2} + \frac{1}{2t}[G_1 - G_2 - p_1 + p_2] \;\;\Leftrightarrow\;\; \frac{t}{2t} \;\leq\; \frac{1}{2t}[G_1 - G_2 - p_1 + p_2]$$

$$\Leftrightarrow\;\; t \;\leq\; G_1 - G_2 - p_1 + p_2 \;\;\Leftrightarrow\;\; p_2 - p_1 \;\geq\; G_2 - G_1 + t\,.$$

All consumers buy the product from Firm 2 if, for all $x \in [0,1]$:

$$G_2 - t[1-x] - p_2 \;>\; G_1 - t\,x - p_1 \;\;\Leftrightarrow\;\; t[1-2x] \;<\; G_2 - G_1 - p_2 + p_1$$

$$\Leftrightarrow\;\; 1 - 2x \;<\; \frac{1}{t}[G_2 - G_1 - p_2 + p_1] \;\;\Leftrightarrow\;\; x \;>\; \frac{1}{2} + \frac{1}{2t}[G_1 - G_2 - p_1 + p_2]\,. \tag{29}$$

(29) holds for all $x \in [0,1]$ if:

$$0 \;>\; \frac{1}{2} + \frac{1}{2t}[G_1 - G_2 - p_1 + p_2] \;\;\Leftrightarrow\;\; \frac{1}{2t}[t + G_1 - G_2 - p_1 + p_2] < 0$$

$$\Leftrightarrow\;\; p_2 - p_1 \;<\; G_2 - G_1 - t\,. \;\; \square$$

Lemma A1.2. When horizontal product characteristics cannot be changed and $t > 3A$, no equilibrium exists in which one firm serves all consumers.

Proof. First suppose Firm 1 serves all consumers. Then Lemma A1.1 implies that for all p_2 that generate nonnegative profit for Firm 2:

$$p_1 \;\leq\; p_2 + G_1 - G_2 - t\,. \tag{30}$$

(30) holds for all such p_2 if:

$$p_1 \;\leq\; c_2 + G_1 - G_2 - t\,. \tag{31}$$

Firm 1's profit when it serves all consumers at a price that satisfies (31) is:

$$\pi_1 \;=\; p_1 - c_1 \;\leq\; c_2 + G_1 - G_2 - t - c_1$$

$$=\; G_1 - c_1 - (G_2 - c_2) - t \;<\; 0 \;\text{ when }\; t \;>\; 3A\,. \tag{32}$$

(32) implies that no equilibrium exists in which Firm 1 serves all consumers.

Now suppose Firm 2 serves all consumers. Then Lemma A1.1 implies that for all p_1 that generate nonnegative profit for Firm 1:

$$p_2 \;<\; p_1 + G_2 - G_1 - t\,. \tag{33}$$

(33) holds for all such p_1 if:

$$p_2 \;<\; c_1 + G_2 - G_1 - t\,. \tag{34}$$

Firm 2's profit when it serves all consumers at a price that satisfies (34) is:

$$\pi_2 \ = \ p_2 - c_2 \ < \ c_1 + G_2 - G_1 - t - c_2$$

$$= \ G_2 - c_2 - (\, G_1 - c_1 \,) - t = \ - A - t \ < \ 0. \tag{35}$$

(35) implies that no equilibrium exists in which Firm 2 serves all consumers. □

Lemma A1.3. When horizontal product characteristics cannot be changed and $p_2 - p_1 \in [\, G_2 - G_1 - t,\ G_2 - G_1 + t \,]$: (i) a consumer located at $x_0 \ \equiv \ \frac{1}{2t}\,[\, t + G_1 - G_2 + p_2 - p_1 \,] \in [\, 0, 1 \,]$ is indifferent between purchasing the product from Firm 1 and from Firm 2; (ii) if $x_0 \ > \ 0$, all consumers located in $[\, 0, x_0 \,]$ buy the product from Firm 1; and (iii) if $x_0 \ < \ 1$, all consumers located in $(\, x_0, 1 \,]$ buy the product from Firm 2.

Proof. A consumer located at x is indifferent between purchasing the product from Firm 1 and from Firm 2 if:

$$G_1 - t\,x - p_1 \ = \ G_2 - t\,[\,1 - x\,] - p_2 \quad \Leftrightarrow \quad t\,[\,1 - 2\,x\,] \ = \ G_2 - G_1 - p_2 + p_1$$

$$\Leftrightarrow \quad 1 - 2\,x \ = \ \frac{1}{t}\,[\, G_2 - G_1 - p_2 + p_1 \,] \quad \Leftrightarrow \quad x \ = \ \frac{1}{2} + \frac{1}{t}\,[\, G_1 - G_2 - p_1 + p_2 \,]$$

$$\Leftrightarrow \quad x \ = \ \frac{1}{2\,t}\,[\, t + G_1 - G_2 + p_2 - p_1 \,] \ \equiv \ x_0$$

$$\in \ [\,0, 1\,] \quad \Leftrightarrow \quad t + G_1 - G_2 + p_2 - p_1 \ \in \ [\,0, 2\,t\,]$$

$$\Leftrightarrow \quad p_2 - p_1 \ \in \ [\, G_2 - G_1 - t,\ G_2 - G_1 + t \,] \,.$$

If $x_0 \ > \ 0$, then a consumer located at $x \in [\,0, x_0\,]$ buys the product from Firm 1 because:

$$G_1 - t\,x - p_1 \ \geq \ G_2 - t\,[\,1 - x\,] - p_2 \quad \Leftrightarrow \quad t\,[\,1 - 2\,x\,] \ \geq \ G_2 - G_1 + p_1 - p_2$$

$$\Leftrightarrow \quad 2\,x \ \leq \ 1 + \frac{1}{t}\,[\, G_1 - G_2 + p_2 - p_1 \,]$$

$$\Leftrightarrow \quad x \ \leq \ \frac{1}{2\,t}\,[\, t + G_1 - G_2 + p_2 - p_1 \,] \ = \ x_0 \,.$$

If $x_0 \ < \ 1$, then a consumer located at $x \in (\, x_0, 1 \,]$ buys the product from Firm 2 because:

$$G_2 - t\,[\,1 - x\,] - p_2 \ > \ G_1 - t\,x - p_1 \quad \Leftrightarrow \quad t\,[\,1 - 2\,x_0\,] \ < \ G_2 - G_1 + p_1 - p_2$$

$$\Leftrightarrow \quad 2\,x \ > \ 1 + \frac{1}{t}\,[\, G_1 - G_2 + p_2 - p_1 \,]$$

$$\Leftrightarrow \quad x \ > \ \frac{1}{2\,t}\,[\, t + G_1 - G_2 + p_2 - p_1 \,] \ = \ x_0 \,. \ \square$$

Lemma A1.4. Suppose horizontal product characteristics cannot be changed and $t > 3A$. Then in equilibrium, there exists a $x_0 \in [0,1]$ such that: (i) a consumer located at x_0 is indifferent between buying the product from Firm 1 and from Firm 2; (ii) all consumers located in $[0, x_0]$ buy the product from Firm 1; and (iii) all consumers located in $(x_0, 1]$ buy the product from Firm 2. Furthermore: $p_1 = c_1 + t + A$; $p_2 = c_2 + t - A$; $\pi_1 = \frac{1}{2t}[t + A]^2$; and $\pi_2 = \frac{1}{2t}[t - A]^2$.

Proof. Lemma A1.2 implies that Firm 1 and Firm 2 both serve some consumers in equilibrium. Therefore, Lemma A1.1 implies that $p_2 - p_1 \in [G_2 - G_1 - t, G_2 - G_1 + t]$. Consequently, Lemma A1.3 implies that a consumer located at

$$x_0 \equiv \frac{1}{2t}[t + G_1 - G_2 + p_2 - p_1] \in [0,1] \tag{36}$$

is indifferent between purchasing the product from Firm 1 and from Firm 2. Furthermore, all consumers located in $[0, x_0]$ buy the product from Firm 1, and all consumers located in $(x_0, 1]$ buy the product from Firm 2. Therefore, (36) implies that Firm 1's profit is:

$$\pi_1 = [p_1 - c_1]\frac{1}{2t}[t + G_1 - G_2 + p_2 - p_1]. \tag{37}$$

The unique value of p_1 that maximizes π_1 in (37) is given by:

$$\frac{\partial \pi_1}{\partial p_1} = 0 \Leftrightarrow -[p_1 - c_1] + t + G_1 - G_2 + p_2 - p_1 = 0$$

$$\Leftrightarrow p_1 = \frac{1}{2}[t + c_1 + G_1 - G_2 + p_2]. \tag{38}$$

(36) and Lemma A1.3 imply that Firm 2's profit is:

$$\pi_2 = [p_2 - c_2]\frac{1}{2t}[t + G_2 - G_1 + p_1 - p_2]. \tag{39}$$

The unique value of p_2 that maximizes π_2 in (39) is given by:

$$\frac{\partial \pi_2}{\partial p_2} = 0 \Leftrightarrow -[p_2 - c_2] + t + G_2 - G_1 + p_1 - p_2 = 0$$

$$\Leftrightarrow p_2 = \frac{1}{2}[t + c_2 + G_2 - G_1 + p_1]. \tag{40}$$

(38) and (40) imply:

$$p_1 = \frac{1}{2}[t + c_1 + G_1 - G_2] + \frac{1}{4}[t + c_2 + G_2 - G_1 + p_1]$$

$$\Rightarrow \frac{3}{4}p_1 = \frac{1}{4}[2t + 2c_1 + 2G_1 - 2G_2 + t + c_2 + G_2 - G_1]$$

$$\Rightarrow \quad p_1 \;=\; \frac{1}{3}\,[\,3\,t + 2\,c_1 + c_2 + G_1 - G_2\,] \;=\; \frac{1}{3}\,[\,3\,t + G_1 - c_1 - (\,G_2 - c_2\,) + 3\,c_1\,]$$

$$= \frac{1}{3}\,[\,3\,t + 3\,A + 3\,c_1\,] \;=\; c_1 + t + A\,. \tag{41}$$

(40) and (41) imply:

$$p_2 \;=\; \frac{1}{2}\,[\,t + c_2 + G_2 - G_1\,] + \frac{1}{6}\,[\,3\,t + 2\,c_1 + c_2 + G_1 - G_2\,]$$

$$= \frac{1}{6}\,[\,3\,t + 3\,c_2 + 3\,G_2 - 3\,G_1 + 3\,t + 2\,c_1 + c_2 + G_1 - G_2\,]$$

$$= \frac{1}{6}\,[\,6\,t + 4\,c_2 + 2\,c_1 + 2\,G_2 - 2\,G_1\,] = \frac{1}{3}\,[\,3\,t + G_2 - c_2 - (\,G_1 - c_1\,) + 3\,c_2\,]$$

$$= \frac{1}{3}\,[\,3\,t - 3\,A + 3\,c_2\,] \;=\; c_2 + t - A\,. \tag{42}$$

(41) implies that Firm 1's profit margin is positive because $p_1 - c_1 = t + A > 0$. (42) implies that Firm 2's profit margin is positive because $p_2 - c_2 = t - A > 0$.

(41) and (42) imply:

$$p_2 - p_1 \;=\; \frac{1}{3}\,[\,c_2 - c_1 + 2\,G_2 - 2\,G_1\,]\,. \tag{43}$$

(41), (43), and Lemma A1.3 imply:

$$\pi_1 \;=\; \frac{1}{3}\,[\,3\,t + 2\,c_1 + c_2 + G_1 - G_2 - 3\,c_1\,]$$

$$\cdot \frac{1}{2\,t}\left[\,t + G_1 - G_2 + \frac{1}{3}\,(\,c_2 - c_1 + 2\,G_2 - 2\,G_1\,)\right]$$

$$= \frac{1}{18\,t}\,[\,3\,t + c_2 - c_1 + G_1 - G_2\,]\,[\,3\,t + 3\,G_1 - 3\,G_2 + c_2 - c_1 + 2\,G_2 - 2\,G_1\,]$$

$$= \frac{1}{18\,t}\,[\,3\,t + c_2 - c_1 + G_1 - G_2\,]^2$$

$$= \frac{1}{18\,t}\,[\,3\,t + G_1 - c_1 - (\,G_2 - c_2\,)\,]^2 \;=\; \frac{1}{18\,t}\,[\,3\,t + 3\,A\,]^2 \;=\; \frac{1}{2\,t}\,[\,t + A\,]^2\,.$$

(42), (43), and Lemma A1.3 imply:

$$\pi_2 \;=\; \frac{1}{3}\,[\,3\,t + 2\,c_2 + c_1 + G_2 - G_1 - 3\,c_2\,]$$

$$\cdot \frac{1}{2\,t}\left[\,t + G_2 - G_1 + \frac{1}{3}\,(\,c_1 - c_2 + 2\,G_1 - 2\,G_2\,)\right]$$

$$= \frac{1}{18\,t}\,[\,3\,t + c_1 - c_2 + G_2 - G_1\,]\,[\,3\,t + 3\,G_2 - 3\,G_1 + c_1 - c_2 + 2\,G_1 - 2\,G_2\,]$$

$$= \frac{1}{18\,t}\,[\,3\,t + c_1 - c_2 + G_2 - G_1\,]^2 \;=\; \frac{1}{18\,t}\,[\,3\,t + G_2 - c_2 - (\,G_1 - c_1\,)\,]^2$$

$$= \frac{1}{18\,t}\,[\,3\,t - 3\,A\,]^2 \;=\; \frac{1}{2\,t}\,[\,t - A\,]^2 . \;\square$$

Lemma A1.5. Suppose horizontal product characteristics cannot be changed and $t \in (\,A,\, 3\,A\,]$. Then in the unique equilibrium, both firms attract customers, Firm 1's profit is $\pi_1 = \frac{1}{2\,t}\,[\,t + A\,]^2 > 0$, and Firm 2's profit is $\pi_2 = \frac{1}{2\,t}\,[\,t - A\,]^2 > 0$.

Proof. First suppose that an equilibrium exists in which all consumers buy the product from Firm 2. Then because the consumer located at 0 buys the product from Firm 2:

$$G_2 - p_2 - t \;>\; G_1 - p_1 \;\;\Leftrightarrow\;\; p_2 \;<\; p_1 + G_2 - G_1 - t\,. \tag{44}$$

(44) must hold for all p_1 for which Firm 1's profit margin is positive. Therefore:

$$p_2 \;<\; c_1 + G_2 - G_1 - t \;\equiv\; \widehat{p}_2\,. \tag{45}$$

Firm 2's profit when it sets a price marginally below $\widehat{p}_2$ is nearly:

$$\pi_2 \;=\; \widehat{p}_2 - c_2 \;=\; c_1 + G_2 - G_1 - t - c_2$$

$$= G_2 - c_2 - (\,G_1 - c_1\,) - t \;=\; -\,3\,A - t \;<\; 0\,. \tag{46}$$

(46) implies that an equilibrium in which all consumers buy the product from Firm 2 does not exist under the specified conditions.

Now suppose that an equilibrium exists in which all consumers buy the product from Firm 1. Then because the consumer located at 1 buys the product from Firm 1:

$$G_1 - p_1 - t \;\geq\; G_2 - p_2 \;\;\Leftrightarrow\;\; p_1 \;\leq\; p_2 + G_1 - G_2 - t\,. \tag{47}$$

(47) must hold for all p_2 for which Firm 2's profit margin is positive. Therefore:

$$p_1 \;\leq\; c_2 + G_1 - G_2 - t \;\equiv\; \widehat{p}_1\,. \tag{48}$$

Firm 1's profit when it sets a price $\widehat{p}_1$ is:

$$\pi_1 \;=\; \widehat{p}_1 - c_1 \;=\; c_2 + G_1 - G_2 - t - c_1$$

$$= G_1 - c_1 - (\,G_2 - c_2\,) - t \;=\; 3\,A - t \;>\; 0\,. \tag{49}$$

If a consumer located at $x \in [\,0, 1\,]$ is indifferent between purchasing the product from Firm 1 and from Firm 2, then:

$$G_1 - t\,x - p_1 \;=\; G_2 - t\,[\,1 - x\,] - p_2 \quad\Leftrightarrow\quad t\,[\,1 - 2\,x\,] \;=\; G_2 - G_1 - p_2 + p_1$$

$$\Leftrightarrow\quad 1 - 2\,x \;=\; \frac{1}{t}\,[\,G_2 - G_1 - p_2 + p_1\,] \quad\Leftrightarrow\quad x \;=\; \frac{1}{2} + \frac{1}{t}\,[\,G_1 - G_2 - p_1 + p_2\,]$$

$$\Leftrightarrow\quad x \;=\; \frac{1}{2\,t}\,[\,t + G_1 - G_2 + p_2 - p_1\,]\,. \tag{50}$$

(50) implies that when $p_2 = c_2$ and $p_1 \in (\,\widehat{p}_1,\, \widehat{p}_1 + 2\,t\,)$, consumers located in $[\,0,\, \widehat{x}_0\,]$ purchase the product from Firm 1 and consumers located in $(\,\widehat{x}_0, 1\,]$ purchase the product from Firm 2, where:

$$\widehat{x}_0 \;=\; \frac{1}{2\,t}\;[\,t + G_1 - G_2 + c_2 - p_1\,] \;\in\; (\,0, 1\,)\,.$$

Firm 1's corresponding profit is:

$$\pi_1(p_1) \;=\; [\,p_1 - c_1\,]\;\frac{1}{2\,t}\;[\,t + G_1 - G_2 + c_2 - p_1\,]\,. \tag{51}$$

Differentiating (51) provides:

$$\pi_1'(p_1) \;=\; \frac{1}{2\,t}\;[\,t + G_1 - G_2 + c_2 - p_1 - (\,p_1 - c_1\,)\,]$$

$$=\; \frac{1}{2\,t}\;[\,t + G_1 - G_2 + c_2 + c_1 - 2\,p_1\,] \quad\Rightarrow\quad \pi_1''(p_1) \;=\; -\,\frac{1}{t} \;<\; 0\,. \tag{52}$$

(48) and (52) imply:

$$\pi_1'(p_1)\,\big|_{p_1 = \widehat{p}_1} \;=\; \frac{1}{2\,t}\;[\,t + G_1 - G_2 + c_2 + c_1 - 2\,(\,c_2 + G_1 - G_2 - t\,)\,]$$

$$=\; \frac{1}{2\,t}\;[\,3\,t - G_1 + G_2 - c_2 + c_1\,] \;=\; \frac{1}{2\,t}\;[\,3\,t + G_2 - c_2 - (\,G_1 - c_1\,)\,]$$

$$=\; \frac{1}{2\,t}\;[\,3\,t - 3\,A\,] \;=\; \frac{3}{2\,t}\;[\;t - A\,] \;>\; 0\,. \tag{53}$$

(52) and (53) imply that when $p_2 = c_2$, Firm 1 will increase p_1 above $\widehat{p}_1$, thereby ensuring that both firms attract customers. Consequently, the profit calculations in the proof of Lemma A1.4 imply that at the unique equilibrium, Firm 1's profit is $\pi_1 = \frac{1}{2\,t}\,[\,t + A\,]^2 > 0$ and Firm 2's profit is $\pi_2 = \frac{1}{2\,t}\,[\,t - A\,]^2 > 0$. $\square$

Lemma 2. *Suppose $K_1 = K_2 = 0$. Then the unique equilibrium is the monopoly equilibrium in which $p_1 = c_1 + 3\,A$, $p_2 = c_2$, $\pi_1 = 3\,A$, $\pi_2 = 0$, and (only) consumers located in $(\,\frac{1}{2}, 1\,]$ change the default horizontal characteristic of the product they purchase.*

<u>Proof</u>. The proof follows directly from the following lemmas (Lemmas A2.1 – A2.3).

Lemma A2.1. Suppose $K_1 = K_2 = 0$. Then: (i) a consumer located in $[\,0, \frac{1}{2}\,)$ will change the default horizontal characteristic of the product she purchases if and only if she purchases the product from Firm 2; (ii) a consumer located in $(\frac{1}{2}, 1]$ will change the default horizontal characteristic of the product she purchases if and only if she purchases the product from Firm 1; and (iii) a consumer located at $\frac{1}{2}$ will not change the default horizontal characteristic of the product she purchases.

Proof. The conclusions follow directly from the proofs of Lemmas A1 – A3. □

Lemma A2.2. Suppose $K_1 = K_2 = 0$. Then: (i) all consumers buy the product from Firm 1 if $p_2 > p_1 + G_2 - G_1$; (ii) all consumers buy the product from Firm 2 if $p_2 < p_1 + G_2 - G_1$; and (iii) all consumers are indifferent between buying the product from Firm 1 and from Firm 2 if $p_2 = p_1 + G_2 - G_1$.

Proof. Lemma A2.1 implies that a consumer located at $x_1 \in [0, \frac{1}{2})$ buys the product from Firm 1 if:

$$G_1 - t\,x_1 - p_1 \;>\; G_2 - t\,x_1 - p_2 \quad\Leftrightarrow\quad p_2 \;>\; p_1 + G_2 - G_1\,.$$

Lemma A2.1 also implies that a consumer located at $x_2 \in (\frac{1}{2}, 1]$ buys the product from Firm 1 if:

$$G_1 - t\,[\,1 - x_2\,] - p_1 \;>\; G_2 - t\,[\,1 - x_2\,] - p_2 \quad\Leftrightarrow\quad p_2 \;>\; p_1 + G_2 - G_1\,.$$

Lemma A2.1 further implies that a consumer located at $\frac{1}{2}$ buys the product from Firm 1 if:

$$G_1 - \frac{1}{2}\,t - p_1 \;>\; G_2 - \frac{1}{2}\,t - p_2 \quad\Leftrightarrow\quad p_2 \;>\; p_1 + G_2 - G_1\,.$$

The proofs of the remaining conclusions are analogous, and so are omitted. □

Lemma A2.3. Suppose $K_1 = K_2 = 0$. Then in equilibrium: (i) all consumers purchase the product from Firm 1 at price $p_1 = c_2 + G_1 - G_2$; (ii) Firm 2's profit is 0; and (iii) Firm 1's profit is $3\,A = G_1 - c_1 - (\,G_2 - c_2\,)$.

Proof. Lemmas A2.1 and A2.2, and the assumption that all consumers who are indifferent between buying the product from Firm 1 and Firm 2 buy the product from Firm 1, imply that Firm 1's profit is:

$$\pi_1 \;=\; \begin{cases} 0 & \text{if } p_1 \;>\; p_2 + G_1 - G_2 \\ p_2 + G_1 - G_2 - c_1 & \text{if } p_1 \;\leq\; p_2 + G_1 - G_2\,. \end{cases} \tag{54}$$

Firm 2 must secure nonnegative profit in equilibrium. Therefore, in any equilibrium in which all consumers either strictly prefer to purchase the product from Firm 2 or are indifferent between purchasing the product from Firm 1 and Firm 2, it must be the case that $p_2 \geq c_2$. Consequently, in any such equilibrium:

$$p_2 + G_1 - G_2 - c_1 \;\geq\; G_1 - c_1 - (G_2 - c_2) \;>\; 0\,. \tag{55}$$

(54) and (55) imply that Firm 1 secures strictly higher profit by setting $p_1 = p_2 + G_1 - G_2$ than by setting $p_1 \neq p_2 + G_1 - G_2$. Therefore, in equilibrium, Firm 1 will set $p_1 = c_2 + G_1 - G_2$ to ensure that Firm 2 cannot profitably attract any customers. Consequently, Firm 2's profit is 0, and Firm 1's profit is:

$$c_2 + G_1 - G_2 - c_1 \;=\; G_1 - c_1 - (G_2 - c_2) \;=\; 3\,A. \quad \square \; \blacksquare$$

Definitions. $K_{1a} \equiv \frac{1}{2}[3A - t]$; $K_{1b} \equiv 3A - \frac{1}{8t}[t + 3A]^2$; $\underline{K}_1 = \max\{K_{1a}, K_{1b}\}$;

$$\overline{K}_1 \;\equiv\; \frac{1}{2t}[t^2 + 2At - A^2]; \qquad \overline{K}_2 \;\equiv\; \frac{1}{2t}[t^2 - 2At - A^2].$$

Lemma 3. *Suppose $A < t$ and $K_2 \geq \overline{K}_2$. Then the duopoly equilibrium identified in Lemma 1 exists if and only if $K_1 \geq \overline{K}_1$. At the unique such equilibrium, no customer changes the horizontal characteristic of the product she purchases.*

Proof. (5) and Lemmas A4 and A11 imply that in the equilibrium identified in Lemma 1, the profits of Firm 1 and Firm 2 are, respectively:

$$\pi_1 \;=\; [p_1 - c_1]\,\frac{1}{2t}\,[t + G_1 - G_2 + p_2 - p_1]; \tag{56}$$

$$\pi_2 \;=\; [p_2 - c_2]\,\frac{1}{2t}\,[t + G_2 - G_1 + p_1 - p_2]. \tag{57}$$

The unique value of p_1 that maximizes π_1 in (56) is given by:

$$\frac{\partial \pi_1}{\partial p_1} = 0 \;\Leftrightarrow\; -[p_1 - c_1] + t + G_1 - G_2 + p_2 - p_1 = 0$$

$$\Leftrightarrow\; p_1 = \frac{1}{2}[t + c_1 + G_1 - G_2 + p_2]. \tag{58}$$

The unique value of p_2 that maximizes π_2 in (57) is given by:

$$\frac{\partial \pi_2}{\partial p_2} = 0 \;\Leftrightarrow\; -[p_2 - c_2] + t + G_2 - G_1 + p_1 - p_2 = 0$$

$$\Leftrightarrow\; p_2 = \frac{1}{2}[t + c_2 + G_2 - G_1 + p_1]. \tag{59}$$

(58) and (59) imply that in any such equilibrium:

$$p_1 = \frac{1}{2}[\,t + c_1 + G_1 - G_2\,] + \frac{1}{4}[\,t + c_2 + G_2 - G_1 + p_1\,]$$

$$\Rightarrow \quad \frac{3}{4}\,p_1 = \frac{1}{4}[\,2\,t + 2\,c_1 + 2\,G_1 - 2\,G_2 + t + c_2 + G_2 - G_1\,]$$

$$\Rightarrow \quad p_1 = \frac{1}{3}\,[\,3\,t + 2\,c_1 + c_2 + G_1 - G_2\,] = \frac{1}{3}\,[\,3\,t + G_1 - c_1 - (\,G_2 - c_2\,) + 3\,c_1\,]$$

$$= \frac{1}{3}\,[\,3\,t + 3\,A + 3\,c_1\,] = c_1 + t + A\,. \tag{60}$$

(59) and (60) imply:

$$p_2 = \frac{1}{2}[\,t + c_2 + G_2 - G_1\,] + \frac{1}{6}[\,3\,t + 2\,c_1 + c_2 + G_1 - G_2\,]$$

$$= \frac{1}{6}[\,3\,t + 3\,c_2 + 3\,G_2 - 3\,G_1 + 3\,t + 2\,c_1 + c_2 + G_1 - G_2\,]$$

$$= \frac{1}{6}[\,6\,t + 4\,c_2 + 2\,c_1 + 2\,G_2 - 2\,G_1\,] = \frac{1}{3}[\,3\,t + 2\,c_2 + c_1 + G_2 - G_1\,]$$

$$= \frac{1}{3}\,[\,3\,t + G_2 - c_2 - (\,G_1 - c_1\,) + 3\,c_2\,] = \frac{1}{3}\,[\,3\,t - 3\,A + 3\,c_2\,] = c_2 + t - A\,. \tag{61}$$

(60) implies that Firm 1's profit margin is positive because $p_1 - c_1 = t + A > 0$. (61) implies that because $t > A$ by assumption, Firm 2's profit margin is positive because $p_2 - c_2 = t - A > 0$.

(60) and (61) imply:

$$p_2 - p_1 = \frac{1}{3}[\,c_2 - c_1 + 2\,G_2 - 2\,G_1\,]\,. \tag{62}$$

(4), (5), and (62) imply that the consumer who is indifferent between purchasing the product from Firm 1 and Firm 2 is located at:

$$x_0 = \frac{1}{2\,t}\left[\,t + G_1 - G_2 + \frac{1}{3}(\,c_2 - c_1 + 2\,G_2 - 2\,G_1\,)\right]$$

$$= \frac{1}{6\,t}[\,3\,t + 3\,G_1 - 3\,G_2 + c_2 - c_1 + 2\,G_2 - 2\,G_1\,] = \frac{1}{6\,t}[\,3\,t + G_1 - G_2 + c_2 - c_1\,]$$

$$= \frac{1}{2} + \frac{1}{6\,t}[\,G_1 - c_1 - (G_2 - c_2)\,] = \frac{1}{2} + \frac{A}{2\,t}\,. \tag{63}$$

If $K_1 \geq \overline{K}_1 \equiv \frac{1}{2\,t}[\,t^2 + 2\,A\,t - A^2\,]$, then (63) implies that $x_0 \in (\,\frac{1}{2} - \frac{K_2}{2\,t}, \frac{1}{2} + \frac{K_1}{2\,t}\,)$ because $K_1 > A$. This is the case because $\overline{K}_1 > A$ since:

$$\overline{K}_1 > A \quad \Leftrightarrow \quad \frac{1}{2\,t}\left[\,t^2 + 2\,A\,t - A^2\,\right] > A \quad \Leftrightarrow \quad t^2 + 2\,A\,t - A^2 > 2\,A\,t \quad \Leftrightarrow \quad t^2 > A^2\,.$$

The last inequality here holds because $t > A$, by assumption. Because $x_0 \in (\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t})$, no customer changes the default horizontal characteristic of the product she purchases (from Lemma A3).

(62) implies:
$$p_1 - p_2 \in (G_1 - G_2 - K_1, G_1 - G_2 + K_2). \tag{64}$$

(64) reflects two conclusions. First:

$$p_1 - p_2 > G_1 - G_2 - K_1 \Leftrightarrow \frac{1}{3}[c_1 - c_2 - 2G_2 + 2G_1] > G_1 - G_2 - K_1$$

$$\Leftrightarrow c_1 - c_2 - 2G_2 + 2G_1 > 3G_1 - 3G_2 - 3K_1 \Leftrightarrow c_1 - c_2 > G_1 - G_2 - 3K_1$$

$$\Leftrightarrow K_1 > \frac{1}{3}[G_1 - G_2 + c_2 - c_1] = A. \tag{65}$$

The inequality in (65) holds because $K_1 > A$.

Second:

$$p_1 - p_2 < G_1 - G_2 + K_2 \Leftrightarrow \frac{1}{3}[c_1 - c_2 - 2G_2 + 2G_1] < G_1 - G_2 + K_2$$

$$\Leftrightarrow c_1 - c_2 - 2G_2 + 2G_1 < 3G_1 - 3G_2 + 3K_2 \Leftrightarrow c_1 - c_2 < G_1 - G_2 + 3K_2$$

$$\Leftrightarrow K_2 > \frac{1}{3}[G_2 - G_1 + c_1 - c_2] = -A. \tag{66}$$

The inequality in (66) holds because $A > 0$ and $K_2 \geq 0$, by assumption.

To prove that profits are positive at the putative equilibrium, observe first that (60) and (63) imply:
$$\pi_1 = [p_1 - c_1]\, x_0 = [t + A]\left[\frac{t + A}{2t}\right] = \frac{1}{2t}[t + A]^2 > 0. \tag{67}$$

Further observe that (61) and (63) imply:
$$\pi_2 = [p_2 - c_2][1 - x_0] = [t - A]\left[\frac{t - A}{2t}\right] = \frac{1}{2t}[t - A]^2 > 0. \tag{68}$$

The foregoing analysis and Lemma A11 imply that the identified putative equilibrium is unique among equilibria in which (64) holds. It remains to verify that when $K_1 \geq \overline{K}_1$, neither firm can strictly increase its profit by unilaterally changing its price so that (64) does not hold.[3] We first show this is the case for Firm 1.

Lemmas A7 and A8 imply that if Firm 1 sets $p_1 > p_2 + G_1 - G_2 + K_2$, then no consumers purchase Firm 1's product. Therefore, Firm 1's profit (0) is less than the profit specified in (67). If Firm 1 sets $p_1 = p_2 + G_1 - G_2 + K_2$ when p_2 is as specified in (61), then:

[3]Recall the maintained assumption that a putative equilibrium is an equilibrium if neither firm can strictly increase its profit by deviating unilaterally from the putative equilibrium.

$$
\begin{aligned}
p_1 &= p_2 + G_1 - G_2 + K_2 \;=\; c_2 + t - A + G_1 - G_2 + K_2 \\
&= G_1 - c_1 - (G_2 - c_2) + c_1 + t - A + K_2 \\
&= 3A + c_1 + t - A + K_2 \;=\; c_1 + 2A + t + K_2 \\
&\Rightarrow\; p_1 - c_1 \;=\; 2A + t + K_2 \;>\; 0\,.
\end{aligned} \tag{69}
$$

When $p_1 = p_2 + G_1 - G_2 + K_2$, Lemmas A7 and A8 imply that all consumers located in $[\,0, \frac{1}{2} - \frac{K_2}{2t}\,]$ buy Firm 1's product whereas all consumers located in $(\,\frac{1}{2} - \frac{K_2}{2t}, 1\,]$ buy Firm 2's product. Therefore, (69) implies that Firm 1's profit is:

$$
\pi_{1D} \;=\; [\,p_1 - c_1\,]\left[\frac{1}{2} - \frac{K_2}{2t}\right] \;=\; [\,2A + t + K_2\,]\left[\frac{1}{2} - \frac{K_2}{2t}\right]. \tag{70}
$$

(67) and (70) imply that Firm 1 cannot increase its profit by setting $p_1 = p_2 + G_1 - G_2 - K_1$ when p_2 is as specified in (61) because:

$$
\begin{aligned}
\pi_1 \;>\; \pi_{1D} \;&\Leftrightarrow\; \frac{1}{2t}\,[\,t + A\,]^2 \;>\; [\,2A + t + K_2\,]\left[\frac{1}{2} - \frac{K_2}{2t}\right] \\
&\Leftrightarrow\; \frac{1}{2t}\left[t^2 + 2At + A^2\right] \;>\; A + \frac{1}{2}t + \frac{1}{2}K_2 - \frac{AK_2}{t} - \frac{K_2}{2} - \frac{(K_2)^2}{2t} \\
&\Leftrightarrow\; t^2 + 2At + A^2 \;>\; 2At + t^2 - 2AK_2 - (K_2)^2 \\
&\Leftrightarrow\; A^2 \;>\; -2AK_2 - (K_2)^2\,.
\end{aligned}
$$

The last inequality here always holds because $A > 0$ and $K_2 \geq 0$.

Lemmas A9 and A10 imply that if Firm 1 sets $p_1 \leq p_2 + G_1 - G_2 - K_1$, then all consumers purchase the Firm 1's product. (4) implies that the maximum profit Firm 1 can secure by setting such a price when p_2 is as specified in (61) is:

$$
\begin{aligned}
\pi'_{1D} &= p_2 + G_1 - G_2 - K_1 - c_1 \;=\; c_2 + t - A + G_1 - G_2 - K_1 - c_1 \\
&= G_1 - c_1 - (G_2 - c_2) - A + t - K_1 \\
&= 3A - A + t - K_1 \;=\; 2A + t - K_1\,.
\end{aligned} \tag{71}
$$

(67) and (71) imply that Firm 1 cannot increase its profit by setting $p_1 \leq p_2 + G_1 - G_2 - K_1$ when p_2 is as specified in (61) when the maintained conditions hold because:

$$
\pi_1 \;\geq\; \pi'_{1D} \;\Leftrightarrow\; \frac{1}{2t}[\,t + A\,]^2 \;\geq\; 2A + t - K_1 \;\Leftrightarrow\; [\,t + A\,]^2 \;\geq\; 4At + 2t^2 - 2K_1 t
$$

$$
\Leftrightarrow\; t^2 + 2At + A^2 \;\geq\; 4At + 2t^2 - 2K_1 t
$$

$$\Leftrightarrow \quad 2\,K_1\,t \;\geq\; t^2 + 2\,A\,t - A^2 \quad \Leftrightarrow \quad K_1 \;\geq\; \frac{1}{2\,t}\left[\,t^2 + 2\,A\,t - A^2\,\right] \;=\; \overline{K}_1\,. \tag{72}$$

Now we show that Firm 2 cannot increase its profit by unilaterally changing its price so that (64) does not hold when p_1 is as specified in (60).

Lemmas A9 and A10 imply that if Firm 2 sets $p_2 > p_1 + G_2 - G_1 + K_1$, then no consumers purchase Firm 2's product, so Firm 2's profit (0) is no greater than the profit specified in (68).

If Firm 2 sets $p_2 = p_1 + G_2 - G_1 + K_1$ when p_1 is as specified in (60), then:

$$\begin{aligned} p_2 \;&=\; p_1 + G_2 - G_1 + K_1 \;=\; c_1 + t + A + G_2 - G_1 + K_1 \\ &=\; G_2 - c_2 - G_1 + c_1 + t + A + K_1 + c_2 \\ &=\; -\,3\,A + t + A + K_1 + c_2 \;=\; -\,2\,A + t + K_1 + c_2 \;>\; c_2\,. \end{aligned}$$

The last inequality holds here because $K_1 > A$ and because $t > A$, by assumption. Because $p_2 = p_1 + G_2 - G_1 + K_1 > c_2$ when p_1 is as specified in (60), the proof of Lemma A15 implies that Firm 2 can increase its profit by setting p_2 to ensure $p_1 - p_2 \in (G_1 - G_2 - K_1, G_1 - G_2 + K_2)$. Therefore, Firm 2 cannot increase its profit by setting $p_2 = p_1 + G_2 - G_1 + K_1$ when p_1 is as specified in (60).

Lemmas A7 and A8 imply that if Firm 2 sets $p_2 < p_1 + G_2 - G_1 - K_2$, then all consumers purchase Firm 2's product. (4) implies that the maximum profit Firm 2 can secure by setting such a price when p_1 is as specified in (60) is nearly:

$$\begin{aligned} \pi_{2D} \;&=\; p_1 + G_2 - G_1 - K_2 - c_2 \;=\; c_1 + t + A + G_2 - G_1 - K_2 - c_2 \\ &=\; G_2 - c_2 - (\,G_1 - c_1\,) + A + t - K_2 \\ &=\; -\,3\,A + A + t - K_2 \;=\; -\,2\,A + t - K_2. \end{aligned} \tag{73}$$

(68) and (73) imply that Firm 2 cannot increase its profit by setting $p_2 < p_1 + G_2 - G_1 - K_2$ when p_1 is as specified in (60) and the maintained conditions hold because:

$$\begin{aligned} \pi_2 \;\geq\; \pi_{2D} \quad &\Leftrightarrow \quad \frac{1}{2\,t}\,[\,t - A\,]^2 \;\geq\; -\,2\,A + t - K_2 \quad \Leftrightarrow \quad [\,t - A\,]^2 \;\geq\; -\,4\,A\,t + 2\,t^2 - 2\,K_2\,t \\ &\Leftrightarrow \quad t^2 - 2\,A\,t + A^2 \;\geq\; -\,4\,A\,t + 2\,t^2 - 2\,K_2\,t \\ &\Leftrightarrow \quad 2\,K_2\,t \;\geq\; t^2 - 2\,A\,t - A^2 \quad \Leftrightarrow \quad K_2 \;\geq\; \frac{1}{2\,t}\left[\,t^2 - 2\,A\,t - A^2\,\right]. \end{aligned} \tag{74}$$

If Firm 2 sets $p_2 = p_1 + G_2 - G_1 - K_2$ when p_1 is as specified in (60), (6) and Lemma A5 imply that all consumers located in $[\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,]$ buy Firm 1's product, whereas all consumers located in $(\,\frac{1}{2} - \frac{K_2}{2\,t}, 1\,]$ buy Firm 2's product. Therefore, Firm 2's profit is:

$$\pi'_{2D} \;=\; [\,p_2 - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] \;<\; p_2 - c_2 \;=\; \pi_{2D} \;<\; \frac{1}{2\,t}\,[\,t - A\,]^2 \;=\; \pi_2\,. \tag{75}$$

The first inequality in (75) holds because $K_2 < t$ by assumption and because $p_2 - c_2$ must be strictly positive if Firm 2 is to secure positive profit in this case. The last inequality in (75) reflects (74). (68) and (75) imply that Firm 2 will not set $p_2 = p_1 + G_2 - G_1 - K_2$ when p_1 is as specified in (60).

It remains to show that the putative equilibrium identified above does not exist when $K_1 < \overline{K}_1 \equiv \frac{1}{2t}\,[\,t^2 + 2\,A\,t - A^2\,]$. (67) establishes that Firm 1's profit is $\frac{1}{2t}\,[\,t + A\,]^2$ at this putative equilibrium. (71) implies that if Firm 1 reduces its price to ensure that all consumers purchase its product, it can secure profit $2\,A + t - K_1$. (72) establishes that:

$$2\,A + t - K_1 \;>\; \frac{1}{2\,t}\,[\,t + A\,]^2 \;\text{ if }\; K_1 \;<\; \overline{K}_1\,.$$

Therefore, the putative equilibrium identified above is not an equilibrium when $K_1 < \overline{K}_1$. ■

Observation A1.

$$\underline{K}_1 \;=\; \begin{cases} K_{1a} & \text{if } \; t \;<\; A \\ K_{1b} & \text{if } \; t \;\geq\; A\,. \end{cases}$$

Proof. Observe that:

$$K_{1a} \;\lesseqgtr\; K_{1b} \;\Leftrightarrow\; \frac{1}{2}\,[\,3\,A - t\,] \;\lesseqgtr\; 3\,A - \frac{1}{8\,t}\,[\,t + 3\,A\,]^2$$

$$\Leftrightarrow\; 24\,A\,t - [\,t + 3\,A\,]^2 \;\gtreqless\; 4\,t\,[\,3\,A - t\,]$$

$$\Leftrightarrow\; 24\,A\,t - \left[\,t^2 + 6\,A\,t + 9\,A^2\,\right] \;\gtreqless\; 12\,A\,t - 4\,t^2$$

$$\Leftrightarrow\; 18\,A\,t - t^2 - 9\,A^2 \;\gtreqless\; 12\,A\,t - 4\,t^2 \;\;\Leftrightarrow\;\; 3\,t^2 + 6\,A\,t - 9\,A^2 \;\gtreqless\; 0$$

$$\Leftrightarrow\; t^2 + 2\,A\,t - 3\,A^2 \;\gtreqless\; 0 \;\;\Leftrightarrow\;\; t^2 + 3\,A\,t - A\,t - 3\,A^2 \;\gtreqless\; 0$$

$$\Leftrightarrow\; t\,[\,t + 3\,A\,] - A\,[\,t + 3\,A\,] \;\gtreqless\; 0 \;\;\Leftrightarrow\;\; [\,t - A\,]\,[\,t + 3\,A\,] \;\gtreqless\; 0\,. \tag{76}$$

The Observation follows from (76) because $\underline{K}_1 = \max\{\,K_{1a},\, K_{1b}\,\}$. ■

Lemma 4. *The monopoly equilibrium exists if and only if* $K_1 \leq \underline{K}_1$*. At the unique monopoly equilibrium,* $p_1 = c_1 + 3\,A - K_1$*,* $p_2 = c_2$*,* $\pi_1 = 3\,A - K_1 > 0$*, and* $\pi_2 = 0$*. Furthermore, all consumers located in* $(\,\frac{1}{2} + \frac{K_1}{2t},\, 1\,]$ *(and only these consumers) change the default horizontal characteristic of the product they purchase.*

Proof. Case (i). $t < A$ and $K_1 \leq \underline{K}_1$. Observation A1 implies that $\underline{K}_1 = K_{1a} \equiv \frac{1}{2}[3A - t]$ because $t < A$ in this case.

We first show that when $p_2 = c_2$ in this case, Firm 1 maximizes its profit by setting $p_1 = c_1 + 3A - K_1$, which ensures that all consumers buy Firm 1's product.

Lemmas A9 and A10 imply that if $p_2 = c_2$, then among all values of p_1 that ensure all consumers buy Firm 1's product (i.e., among all $p_1 \leq p_2 + G_1 - G_2 - K_1$), the unique value of p_1 that maximizes Firm 1's profit is:

$$p_1 = c_2 + G_1 - G_2 - K_1 = G_1 - c_1 - (G_2 - c_2) + c_1 - K_1 = c_1 + 3A - K_1 > 0. \quad (77)$$

The inequality in (77) holds because, by assumption:

$$3A - K_1 \geq 3A - \underline{K}_1 = 3A - \frac{1}{2}[3A - t] = \frac{3A}{2} + \frac{t}{2} > 0.$$

(77) implies that Firm 1's corresponding profit in the putative monopoly equilibrium is:

$$\pi_1 = p_1 - c_1 = 3A - K_1 > 0. \quad (78)$$

We now show that when $p_2 = c_2$, Firm 1 cannot increase its profit by setting $p_1 \in (p_2 + G_1 - G_2 - K_1, p_2 + G_1 - G_2 + K_2)$ or $p_1 \geq p_2 + G_1 - G_2 + K_2$.

From (56), when $p_2 = c_2$ and $p_1 = c_2 + G_1 - G_2 - K_1$:

$$\begin{aligned} \frac{\partial \pi_1}{\partial p_1} &= -[p_1 - c_1] + t + G_1 - G_2 + p_2 - p_1 \\ &= -[c_2 + G_1 - G_2 - K_1 - c_1] + t + G_1 - G_2 + c_2 - c_2 - G_1 + G_2 + K_1 \\ &= -[3A - K_1] + t + K_1 = t + 2K_1 - 3A \leq 0. \end{aligned} \quad (79)$$

The inequality in (79) holds because $K_1 \leq \frac{1}{2}[3A - t]$, by assumption. (79) implies that $\frac{\partial^2 \pi_1}{\partial p_1^2} = -2 < 0$ when $p_1 \in (p_2 + G_1 - G_2 - K_1, p_2 + G_1 - G_2 + K_2)$. Therefore, (79) implies that when $p_2 = c_2$, Firm 1 cannot increase its profit by increasing p_1 from $c_1 + 3A - K_1$ to some $p_1 \in (p_2 + G_1 - G_2 - K_1, p_2 + G_1 - G_2 + K_2)$.

Lemmas A7 and A8 imply that if Firm 1 sets $p_1 > p_2 + G_1 - G_2 + K_2$, it will not sell any of its product, so its profit will be 0. Therefore, among all $p_1 \geq p_2 + G_1 - G_2 + K_2$, the price that maximizes Firm 1's profit is $p_1 = p_2 + G_1 - G_2 + K_2$. When $p_2 = c_2$, this price is $p_1 = c_2 + G_1 - G_2 + K_2$. (6) and Lemma A5 imply that when $p_1 = p_2 + G_1 - G_2 + K_2$, all consumers located in $[0, \frac{1}{2} - \frac{K_2}{2t}]$ buy Firm 1's product, whereas all consumers located in $(\frac{1}{2} - \frac{K_2}{2t}, 1]$ buy Firm 2's product. Therefore, Firm 1's profit is:

$$\begin{aligned} \widetilde{\widetilde{\pi}}_1 &= [p_1 - c_1]\left[\frac{1}{2} - \frac{K_2}{2t}\right] = [G_1 - G_2 - c_1 + c_2 + K_2]\left[\frac{1}{2} - \frac{K_2}{2t}\right] \\ &= [3A + K_2]\left[\frac{1}{2} - \frac{K_2}{2t}\right]. \end{aligned} \quad (80)$$

(78) and (80) imply that Firm 1 cannot increase its profit by setting $p_1 \geq p_2 + G_1 - G_2 + K_2$

because:

$$\pi_1 \;\geq\; \tilde{\tilde{\pi}}_1 \quad \Leftrightarrow \quad 3\,A - K_1 \;\geq\; [\,3\,A + K_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right]. \tag{81}$$

The last inequality in (81) holds because $3\,A - K_1 \;\geq\; 3\,A - \underline{K}_1$, by assumption, and because:

$$\begin{aligned}
& 3\,A - \frac{1}{2}\,[\,3\,A - t\,] \;\geq\; [\,3\,A + K_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \\
\Leftrightarrow \quad & \frac{1}{2}\,[\,3\,A + t\,] \;\geq\; \frac{1}{2}[\,3\,A + K_2\,]\left[1 - \frac{K_2}{t}\right] \\
\Leftrightarrow \quad & 3\,A + t \;\geq\; [\,3\,A + K_2\,]\left[1 - \frac{K_2}{t}\right].
\end{aligned} \tag{82}$$

The inequality in (82) holds because $K_2 \in [\,0, t\,)$.

In summary, we have established that when $K_1 \leq \underline{K}_1$ and $p_2 = c_2$, Firm 1 maximizes its profit by setting $p_1 = c_2 + G_1 - G_2 - K_1$, thereby ensuring that all consumers buy Firm 1's product.

We now show that when Firm 1 sets $p_1 = c_2 + G_1 - G_2 - K_1$, Firm 2 cannot secure strictly more profit than it secures by setting $p_2 = c_2$. Lemmas A9 and A10 imply that when Firm 1 sets $p_1 = c_2 + G_1 - G_2 - K_1$, Firm 2 sells none of its product (so Firm 2 secures no profit) if it sets $p_2 = c_2$. Firm 2 continues to sell none of its product (so Firm 2 continues to secure no profit) if it sets $p_2 > c_2$. Firm 2 incurs negative profit if it sets $p_2 < c_2$. Therefore, Firm 2 cannot increase its profit by setting $p_2 \neq c_2$ when Firm 1 sets $p_1 = c_2 + G_1 - G_2 - K_1$.

Finally, Lemma A2 implies that all consumers located in the interval $(\,\frac{1}{2} + \frac{K_1}{2\,t},\, 1\,]$ (and only these consumers) change the default setting on the product they purchase from Firm 1.

Case (ii). $t \;\geq\; A$ and $K_1 \;\leq\; \underline{K}_1$. Observation A1 implies that $\underline{K}_1 \;=\; K_{1b} \;\equiv\; 3\,A - \frac{1}{8\,t}\,[\,t + 3\,A\,]^2$ because $t \geq A$ in this case.

The proof of Case (i) implies that the identified monopoly equilibrium exists if $t \;\geq\; A$ and $K_1 \;\leq\; K_{1a}$. The remainder of the present proof establishes the corresponding existence when $t \geq A$ and $K_1 \in (\,K_{1\,a},\, \underline{K}_1\,]$.

We first show that when $p_2 = c_2$ in this case, Firm 1 maximizes its profit by setting $p_1 = c_1 + 3\,A - K_1$, thereby ensuring that all consumers buy its product.

The proof that $p_1 = c_1 + 3\,A - K_1$ is the unique value of $p_1 \leq p_2 + G_1 - G_2 - K_1$ that maximizes Firm 1's profit (when $p_2 = c_2$) is analogous to the corresponding proof in Case (i). The inequality in (77) holds in the present case because, by assumption:

$$K_1 \;\leq\; \underline{K}_1 \quad \Leftrightarrow \quad 3\,A - K_1 \;\geq\; \frac{1}{8\,t}\,[\,t + 3\,A\,]^2\,. \tag{83}$$

(78) and (83) imply that Firm 1's corresponding profit is:

$$\pi_1 \; = \; p_1 - c_1 \; = \; 3\,A - K_1 \; \geq \; \frac{1}{8\,t}\,[\,t + 3\,A\,]^2 \;\; > \; 0\,. \tag{84}$$

We now show that when $p_2 = c_2$, Firm 1 cannot increase its profit by setting $p_1 \in (\,p_2 + G_1 - G_2 - K_1,\, p_2 + G_1 - G_2 + K_2\,)$ or $p_1 \geq p_2 + G_1 - G_2 + K_2$.

(58) implies that when $p_1 \in (\,p_2 + G_1 - G_2 - K_1,\, p_2 + G_1 - G_2 + K_2\,)$, the price that maximizes Firm 1's profit when $p_2 = c_2$ is:

$$p_1 \; = \; \frac{1}{2}\,[\,t + c_1 + G_1 - G_2 + p_2\,] \; = \; \frac{1}{2}\,[\,t + G_1 - G_2 + c_1 + c_2\,]\,. \tag{85}$$

(56) and (85) imply that Firm 1's corresponding profit is:

$$\begin{aligned}
\pi_1' \; &= \; [\,p_1 - c_1\,]\,\frac{1}{2\,t}\,[\,t + G_1 - G_2 + p_2 - p_1\,] \\
&= \; \frac{1}{2}\,[\,t + G_1 - G_2 + c_1 + c_2 - 2\,c_1\,]\,\frac{1}{4\,t}\,[\,2\,t + 2\,G_1 - 2\,G_2 - G_1 + G_2 - c_1 + c_2 - t\,] \\
&= \; \frac{1}{8\,t}\,[\,t + G_1 - G_2 + c_2 - c_1\,]^2 \; = \; \frac{1}{8\,t}\,[\,t + 3\,A\,]^2\,. \qquad (86)
\end{aligned}$$

(84) and (86) imply that Firm 1 cannot increase its profit by setting $p_1 \in (\,p_2 + G_1 - G_2 - K_1,\, p_2 + G_1 - G_2 + K_2\,)$ because:

$$\pi_1 \; \gtreqless \; \pi_1' \;\; \Leftrightarrow \;\; 3\,A - K_1 \; \gtreqless \; \frac{1}{8\,t}\,[\,t + 3\,A\,]^2 \;\; \Leftrightarrow \;\; K_1 \; \lesseqgtr \; 3\,A - \frac{1}{8\,t}\,[\,t + 3\,A\,]^2\,. \tag{87}$$

The last inequality in (87) holds because, by assumption, $K_1 \leq \underline{K}_1$ in the present case.

The analysis in Case (i) implies that when $p_2 = c_2$, $p_1 = p_2 + G_1 - G_2 + K_2$ is the unique $p_1 \geq p_2 + G_1 - G_2 + K_2$ that maximizes Firm 1's profit. Furthermore, Firm 1's profit when it sets this price (and when $p_2 = c_2$) is $\tilde{\tilde{\pi}}_1$, as specified in (80). (78) and (80) imply Firm 1 cannot increase its profit by setting $p_1 \geq p_2 + G_1 - G_2 + K_2$ because:

$$\pi_1 \; \geq \; \tilde{\tilde{\pi}}_1 \;\; \Leftrightarrow \;\; 3\,A - K_1 \; \geq \; [\,3\,A + K_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right]. \tag{88}$$

The last inequality in (88) holds because $3\,A - K_1 \geq \frac{1}{8\,t}\,[\,t + 3\,A\,]^2$ (since $K_1 \leq \underline{K}_1$, by assumption, in the present case) and because:

$$\begin{aligned}
& \frac{1}{8\,t}\,[\,t + 3\,A\,]^2 \; \geq \; [\,3\,A + K_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \\
\Leftrightarrow \;\; & \frac{1}{8\,t}\left[\,t^2 + 6\,A\,t + 9\,A^2\,\right] \; \geq \; \frac{1}{2\,t}\,[\,3\,A + K_2\,]\,[\,t - K_2\,] \\
\Leftrightarrow \;\; & t^2 + 6\,A\,t + 9\,A^2 \; \geq \; [\,12\,A + 4\,K_2\,]\,[\,t - K_2\,] \\
\Leftrightarrow \;\; & t^2 + 6\,A\,t + 9\,A^2 \; \geq \; 12\,A\,t - 12\,A\,K_2 + 4\,K_2\,t - 4\,K_2^2 \\
\Leftrightarrow \;\; & t^2 - 6\,A\,t - 4\,K_2\,t + 9\,A^2 + 12\,A\,K_2 + 4\,K_2^2 \; \geq \; 0
\end{aligned}$$

$$\Leftrightarrow \quad t^2 - 2\,[\,3\,A + 2\,K_2\,]\;t + [\,3\,A + 2\,K_2\,]^2 \;\geq\; 0 \quad \Leftrightarrow \quad [\,t - (\,3\,A + 2\,K_2\,)\,]^2 \;\geq\; 0\,. \qquad (89)$$

The inequality in (89) always holds, so the first inequality in (89) holds under the specified conditions.

In summary, we have established that when $p_2 = c_2$ in the present case, Firm 1 maximizes its profit by setting $p_1 = c_2 + G_1 - G_2 - K_1$.

When Firm 1 sets $p_1 = c_2 + G_1 - G_2 - K_1$, Firm 2 maximizes its profit by setting $p_2 = c_2$, for the reasons explained in Case (i).

Finally, Lemma A2 implies that all consumers located in the interval $(\,\frac{1}{2} + \frac{K_1}{2\,t},\,1\,]$ (and only these consumers) change the default setting on the product they purchase from Firm 1.

Case (iii). $K_1 > \underline{K}_1$. It remains to prove that the putative equilibrium identified in Case (i) and Case (ii) is not an equilibrium in Case (iii). $K_1 > K_{1a}$ and $K_1 > K_{1b}$ in the present case because $K_1 > \underline{K}_1 \equiv \max\{\,K_{1a},\,K_{1b}\,\}$. (78) establishes that Firm 1 secures profit $\pi_1 = 3\,A - K_1$ at the putative equilibrium identified in Case (i) and Case (ii). (85) and (86) establish that when Firm 2 sets $p_2 = c_2$, Firm 1 can secure profit $\pi_1^{'} = \frac{1}{8\,t}\,[\,t + 3\,A\,]^2$ by setting $p_1 = \frac{1}{2}\,[\,t + G_1 - G_2 + c_1 + c_2\,]$. (87) establishes that $\pi_1^{'} > \pi_1$ when $K_1 > \underline{K}_1$. Therefore, the putative equilibrium is not an equilibrium in Case (iii). ■

Lemma A18. Suppose $K_1 \in (0, t)$ and $K_2 \in (0, t)$. Then the duopoly equilibrium identified in Lemma 3 (and Lemma 1) is the unique duopoly equilibrium.

Proof. In any duopoly equilibrium, there is a consumer who is indifferent between buying Firm 1's product and buying Firm 2's product. (Lemma A17.) Let x_0 denote the location of this consumer. A duopoly equilibrium can have $x_0 \in (\,0,\,\frac{1}{2} - \frac{K_2}{2\,t}\,)$, $x_0 \in [\,\frac{1}{2} - \frac{K_2}{2\,t},\,\frac{1}{2} + \frac{K_1}{2\,t}\,]$, or $x_0 \in (\,\frac{1}{2} + \frac{K_1}{2\,t},\,1\,)$.

First suppose that $x_0 \in (\,0,\,\frac{1}{2} - \frac{K_2}{2\,t}\,)$. Lemma A1 implies that because $x_0 < \frac{1}{2} - \frac{K_2}{2\,t}$, the consumer located at x_0 will change the default characteristic if and only if she buys Firm 2's product. Therefore, the definition of x_0 implies:

$$G_1 - p_1 - t\,x_0 \;=\; G_2 - p_2 - t\,x_0 - K_2 \quad \Leftrightarrow \quad p_1 \;=\; p_2 + G_1 - G_2 + K_2\,. \qquad (90)$$

Lemma A14 implies that an equilibrium does not exist when (90) holds under the maintained conditions. Consequently, a duopoly equilibrium in which $x_0 \in (\,0,\,\frac{1}{2} - \frac{K_2}{2\,t}\,)$ does not exist in this case.

Now suppose that $x_0 \in (\,\frac{1}{2} + \frac{K_1}{2\,t},\,1\,)$. Lemma A2 implies that because $x_0 > \frac{1}{2} + \frac{K_1}{2\,t}$, the consumer located at x_0 will change the default characteristic if and only if she buys Firm 1's product. Therefore, the definition of x_0 implies:

$$G_1 - p_1 - t\,[1 - x_0\,] - K_1 \;=\; G_2 - p_2 - t\,[1 - x_0\,] \quad \Leftrightarrow \quad p_2 \;=\; p_1 + G_2 - G_1 + K_1\,. \qquad (91)$$

Lemmas A9 and A10 imply that all consumers buy Firm 1's product when the equality in (91) holds. Therefore, a duopoly equilibrium does not exist when $x_0 \in (\,\frac{1}{2} + \frac{K_1}{2\,t},\,1\,)$ under the maintained conditions.

The proof of Lemma 3 establishes that when $x_0 \in (\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t})$, the unique duopoly equilibrium is the one characterized in Lemma 3 (and Lemma 1).

Next suppose that $x_0 = \frac{1}{2} - \frac{K_2}{2t}$. Lemmas A1 – A3 imply that no consumer changes the default characteristic of the product she purchases in this case. Consequently, x_0 is determined by:

$$G_1 - p_1 - t\, x_0 \;=\; G_2 - p_2 - t\,[\,1 - x_0\,] \;\;\Leftrightarrow\;\; 2\,t\,x_0 \;=\; G_1 - G_2 + p_2 - p_1 + t$$

$$\Leftrightarrow \;\; x_0 \;=\; \frac{1}{2} - \frac{1}{2\,t}\,[\,G_2 - G_1 + p_1 - p_2\,]\,. \tag{92}$$

(92) implies that because $x_0 = \frac{1}{2} - \frac{K_2}{2t}$ by assumption:

$$K_2 \;=\; G_2 - G_1 + p_1 - p_2 \;\;\Leftrightarrow\;\; p_1 \;=\; p_2 + G_1 - G_2 + K_2\,. \tag{93}$$

Lemma A14 implies that an equilibrium does not exist when (93) holds under the maintained conditions. Consequently, a duopoly equilibrium in which $x_0 = \frac{1}{2} - \frac{K_2}{2t}$ does not exist in this case.

Finally, suppose that $x_0 = \frac{1}{2} + \frac{K_1}{2t}$ in a putative duopoly equilibrium. Lemma A1 – A3 imply that no consumer changes the default characteristic of the product she purchases in this case. Consequently, x_0 is determined by:

$$G_1 - p_1 - t\, x_0 \;=\; G_2 - p_2 - t\,[\,1 - x_0\,] \;\;\Leftrightarrow\;\; 2\,t\,x_0 \;=\; G_1 - G_2 + p_2 - p_1 + t$$

$$\Leftrightarrow \;\; x_0 \;=\; \frac{1}{2} + \frac{1}{2\,t}\,[\,G_1 - G_2 + p_2 - p_1\,]\,. \tag{94}$$

(94) implies that because $x_0 = \frac{1}{2} + \frac{K_1}{2t}$, by assumption:

$$K_1 \;=\; G_1 - G_2 + p_2 - p_1 \;\;\Leftrightarrow\;\; p_2 \;=\; p_1 + G_2 - G_1 + K_1\,. \tag{95}$$

Lemmas A9 and A10 imply that if $p_2 = p_1 + G_2 - G_1 + K_1$ (or equivalently, $p_1 = p_2 + G_1 - G_2 - K_1$), all consumers buy the product from Firm 1. Consequently, a duopoly equilibrium in which $x_0 = \frac{1}{2} + \frac{K_1}{2t}$ does not exist in this case. ■

Observation A2. *Suppose $A < t$. Then the conditions identified in Lemma 3 and the conditions identified in Lemma 4 are mutually exclusive because*

$$\overline{K}_1 \;\equiv\; \frac{1}{2\,t}\left[\,t^2 + 2\,A\,t - A^2\,\right] \;>\; 3\,A - \frac{1}{8\,t}\,[\,t + 3\,A\,]^2 \;\equiv\; \underline{K}_1\,. \tag{96}$$

Proof. The conditions in Lemma 3 require $K_1 \geq \overline{K}_1$. Observation A1 implies $\underline{K}_1 = K_{1a} \equiv 3\,A - \frac{1}{8t}\,[\,t + 3\,A\,]^2$ if $t > A$. Therefore, the conditions in Lemma 4 require $K_1 \leq \underline{K}_1 = 3\,A - \frac{1}{8t}\,[\,t + 3\,A\,]^2$ if $t > A$. These conditions cannot both hold because:

$$\overline{K}_1 \;>\; \underline{K}_1 \;\;\Leftrightarrow\;\; \frac{1}{2\,t}\left[\,t^2 + 2\,A\,t - A^2\,\right] \;>\; 3\,A - \frac{1}{8\,t}\,[\,t + 3\,A\,]^2$$

$$\Leftrightarrow \;\; 4\,\left[\,t^2 + 2\,A\,t - A^2\,\right] \;>\; 24\,A\,t - [\,t + 3\,A\,]^2$$

$$\Leftrightarrow \quad 4t^2 + 8At - 4A^2 \ > \ 24At - \left[t^2 + 6At + 9A^2 \right]$$

$$\Leftrightarrow \quad 4t^2 + 8At - 4A^2 \ > \ 18At - t^2 - 9A^2$$

$$\Leftrightarrow \quad 5t^2 - 10At + 5A^2 \ > \ 0 \quad \Leftrightarrow \quad 5\left[t^2 - 2At + A^2 \right] \ > \ 0 \quad \Leftrightarrow \quad 5\left[t - A \right]^2 \ > \ 0.$$

The last inequality here holds (so $\overline{K}_1 > \underline{K}_1$) because $t > A$, by assumption. ■

Observation A3. *An equilibrium in which all consumers buy Firm 2's product does not exist when* $A > 0$.

Proof. In any equilibrium in which all consumers buy Firm 2's product: (i) Firm 1's profit is 0 (because no consumers buy its product); and (ii) $p_2 \geq c_2$ (because Firm 2 must secure nonnegative profit). When $A > 0$, Firm 1 can secure strictly positive profit whenever Firm 2 sets $p_2 \geq c_2$. This is the case because Lemmas A1 and A2 imply that the consumer located at 0 strictly prefers to buy Firm 1's product than to buy Firm 2's product if:

$$G_1 - p_1 \ \geq \ G_2 - p_2 - K_2 \, . \tag{97}$$

Because $p_2 \geq c_2$, the inequality in (97) holds if:

$$G_1 - p_1 \ \geq \ G_2 - c_2 - K_2 \quad \Leftrightarrow \quad p_1 \ \leq \ c_1 + K_2 + G_1 - c_1 - (G_2 - c_2)$$

$$\Leftrightarrow \quad p_1 \ \leq \ c_1 + 3A + K_2 \, . \tag{98}$$

Because $K_2 \geq 0$, the inequality in (98) (and thus the inequality in (97)) holds if $p_1 \leq c_1 + 3A$. Therefore, if Firm 1 sets $p_1 \in (c_1, \, c_1 + 3A)$, it can secure strictly positive profit by ensuring the patronage of consumers located close to 0. ■

Proposition 1. *(i) The duopoly equilibrium identified in Lemma 3 is the unique equilibrium if Firm 1's competitive advantage is sufficiently limited (i.e.,* $A < t$*) and default-switching costs are sufficiently pronounced (i.e.,* $K_1 \geq \overline{K}_1$ *and* $K_2 \geq \overline{K}_2$*). (ii) The monopoly equilibrium identified in Lemma 4 is the unique equilibrium if Firm 1's default-switching cost is sufficiently small (i.e.,* $K_1 \leq \underline{K}_1$*). (iii) No equilibrium exists if: (a) Firm 1's competitive advantage and default-switching costs are sufficiently pronounced (i.e.* $A > t$ *and* $K_1 > \underline{K}_1$*); (b) Firm 1's competitive advantage is sufficiently limited and default-switching costs are intermediate in magnitude (i.e.,* $A \leq t$ *and* $K_1 \in \left(\underline{K}_1, \, \overline{K}_1 \right)$*); or (c) Firm 2's default-switching cost is sufficiently small and Firm 1's default-switching cost is sufficiently large (i.e.,* $K_2 < \overline{K}_2$ *and* $K_1 > \underline{K}_1$*).*

Proof. The proof employs the following four conclusions.

Conclusion 1.1. *A monopoly equilibrium in which all consumers purchase the product from Firm 1 does not exist if* $K_1 > \underline{K}_1$.

Proof. The Conclusion follows from Lemma 4. $\square$

Conclusion 1.2. *No duopoly equilibrium exists if* $A > t$.

Proof. (42) and Lemma A18 imply that Firm 2's profit margin in a duopoly equilibrium is $p_2 - c_2 = t - A$. This profit margin is strictly negative when $A > t$. Therefore, a duopoly equilibrium cannot exist when $A > t$. $\square$

Conclusion 1.3. *No duopoly equilibrium exists if* $K_1 < \overline{K}_1$.

Proof. The Conclusion follows from Lemma 3 and Lemma A18. $\square$

Conclusion 1.4. No equilibrium exists if $K_2 < \overline{K}_2$ and $K_1 > \underline{K}_1$.

Proof. Observation A3 establishes that when $A > 0$, no equilibrium exists in which all consumers buy Firm 2's product.

Lemma 4 establishes that a monopoly equilibrium in which all consumers purchase Firm 1's product does not exist when $K_1 > \underline{K}_1$.

Lemma A18 and the proof of Lemma 3 establish that if a putative duopoly equilibrium exists, then $p_1 = c_1 + t + A$, $p_2 = c_2 + t - A$, and $\pi_2 = \frac{1}{2t}[t - A]^2$ in this putative equilibrium. (See (60), (61), and (68).) The proof of Lemma 3 also establishes that when $p_1 = c_1 + t + A$, Firm 2 can secure profit that strictly exceeds $\frac{1}{2t}[t - A]^2$ by reducing p_2 sufficiently far below $c_2 + t - A$ to ensure that all consumers prefer to buy Firm 2's product than to buy Firm 1's product if $K_2 < \overline{K}_2$. (See (74).) Therefore, the putative equilibrium is not an equilibrium when $K_2 < \overline{K}_2$. $\square$

The findings in Proposition 1(i) and Proposition 1(ii) follow from Observation A2 and Lemmas 3, 4, and A18. The finding in Proposition 1(iii)(a) follows from Observation A2 and Conclusions 1.1 and 1.2. The finding in Proposition 1(iii)(b) follows from Observation A2, and Conclusions 1.1 and 1.3. The finding in Proposition 1(iii)(c) follows from Observation A2 and Conclusion 1.4. $\blacksquare$

Lemma 5. *Consumer welfare in the duopoly equilibrium identified in Lemma 3 is:*

$$W^{Cd} \equiv \int_0^{x_0} [G_1 - p_1 - t\,x]\,dx + \int_{x_0}^{1} [G_2 - p_2 - t\,(1-x)]\,dx = G_2 - c_2 - \frac{5\,t}{4} + \frac{3\,A}{2} + \frac{A^2}{4\,t}.$$

Consumer welfare in the monopoly equilibrium identified in Lemma 2 is:

$$W_0^{Cm} \equiv \int_0^{\frac{1}{2}} [G_1 - p_1 - t\,x]\,dx + \int_{\frac{1}{2}}^{1} [G_1 - p_1 - t\,(1-x)]\,dx = G_2 - c_2 - \frac{t}{4}.$$

Proof. Because no consumer changes the default horizontal characteristic of the product she purchases in the duopoly equilibrium identified in Lemma 3, consumer welfare in this equilibrium is:

$$
\begin{aligned}
W^{Cd} &= \int_0^{x_0} [\, G_1 - p_1 - t\,x \,]\, dx + \int_{x_0}^1 [\, G_2 - p_2 - t\,(1 - x\,)\,]\, dx \\
&= \left[(\, G_1 - p_1\,)\, x - \frac{t\,x^2}{2} \right]_0^{x_0} + \left[(\, G_2 - p_2 - t\,)\, x + \frac{t\,x^2}{2} \right]_{x_0}^1 \\
&= [\, G_1 - p_1\,]\, x_0 - \frac{t\,x_0^2}{2} + [\, G_2 - p_2\,]\,[\, 1 - x_0\,] - t\,[\, 1 - x_0\,] + \frac{t}{2}\,[\, 1 - x_0^2\,] \\
&= [\, G_1 - p_1\,]\, x_0 + [\, G_2 - p_2\,]\,[\, 1 - x_0\,] + \frac{t}{2}\,[\, 1 - x_0^2 - x_0^2 - 2\,(\, 1 - x_0\,)\,] \\
&= [\, G_1 - p_1\,]\, x_0 + [\, G_2 - p_2\,]\,[\, 1 - x_0\,] + \frac{t}{2}\,[\, -1 - 2\,x_0^2 + 2\,x_0\,] \\
&= G_2 - p_2 + [\, G_1 - G_2 + p_2 - p_1\,]\, x_0 - \frac{t}{2}\,[\, 1 - 2\,x_0 + 2\,x_0^2\,]\,. \qquad (99)
\end{aligned}
$$

Lemma 3 further implies that at the identified equilibrium:

$$
\begin{aligned}
& p_1 = c_1 + t + A \quad \text{and} \quad p_2 = c_2 + t - A \\
\Leftrightarrow \quad & p_2 - p_1 = c_2 + t + A - c_1 - t + A = c_2 - c_1 - 2\,A\,. \qquad (100)
\end{aligned}
$$

(4), (5), (100), and Lemma 3 imply:

$$
\begin{aligned}
x_0 &= \frac{1}{2\,t}\,[\, t + G_1 - G_2 + p_2 - p_1\,] = \frac{1}{2\,t}\,[\, t + G_1 - G_2 + c_2 - c_1 - 2\,A\,] \\
&= \frac{1}{2\,t}\,[\, t + 3\,A - 2\,A\,] = \frac{1}{2\,t}\,[\, t + A\,]\,. \qquad (101)
\end{aligned}
$$

(4) and (99) – (101) imply:

$$
\begin{aligned}
W^{Cd} &= G_2 - c_2 - t + A + [\, G_1 - G_2 + c_2 - c_1 - 2\,A\,]\, \frac{1}{2\,t}\,[\, t + A\,] - \frac{t}{2}\,[\, 1 - 2\,x_0 + 2\,x_0^2\,] \\
&= G_2 - c_2 - t + A + [\, 3\,A - 2\,A\,]\, \frac{1}{2\,t}\,[\, t + A\,] - \frac{t}{2} + t\,x_0 - t\,x_0^2 \\
&= G_2 - c_2 - t + A + \frac{A}{2\,t}\,[\, t + A\,] - \frac{t}{2} + t\,x_0\,[\, 1 - x_0\,] \\
&= G_2 - c_2 - \frac{3\,t}{2} + A + \frac{A}{2} + \frac{A^2}{2\,t} + t\,\left[\frac{1}{2} + \frac{A}{2\,t} \right] \left[\frac{1}{2} - \frac{A}{2\,t} \right] \\
&= G_2 - c_2 - \frac{3\,t}{2} + \frac{3\,A}{2} + \frac{A^2}{2\,t} + t\,\left[\frac{1}{4} - \frac{A^2}{4\,t^2} \right]
\end{aligned}
$$

$$= \; G_2 - c_2 - \frac{3\,t}{2} + \frac{3\,A}{2} + \frac{A^2}{2\,t} + \frac{t}{4} - \frac{A^2}{4\,t} \; = \; G_2 - c_2 - \frac{5}{4}\,t + \frac{3}{2}\,A + \frac{A^2}{4\,t}\,. \tag{102}$$

The (only) consumers who change the default horizontal characteristic of the product they purchase in the monopoly equilibrium identified in Lemma 2 are those located in $(\frac{1}{2}, 1\,]$. Therefore, because $K_1 = 0$, consumer welfare in this equilibrium is:

$$\begin{aligned}
W_0^{Cm} \; &= \; \int_0^{\frac{1}{2}} [\, G_1 - p_1 - t\,x\,]\,dx + \int_{\frac{1}{2}}^1 [\, G_1 - p_1 - t\,(\,1 - x\,)\,]\,dx \\
&= \; \int_0^1 [\, G_1 - p_1\,]\,dx - t\int_0^{\frac{1}{2}} x\,dx - t\int_{\frac{1}{2}}^1 [\,1 - x\,]\,dx \\
&= \; [\,(G_1 - p_1\,)\,x\,]_0^1 - \frac{t}{2}\,[\,x^2\,]_0^{\frac{1}{2}} - t\left[x - \frac{x^2}{2}\right]_{\frac{1}{2}}^1 \\
&= \; G_1 - p_1 - \frac{t}{8} - t\left[\left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{8}\right)\right] \\
&= \; G_1 - p_1 - \frac{t}{8} - t\left[\frac{1}{8}\right] \; = \; G_1 - p_1 - \frac{t}{4} \\
&= \; G_1 - [\,c_2 + G_1 - G_2\,] - \frac{t}{4} \; = \; G_2 - c_2 - \frac{t}{4}\,.
\end{aligned} \tag{103}$$

The penultimate equality in (103) holds because $p_1 = c_2 + G_1 - G_2$ in the equilibrium identified in Lemma 2. ■

Proposition 2. *Suppose* $A < \frac{t}{2+\sqrt{3}}$, $K_1 \geq \overline{K}_1$, *and* $K_2 \geq \overline{K}_2$.[4] *Then Firm 1 and Firm 2 both secure more profit in the duopoly equilibrium identified in Lemma 3 (where* $K_1 > 0$ *and* $K_2 > 0$*) than in the monopoly equilibrium identified in Lemma 2 (where* $K_1 = K_2 = 0$*). Consumer welfare and total welfare are both lower in the duopoly equilibrium than in the monopoly equilibrium.*

Proof. Lemma 2 implies that when $K_1 = K_2 = 0$, Firm 1's equilibrium profit is:

$$\pi_{1a} \; = \; 3\,A. \tag{104}$$

$A < t$ when $A < \frac{t}{2+\sqrt{3}}$. Therefore, the maintained assumptions ensure that the conditions in Lemma 3 are all satisfied. In the equilibrium identified in Lemma 3 (and Lemma

[4] These three conditions all hold simultaneously if $K_1 > A\left[\frac{5+3\sqrt{3}}{2+\sqrt{3}}\right]$, $K_2 > A\left[\frac{1+\sqrt{3}}{2+\sqrt{3}}\right]$, and $t \in \left(t_H, \min\{\,\bar{t}_1, \bar{t}_2\,\}\right]$, where $t_H = A\left[2+\sqrt{3}\right]$, $\bar{t}_1 \equiv K_1 - A + \sqrt{[K_1 - A]^2 + A^2}$ and $\bar{t}_2 \equiv A + K_2 + \sqrt{(A+K_2)^2 + A^2}$. See the proof of Proposition 2 in Chakravorty and Sappington (2025).

1), Firm 1's profit is:

$$\pi_{1b} \;=\; \frac{[\,t + A\,]^2}{2\,t}\,. \tag{105}$$

(104) and (105) imply:

$$\pi_{1b} \;>\; \pi_{1a} \;\;\Leftrightarrow\;\; t^2 + 2\,A\,t + A^2 \;>\; 6\,A\,t \;\;\Leftrightarrow\;\; t^2 - 4\,A\,t + A^2 \;>\; 0\,. \tag{106}$$

The roots of the quadratic equation associated with (106) are:

$$t^* \;=\; \frac{1}{2}\left[\,4\,A \,\pm\, \sqrt{16\,A^2 - 4\,A^2}\;\right] \;=\; \frac{1}{2}\left[\,4\,A \,\pm\, \sqrt{12\,A^2}\;\right] \;=\; A\left[\,2 \pm \sqrt{3}\,\right]. \tag{107}$$

(106) and (107) imply that $\pi_{1b} > \pi_{1a}$ when $t > A\left[\,2 + \sqrt{3}\,\right] \;\Leftrightarrow\; A < \frac{t}{2+\sqrt{3}}$.

Lemma 2 implies that Firm 2's equilibrium profit is 0 when $K_1 = K_2 = 0$. Firm 2's profit in the duopoly equilibrium identified in Proposition 3 is $\frac{1}{2\,t}\,[\,t - A\,]^2 > 0$.

(102) and (103) imply:

$$\begin{aligned} W^{Cd} \gtreqless W_0^{Cm} \;\;&\Leftrightarrow\;\; G_2 - c_2 - \frac{5\,t}{4} + \frac{3\,A}{2} + \frac{A^2}{4\,t} \gtreqless G_2 - c_2 - \frac{t}{4} \\ &\Leftrightarrow\;\; -\frac{5\,t}{4} + \frac{t}{4} + \frac{3\,A}{2} + \frac{A^2}{4\,t} \gtreqless 0 \;\;\Leftrightarrow\;\; -\,t + \frac{3\,A}{2} + \frac{A^2}{4\,t} \gtreqless 0 \\ &\Leftrightarrow\;\; A^2 + 6\,t\,A - 4\,t^2 \gtreqless 0\,. \end{aligned} \tag{108}$$

The ("A") roots of the quadratic equation in (108) are:

$$\frac{1}{2}\left[\,-\,6\,t \,\pm\, \sqrt{36\,t^2 + 16\,t^2}\;\right] \;=\; \frac{1}{2}\left[\,-\,6\,t \,\pm\, \sqrt{52\,t^2}\;\right] \;=\; -\,3\,t \,\pm\, t\sqrt{13}\,. \tag{109}$$

The smaller root in (109) is:

$$-\,3\,t - t\sqrt{13} \;<\; 0\,. \tag{110}$$

The larger root in (109) is:

$$-\,3\,t + t\sqrt{13} \;=\; t\left[\,\sqrt{13} - 3\,\right] \;>\; 0\,. \tag{111}$$

(108) – (111) imply that $W^{Cd} < W_0^{Cm}$ if $A \in (0, \left[\,\sqrt{13} - 3\,\right] t)$.

Observe that $\sqrt{13} - 3 \approx 0.606 > \frac{1}{2+\sqrt{3}} \approx 0.268$. Therefore, $A \in (0, \left[\,\sqrt{13} - 3\,\right] t)$ when $A < \frac{t}{2+\sqrt{3}}$. Consequently, $W^{Cd} < W_0^{Cm}$ when $A < \frac{t}{2+\sqrt{3}}$.

To compare total welfare in the two equilibria, let T^d denote total welfare in the duopoly equilibrium identified in Lemma 3. Also let T_0^m denote total welfare in the monopoly equilibrium identified in Lemma 2, where $K_1 = K_2 = 0$.

(67) and (68) imply that at the equilibrium identified in Lemma 3, industry profit is:

$$\frac{1}{2\,t}\,[\,t + A\,]^2 + \frac{1}{2\,t}\,[\,t - A\,]^2 \;=\; \frac{1}{2\,t}\left[\,2\,t^2 + 2\,A^2\,\right] \;=\; \frac{1}{t}\left[\,t^2 + A^2\,\right]. \tag{112}$$

(102) and (112) imply:

$$T^d \;=\; G_2 - c_2 - \frac{5}{4}\,t + \frac{3}{2}\,A + \frac{A^2}{4\,t} + \frac{1}{t}\,\left[\,t^2 + A^2\,\right] \;=\; G_2 - c_2 - \frac{t}{4} + \frac{3\,A}{2} + \frac{5\,A^2}{4\,t}\,. \qquad (113)$$

Lemma 2 implies that when $K_1 = K_2 = 0$, equilibrium industry profit is $\pi_1 + \pi_2 = 3\,A + 0 = 3\,A$. Therefore, (103) implies:

$$T_0^m \;=\; G_2 - c_2 - \frac{t}{4} + 3\,A\,. \qquad (114)$$

(113) and (114) imply:

$$T_0^m \;>\; T^d \;\Leftrightarrow\; G_2 - c_2 - \frac{t}{4} + 3\,A \;>\; G_2 - c_2 - \frac{t}{4} + \frac{3\,A}{2} + \frac{5\,A^2}{4\,t}$$

$$\Leftrightarrow\;\; 3\,A \;>\; \frac{3\,A}{2} + \frac{5\,A^2}{4\,t} \;\;\Leftrightarrow\;\; \frac{3\,A}{2} \;>\; \frac{5\,A^2}{4\,t} \;\;\Leftrightarrow\;\; 3 \;>\; \frac{5\,A}{2\,t} \;\;\Leftrightarrow\;\; A \;<\; \frac{6}{5}\,t\,.$$

The last inequality here holds because $A < t$, by assumption.

Finally, to establish that the three conditions in the Proposition can all hold simultaneously, observe that:

$$K_2 \,\geq\, \overline{K}_2 \;\equiv\; \frac{1}{2\,t}\left[\,t^2 - 2\,A\,t - A^2\,\right] \;\Leftrightarrow\; t^2 - 2\,[\,A + K_2\,]\,t - A^2 \;\leq\; 0\,. \qquad (115)$$

The roots of the quadratic equation associated with (115) are:

$$t^* \;=\; \frac{1}{2}\left[\,2\,(A + K_2) \pm \sqrt{4\,(A + K_2)^2 + 4\,A^2}\,\right] \;=\; A + K_2 \pm \sqrt{(A + K_2)^2 + A^2}\,. \qquad (116)$$

Because $A + K_2 \leq \sqrt{(A + K_2)^2 + A^2}$, (115) and (116) imply:

$$K_2 \,\geq\, \overline{K}_2 \;\text{ if }\; t \in \left[\,0,\, \bar{t}_2\,\right] \;\text{ where }\; \bar{t}_2 \;\equiv\; A + K_2 + \sqrt{(A + K_2)^2 + A^2}\,. \qquad (117)$$

The conditions in the Proposition include $A < \frac{t}{2+\sqrt{3}} \Leftrightarrow t > A\left[\,2 + \sqrt{3}\,\right]$. Therefore, (117) implies that if the conditions in Proposition all hold, it must be the case that:

$$A\left[\,2 + \sqrt{3}\,\right] \;<\; A + K_2 + \sqrt{(A + K_2)^2 + A^2}$$

$$\Leftrightarrow\;\; \sqrt{(A + K_2)^2 + A^2} \;>\; A\left[\,1 + \sqrt{3}\,\right] - K_2$$

$$\Leftrightarrow\;\; [\,A + K_2\,]^2 + A^2 \;>\; A^2\left[\,1 + \sqrt{3}\,\right]^2 - 2\,A\,K_2\left[\,1 + \sqrt{3}\,\right] + (K_2)^2$$

$$\Leftrightarrow\;\; 2\,A^2 + 2\,A\,K_2 + (K_2)^2 \;>\; A^2\left[\,1 + 2\sqrt{3} + 3\,\right] - 2\,A\,K_2\left[\,1 + \sqrt{3}\,\right] + (K_2)^2$$

$$\Leftrightarrow\;\; 2\,A^2 + 2\,A\,K_2 \;>\; A^2\left[\,4 + 2\sqrt{3}\,\right] - 2\,A\,K_2\left[\,1 + \sqrt{3}\,\right]$$

$$\Leftrightarrow\;\; A^2\left[\,2 + 2\sqrt{3}\,\right] - 2\,A\,K_2\left[\,2 + \sqrt{3}\,\right] \;<\; 0$$

$$\Leftrightarrow \; A\left[\,2+2\sqrt{3}\,\right] - 2\,K_2\left[\,2+\sqrt{3}\,\right] \;<\; 0 \;\;\Leftrightarrow\;\; K_2 \;>\; A\left[\,\frac{1+\sqrt{3}}{2+\sqrt{3}}\,\right]. \tag{118}$$

Further observe that:

$$K_1 \ge \overline{K}_1 \;\equiv\; \frac{1}{2\,t}\left[\,t^2 + 2\,A\,t - A^2\,\right] \;\;\Leftrightarrow\;\; t^2 + 2\,[\,A - K_1\,]\,t - A^2 \;\le\; 0\,. \tag{119}$$

The roots of the quadratic equation associated with (119) are:

$$t^* \;=\; \frac{1}{2}\left[\,2\,(K_1 - A) \pm \sqrt{4\,[\,A - K_1\,]^2 + 4\,A^2}\;\right] \;=\; K_1 - A \pm \sqrt{[\,A - K_1\,]^2 + A^2}\,. \tag{120}$$

(119) and (120) imply:

$$K_1 \ge \overline{K}_1 \;\;\Leftrightarrow\;\; t \in [\,\underline{t}_1,\, \bar{t}_1\,] \;\text{ where }\; \underline{t}_1 \;\equiv\; K_1 - A - \sqrt{[\,K_1 - A\,]^2 + A^2}$$

$$\text{and }\; \bar{t}_1 \;\equiv\; K_1 - A + \sqrt{[\,K_1 - A\,]^2 + A^2}\;\,]. \tag{121}$$

Because $K_1 - A \le \sqrt{[\,K_1 - A\,]^2 + A^2}$, (121) implies:

$$K_1 \ge \overline{K}_1 \;\text{ if }\; t \in [\,0,\, \bar{t}_1\,]. \tag{122}$$

Because the conditions identified in the Proposition require $t > A\left[\,2+\sqrt{3}\,\right]$, (122) implies that if the conditions in the Proposition all hold, it must be the case that:

$$A\left[\,2+\sqrt{3}\,\right] \;<\; K_1 - A + \sqrt{[\,A - K_1\,]^2 + A^2}$$

$$\Leftrightarrow\;\; \sqrt{[\,A - K_1\,]^2 + A^2} \;>\; A\left[\,3+\sqrt{3}\,\right] - K_1$$

$$\Leftrightarrow\;\; [\,A - K_1\,]^2 + A^2 \;>\; A^2\left[\,3+\sqrt{3}\,\right]^2 - 2\,A\,K_1\left[\,3+\sqrt{3}\,\right] + (K_1)^2$$

$$\Leftrightarrow\;\; 2\,A^2 - 2\,A\,K_1 + (K_1)^2 \;>\; A^2\left[\,9+6\sqrt{3}+3\,\right] - 2\,A\,K_1\left[\,3+\sqrt{3}\,\right] + (K_1)^2$$

$$\Leftrightarrow\;\; 2\,A^2 - 2\,A\,K_1 \;>\; A^2\left[\,12+6\sqrt{3}\,\right] - 2\,A\,K_1\left[\,3+\sqrt{3}\,\right]$$

$$\Leftrightarrow\;\; A^2 - A\,K_1 \;>\; A^2\left[\,6+3\sqrt{3}\,\right] - A\,K_1\left[\,3+\sqrt{3}\,\right]$$

$$\Leftrightarrow\;\; A - K_1 \;>\; A\left[\,6+3\sqrt{3}\,\right] - K_1\left[\,3+\sqrt{3}\,\right]$$

$$\Leftrightarrow\;\; K_1\left[\,2+\sqrt{3}\,\right] \;>\; A\left[\,5+3\sqrt{3}\,\right] \;\;\Leftrightarrow\;\; K_1 \;>\; A\left[\,\frac{5+3\sqrt{3}}{2+\sqrt{3}}\,\right]. \tag{123}$$

In summary, the conditions in the Proposition all hold if (118) and (123) hold and $t \in \left(\,A\,[\,2+\sqrt{3}\,]\,,\, \min\{\,\bar{t}_1,\, \bar{t}_2\,\}\,\right]$. ■

Proposition 3. *In the setting with endogenous* K*: (i) If* $A > \frac{t}{2+\sqrt{3}}$*, then the only equilibria are monopoly equilibria in which* $K_1 = 0$ *and all consumers buy Firm 1's product. (ii) If* $A < \frac{t}{2+\sqrt{3}}$*, then duopoly equilibria in which* $K_1 \geq \overline{K}_1$ *and* $K_2 \geq \overline{K}_2$ *exist, as can monopoly equilibria in which* $K_1 = 0$*,* $K_2 \in \big[0, \overline{K}_2\big)$*, and all consumers buy Firm 1's product.*

Proof of (i). Observation A3 implies that no equilibrium exists in which all consumers buy Firm 2's product because $A > 0$, by assumption.

(42) and Lemma A18 imply that if $A > t$, then Firm 2's profit margin in a duopoly equilibrium ($p_2 - c_2 = t - A$) is negative. Therefore, a duopoly equilibrium cannot exist when $A > t$.

(63) and Lemma A18 imply that if $A = t$, then in a putative duopoly equilibrium, the consumer who is indifferent between buying Firm 1's product and Firm 2's product is located at $x_0 = \frac{1}{2} + \frac{A}{2t} = \frac{1}{2} + \frac{A}{2A} = 1$. Therefore, no duopoly equilibrium exists if $A = t$.

Lemmas 2 and 4 establish that Firm 1's profit in any monopoly equilibrium in which all consumers buy Firm 1's product is $3A - K_1 \leq 3A$. Therefore, Firm 1's profit is highest in such an equilibrium when $K_1 = 0$.

Lemmas 1, 3, and A18 imply that Firm 1's profit in a duopoly equilibrium is $\frac{1}{2t}[t+A]^2$. Observe that:

$$\begin{aligned} &\frac{1}{2t}[t+A]^2 > 3A \quad \Leftrightarrow \quad [t+A]^2 > 6At \\ &\Leftrightarrow \quad t^2 + 2At + A^2 > 6At \quad \Leftrightarrow \quad t^2 - 4At + A^2 > 0. \end{aligned} \tag{124}$$

(124) holds when $t < t_L$ or $t > t_H$, where:

$$\begin{aligned} t_L &= \frac{1}{2}\Big[4A - \sqrt{16A^2 - 4A^2}\,\Big] = \frac{1}{2}\Big[4A - \sqrt{12A^2}\,\Big] = \Big[2 - \sqrt{3}\,\Big]A \quad \text{and} \\ t_H &= \frac{1}{2}\Big[4A + \sqrt{16A^2 - 4A^2}\,\Big] = \frac{1}{2}\Big[4A + \sqrt{12A^2}\,\Big] = \Big[2 + \sqrt{3}\,\Big]A. \end{aligned} \tag{125}$$

(125) implies:

$$\begin{aligned} t < t_L \quad &\Leftrightarrow \quad t < \Big[2 - \sqrt{3}\,\Big]A \quad \Leftrightarrow \quad A > \frac{t}{2-\sqrt{3}}; \\ t > t_H \quad &\Leftrightarrow \quad t > \Big[2 + \sqrt{3}\,\Big]A \quad \Leftrightarrow \quad A < \frac{t}{2+\sqrt{3}}. \end{aligned} \tag{126}$$

Neither of the final inequalities in (126) hold when $A \in (\frac{t}{2+\sqrt{3}}, t)$ (because $\frac{t}{2-\sqrt{3}\,]} \approx 3.732\,t > t$). Therefore, the analysis in (124) – (126) implies that if Firm 1 sets $K_1 > 0$ and a duopoly equilibrium ensues, Firm 1's profit does not exceed $3A$ when $A \in (\frac{t}{2+\sqrt{3}}, t)$.

Finally, suppose a duopoly equilibrium exists when $A \in (\frac{t}{2+\sqrt{3}}, t)$. Lemmas 1, 3, and A18 imply that $p_2 = c_2 + t - A \geq c_2$ and $\pi_1 = \frac{1}{2t}[t+A]^2$ in this putative equilibrium. The analysis in (124) – (126) implies that when $A \in (\frac{t}{2+\sqrt{3}}, t)$, Firm 1 can secure strictly more profit by setting $K_1 = 0$ and reducing p_1 sufficiently to ensure that all consumers buy its product. Therefore, the putative duopoly equilibrium is not an equilibrium.

Proof of (ii). We first prove that the identified duopoly equilibria exist when $A < \frac{t}{2+\sqrt{3}}$, $K_1 \geq \overline{K}_1$ and $K_2 \geq \overline{K}_2$.

Observe that $A < t$ when $A < \frac{t}{2+\sqrt{3}}$ because $\frac{1}{2+\sqrt{3}} \approx 0.268 < 1$. Therefore, Lemma 3 establishes that a duopoly equilibrium exists under the specified conditions. Lemmas 3 and A18 imply that the profits of Firm 1 and Firm 2 in this equilibrium are $\pi_1^d = \frac{1}{2t}[t + A]^2 > 0$ and $\pi_2^d = \frac{1}{2t}[t - A]^2 > 0$, respectively. Observe that these profits do not vary with K_1 or K_2. Therefore, neither firm can increase its profit by unilaterally varying its default-switching cost if a duopoly equilibrium ensues.

Observation A3 implies that no equilibrium exists in which all consumers buy Firm 2's product because $A > 0$, by assumption.

The proof of Observation A2 implies that a monopoly equilibrium in which all consumers buy Firm 1's product does not exist when $A < t$ and $K_1 \geq \overline{K}_1$. Therefore, such an equilibrium does not exist under the specified conditions.

The analysis in (124) – (126) implies that a monopoly equilibrium in which all consumers buy Firm 1's product does not exist when $A < \frac{t}{2+\sqrt{3}}$ and $K_1 \leq \underline{K}_1$. This is the case because when $p_2 = c_2$, Firm 1's profit in a monopoly equilibrium is at most $3A$, which is strictly less than Firm 1's profit in the identified duopoly equilibrium ($\frac{1}{2t}[t + A]^2$).

Now we prove that the identified monopoly equilibria can exist when $A < \frac{t}{2+\sqrt{3}}$, $K_1 = 0$, and $K_2 \in [0, \overline{K}_2)$. Lemmas 2 and 4 imply that in any monopoly equilibrium with the specified properties, Firm 1's profit is $\pi_1^m = 3A$ and Firm 2's profit is $\pi_2^m = 0$.

Lemma 2 implies that the identified monopoly equilibrium exists when $K_1 = 0$ and $K_2 = 0$. To determine when a monopoly equilibrium with $K_1 = 0$ and $K_2 \in (0, \overline{K}_2)$ exists, observe that $\frac{t}{3[3+2\sqrt{2}]} < \frac{t}{2+\sqrt{3}}$. Also observe that $A \leq t$ (so $\underline{K}_1 = K_{1b} \equiv 3A - \frac{1}{8t}[t + 3A]^2$, from Observation A1) when $A < \frac{t}{2+\sqrt{3}}$. Furthermore:

$$K_{1b} \geq 0 \quad \Leftrightarrow \quad A \in \left[\frac{t}{3(3+2\sqrt{2})}, \frac{t}{3(3-2\sqrt{2})}\right]. \tag{127}$$

(127) holds because:

$$3A \geq \frac{1}{8t}[t+3A]^2 \quad \Leftrightarrow \quad [t+3A]^2 \leq 24At \quad \Leftrightarrow \quad t^2 - 18At + 9A^2 \leq 0. \tag{128}$$

The ("t") roots of the quadratic equation associated with (128) are:

$$\frac{1}{2}\left[18A \pm \sqrt{324A^2 - 36A^2}\right] = 9A \pm A\sqrt{72} = 3\left[3 \pm 2\sqrt{2}\right]A. \tag{129}$$

(128) and (129) imply that $3A \geq \frac{1}{8t}[t + 3A]^2$ if and only if (127) holds. Consequently,

Lemma 4 implies that the identified monopoly equilibria with $K_1 = 0$ and $K_2 \in (0, \overline{K}_2)$ exist when $A \in [\frac{t}{3(3+2\sqrt{2})}, \frac{t}{2+\sqrt{3}})$.

Lemma A18 and the proof of Lemma 3 establish that if Firm 1 increases K_1 above 0, a duopoly equilibrium does not ensue because $K_2 < \overline{K}_2$. Furthermore, the increase in K_1 would reduce Firm 1's profit (π_1) if a monopoly equilibrium in which all consumers buy Firm 1's product ensues (because $\pi_1 = 3A - K_1$ in any such equilibrium). Consequently, Firm 1 cannot secure a strict increase in profit by setting $K_1 > 0$ when $K_2 \in [0, \overline{K}_2)$.

If Firm 2 changes K_2, a duopoly equilibrium will not ensue because $K_1 = 0 < \overline{K}_1$. (Recall Lemmas 3 and A18.) $\overline{K}_1 > 0$ when $A < \frac{t}{2+\sqrt{3}}$ because:

$$\overline{K}_1 = \frac{1}{2t}\left[t^2 + 2At - A^2\right] > 0 \Leftrightarrow t^2 + 2At - A^2 > 0$$

$$\Leftrightarrow t^2 + 2At + A^2 > 2A^2 \Leftrightarrow [t + A]^2 > 2A^2$$

$$\Leftrightarrow t + A > \sqrt{2}A \Leftrightarrow t > \left[\sqrt{2} - 1\right]A \Leftrightarrow A < \frac{t}{\sqrt{2} - 1}. \qquad (130)$$

The last inequality in (130) holds when $A < \frac{t}{2+\sqrt{3}}$ because $\frac{1}{\sqrt{2}-1} \approx 2.415 > 0.268 \approx \frac{1}{2+\sqrt{3}}$.

Furthermore, a change in K_2 will not change Firm 2's profit if a monopoly equilibrium in which all consumers buy Firm 1's product ensues. Therefore, when $K_1 = 0$ and $K_2 \in [0, \overline{K}_2)$, neither firm can strictly increase its profit above the level it secures in the monopoly equilibrium where all consumers buy Firm 1's product by changing its default-switching cost.
■