Technical Appendix to Accompany
"On the Design of Price Caps as Sanctions"

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Part A of this Technical Appendix provides detailed proofs of the formal conclusions in the paper.[1] Part B analyzes the benchmark setting in which R is a monopoly supplier. Part C considers the setting where R's profit replaces R's revenue in the welfare function. Part D analyzes a benchmark setting with exogenous prices. Part E explores another benchmark setting in which R has a different cost structure. Part F examines how equilibrium outcomes change as parameter values change in the modified baseline setting, where iso-elastic demand prevails. Part G presents two supplemental figures.

<u>Equations from the Text</u>

$$C^R(q_A, q_N) \;=\; c_A\, q_A + \frac{k_A}{2}\,[\,q_A\,]^2 + c_N\, q_N + \frac{k_N}{2}\,[\,q_N\,]^2 + \frac{k^R}{2}\,[\,q_A + q_N\,]^2\,. \tag{1}$$

$$D \;\equiv\; [\,2\,b + k\,]\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R\,\right] + b\,k_A\,[\,3\,b + 2\,k\,] - b^2\,[\,b + k\,] \;>\; 0\,. \tag{2}$$

$$k_A\,[\,(a - c_N)\,(2\,b + k) - b\,(a - c)\,] \;>\; [\,c_N - c_A\,]\left[\,3\,b^2 + 2\,b\left(k + k^R\right) + k\,k^R\,\right]. \tag{3}$$

R's problem is:

$$\underset{q_A \geq 0,\; q_N \geq 0}{\text{Maximize}} \quad P_A(\,q_A + q_N + q\,)\,q_A + [\,a - b\,(\,q_A + q_N + q\,)\,]\,q_N - C^R(q_A, q_N)$$

$$\text{where} \quad P_A(Q) \;=\; \begin{cases} \overline{p} & \text{if } P(Q) \;\geq\; \overline{p} \\ P(Q) & \text{if } P(Q) \;<\; \overline{p}\,. \end{cases} \tag{4}$$

The rival's problem is:

$$\underset{q \geq 0}{\text{Maximize}} \quad [\,a - b\,(\,q_A + q_N + q\,)\,]\,q \;-\; C(q)\,. \tag{5}$$

The inequality in (2) holds because:

$$\begin{aligned} D \;&=\; [\,2\,b + k\,]\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R\,\right] + b\,k_A\,[\,3\,b + 2\,k\,] - b^2\,[\,b + k\,] \\ &=\; b\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R\,\right] + [\,b + k\,]\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R\,\right] \\ &\qquad + 2\,b\,k_A\,[\,b + k\,] + b^2\,k_A - b^2\,[\,b + k\,] \\ &=\; b\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R\,\right] \\ &\qquad + [\,b + k\,]\left[\,k_A\,k_N + k_N\,k^R + k_A\,k^R + 2\,b\,k_A - b^2\,\right] + b^2\,k_A \\ &>\; [\,b + k\,]\left[\,k_A\,k_N + k_N\,k^R + k_A\,k^R + 2\,b\,k_A - b^2\,\right] \end{aligned}$$

[1] Some of the formal conclusions below generalize their counterparts in the paper.

$$= \; [\, b + k \,] \left\{ k_A \,[\, 2\, b + k_N \,] + k^R \,[\, k_A + k_N \,] - b^2 \right\} \; > \; 0 \, .$$

The final inequality here holds because $k_A \,[\, 2\, b + k_N \,] + k^R \,[\, k_A + k_N \,] > \; b^2$, by assumption.

A. Proofs of Formal Conclusions in the Paper.

Proposition 1. *There exist values of the price cap, $0 < \overline{p}_0 < \overline{p}_d < \overline{p}_b$, such that, in equilibrium, $q_A = 0$ if and only if $\overline{p} \leq \overline{p}_0$. Furthermore: (i) $\overline{p} < P(Q)$ if $\overline{p} \leq \overline{p}_d$; (ii) $\overline{p} = P(Q)$ if $\overline{p} \in (\overline{p}_d, \overline{p}_b]$; and (iii) $\overline{p} > P(Q)$ if $\overline{p} > \overline{p}_b$.*

Proof. The proof follows directly from Lemmas A1 – A6 (below), which refer to the following definitions.

$$\overline{p}_0 \; \equiv \; c_A + \frac{[\, a - c_N \,]\,[\, 2\, b + k \,] \; - b \,[\, a - c \,]}{[\, 2\, b + k_N + k^R \,]\,[\, 2\, b + k \,] - b^2} \; [\, b + k^R \,] \, . \tag{6}$$

$$\overline{p}_d \; \equiv \; \frac{1}{D_2} \; \{ \,[\, a\,(b + k) + b\, c \,] \left[\left(b + k^R\right) (k_N + k_A) + k_N \, k_A - b \, k_N \right] \\ + \; b \,[\, b + k \,]\,[\, k_A - b \,]\, c_N + b \,[\, k_N + b \,]\,[\, b + k \,]\, c_A \, \}$$

$$\text{where} \;\; D_2 \; \equiv \; b \,[\, b + k \,]\,[\, k_N + k_A \,] + k_N \,[\, k_A - b \,]\,[\, 2\, b + k \,] \\ + \; [\, k_N + k_A \,]\,[\, 2\, b + k \,] \left[\, b + k^R \,\right] . \tag{7}$$

$$\overline{p}_b \; \equiv \; \frac{1}{D_3} \; \{ \,[\, a\,(b + k) + b\, c \,] \left[\left(b + k^R\right) (k_N + k_A) + k_N \, k_A \right] \\ + \; b\, c_N \,[\, b + k \,]\, k_A + b \, k_N \,[\, b + k \,]\, c_A \, \}$$

$$\text{where} \;\; D_3 \; \equiv \; b \,[\, b + k \,]\,[\, k_N + k_A \,] + k_N \, k_A \,[\, 2\, b + k \,] \\ + \; [\, k_N + k_A \,]\,[\, 2\, b + k \,] \left[\, b + k^R \,\right] \; = \; D_2 + b \, k_N \,[\, 2\, b + k \,] \, . \tag{8}$$

Lemma A1. Suppose $\overline{p} \leq \overline{p}_0$. Then in equilibrium:

$$q_A \; = \; 0 \, , \quad q_N \; = \; \frac{[\, a - c_N \,]\,[\, 2\, b + k \,] - b \,[\, a - c \,]}{[\, 2\, b + k_N + k^R \,]\,[\, 2\, b + k \,] - b^2} \, ,$$

$$q \; = \; \frac{[\, a - c \,] \left[\, 2\, b + k_N + k^R \,\right] - b \,[\, a - c_N \,]}{[\, 2\, b + k_N + k^R \,]\,[\, 2\, b + k \,] - b^2} \, , \;\; \text{and}$$

$$Q \; = \; q_A + q_N + q \; = \; \frac{[\, a - c \,] \left[\, b + k_N + k^R \,\right] + [\, a - c_N \,]\,[\, b + k \,]}{[\, 2\, b + k_N + k^R \,]\,[\, 2\, b + k \,] - b^2} \, . \tag{9}$$

Proof. (4) implies that R's problem when $q_A = 0$ is:

$$\underset{q_N \, \geq 0}{\text{Maximize}} \;\; [\, a - b\,(q_N + q) - c_N \,]\, q_N - \frac{k_N}{2} \, (q_N)^2 - \frac{k^R}{2} \, (q_N)^2 \, . \tag{10}$$

(10) implies that R's profit-maximizing choice of $q_N > 0$ is determined by:

$$a - 2\,b\,q_N - b\,q - c_N - k_N\,q_N - k^R\,q_N \;=\; 0 \;\;\Rightarrow\;\; q_N \;=\; \frac{a - c_N - b\,q}{2\,b + k_N + k^R}\,. \tag{11}$$

(5) implies that the necessary condition for an interior solution to the rival's problem is:

$$a - b\,[\,q_A + q_N + q\,] - c - b\,q - k\,q \;=\; 0 \;\;\Leftrightarrow\;\; [\,2\,b + k\,]\,q \;=\; a - b\,[\,q_A + q_N\,] - c$$

$$\Leftrightarrow\;\; q \;=\; \frac{a - c}{2\,b + k} - \frac{b}{2\,b + k}\,[\,q_A + q_N\,]\,. \tag{12}$$

(11) and (12) imply that when $q_A = 0$:

$$q_N \;=\; \frac{a - c_N}{2\,b + k_N + k^R} - \frac{b}{2\,b + k_N + k^R}\left[\frac{a - c - b\,q_N}{2\,b + k}\right]$$

$$=\; \frac{[\,a - c_N\,]\,[\,2\,b + k\,] - b\,[\,a - b\,q_N - c\,]}{[\,2\,b + k_N + k^R]\,[\,2\,b + k\,]}$$

$$\Rightarrow\;\; q_N\left[1 - \frac{b^2}{[\,2\,b + k_N + k^R]\,[\,2\,b + k\,]}\right] \;=\; \frac{[\,a - c_N\,]\,[\,2\,b + k\,] \;-\; b\,[\,a - c\,]}{[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,]}$$

$$\Rightarrow\;\; q_N\left[\,(2\,b +\; k_N)\,(2\,b \;+\; k) - b^2\,\right] \;=\; [\,a - c_N\,]\,[\,2\,b + k\,] - b\,[\;a - c\,]$$

$$\Rightarrow\;\; q_N \;=\; \frac{[\,a - c_N\,]\,[\,2\,b + k\,] - b\,[\,a - c\,]}{[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,] - b^2}\,. \tag{13}$$

(12) and (13) imply:

$$q \;=\; \frac{a - c}{2\,b + k} - \left[\frac{b}{2\,b + k}\right]\frac{[\,a - c_N\,]\,[\,2\,b + k\,] - b\,[\,a - c\,]}{[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,] - b^2}$$

$$=\; \frac{[\,a - c\,]\left[\,\left(2\,b + k_N + k^R\right)(2\,b + k) - b^2\,\right] - b\,[\,a - c_N\,]\,[\,2\,b + k\,] + b^2\,[\,a - c\,]}{[\,2\,b + k\,]\,[\,[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,] - b^2\,]}$$

$$=\; \frac{[\,a - c\,]\left[\,2\,b + k_N + k^R\,\right][\,2\,b + k\,] - b\,[\,a - c_N\,]\,[\,2\,b + k\,]}{[\,2\,b \;+\; k\,]\,[\,[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,] \;-\; b^2\,]}$$

$$=\; \frac{[\,a - c\,]\left[\,2\,b + k_N + k^R\,\right] - b\,[\,a - c_N\,]}{[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,] - b^2}\,. \tag{14}$$

(13) and (14) imply:

$$Q \;=\; q + q_N \;=\; \frac{[\,a - c\,]\left[\,b + k_N + k^R\,\right] + [\,a - c_N\,]\,[\,b + k\,]}{[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,] - b^2}\,. \tag{15}$$

From (6):

$$\overline{p}_0 \;=\; \frac{1}{[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,] - b^2}$$

$$\cdot \left\{ [\, a - c_N \,] [\, 2\, b + k \,] \left[b + k^R \right] - b [\, a - c \,] \left[b + k^R \right] \right.$$
$$\left. + \; c_A \left[\left(2\, b + k_N^R + k^R \right) (2\, b \; + \; k) - b^2 \right] \right\} . \tag{16}$$

(15) implies:

$$P(Q) \; = \; a - b \, \frac{[\, a - c \,] \left[b + k_N + k^R \right] + [\, a - c_N \,] [\, b + k \,]}{[\, 2\, b + k_N + k^R \,] [\, 2\, b + k \,] - b^2}$$
$$= \; \frac{a \left[\left(2\, b + k_N + k^R \right) (2\, b + k) - b^2 \right] - b [\, a - c \,] \left[b + k_N + k^R \right] - b [\, a - c_N \,] [\, b + k \,]}{[\, 2\, b + k_N + k^R \,] [\, 2\, b + k \,] - b^2} . \tag{17}$$

Observe that: $\left[2\, b + k_N + k^R \right] [\, 2\, b + k \,] \; > \; 4\, b^2 \; > \; b^2 .$

Therefore, (16) and (17) imply:

$$\overline{p}_0 \; < \; P(Q) \quad \Leftrightarrow \quad [\, a - c_N \,] [\, 2\, b + k \,] \left[b + k^R \right] - b [\, a - c \,] \left[b + k^R \right]$$
$$+ \; c_A \left[\left(2\, b + k_N + k^R \right) (2\, b \; + \; k) - b^2 \right]$$
$$< \; a \left[\left(2\, b + k_N + k^R \right) (2\, b + k) - b^2 \right]$$
$$- \; b [\, a - c \,] \left[b + k_N + k^R \right] - b [\, a - c_N \,] [\, b + k \,]$$
$$\Leftrightarrow \quad [\, a - c_N \,] [\, 2\, b + k \,] \left[b + k^R \right] - b [\, a - c \,] \left[b + k^R \right]$$
$$+ \; c_A \left[\left(2\, b + k_N + k^R \right) (2\, b + k) - b^2 \right]$$
$$< \; a \left[\left(2\, b + k_N + k^R \right) (2\, b + k) - b^2 \right]$$
$$- \; b [\, a - c \,] \left[b + k_N + k^R \right] - b [\, a - c_N \,] [\, b + k \,]$$
$$\Leftrightarrow \quad 0 \; < \; [\, a - c_A \,] \left[\left(2\, b + k_N \; + \; k^R \right) (2\, b + k) - b^2 \right] - b [\, a - c \,] \, k_N$$
$$- \; [\, a - c_N \,] \left[(2\, b + k) \left(b + k^R \right) + b\, (b + k) \right]$$
$$\Leftrightarrow \quad 0 \; < \; [\, a \; - c_A \,] \left[2\, b\, k + 2\, b\, k^R + k\, k^R + 3\, b^2 + 2\, b\, k_N + k\, k_N \right]$$
$$- \; b [\, a - c \,] \, k_N - [\, a - c_N \,] \left[2\, b\, k + 2\, b\, k^R + k\, k^R + 3\, b^2 \right]$$
$$\Leftrightarrow \quad [\, c_N - c_A \,] \left[2\, b\, k + 2\, b\, k^R + k\, k^R + 3\, b^2 \right]$$
$$+ \; k_N \left[(a - c_A) (2\, b + k) - b\, (a - c) \right] \; > \; 0 . \tag{18}$$

The last inequality in (18) reflects (3). Therefore, $\overline{p} \, < \, P(Q)$ when $\overline{p} \, \leq \, \overline{p}_0$.

It remains to show that $q_A \, = \, 0$ when $\overline{p} \, \leq \, \overline{p}_0$. Because $\overline{p} < P(Q)$ when $\overline{p} \, \leq \, \overline{p}_0$, $q_A \, = \, 0$

when:

$$\frac{\partial}{\partial q_A} \left\{ [\,\overline{p} - c_A\,]\, q_A + [\,a - b\,(q_A + q_N + q) - c_N\,]\, q_N - \frac{k_A}{2}\,[\,q_A\,]^2 - \frac{k_N}{2}\,[\,q_N\,]^2 - \frac{k^R}{2}\,[\,q_N + q_A\,]^2 \right\} \Bigg|_{q_A = 0} \leq 0$$

$$\Leftrightarrow \quad \overline{p} - c_A - b\, q_N - k^R q_N \leq 0$$

$$\Leftrightarrow \quad \overline{p} \leq c_A + \frac{[\,a - c_N\,]\,[\,2\,b + k\,] - b\,[\,a - c\,]}{[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,] - b^2} \left[\, b + k^R \,\right] = \overline{p}_0\,. \tag{19}$$

The equality in (19) reflects (13). □

Lemma A2. *Suppose* $\overline{p} \in (\,\overline{p}_0, \overline{p}_d\,]$. *Then in equilibrium:*

$$q_A = \frac{1}{D}\{\left[\, 3\,b^2 + 2\,b\left(k + k_N + k^R\right) + k\left(k_N + k^R\right)\right][\,\overline{p} - c_A\,] + b\left[\, b + k^R\,\right][\,a - c\,] - [\,2\,b + k\,]\left[\, b + k^R\,\right][\,a - c_N\,]\,\}\,; \tag{20}$$

$$q_N = \frac{1}{D}\{\,[\,2\,b + k\,]\left[\,k_A + k^R\,\right][\,a - c_N\,] - b\left[\,k_A + k^R\,\right][\,a - c\,] - \left[\, b\left(b + 2\,k^R\right) + k\left(b + k^R\right)\right][\,\overline{p} - c_A\,]\,\}\,; \tag{21}$$

$$Q^R \equiv q_A + q_N = \frac{1}{D}\{\,[\,2\,b + k\,]\,[\,b + k_N\,]\,[\,\overline{p} - c_A\,] + [\,2\,b + k\,]\,[\,k_A - b\,]\,[\,a - c_N\,] - b\,[\,k_A - b\,]\,[\,a - c\,]\,\}\,; \tag{22}$$

$$q = \frac{1}{D}\{\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R + 2\,b\,k_A - b^2\,\right][\,a - c\,] - b\,[\,k_A - b\,]\,[\,a - c_N\,] - b\,[\,b + k_N\,]\,[\,\overline{p} - c_A\,]\,\}\,; \text{ and} \tag{23}$$

$$Q = q + q_A + q_N = \frac{1}{D}\{\,[\,b + k\,]\,[\,b + k_N\,]\,[\,\overline{p} - c_A\,] + [\,b + k\,]\,[\,k_A - b\,]\,[\,a - c_N\,] + \left[\,k^R\,(\,k_A + k_N\,) + k_A\,(\,b + k_N\,)\right][\,a - c\,]\,\}\,. \tag{24}$$

Proof. (4) implies that if $q_A > 0$ and $\overline{p} < P(Q)$, R's problem, [P-R], is:

$$\underset{q_A,\,q_N}{\text{Maximize}} \;\; \overline{p}\, q_A + [\,a - b\,(\,q_A + q_N + q\,)\,]\, q_N - c_A\, q_A - \frac{k_A}{2}\,[\,q_A\,]^2 - c_N\, q_N - \frac{k_N}{2}\,[\,q_N\,]^2 - \frac{k^R}{2}\,[\,q_A + q_N\,]^2\,.$$

The necessary conditions for a solution to [P-R] in this case are:[2]

$$q_A: \qquad \overline{p} - b\, q_N - c_A - k_A\, q_A - k^R\, [\, q_A + q_N\,] \;=\; 0\,; \tag{25}$$

$$q_N: \qquad a - b\, [\, q_A + q_N + q\,] - b\, q_N - c_N - k_N\, q_N - k^R\, [\, q_A + q_N\,] \;=\; 0\,. \tag{26}$$

(25) implies:

$$\overline{p} - b\, q_N - c_A - k^R\, q_N \;=\; \left[\, k_A + k^R \,\right] q_A \;\;\Rightarrow\;\; q_A \;=\; \frac{\overline{p} - c_A}{k_A + k^R} - \left[\frac{b + k^R}{k_A + k^R}\right] q_N\,. \tag{27}$$

(26) implies:

$$a - b\, [\, q_A + q\,] - c_N - k^R\, q_A \;=\; \left[\, 2\, b + k_N + k^R \,\right] q_N$$

$$\Rightarrow\;\; q_N \;=\; \frac{a - c_N}{2\, b + k_N + k^R} - \frac{\left[\, b + k^R \,\right] q_A + b\, q}{2\, b + k_N + k^R}\,. \tag{28}$$

(25) also implies:

$$\overline{p} - c_A - k_A\, q_A - k^R\, q_A \;=\; \left[\, b + k^R \,\right] q_N \;\;\Rightarrow\;\; q_N \;=\; \frac{\overline{p} - c_A}{b + k^R} - \left[\frac{k_A + k^R}{b + k^R}\right] q_A\,. \tag{29}$$

(28) and (29) imply:

$$\frac{a - c_N}{2\, b + k_N + k^R} - \frac{\left[\, b + k^R \,\right] q_A + b\, q}{2\, b + k_N + k^R} \;=\; \frac{\overline{p} - c_A}{b + k^R} - \left[\frac{k_A + k^R}{b + k^R}\right] q_A$$

$$\Rightarrow\;\; \left[\frac{b + k^R}{2\, b + k_N + k^R} - \frac{k_A + k^R}{b + k^R}\right] q_A \;=\; \frac{a - c_N}{2\, b + k_N + k^R} - \frac{\overline{p} - c_A}{b + k^R} - \frac{b\, q}{2\, b + k_N + k^R}$$

$$\Rightarrow\;\; \left\{ \left[\, b + k^R \,\right]^2 - \left[\, k_A + k^R \,\right] \left[\, 2\, b + k_N + k^R \,\right] \right\} q_A$$

$$= \;\; \left[\, b + k^R \,\right] [\, a - c_N\,] - \left[\, 2\, b + k_N + k^R \,\right] [\, \overline{p} - c_A\,] - b\left[\, b + k^R \,\right] q$$

$$\Rightarrow\;\; q_A \;=\; \frac{\left[\, b + k^R \,\right] [\, a - c_N\,] - \left[\, 2\, b + k_N + k^R \,\right] [\, \overline{p} - c_A\,] - b\left[\, b + k^R \,\right] q}{\left[\, b + k^R \,\right]^2 - \left[\, k_A + k^R \,\right] \left[\, 2\, b + k_N + k^R \,\right]}\,. \tag{30}$$

(5) implies that the rival's problem in this setting, [P], is:

$$\underset{q}{\text{Maximize}} \quad [\, a - b\, (\, q_A + q_N + q\,) - c\,]\, q - \frac{k}{2}\, (q)^2\,. \tag{31}$$

The necessary condition for an interior solution to [P] is:

$$a - b\, [\, q_A + q_N + q\,] - c - b\, q - k\, q \;=\; 0 \;\;\Leftrightarrow\;\; [\, 2\, b + k\,]\, q \;=\; a - b\, [\, q_A + q_N\,] - c$$

[2]It is readily verified that the determinant of the Hessian associated with [P-R] in this setting is $\left[\, k_A + k^R \,\right] \left[\, 2\, b + k_N + k^R \,\right] - \left[\, b + k^R \,\right]^2$, which is strictly positive if $k_A \geq \frac{b}{2}$.

$$\Leftrightarrow \quad q \;=\; \frac{a-c}{2\,b+k} - \frac{b}{2\,b+k}\;[\,q_A + q_N\,]\,. \tag{32}$$

(29) and (32) imply:

$$\begin{aligned} q \;&=\; \frac{a-c}{2\,b+k} - \frac{b}{2\,b+k}\left[\,q_A + \frac{\overline{p}-c_A}{b+k^R} - \left(\frac{k_A+k^R}{b+k^R}\right) q_A\,\right] \\ &=\; \frac{a-c}{2\,b+k} - \frac{b}{2\,b+k}\left[\frac{\overline{p}-c_A}{b+k^R}\right] - \frac{b}{2\,b+k}\left[1 - \frac{k_A+k^R}{b+k^R}\right] q_A \\ &=\; \frac{a-c}{2\,b+k} - \frac{b}{2\,b+k}\left[\frac{\overline{p}-c_A}{b+k^R}\right] - \frac{b}{2\,b+k}\left[\frac{b-k_A}{b+k^R}\right] q_A\,. \end{aligned} \tag{33}$$

(30) and (33) imply:

$$\begin{aligned} q_A \;&=\; \frac{\left[\,b+k^R\,\right][\,a-c_N\,] - \left[\,2\,b+k_N+k^R\,\right][\,\overline{p}-c_A\,]}{[\,b+k^R\,]^2 - [\,k_A+k^R\,][\,2\,b+k_N+k^R\,]} \\ &\quad - \frac{b\left[\,b+k^R\,\right]}{[\,b+k^R\,]^2 - [\,k_A+k^R\,][\,2\,b+k_N+k^R\,]} \\ &\qquad \cdot \left\{ \frac{a-c}{2\,b+k} - \frac{b}{2\,b+k}\left[\frac{\overline{p}-c_A}{b+k^R}\right] - \frac{b}{2\,b+k}\left[\frac{b-k_A}{b+k^R}\right] q_A \right\} \end{aligned}$$

$$\Rightarrow \quad q_A\left[1 - \left(\frac{b\left[\,b+k^R\,\right]}{[\,b+k^R\,]^2 - [\,k_A+k^R\,][\,2\,b+k_N+k^R\,]}\right)\left(\frac{b}{2\,b+k}\right)\left(\frac{b-k_A}{b+k^R}\right)\right]$$

$$\begin{aligned} &= \frac{\left[\,b+k^R\,\right][\,a-c_N\,] - \left[\,2\,b+k_N+k^R\,\right][\,\overline{p}-c_A\,]}{[\,b+k^R\,]^2 - [\,k_A+k^R\,][\,2\,b+k_N+k^R\,]} \\ &\quad - \frac{b\left[\,b+k^R\,\right]}{[\,b+k^R\,]^2 - [\,k_A+k^R\,][\,2\,b+k_N+k^R\,]}\left[\frac{a-c}{2\,b+k} - \frac{b}{2\,b+k}\left(\frac{\overline{p}-c_A}{b+k^R}\right)\right] \end{aligned}$$

$$\Rightarrow \quad q_A\left[1 - \frac{b^2\,[\,b-k_A\,]}{[\,2\,b+k\,]\left\{[\,b+k^R\,]^2 - [\,k_A+k^R\,][\,2\,b+k_N+k^R\,]\right\}}\right]$$

$$\begin{aligned} &= \frac{[\,2\,b+k\,]\left\{\left[\,b+k^R\,\right][\,a-c_N\,] - \left[\,2\,b+k_N+k^R\,\right][\,\overline{p}-c_A\,]\right\}}{[\,2\,b+k\,]\left\{[\,b+k^R\,]^2 - [\,k_A+k^R\,][\,2\,b+k_N+k^R\,]\right\}} \\ &\quad - \frac{b\left[\,b+k^R\,\right]\left[\,a-c-b\left(\frac{\overline{p}-c_A}{b+k^R}\right)\right]}{[\,2\,b+k\,]\left\{[\,b+k^R\,]^2 - [\,k_A+k^R\,][\,2\,b+k_N+k^R\,]\right\}} \end{aligned}$$

$$\Rightarrow \quad q_A\left\{[\,2\,b+k\,]\left(\left[\,b+k^R\,\right]^2 - \left[\,k_A+k^R\,\right]\left[\,2\,b+k_N+k^R\,\right]\right) - b^2\,[\,b-k_A\,]\right\}$$

$$= \left[2b + k \right] \left\{ \left[b + k^R \right] \left[a - c_N \right] - \left[2b + k_N + k^R \right] \left[\overline{p} - c_A \right] \right\}$$
$$- b \left[(a - c) \left(b + k^R \right) - b \left(\overline{p} - c_A \right) \right] . \tag{34}$$

Observe that:

$$\left[2b + k \right] \left\{ \left[b + k^R \right]^2 - \left[k_A + k^R \right] \left[2b + k_N + k^R \right] \right\} - b^2 \left[b - k_A \right]$$
$$= \left[2b + k \right] \left\{ b^2 + 2b k^R + \left(k^R \right)^2 - 2b k_A - 2b k^R - k_A k_N - k^R k_N - k_A k^R - \left(k^R \right)^2 \right\}$$
$$- b^3 + b^2 k_A$$
$$= \left[2b + k \right] \left[b^2 - 2b k_A - k_A k_N - k^R k_N - k_A k^R \right] - b^3 + b^2 k_A$$
$$= 2b^3 - 4b^2 k_A - 2b k_A k_N - 2b k^R k_N - 2b k_A k^R$$
$$+ b^2 k - 2b k k_A - k k_A k_N - k k^R k_N - k k_A k^R - b^3 + b^2 k_A$$
$$= b^3 - 3b^2 k_A - 2b k_A k_N - 2b k^R k_N - 2b k_A k^R$$
$$+ b^2 k - 2b k k_A - k k_A k_N - k k^R k_N - k k_A k^R$$
$$= b^2 \left[b + k \right] - b k_A \left[3b + 2k \right] - \left[2b + k \right] \left[k_N \left(k_A + k^R \right) + k_A k^R \right] . \tag{35}$$

Further observe that:

$$\left[2b + k \right] \left[2b + k_N + k^R \right] - b^2 = 2b \left[2b + k_N + k^R \right] + k \left[2b + k_N + k^R \right] - b^2$$
$$= 3b^2 + 2b \left[k_N + k^R \right] + k \left[2b + k_N + k^R \right]$$
$$= 3b^2 + 2b \left[k + k_N + k^R \right] + k \left[k_N + k^R \right] . \tag{36}$$

(2) and (34) – (36) imply:

$$q_A = \frac{1}{D} \left\{ \left[3b^2 + 2b \left(k + k_N + k^R \right) + k \left(k_N + k^R \right) \right] \left[\overline{p} - c_A \right] \right.$$
$$\left. + \; b \left[b + k^R \right] \left[a - c \right] - \left[2b + k \right] \left[b + k^R \right] \left[a - c_N \right] \right\} . \tag{37}$$

(2), (29), and (37) imply:

$$q_N = \frac{\overline{p} - c_A}{b + k^R} - \left[\frac{k_A + k^R}{b + k^R} \right] \frac{1}{D} \left\{ \left[3b^2 + 2b \left(k + k_N + k^R \right) + k \left(k_N + k^R \right) \right] \left[\overline{p} - c_A \right] \right.$$
$$\left. + \; b \left[b + k^R \right] \left[a - c \right] - \left[2b + k \right] \left[b + k^R \right] \left[a - c_N \right] \right\}$$
$$= \frac{1}{D \left[b + k^R \right]} \left\{ \left[\overline{p} - c_A \right] D \right.$$
$$- \left[k_A + k^R \right] \left[3b^2 + 2b \left(k + k_N + k^R \right) + k \left(k_N + k^R \right) \right] \left[\overline{p} - c_A \right]$$

$$
\begin{aligned}
&- b \left[b + k^R \right] \left[k_A + k^R \right] [a - c] \\
&+ [2 b + k] \left[k_A + k^R \right] \left[b + k^R \right] [a - c_N] \}. \qquad (38)
\end{aligned}
$$

(2) implies:

$$
\begin{aligned}
D &- \left[k_A + k^R \right] \left[3 b^2 + 2 b \left(k + k_N + k^R \right) + k \left(k_N + k^R \right) \right] \\
&= [2 b + k] \left[k_N \left(k_A + k^R \right) + k_A k^R \right] + b k_A [3 b + 2 k] - b^2 [b + k] \\
&\quad - \left[k_A + k^R \right] \left[3 b^2 + 2 b \left(k + k_N + k^R \right) + k \left(k_N + k^R \right) \right] \\
&= \left[k_A + k^R \right] \left[(2 b + k) k_N - 3 b^2 - 2 b \left(k + k_N + k^R \right) - k \left(k_N + k^R \right) \right] \\
&\quad + [2 b + k] k_A k^R + 3 b^2 k_A + 2 b k k_A - b^3 - b^2 k \\
&= \left[k_A + k^R \right] \left[- 3 b^2 - 2 b k - 2 b k^R - k k^R \right] \\
&\quad + 2 b k_A k^R + k k_A k^R + 3 b^2 k_A + 2 b k k_A - b^3 - b^2 k \\
&= - 3 b^2 k_A - 3 b^2 k^R - 2 b k k_A - 2 b k k^R - 2 b k_A k^R - 2 b \left(k^R \right)^2 - k k_A k^R \\
&\quad - k \left(k^R \right)^2 + 2 b k_A k^R + k k_A k^R + 3 b^2 k_A + 2 b k k_A - b^3 - b^2 k \\
&= - 3 b^2 k^R - 2 b k k^R - 2 b k_A k^R - 2 b \left(k^R \right)^2 - k \left(k^R \right)^2 \\
&\quad + 2 b k_A k^R - b^3 - b^2 k \\
&= - b^2 k^R - 2 b^2 k^R - b k k^R - b k k^R - 2 b \left(k^R \right)^2 - k \left(k^R \right)^2 - b^2 b - b k b \\
&= - b^2 \left[b + k^R \right] - 2 b^2 k^R - b k \left[b + k^R \right] - b k k^R - 2 b \left(k^R \right)^2 - k \left(k^R \right)^2 \\
&= - b^2 \left[b + k^R \right] - 2 b k^R \left[b + k^R \right] - b k \left[b + k^R \right] - k k^R \left[b + k^R \right] \\
&= - \left[b + k^R \right] \left[b^2 + 2 b k^R + b k + k k^R \right] \\
&= - \left[b + k^R \right] \left[b \left(b + 2 k^R \right) + k \left(b + k^R \right) \right]. \qquad (39)
\end{aligned}
$$

(38) and (39) imply:

$$
\begin{aligned}
q_N = \frac{1}{D} \{ & [2 b + k] \left[k_A + k^R \right] [a - c_N] - b \left[k_A + k^R \right] [a - c] \\
& - \left[b \left(b + 2 k^R \right) + k \left(b + k^R \right) \right] [\overline{p} - c_A] \}. \qquad (40)
\end{aligned}
$$

Observe that:

$$
3 b^2 + 2 b \left[k + k_N + k^R \right] + k \left[k_N + k^R \right] - \left[b \left(b + 2 k^R \right) + k \left(b + k^R \right) \right]
$$

$$= \; 3\,b^2 + 2\,b\,k + 2\,b\,k_N + 2\,b\,k^R + k\,k_N + k\,k^R - b^2 - 2\,b\,k^R - b\,k - k\,k^R$$

$$= \; 2\,b^2 + b\,k + 2\,b\,k_N + k\,k_N \; = \; b\,[\,2\,b + k\,] + k_N\,[\,2\,b + k\,] \; = \; [\,2\,b + k\,]\,[\,b + k_N\,]\,. \quad (41)$$

Further observe that:

$$b\left[\,b + k^R\,\right] - b\left[\,k_A + k^R\,\right] \; = \; b\,[\,b - k_A\,] \quad \text{and}$$

$$[\,2\,b + k\,]\left[\,k_A + k^R\,\right] - [\,2\,b + k\,]\left[\,b + k^R\,\right] \; = \; [\,2\,b + k\,]\,[\,k_A - b\,]\,. \quad (42)$$

(37) and (40) – (42) imply:

$$q_A + q_N \; = \; \frac{1}{D}\,\{\,[\,2\,b + k\,]\,[\,b + k_N\,]\,[\,\overline{p} - c_A\,] - b\,[\,k_A - b\,]\,[\,a - c\,]$$
$$+ \; [\,2\,b + k\,]\,[\,k_A - b\,]\,[\,a - c_N\,]\,\}\,. \quad (43)$$

(32) and (43) imply:

$$q \; = \; \frac{a - c}{2\,b + k} - \left[\frac{b}{2\,b + k}\right]\frac{1}{D}\,\{\,[\,2\,b + k\,]\,[\,b + k_N\,]\,[\,\overline{p} - c_A\,] - b\,[\,k_A - b\,]\,[\,a - c\,]$$
$$+ \; [\,2\,b + k\,]\,[\,k_A - b\,]\,[\,a - c_N\,]\,\}$$
$$= \; \frac{D + b^2\,[\,k_A - b\,]}{D\,[\,2\,b + k\,]}\,[\,a - c\,]$$
$$- \; \frac{b}{D}\,\{\,[\,b + k_N\,]\,[\,\overline{p} - c_A\,] + [\,2\,b + k\,]\,[\,k_A - b\,]\,[\,a - c_N\,]\,\}\,. \quad (44)$$

(2) implies:

$$D + b^2\,[\,k_A - b\,] \; = \; [\,2\,b + k\,]\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R\,\right] + b\,k_A\,[\,3\,b + 2\,k\,]$$
$$- \; b^2\,[\,b + k\,] + b^2\,[\,k_A - b\,]$$
$$= \; [\,2\,b + k\,]\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R\,\right] + 4\,b^2 k_A + 2\,b\,k\,k_A - b^2\,[\,2\,b + k\,]$$
$$= \; [\,2\,b + k\,]\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R\,\right] + 2\,b\,k_A\,[\,2\,b + k\,] - b^2\,[\,2\,b + k\,]$$
$$= \; [\,2\,b + k\,]\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R + 2\,b\,k_A - b^2\,\right]. \quad (45)$$

(44) and (45) imply:

$$q \; = \; \frac{1}{D}\,\{\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R + 2\,b\,k_A - b^2\,\right][\,a - c\,]$$
$$- \; b\,[\,k_A - b\,]\,[\,a - c_N\,] - b\,[\,b + k_N\,]\,[\,\overline{p} - c_A\,]\,\}\,. \quad (46)$$

Observe that:

$$[\,2\,b + k\,]\,[\,b + k_N\,] - b\,[\,b + k_N\,] \; = \; [\,b + k\,]\,[\,b + k_N\,]\,;$$

$$[\,2\,b + k\,]\,[\,k_A - b\,] - b\,[\,k_A - b\,] \; = \; [\,b + k\,]\,[\,k_A - b\,]\,; \quad \text{and}$$

$$k_N \left[k_A + k^R \right] + k_A k^R + 2 b k_A - b^2 - b [k_A - b]$$

$$= k_N \left[k_A + k^R \right] + k_A k^R + b k_A = k^R [k_A + k_N] + k_A [b + k_N] . \tag{47}$$

(43), (46), and (47) imply:

$$Q = q + q_A + q_N = \frac{1}{D} \{ [b + k] [b + k_N] [\overline{p} - c_A] + [b + k] [k_A - b] [a - c_N]$$

$$+ \left[k^R (k_A + k_N) + k_A (b + k_N) \right] [a - c] \} . \tag{48}$$

It remains to show that $q_A > 0$ and $\overline{p} \le P(Q)$ when $\overline{p} \in (\overline{p}_0, \overline{p}_d]$. (37) implies that $q_A > 0$ if:

$$b [b + k_N] [a - c] + [\overline{p} - c_A] \left[2 b k + 2 b k_N + 2 b k^R + k k_N + k k^R + 3 b^2 \right]$$

$$- [a - c_N] \left[b k + 2 b k^R + k k^R + 2 b^2 \right] > 0$$

$$\Leftrightarrow \quad c_A + \frac{[a - c_N] \left[b k + 2 b k^R + k k^R + 2 b^2 \right] - b [b + k_N] [a - c]}{2 b k + 2 b k_N + 2 b k^R + k k_N + k k^R + 3 b^2} < \overline{p}$$

$$\Leftrightarrow \quad \overline{p} > c_A + \frac{[a - c_N] [2 b + k] - b [a - c]}{[2 b + k_N + k^R] [2 b + k] - b^2} \left[b + k^R \right] = \overline{p}_0 .$$

The equality here reflects (6). (48) implies:

$$Q = \frac{1}{D} [C_1 a + C_2 c + C_3 c_N + C_4 c_A - C_4 \overline{p}] \tag{49}$$

$$\text{where } C_1 \equiv [b + k] [k_A - b] + k^R [k_A + k_N] + k_A [b + k_N]$$

$$= b k_A + b k_A + k k_A - b k + k_A k^R + k_N k^R + b k_A + k_A k_N$$

$$= 2 b k_A + k k_A + k_A k_N + k_A k^R + k_N k^R - b^2 - b k ;$$

$$C_2 \equiv - k^R [k_A + k_N] - k_A [b + k_N] ; \;\; C_3 \equiv - [b + k] [k_A - b] ; \text{ and}$$

$$C_4 \equiv - [b + k] [b + k_N] . \tag{50}$$

(49) implies:

$$P(Q) = a - b Q = \frac{[D - b C_1] a - b c C_2 - b C_3 c_N - b C_4 c_A + b C_4 \overline{p}}{D} . \tag{51}$$

(2) and (50) imply:

$$D - b C_1 = [2 b + k] \left[k_N \left(k_A + k^R \right) + k_A k^R \right] + b k_A [3 b + 2 k] - b^2 [b + k]$$

$$- b \left[2 b k_A + k k_A + k_A k_N + k_A k^R + k_N k^R - b^2 - b k \right]$$

$$= 2 b k_A k_N + 2 b k_N k^R + 2 b k_A k^R + k k_A k_N + k k_N k^R + k k_A k^R + 3 b^2 k_A$$

$$
\begin{aligned}
&+\ 2\,b\,k\,k_A - b\,k - 2\,b^2 k_A - b\,k\,k_A - b\,k_A\,k_N - b\,k_A\,k^R - b\,k_N\,k^R + b^2 k \\
&= b^2 k_A + b\,k\,k_A + b\,k_A\,k_N + b\,k_A\,k^R + b\,k_N\,k^R + k\,k_A\,k_N + k\,k_A\,k^R + k\,k_N\,k^R \\
&= [\,b + k\,]\left[\,b\,k_A + k_A\,k_N + k_A\,k^R + k_N\,k^R\,\right] \\
&= [\,b + k\,]\,[\,(b + k_A)\,(k_N + k_A) + k_N\,k_A - b\,k_N\,]\,. \qquad (52)
\end{aligned}
$$

(2) and (50) imply:

$$
\begin{aligned}
D\ -\ b\,C_4\ &=\ [\,2\,b + k\,]\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R\,\right] + b\,k_A\,[\,3\,b + 2\,k\,] - b^2\,[\,b + k\,] \\
&\quad - b\,[\,b + k\,]\,[\,b + k_N\,] \\
&= 3\,b^2 k_A - b^2 k - b^3 + 2\,b\,k\,k_A + 2\,b\,k_A\,k_N + 2\,b\,k_A\,k^R + 2\,b\,k_N\,k^R + k\,k_A\,k_N \\
&\quad +\ k\,k_A\,k^R + k\,k_N\,k^R + k_N\,b^2 + k_N\,k\,b + b^3 + b^2 k \\
&= 3\,b^2 k_A + 2\,b\,k\,k_A + 2\,b\,k_A\,k_N + 2\,b\,k_A\,k^R + 2\,b\,k_N\,k^R + k\,k_A\,k_N \\
&\quad + k\,k_A\,k^R + k\,k_N\,k^R + k_N\,b^2 + k_N\,k\,b \\
&= b\,[\,b + k\,]\,[\,k_N + k_A\,] + [\,k_A\,k_N - k_N\,b\,]\,[\,2\,b + k\,] \\
&\quad +\ [\,k_N + k_A\,]\,[\,2\,b + k\,]\left[\,b + k^R\,\right]. \qquad (53)
\end{aligned}
$$

(51) implies:

$$
\begin{aligned}
\overline{p}\ \leq\ P(Q)\ &=\ \frac{[\,D - b\,C_1\,]\,a\ - b\,c\,C_2 -\ b\,C_3\,c_N\ -\ b\,C_4\,c_A + b\,C_4\,\overline{p}}{D} \\
\Leftrightarrow\ \ \overline{p} - \frac{b\,C_4}{D}\,\overline{p}\ &\leq\ \frac{[\,D - b\,C_1\,]\,a - b\,c\,C_2 - b\,C_3\,c_N - b\,C_4\,c_A}{D} \\
\Leftrightarrow\ \ \overline{p}\left[1\ -\ \frac{b\,C_4}{D}\right]\ &\leq\ \frac{[\,D - b\,C_1\,]\,a - b\,c\,C_2 - b\,C_3\,c_N - b\,C_4\,c_A}{D} \qquad (54) \\
\Leftrightarrow\ \ \overline{p}\,[\,D - b\,C_4\,]\ &\leq\ [\,D - b\,C_1\,]\,a - b\,c\,C_2 - b\,C_3\,c_N - b\,C_4\,c_A \\
\Leftrightarrow\ \ \overline{p}\ &\leq\ \frac{[\,D - b\,C_1\,]\,a - b\,c\,C_2 - b\,C_3\,c_N - b\,C_4\,c_A}{D - b\,C_4}\,. \qquad (55)
\end{aligned}
$$

(54) reflects the fact that $D - b\,C_4 > 0$ because $C_4 < 0$ (from (50)), and because $D > 0$, by assumption.

(50), (52), (53), and (55) imply:

$$
\begin{aligned}
\overline{p}\ \leq\ \frac{1}{D_2}\,\{\,a\,[\,b + k\,]\left[\,\left(b + k^R\right)(k_N + k_A) + k_N\,k_A - b\,k_N\,\right] \\
+\ b\,c\left[\,(k_N + k_A)\left(b + k^R\right) + k_A\,k_N - b\,k_N\,\right]
\end{aligned}
$$

$$+ \, b\,[\,b+k\,]\,[\,k_A - b\,]\,c_N + b\,[\,k_N + b\,]\,[\,b+k\,]\,c_A\,\}$$

$$\Leftrightarrow \quad \overline{p} \;\leq\; \frac{1}{D_2}\,\{\,[\,(b+k)\,a + b\,c\,]\,\left[\,\left(b + k^R\right)(k_N + k_A) + k_N\,k_A - b\,k_N\,\right]$$
$$+ \;\; b\,[\,b+k\,]\,[\,k_A - b\,]\,c_N + b\,[\,k_N + b\,]\,[\,b+k\,]\,c_A\,\} \;=\; \overline{p}_d\,. \tag{56}$$

The equality in (56) reflects (7). (55) and (56) imply that $\overline{p} \leq P(Q)$ (and $q_A > 0$) when $\overline{p} \in (\,\overline{p}_0, \overline{p}_d\,]$. □

Lemma A3. *Suppose* $\overline{p} \in (\,\overline{p}_d, \overline{p}_b\,]$, *where* $\overline{p}_d < \overline{p}_b$. *Then in equilibrium,* $P(Q) = \overline{p}$. *Furthermore*:

$$q_A \;=\; \frac{b\,[\,b+k\,]\,[\,c_N - c_A\,] + k_N\,[\,a - \overline{p}\,]\,[\,b+k\,] - b\,k_N\,[\,\overline{p} - c\,]}{b\,[\,b+k\,]\,[\,k_N + k_A\,]}\,;$$

$$q_N \;=\; \frac{k_A\,[\,b+k\,]\,[\,a - \overline{p}\,] - b\,k_A\,[\,\overline{p} - c\,] - b\,[\,b+k\,]\,[\,c_N - c_A\,]}{b\,[\,b+k\,]\,[\,k_N + k_A\,]}\,;$$

$$Q^R \;\equiv\; q_A + q_N \;=\; \frac{[\,b+k\,]\,[\,a - \overline{p}\,] - b\,[\,\overline{p} - c\,]}{b\,[\,b+k\,]}\,;$$

$$q \;=\; \frac{\overline{p} - c}{b+k}\,; \quad \text{and} \quad Q \;\equiv\; \frac{a - \overline{p}}{b}\,. \tag{57}$$

<u>Proof</u>. (4) implies that R's problem, [P-R], can be written as:

$$\underset{q_A,\,Q^R}{\text{Maximize}} \;\; \Pi_R \;\equiv\; \left[\,P_A(q + Q^R) - c_A\,\right] q_A + \left[\,P(Q^R + q) - c_N\,\right]\left[\,Q^R - q_A\,\right]$$
$$-\;\; \frac{k_A}{2}\,[\,q_A\,]^2 - \frac{k_N}{2}\,\left[\,Q^R - q_A\,\right]^2 - \frac{k^R}{2}\,\left[\,Q^R\,\right]^2$$

$$\text{where} \;\; P_A(q \,+\, Q^R) \;=\; \begin{cases} \overline{p} & \text{if } \; P(\,q + Q^R\,) \,\geq\, \overline{p} \\ P(\,q + Q^R\,) & \text{if } \; \overline{p} \,>\, P(\,q + Q^R\,). \end{cases} \tag{58}$$

(58) implies that when $q_A > 0$ and there exists a range of $\overline{p}$ for which $P(Q) = \overline{p}$, the necessary conditions for a solution to R's problem are:

$$\frac{\partial \Pi_R}{\partial q_A} \;=\; P_A\left(q + Q^R\right) - c_A - k_A\,q_A - \left[\,P\left(q + Q^R\right) - c_N\,\right] + k_N\left[\,Q^R - q_A\,\right] \;=\; 0\,; \tag{59}$$

$$\frac{\partial^+ \Pi^R}{\partial Q^R} \;\leq\; 0 \;\; \text{and} \;\; \frac{\partial^- \Pi^R}{\partial Q^R} \;\geq\; 0 \;\; \text{for all} \;\; \overline{p} \in [\,\overline{p}_d, \overline{p}_b\,]\,, \tag{60}$$

where: (i) $\frac{\partial^- \Pi_R}{\partial Q^R}$ denotes the left-sided derivative of Π_R with respect to Q^R, which is relevant when $P_A(\cdot) = \overline{p}$; and (ii) $\frac{\partial^+ \Pi_R}{\partial Q^R}$ denotes the right-sided derivative of Π_R with respect to Q^R, which is relevant when $P_A(\cdot) = P(Q)$. The first inequality in (60) indicates that R's profit declines if R increases Q^R so as to reduce $P(Q)$ below $\overline{p}$ (thereby rendering the cap nonbinding). The second inequality in (60) indicates that R's profit declines if R reduces Q^R so as to increase $P(Q)$ above $\overline{p}$ (thereby causing the cap to bind). Together, the inequalities in (60) ensure that when $\overline{p} \in [\overline{p}_d, \overline{p}_b]$, R cannot increase its profit by changing Q^R so as to cause $P(Q)$ to differ from $\overline{p}$.

(12) implies:

$$a - b\,Q - b\,q - c - k\,q \;=\; 0$$

$$\Leftrightarrow \quad \overline{p} - b\,q - c - k\,q \;=\; 0 \quad \Leftrightarrow \quad q \;=\; \frac{\overline{p} - c}{b + k}\,. \tag{61}$$

Because $\overline{p} = a - b\left[\,q + Q^R\,\right]$, (61) implies:

$$\overline{p} \;=\; a - b\left[\frac{\overline{p} - c}{b + k} + Q^R\right] \quad \Leftrightarrow \quad b\,Q^R \;=\; a - \overline{p} - b\left[\frac{\overline{p} - c}{b + k}\right]$$

$$\Leftrightarrow \quad Q^R \;=\; \frac{a - \overline{p}}{b} - \frac{\overline{p} - c}{b + k} \;=\; \frac{[\,a - \overline{p}\,]\,[\,b + k\,] - b\,[\,\overline{p} - c\,]}{b\,[\,b + k\,]}\,. \tag{62}$$

Because $\overline{p} = P_A(q + Q^R)$ in equilibrium, by assumption, (59) holds if:

$$\overline{p} - c_A - k_A\,q_A - [\,\overline{p} - c_N\,] + k_N\left[\,Q^R - q_A\,\right] \;=\; 0$$

$$\Leftrightarrow \quad c_N - c_A - k_A\,q_A + k_N\,Q^R - k_N\,q_A \;=\; 0\,. \tag{63}$$

(62) implies that (63) holds if:

$$c_N - c_A - k_A\,q_A + k_N\,\frac{[\,a - \overline{p}\,]\,[\,b + k\,] - b\,[\,\overline{p} - c\,]}{b\,[\,b + k\,]} - k_N\,q_A \;=\; 0$$

$$\Leftrightarrow \quad q_A\,[\,k_N + k_A\,] \;=\; c_N - c_A + k_N\,\frac{[\,a - \overline{p}\,]\,[\,b + k\,] - b\,[\,\overline{p} - c\,]}{b\,[\,b + k\,]}$$

$$\Leftrightarrow \quad q_A \;=\; \frac{c_N - c_A}{k_N + k_A} + k_N\,\frac{[\,a - \overline{p}\,]\,[\,b + k\,] - b\,[\,\overline{p} - c\,]}{b\,[\,b + k\,]\,[\,k_N + k_A\,]}$$

$$=\; \frac{b\,[\,b + k\,]\,[\,c_N - c_A\,] + k_N\,[\,a - \overline{p}\,]\,[\,b + k\,] - b\,k_N\,[\,\overline{p} - c\,]}{b\,[\,b + k\,]\,[\,k_N + k_A\,]}\,. \tag{64}$$

(62) and (64) imply:

$$q_N \;=\; Q^R - q_A \;=\; \frac{[\,a - \overline{p}\,]\,[\,b + k\,] - b\,[\,\overline{p} - c\,]}{b\,[\,b + k\,]}$$

$$-\; \frac{b\,[\,b + k\,]\,[\,c_N - c_A\,] + k_N\,[\,a - \overline{p}\,]\,[\,b + k\,] - b\,k_N\,[\,\overline{p} - c\,]}{b\,[\,b + k\,]\,[\,k_N + k_A\,]}$$

$$=\; \frac{[\,a - \overline{p}\,]\,[\,b + k\,]\,[\,k_N + k_A\,] - b\,[\,\overline{p} - c\,]\,[\,k_N + k_A\,]}{b\,[\,b + k\,]\,[\,k_N + k_A\,]}$$

$$- \frac{b\,[\,b+k\,]\,[\,c_N - c_A\,] + k_N\,[\,a-\overline{p}\,]\,[\,b+k\,] - b\,k_N\,[\,\overline{p}-c\,]}{b\,[\,b+k\,]\,[\,k_N + k_A\,]}$$

$$= \frac{k_A\,[\,b+k\,]\,[\,a-\overline{p}\,] - b\,k_A\,[\,\overline{p}-c\,] \;-\; b\,[\,b+k\,]\,[\,c_N - c_A\,]}{b\,[\,b+k\,]\,[\,k_N + k_A\,]}\,. \tag{65}$$

(64) and (65) imply:

$$Q^R \;\equiv\; q_A \,+\, q_N \;=\; \frac{1}{b\,[\,b+k\,]\,[\,k_N + k_A\,]}\;\{\,[\,k_N + k_A\,]\,[\,b+k\,]\,[\,a-\overline{p}\,] \;-\; b\,[\,k_N + k_A\,]\,[\,\overline{p}-c\,]\,\}$$

$$= \frac{[\,b+k\,]\,[\,a-\,\overline{p}\,\,] - b\,[\,\overline{p}-c\,]}{b\,[\,b+k\,]}\,. \tag{66}$$

(61) and (66) imply:

$$Q \;\equiv\; Q^R + q \;=\; \frac{[\,b+k\,]\,[\,a-\,\overline{p}\,\,] - b\,[\,\overline{p}-c\,]}{b\,[\,b+k\,]} + \frac{b\,[\,\overline{p}-c\,]}{b\,[\,b+k\,]} \;=\; \frac{a-\overline{p}}{b}\,.$$

(58) implies:

$$\frac{\partial^+ \Pi_R}{\partial Q^R} \;=\; -\,b\,q_A + a - 2\,b\,Q^R - b\,q - c_N + b\,q_A - k_N\left[\,Q^R - q_A\,\right] - k^R\,Q^R$$

$$= \; a - 2\,b\,Q^R - b\,q - c_N - k_N\left[\,Q^R - q_A\,\right] - k^R\,Q^R$$

$$= \; \overline{p} - b\,Q^R - c_N - k_N\,q_N - k^R\,Q^R \;=\; \overline{p} - \left[\,b + k^R\,\right]Q^R - c_N - k_N\,q_N\,; \tag{67}$$

$$\frac{\partial^- \Pi_R}{\partial Q^R} \;=\; a - 2\,b\,Q^R - b\,q - c_N + b\,q_A - k_N\left[\,Q^R - q_A\,\right] - k^R\,Q^R$$

$$= \; a - 2\,b\,Q^R - b\,q - c_N + b\,q_A - k_N\,q_N - k^R\,Q^R$$

$$= \; \overline{p} - b\,Q^R - c_N + b\,q_A - k_N\,q_N - k^R\,Q^R$$

$$= \; \overline{p} - \left[\,b + k^R\,\right]Q^R - c_N + b\,q_A - k_N\,q_N\,. \tag{68}$$

(67) and (68) imply that (60) can be written as:

$$\overline{p} - \left[\,b + k^R\,\right]Q^R - c_N - k_N\,q_N \;\le\; 0 \;<\; \overline{p} - \left[\,b + k^R\,\right]Q^R - c_N + b\,q_A - k_N\,q_N$$

$$\Leftrightarrow \quad \left[\,b + k^R\,\right]Q^R + c_N + k_N\,q_N - b\,q_A \;<\; \overline{p} \;\le\; \left[\,b + k^R\,\right]Q^R + c_N + k_N\,q_N\,. \tag{69}$$

(62) and (65) imply:

$$\overline{p} \;\le\; \left[\,b + k^R\right]Q^R + c_N + k_N\,q_N$$

$$\Leftrightarrow \quad \left[\,b + k^R\right]\;\frac{[\,a-\overline{p}\,]\,[\,b+k\,] \;-\; b\,[\,\overline{p}-c\,]}{b\,[\,b+k\,]} + c_N$$

$$+\ k_N\ \frac{k_A\,[\,b+k\,]\,[\,a-\overline{p}\,]-b\,k_A\,[\,\overline{p}-c\,]-b\,[\,b+k\,]\,[\,c_N-c_A\,]}{b\,[\,b+k\,]\,[\,k_N+k_A\,]}\ \geq\ \overline{p}$$

$$\Leftrightarrow\quad \left[\,b+k^R\,\right]\ \frac{a\,[\,b+k\,]-\overline{p}\ [\,2\,b+k\,]+b\,c}{b\,[\,b+k\,]}+c_N$$

$$+\ k_N\ \frac{k_A\,[\,b+k\,]\,a-\overline{p}\ k_A\,[\,2\,b+k\,]+b\,k_A\,c-b\,[\,b+k\,]\,[\,c_N-c_A\,]}{b\,[\,b+k\,]\,[\,k_N+k_A\,]}\ \geq\ \overline{p}$$

$$\Leftrightarrow\quad \left[\,b+k^R\,\right]\ \frac{a\,[\,b+\ k]+b\,c}{b\,[\,b+k\,]}+c_N+k_N\ \frac{k_A\,[\,b+k\,]\,a+b\,k_A\,c-b\,[\,b+k\,]\,[\,c_N-c_A\,]}{b\,[\,b+k\,]\,[\,k_N+k_A\,]}$$

$$\geq\ \overline{p}+\overline{p}\ \frac{k_N\,k_A\,[\,2\,b+k\,]}{b\,[\,b+k\,]\,[\,k_N+k_A\,]}+\overline{p}\ \frac{[\,2\,b+k\,]\,\left[\,b+k^R\,\right]}{b\,[\,b+k\,]}$$

$$\Leftrightarrow\quad \left[\,b+k^R\,\right]\ \frac{a\,[\,b+k\,]+b\,c}{b\,[\,b+k\,]}+c_N+k_N\ \frac{k_A\,[\,b+k\,]\,a+b\,k_A\,c-b\,[\,b+k\,]\,[\,c_N-c_A\,]}{b\,[\,b+k\,]\,[\,k_N+k_A\,]}$$

$$\geq\ \overline{p}\left[1+\frac{k_N\,k_A\,[\,2\,b+k\,]}{b\,[\,b+k\,]\,[\,k_N+k_A\,]}+\frac{[\,2\,b+k]\,\left[b+k^R\right]}{b\,[\,b+k\,]}\right]$$

$$\Leftrightarrow\quad \left[\,b+k^R\,\right]\,[\,a\,(b+k)+b\,c]\,[\,k_N+k_A\,]+c_N\,b\,[\,b+k\,]\,[\,k_N+k_A\,]$$

$$+\ k_N\,[\,k_A\,(b\ +\ k)\,a+b\,k_A\,c-b\,(b+k)\,(c_N-c_A)\,]$$

$$\geq\ \overline{p}\ [\,b\,(b+k)\,(k_N+k_A)+k_N\,k_A\,(2\,b+k)$$

$$+\ (k_N+k_A)\,(2\,b+k)\,\left(b+k^R\right)\,]\ =\ \overline{p}\ D_3\,. \tag{70}$$

The last equality in (70) reflects (8). (70) implies:

$$\overline{p}\ \leq\ \left[\,b+k^R\,\right]Q^R+c_N+k_N\,q_N$$

$$\Leftrightarrow\quad \overline{p}\ \leq\ \frac{1}{D_3}\left\{\,\left[\,b+k^R\,\right]\,[\,a\,(b+k)+b\,c]\,[k_N+k_A\,]+c_N\,b\,[\,b+k\,]\,[\,k_N+k_A\,]\right.$$

$$\left.+\ \ k_N\,[\,k_A\,(b+k)\,a+b\,k_A\,c-b\,(b+k)\,(c_N-c_A)\,]\,\right\}$$

$$\Leftrightarrow\quad \overline{p}\ \leq\ \frac{1}{D_3}\left\{\,\left[\,b+k^R\,\right]\,[\,a\,(b+k)+b\,c]\,[\,k_N+k_A\,]+c_N\,b\,[\,b+k\,]\,[\,k_N+k_A\,]\right.$$

$$\left.+\ \ k_N\,k_A\,[\,b+k\,]\,a+b\,k_N\,k_A\,c-k_N\,b\,[\,b+k\,]\,[\,c_N-c_A\,]\,\right\}$$

$$\Leftrightarrow\quad \overline{p}\ \leq\ \frac{1}{D_3}\left\{\,\left[\,b+k^R\,\right]\,[\,a\,(b+k)+b\,c]\,[\,k_N+k_A\,]+c_N\,b\,[\,b+k\,]\,k_A\right.$$

$$+\ k_N\,k_A\,[\,b+k\,]\,a+b\,k_A\,k_N\,c+k_N\,b\,[\,b+k\,]\,c_A$$

$$+ \, k_N \, k_A \, [\, b + k \,] \, a + b \, k_A \, k_N \, c + k_N \, b \, [\, b + k \,] \, c_A \, \}$$

$$\Leftrightarrow \quad \overline{p} \; \leq \; \frac{1}{D_3} \, \{ \, a \, [\, b + k \,] \, \Big[\, \big(b + k^R \big) \, (k_N + k_A) + k_N \, k_A \, \Big]$$

$$+ \;\; c \, \Big[\, b \, (k_N + k_A) \, \big(b + k^R \big) + b \, k_A \, k_N \, \Big] + c_N \, b \, [\, b + k \,] \, k_A + k_N \, b \, [\, b + k \,] \, c_A \, \}$$

$$\Leftrightarrow \quad \overline{p} \; \leq \; \frac{1}{D_3} \, \{ \, a \, [\, b + k \,] \, \Big[\, \big(b + k^R \big) \, (k_N + k_A) + k_N \, k_A \, \Big]$$

$$+ \;\; b \, c \, \Big[\, (k_N + k_A) \, \big(b + k^R \big) + k_A \, k_N \, \Big] + c_N \, b \, [\, b + k \,] \, k_A + k_N \, b \, [\, b + k \,] \, c_A \, \}$$

$$\Leftrightarrow \quad \overline{p} \; \leq \; \overline{p}_b \, . \tag{71}$$

(62), (64), and (65) imply:

$$\Big[\, b + k^R \, \Big] \, Q^R + c_N + k_N \, q_N - b \, q_A \; < \; \overline{p}$$

$$\Leftrightarrow \quad \Big[\, b + k^R \, \Big] \; \frac{[\, a - \overline{p} \,] \, [\, b + k \,] - b \, [\, \overline{p} - c \,]}{b \, [\, b + k \,]} + c_N$$

$$+ \; k_N \; \frac{k_A \, [\, b + k \,] \, [\, a - \overline{p} \,] - b \, k_A \, [\, \overline{p} - c \,] - b \, [\, b + k \,] \, [\, c_N - c_A \,]}{b \, [\, b + k \,] \, [\, k_N \, + \, k_A \,]}$$

$$- \; b \; \frac{b \, [\, b + k \,] \, [\, c_N - c_A \,] + k_N \, [\, a - \overline{p} \,] \, [\, b + k \,] - b \, k_N \, [\, \overline{p} - c \,]}{b \, [\, b + k \,] \, [\, k_N \, + \, k_A \,]} \; < \; \overline{p}$$

$$\Leftrightarrow \quad \Big[\, b + k^R \, \Big] \; \frac{a \, [\, b + k \,] - \overline{p} \, [\, 2 \, b + k \,] + b \, c}{b \, [\, b + k \,]} + c_N$$

$$+ \; k_N \; \frac{k_A \, [\, b + k \,] \, a - \overline{p} \, k_A \, [\, 2 \, b + k \,] + b \, k_A \, c - b \, [\, b + k \,] \, [\, c_N - c_A \,]}{b \, [\, b + k \,] \, [\, k_N + k_A \,]}$$

$$- \;\; b \; \frac{b \, [\, b + k \,] \, [\, c_N - c_A \,] + k_N \, a \, [\, b + k \,] - \overline{p} \, k_N \, [\, 2 \, b + k \,] + b \, k_N \, c}{b \, [\, b + k \,] \, [\, k_N + k_A \,]} \; < \; \overline{p}$$

$$\Leftrightarrow \quad \Big[\, b + k^R \, \Big] \; \frac{a \, [\, b + k \,] + b \, c}{b \, [b \, + \, k]} + c_N + k_N \; \frac{k_A \, [\, b + k \,] \, a + b \, k_A \, c - b \, [\, b + k \,] \, [\, c_N - c_A \,]}{b \, [\, b + k \,] \, [\, k_N + k_A \,]}$$

$$- \; b \; \frac{b \, [\, b + k \,] \, [\, c_N - c_A \,] + k_N \, a \, [\, b + k \,] + b \, k_N \, c}{b \, [\, b + k \,] \, [\, k_N + k_A \,]}$$

$$< \; \overline{p} + \overline{p} \; \frac{[\, k_N \, k_A - k_N \, b \,] \, [\, 2 \, b + k \,]}{b \, [\, b + k \,] \, [\, k_N + k_A \,]} + \overline{p} \; \frac{[\, 2 \, b + k \,] \, \Big[\, b + k^R \, \Big]}{b \, [\, b + k \,]}$$

$$\Leftrightarrow \quad \Big[\, b + k^R \, \Big] \; \frac{a \, [\, b + k \,] + b \, c}{b \, [\, b + k \,]} + c_N + k_N \; \frac{k_A \, [\, b + k \,] \, a + b \, k_A \, c - b \, [\, b + k \,] \, [\, c_N - c_A \,]}{b \, [\, b + k \,] \, [k_N \, + \, k_A]}$$

$$
\begin{aligned}
& - \, b \, \frac{b\,[\,b+k\,]\,[\,c_N - c_A\,] + k_N\, a\,[\,b+k\,]\; b\, k_N\, c}{b\,[\,b+k\,]\,[\,k_N + k_A\,]} \\
& < \; \overline{p} \left[1 + \frac{[\,k_N\, k_A - k_N\, b\,]\,[\,2\,b+k\,]}{b\,[\,b+k\,]\,[\,k_N + k_A\,]} + \frac{[\,2\,b+k\,]\left[\,b+k^R\,\right]}{b\,[\,b+k\,]} \right] \\
\Leftrightarrow \quad & \left[\,b+k^R\,\right][\,a\,(b+k)+b\,c\,]\,[\,k_N+k_A\,] + c_N\, b\,[\,b+k\,]\,[\,k_N+k_A\,] \\
& + \, k_N\,[\,k_A\,(b+k)\,a + b\,k_A\,c - b\,(b+k)\,(c_N - c_A)\,] \\
& - \, b\,[\,b\,(b+k)\,(c_N - c_A) + k_N\, a\,(b+k) + b\,k_N\, c\,] \\
& < \; \overline{p}\;[\,b\,(b+k)\,(k_N+k_A) + k_N\,(k_A - b)\,(2\,b+k) \\
& \qquad + \,(k_N+k_A)\,(2\,b+k)\left(b+k^R\right)] \; = \; \overline{p}\; D_2\,. \qquad (72)
\end{aligned}
$$

The last equality in (72) reflects (7). (7) and (72) imply:

$$
\begin{aligned}
& \left[\,b+k^R\,\right] Q^R + c_N + k_N\, q_N - b\, q_A \; < \; \overline{p} \\
\Leftrightarrow \quad & \overline{p} \; > \; \frac{1}{D_2} \left\{ \left[\,b+k^R\,\right][\,a\,(b+k)+b\,c\,]\,[\,k_N+k_A\,] + c_N\, b\,[\,b+k\,]\,[\,k_N+k_A\,] \right. \\
& \qquad + \; k_N\,[\,k_A\,(b+k)\,a + b\,k_A\,c - b\,(b+k)\,(c_N - c_A)\,] \\
& \qquad \left. - \; b\,[\,b\,(b+k)\,(c_N - c_A) + k_N\, a\,(b+k) + b\,k_N\, c\,] \right\} \\
\Leftrightarrow \quad & \overline{p} \; > \; \frac{1}{D_2} \left\{ a \left[\left(b+k^R\right)(b+k)\,(k_N+k_A) + k_N\, k_A\,(b+k) - b\,(b+k)\,k_N \right] \right. \\
& \qquad + \; c \left[\,b\,(k_N+k_A)\left(b+k^R\right) + b\,k_A\,k_N - b^2 k_N \right] \\
& \qquad \left. + \; c_N\, b\,[\,b+k\,]\,[\,k_A - b\,] + b\,[\,k_N + b\,]\,[\,b+k\,]\,c_A \right\} \\
\Leftrightarrow \quad & \overline{p} \; > \; \frac{1}{D_2} \left\{ a\,[\,b+k\,] \left[\left(b+k^R\right)(k_N+k_A) + k_N\, k_A - b\,k_N \right] \right. \\
& \qquad + \; c\,b \left[\,(k_N+k_A)\left(b+k^R\right) + k_A\,k_N - b\,k_N \right] \\
& \qquad \left. + \; c_N\, b\,[\,b+k\,]\,[\,k_A - b\,] + b\,[\,k_N + b\,]\,[\,b+k\,]\,c_A \right\} \; = \; \overline{p}_d\,.
\end{aligned}
$$

(7), (8), (67), (68), and (71) imply:

$$
\begin{aligned}
\overline{p}_d \; &= \; \left[\,b+k^R\,\right] Q^R + c_N + k_N\, q_N - b\, q_A \quad \text{and} \\
\overline{p}_b \; &= \; \left[\,b+k^R\,\right] Q^R + c_N + k_N\, q_N\,. \qquad (73)
\end{aligned}
$$

(73) implies that $\overline{p}_d < \overline{p}_b$ because $q_A > 0$ when $\overline{p} > \overline{p}_0$. $\square$

Lemma A4. *Suppose $\overline{p} > \overline{p}_b$. Then in equilibrium:*

$$q_A = \frac{1}{D_3}\{[a-c_A]\left[2bk+2bk_N+2bk^R+kk_N+kk^R+3b^2\right] - [a-c_N]\left[2bk+2bk^R+kk^R+3b^2\right] - bk_N[a-c]\}; \tag{74}$$

$$q_N = \frac{1}{D_3}\{[a-c_N]\left[2bk+2bk_A+2bk^R+kk_A+kk^R+3b^2\right] - [a-c_A]\left[2bk+2bk^R+kk^R+3b^2\right] - bk_A[a-c]\}; \tag{75}$$

$$q = \frac{1}{D_3}\{[a-c]\left[2bk_A+2bk_N+k_A k_N+k_A k^R+k_N k^R\right] - bk_A[a-c_N] - bk_N[a-c_A]\}; \text{ and} \tag{76}$$

$$Q^R \equiv q_A+q_N = \frac{1}{D_3}\{[a-c_A]k_N[2b+k]+[a-c_N]k_A[2b+k] - b[k_A+k_N][a-c]\} \tag{77}$$

where D_3 is as specified in (8).

<u>Proof</u>. (4) implies that when the price cap does not bind, [P-R] is:

$$\underset{q_A,\,q_N}{\text{Maximize}} \quad [a-b(q_A+q_N+q)][q_A+q_N] - c_A q_A - \frac{k_A}{2}[q_A]^2 - c_N q_N - \frac{k_N}{2}[q_N]^2 - \frac{k^R}{2}[q_A+q_N]^2. \tag{78}$$

Differentiating (78) with respect to q_A provides:

$$a-b[q_A+q_N+q]-b[q_A+q_N]-c_A-k_A q_A-k^R[q_A+q_N] = 0$$

$$\Leftrightarrow \quad a-b[q_N+q]-bq_N-c_A-k^R q_N = q_A\left[2b+k_A+k^R\right]$$

$$\Leftrightarrow \quad q_A = \frac{a-c_A-\left[2b+k^R\right]q_N-bq}{2b+k_A+k^R}. \tag{79}$$

Corresponding differentiation of (78) with respect to q_N provides:

$$q_N = \frac{a-c_N-\left[2b+k^R\right]q_A-bq}{2b+k_N+k^R}. \tag{80}$$

(32) implies:

$$q = \frac{a-c}{2b+k} - \frac{b}{2b+k}[q_A+q_N]. \tag{81}$$

<u>Definitions</u>. $K_A \equiv 2b+k_A+k^R$ and $K_N \equiv 2b+k_N+k^R$. (82)

(79), (81), and (82) imply:

$$q_A \;=\; \frac{a-c_A}{K_A} - \frac{\left[\,2\,b+k^R\,\right] q_N}{K_A} - \frac{b}{K_A}\left[\frac{a-c-b\,(q_A+q_N)}{2\,b+k}\right]$$

$$\Rightarrow\quad q_A\left[1-\frac{b^2}{[\,2\,b+k\,]\,K_A}\right]$$

$$=\; \frac{[\,2\,b+k\,]\,[\,a-c_A\,] - \left[\,2\,b+k^R\,\right][\,2\,b+k\,]\,q_N - b\,[\,a-c\,] + b^2\,q_N}{[\,2\,b+k\,]\,K_A}$$

$$\Rightarrow\quad q_A\left[\frac{[\,2\,b+k\,]\,K_A-b^2}{[\,2\,b+k\,]\,K_A}\right]$$

$$=\; \frac{[\,2\,b+k\,]\,[\,a-c_A\,] - b\,[\,a-c\,] - \left(\left[\,2\,b+k^R\,\right][\,2\,b+k\,]-b^2\right) q_N}{[\,2\,b+k\,]\,K_A}$$

$$\Rightarrow\quad q_A \;=\; \frac{[\,2\,b+k\,]\,[\,a-c_A\,] - b\,[\,a-c\,]}{D_A} - \frac{B}{D_A}\,q_N$$

where $D_A \;\equiv\; [\,2\,b+k\,]\,K_A-b^2$ and $B \;\equiv\; \left[\,2\,b+k^R\,\right][\,2\,b+k\,]-b^2\,.$ (83)

(80) – (82) imply:

$$q_N \;=\; \frac{a-c_N}{K_N} - \frac{\left[\,2\,b+k^R\,\right] q_A}{K_N} - \frac{b}{K_N}\left[\frac{a-c-b\,(q_A+q_N)}{2\,b+k}\right]$$

$$\Rightarrow\quad q_N\left[1-\frac{b^2}{[\,2\,b+k\,]\,K_N}\right]$$

$$=\; \frac{[\,2\,b+k\,]\,[\,a-c_N\,] - \left[\,2\,b+k^R\,\right][\,2\,b+k\,]\,q_A - b\,[\,a-c\,] + b^2\,q_A}{[\,2\,b+k\,]\,K_N}$$

$$\Rightarrow\quad q_N\left[\frac{[\,2\,b+k\,]\,K_N-b^2}{[\,2\,b+k\,]\,K_N}\right]$$

$$=\; \frac{[\,2\,b+k\,]\,[\,a-c_N\,] - b\,[\,a-c\,] - \left(\left[\,2\,b+k^R\,\right][\,2\,b+k\,]-b^2\right) q_A}{[\,2\,b+k\,]\,K_N}$$

$$\Rightarrow\quad q_N \;=\; \frac{[\,2\,b+k\,]\,[\,a-c_N\,] - b\,[\,a-c\,]}{D_N} - \frac{B}{D_N}\,q_A$$

where $K_N \;\equiv\; 2\,b+k_N+k^R$ and $D_N \;\equiv\; [\,2\,b+k\,]\,K_N-b^2\,.$ (84)

(83) and (84) imply:

$$q_A \;=\; \frac{[\,2\,b+k\,]\,[\,a-c_A\,]-b\,[\,a-c\,]}{D_A} - \frac{B}{D_A\,D_N}\,\left\{[\,2\,b+k\,]\,[\,a-c_N\,]-b\,[\,a-c\,]-B\,q_A\right\}$$

$$\Rightarrow \quad q_A \left[1 - \frac{B^2}{D_A\, D_N} \right] \;=\; \frac{1}{D_A\, D_N} \Big\{ \,[\,2\,b + k\,]\, D_N\,[\,a - c_A\,] - b\, D_N\,[\,a - c\,]$$

$$- \;B\,[\,2\,b + k\,]\, D_N\,[\,a - c_N\,] + b\,B\,[\,a - c\,] \Big\}$$

$$\Rightarrow \quad q_A \left[D_A\, D_N - B^2 \right] \;=\; [\,2\,b + k\,]\, D_N\,[\,a - c_A\,] + b\,[\,B - D_N\,]\,[\,a - c\,]$$

$$+ \;[\,2\,b + k\,]\,B\,[\,a - c_N\,]\,. \tag{85}$$

(83) and (84) imply:

$$D_A\, D_N - B^2 \;=\; \left[(2\,b + k)\, K_A - b^2 \right] \left[(2\,b + k)\, K_N - b^2 \right] - \left[\left(2\,b + k^R\right)(2\,b + k) - b^2 \right]^2$$

$$=\; [\,2\,b + k\,]^2\; K_A\, K_N - b^2\,[\,2\,b + k\,]\, K_A - b^2\,[\,2\,b + k\,]\, K_N + b^4$$

$$-\;\; [\,2\,b + k\,]^2 \left[2\,b + k^R \right]^2 + 2\,b^2\,[\,2\,b + k\,] \left[2\,b + k^R \right] - b^4$$

$$=\; [\,2\,b + k\,]\; \{\, [\,2\,b + k\,]\, K_A\, K_N - b^2\,[\,K_A + K_N\,] + 2\,b^2 \left[2\,b + k^R \right]$$

$$-\; [\,2\,b + k\,] \left[2\,b + k^R \right]^2 \}\,. \tag{86}$$

(82) implies that the term in $\{\cdot\}$ in (86) is:

$$[\,2\,b + k\,] \left[2\,b + k^R + k_A \right] \left[2\,b + k^R + k_N \right] - b^2 \left[4\,b + k_A + k_N + 2\,k^R \right]$$

$$+\; 2\,b^2 \left[2\,b + k^R \right] - [\,2\,b + k\,] \left[2\,b + k^R \right]^2$$

$$=\; [\,2\,b + k\,] \Big\{ \left[2\,b + k^R \right]^2 + [\,k_A + k_N\,] \left[2\,b + k^R \right] + k_A\, k_N \Big\}$$

$$+\;\; 2\,b^2 \left[2\,b + k^R \right] - [\,2\,b + k\,] \left[2\,b + k^R \right]^2 - b^2 \left[2 \left(2\,b + k^R\right) + k_A + k_N \right]$$

$$=\; \left[2\,b + k^R \right] \left\{ [\,2\,b + k\,]\,[\,k_A + k_N\,] + 2\,b^2 - 2\,b^2 \right\} + [\,2\,b + k\,]\, k_A\, k_N$$

$$+\; [\,2\,b + k\,]\, k_A\, k_N - b^2\,[\,k_A + k_N\,]$$

$$=\; [\,k_A + k_N\,] \left\{ [\,2\,b + k\,] \left[2\,b + k^R \right] - b^2 \right\} + [\,2\,b + k\,]\, k_A\, k_N$$

$$=\; [\,k_A + k_N\,] \left\{ [\,2\,b + k\,] \left[b + k^R \right] + b\,[\,2\,b + k\,] - b^2 \right\} + [\,2\,b + k\,]\, k_A\, k_N$$

$$=\; [\,k_A + k_N\,] \left\{ [\,2\,b + k\,] \left[b + k^R \right] + b\,[\,b + k\,] \right\} + [\,2\,b + k\,]\, k_A\, k_N \;=\; D_3\,. \tag{87}$$

The last equality in (87) reflects (8).

(82) and (84) imply:

$$D_N \;=\; [\,2\,b + k\,] \left[2\,b + k_N + k^R \right] - b^2 \;=\; 2\,b\,[\,2\,b + k\,] - b^2 + [\,2\,b + k\,] \left[k_N + k^R \right]$$

$$=\; 3\,b^2 + 2\,b\,k + [\,2\,b + k\,] \left[k_N + k^R \right]. \tag{88}$$

(82) and (84) imply:

$$B - D_N = [\,2\,b + k\,]\left[\,2\,b + k^R\,\right] - b^2 - \left\{\,[\,2\,b + k\,]\left[\,2\,b + k_N + k^R\,\right] - b^2\,\right\}$$
$$= [\,2\,b + k\,]\left[\,2\,b + k^R - 2\,b - k_N - k^R\,\right] = -\,[\,2\,b + k\,]\,k_N\,. \tag{89}$$

(83) and (85) – (89) imply that (74) holds. Furthermore, (74) and the symmetry of q_A and q_N in the analysis imply that (75) holds.

Observe that:

$$3\,b^2 + 2\,b\,k + [\,2\,b + k\,]\left[\,k_N + k^R\,\right] - \left[\,(2\,b + k)\left(2\,b + k^R\right) - b^2\,\right]$$
$$= 4\,b^2 + 2\,b\,k + [\,2\,b + k\,]\left[\,k_N + k^R - \left(2\,b + k^R\right)\,\right]$$
$$= 2\,b\,[\,2\,b + k\,] + [\,2\,b + k\,]\,[\,k_N - 2\,b\,] = [\,2\,b + k\,]\,k_N\,; \text{ and}$$

$$3\,b^2 + 2\,b\,k + [\,2\,b + k\,]\left[\,k_A + k^R\,\right] - \left[\,(2\,b + k)\left(2\,b + k^R\right) - b^2\,\right]$$
$$= 4\,b^2 + 2\,b\,k + [\,2\,b + k\,]\left[\,k_A + k^R - \left(2\,b + k^R\right)\,\right]$$
$$= 2\,b\,[\,2\,b + k\,] + [\,2\,b + k\,]\,[\,k_A - 2\,b\,] = [\,2\,b + k\,]\,k_A\,. \tag{90}$$

(74), (75), and (90) imply that $Q^R = q_A + q_N$ is as specified in (77).

(77) and (81) imply:

$$q = \frac{[\,a - c\,]\,D_3}{[\,2\,b + k\,]\,D_3}$$
$$- \frac{b}{[\,2\,b + k\,]\,D_3}\left\{\,[\,a - c_A\,]\,k_N\,[\,2\,b + k\,] + [\,a - c_N\,]\,k_A\,[\,2\,b + k\,]\right.$$
$$\left. -\; b\,[\,k_A + k_N\,]\,[\,a - c\,]\,\right\}$$
$$= \frac{1}{[\,2\,b + k\,]\,D_3}\left\{\,[\,a - c\,]\left[\,D_3 + b^2\,(k_A + k_N)\,\right] - [\,2\,b + k\,]\,b\,k_A\,[\,a - c_N\,]\right.$$
$$\left. -\;[\,2\,b + k\,]\,b\,k_N\,[\,a - c_A\,]\,\right\}. \tag{91}$$

(8) implies:

$$D_3 + b^2\,[\,k_A + k_N\,] = [\,2\,b + k\,]\,k_A\,k_N + [\,k_A + k_N\,]\left[\,b^2 + b\,(b + k) + (2\,b + k)\left(b + k^R\right)\,\right]$$
$$= [\,2\,b + k\,]\,k_A\,k_N + [\,k_A + k_N\,]\left[\,2\,b^2 + b\,k + 2\,b^2 + 2\,b\,k^R + b\,k + k\,k^R\,\right]$$
$$= [\,2\,b + k\,]\,k_A\,k_N + [\,k_A + k_N\,]\left[\,4\,b^2 + 2\,b\,k + 2\,b\,k^R + k\,k^R\,\right]$$
$$= [\,2\,b + k\,]\,k_A\,k_N + [\,k_A + k_N\,]\left[\,2\,b\,(2\,b + k) + k^R\,(2\,b + k)\,\right]$$
$$= [\,2\,b + k\,]\left\{\,k_A\,k_N + [\,k_A + k_N\,]\left[\,2\,b + k^R\,\right]\,\right\}. \tag{92}$$

(91) and (92) imply that q is as specified in (76).

(74) – (76) imply:

$$\begin{aligned} P(Q) &= a - b\,[\,q_A + q_N + q\,] \\ &= a - \frac{b}{D_3}\,[\,B_1\,(a - c_A) + B_2\,(a - c_N) + B_3\,(a - c\,)\,] \end{aligned} \tag{93}$$

$$\text{where } B_1 = k_N\,[\,b + k\,]\,; \quad B_2 = k_A\,[\,b + k\,]\,; \text{ and}$$

$$B_3 = \left[\,b + k^R\right][\,k_A + k_N\,] + k_A\,k_N\,. \tag{94}$$

(93) implies:

$$P(Q) = \frac{[\,D_3 - b\,(B_1 + B_2 + B_3)\,]\,a + b\,B_1\,c_A + b\,B_2\,c_N + b\,B_3\,c}{D_3}. \tag{95}$$

(94) implies:

$$\begin{aligned} B_1 + B_2 + B_3 &= [\,k_A + k_N\,]\,[\,b + k\,] + \left[\,b + k^R\right][\,k_A + k_N\,] + k_A\,k_N \\ &= \left[\,2\,b + k + k^R\right][\,k_A + k_N\,] + k_A\,k_N. \end{aligned}$$

(8) and (94) imply:

$$\begin{aligned} & D_3 - b\,[\,B_1 + B_2 + B_3\,] \\ &= b\,[\,b + k\,]\,[\,k_N + k_A\,] + k_N\,k_A\,[\,2\,b + k\,] + [\,k_N + k_A\,]\,[\,2\,b + k\,]\left[\,b + k^R\right] \\ &\qquad - b\left[\,2\,b + k + k^R\right][\,k_A + k_N\,] - b\,k_N\,k_A \\ &= b\,[\,b + k\,]\,[\,k_N + k_A\,] + k_N\,k_A\,[\,2\,b + k\,] + [\,k_N + k_A\,]\left[\,2\,b^2 + k\,b + 2\,k^R b + k\,k^R\right] \\ &\qquad - \left[\,2\,b^2 + k\,b + b\,k^R\right][\,k_A + k_N\,] - b\,k_N\,k_A \\ &= b\,[\,b + k\,]\,[\,k_N + k_A\,] + k_N\,k_A\,[\,b + k\,] + [\,k_N + k_A\,]\left[\,b\,k^R + k\,k^R\right] \\ &= b\,[\,b + k\,]\,[\,k_N + k_A\,] + k_N\,k_A\,[\,b + k\,] + k^R\,[\,k_N + k_A\,]\,[\,b + k\,] \\ &= [\,b + k\,]\left[\,b\,(k_N + k_A) + k_N\,k_A + k^R\,(k_N + k_A)\right] \\ &= [\,b + k\,]\left[\,\left(b + k^R\right)(k_N + k_A) + k_N\,k_A\right]. \end{aligned} \tag{96}$$

(94), (95), and (96) imply that the price cap does not bind if:

$$\begin{aligned} \overline{p} &> \frac{a\,[\,D_3 - b\,(B_1 + B_2 + B_3)\,] + b\,B_1\,c_A + b\,B_2\,c_N + b\,B_3\,c}{D_3} \\ &= \frac{1}{D_3}\,\Big\{\,a\,[\,b + k\,]\left[\,\left(b + k^R\right)(k_N + k_A) + k_N\,k_A\right] + b\,[\,b + k\,]\,k_N\,c_A \\ &\qquad + \; b\,[\,b + k\,]\,k_A\,c_N + b\,c\left[\,\left(b + k^R\right)(k_A + k_N) + k_A\,k_N\right]\Big\} \end{aligned}$$

$$= \frac{1}{D_3} \{ [(b+k)\, a + b\, c] \left[\left(b + k^R\right) (k_N + k_A) + k_N\, k_A \right]$$

$$+ \; b\, [\, b + k\,]\, k_N\, c_A + b\, [\, b + k\,]\, k_A\, c_N \,\} \; = \; \overline{p}_b \, . \tag{97}$$

The last equality in (97) reflects (8). □

Definitions

$q_{A1}(\overline{p}_0)$, $q_{N1}(\overline{p}_0)$, and $q_1(\overline{p}_0)$, respectively, denote the values of q_A, q_N, and q specified in Lemma A1, where $\overline{p} \leq \overline{p}_0$.

$q_{A2}(\overline{p}_0)$, $q_{N2}(\overline{p}_0)$, and $q_2(\overline{p}_0)$, respectively, denote the values of q_A, q_N, and q specified in Lemma A2, where $\overline{p} \in (\, \overline{p}_0, \overline{p}_d \,]$.

Lemma A5. $\lim\limits_{\overline{p} \to \overline{p}_0} q_{A2}(\overline{p}\,) \; = \; q_{A1}(\overline{p}_0)$, $\lim\limits_{\overline{p} \to \overline{p}_0} q_{N2}(\overline{p}) \; = \; q_{N1}(\overline{p}_0)$, *and* $\lim\limits_{\overline{p} \to \overline{p}_0} q_2(\overline{p}\,) \; = \; q_1(\overline{p}_0)$.

Proof. (11), (12), and (19) imply that when $\overline{p} \leq \overline{p}_0$, q_N, q, and q_A are determined by:

$$\frac{\partial \pi^R}{\partial q_N} \; = \; a - 2\, b\, q_N - b\, q - c_N - k_N\, q_N - k^R\, q_N \; = \; 0\, ;$$

$$\frac{\partial \pi}{\partial q} \; = \; a - b\, q_N - 2\, b\, q - c - k\, q \; = \; 0\, ;$$

$$q_A \; = \; 0; \;\; \text{and} \;\; \frac{\partial \pi^R}{\partial q_A} \; = \; \overline{p} - c_A - b\, q_N - k^R\, q_N \; \leq \; 0\, . \tag{98}$$

(19) implies that the weak inequality in (98) holds as an equality when $\overline{p} = \overline{p}_0$.

(25), (26), and (32) imply that when $\overline{p} \in (\, \overline{p}_0, \overline{p}_d \,]$, q_N, q, and q_A are determined by:

$$\frac{\partial \pi^R}{\partial q_N} \; = \; a - 2\, b\, q_N - b\, q - b\, q_A - c_N - k_N\, q_N - k^R\, [\, q_N + q_A \,] \; = \; 0\, ;$$

$$\frac{\partial \pi}{\partial q} \; = \; a - b\, q_N - b\, q_A - 2\, b\, q - c - k\, q \; = \; 0\, ;$$

$$\frac{\partial \pi^R}{\partial q_A} \; = \; \overline{p} - c_A - k_A\, q_A - b\, q_N - k^R\, [\, q_N + q_A \,] \; = \; 0\, . \tag{99}$$

(6) and (25) imply:

$$\lim_{\overline{p} \to \overline{p}_0} q_{A2}(\overline{p}) \; = \; \frac{1}{D} \left\{ \left[\, 3\, b^2 + 2\, b \left(k + k_N + k^R\right) + k \left(k_N + k^R\right) \right] [\, \overline{p}_0 - c_A \,] \right.$$

$$\left. + \;\; b \left[\, b + k^R \,\right] [\, a - c \,] - [\, 2\, b + k \,] \left[\, b + k^R \,\right] [\, a - c_N \,] \right\}$$

$$= \; \frac{1}{D} \left\{ \left[\, 3\, b^2 + 2\, b \left(k + k_N + k^R\right) + k \left(k_N + k^R\right) \right] \right.$$

$$\cdot \left[b + k^R \right] \frac{[a - c_N][2b+k] - b[a-c]}{[2b + k_N + k^R][2b+k] - b^2}$$

$$+ \; b \left[b + k^R \right] [a - c] - [2b+k] \left[b + k^R \right] [a - c_N] \}$$

$$= \frac{1}{D} \{ \left[b + k^R \right] [a - c_N][2b+k] - b \left[b + k^R \right] [a - c]$$

$$+ \; b \left[b + k^R \right] [a - c] - [2b+k] \left[b + k^R \right] [a - c_N] \} = 0 . \tag{100}$$

(100) reflects the fact that:

$$\left[2b + k_N + k^R \right] [2b+k] - b^2 = 3b^2 + 2bk + \left[k_N^R + k^R \right] [2b+k]$$

$$= 3b^2 + 2b \left[k + k_N + k^R \right] + k \left[k_N + k^R \right] .$$

(100) implies that $\lim_{\overline{p} \to \overline{p}_0} q_{A2}(\overline{p}) = q_{A1}(\overline{p}_0)$. The equations in (99) coincide with the equations in (98) when $\overline{p} = \overline{p}_0$. Therefore, because (20), (21), and (23) imply that q_A, q_N, and q are continuous functions of $\overline{p}$, $\lim_{\overline{p} \to \overline{p}_0} q_{A2}(\overline{p}) = q_{A1}(\overline{p}_0)$, $\lim_{\overline{p} \to \overline{p}_0} q_{N2}(\overline{p}) = q_{N1}(\overline{p}_0)$, and $\lim_{\overline{p} \to \overline{p}_0} q_2(\overline{p}) = q_1(\overline{p}_0)$. □

Lemma A6. $0 < \overline{p}_0 < \overline{p}_d < \overline{p}_b$.

Proof. The proof of Lemma A3 establishes that $\overline{p}_d < \overline{p}_b$. From (6):

$$\overline{p}_0 = \frac{[a - c_N][2b+k] - b[a-c]}{[2b + k_N + k^R][2b+k] - b^2} \left[b + k^R \right] + c_A > 0 . \tag{101}$$

The inequality in (101) holds because $[a - c_N][2b+k] - b[a-c] > 0$, from (3).

To prove that $\overline{p}_0 < \overline{p}_d$, let $Q_1(\overline{p})$ denote the value of $Q(\overline{p})$ specified in Lemma A1, and let $Q_2(\overline{p})$ denote the value of $Q(\overline{p})$ specified in Lemma A2. Lemma A5 implies:

$$Q_1(\overline{p}_0) = Q_2(\overline{p}_0) . \tag{102}$$

Lemma A2 implies:

$$\overline{p} < P(Q_2(\overline{p})) \Leftrightarrow \overline{p} < \overline{p}_d . \tag{103}$$

(102) and (103) imply that if $\overline{p}_0 < P(Q_1(\overline{p}_0))$, then:

$$\overline{p}_0 < P(Q_2(\overline{p}_0)) \Leftrightarrow \overline{p}_0 < \overline{p}_d . \tag{104}$$

The first inequality in (104) holds because (102) implies that $P(Q_1(\overline{p}_0)) = P(Q_2(\overline{p}_0))$. The equivalence in (104) reflects (103). (104) implies that to establish that $\overline{p}_0 < \overline{p}_d$, it suffices to show that $\overline{p}_0 < P(Q_1(\overline{p}_0))$.

(6) implies:

$$\overline{p}_0 = c_A + \frac{[a - c_N][2b+k] - b[a-c]}{[2b + k_N + k^R][2b+k] - b^2} \left[b + k^R \right]$$

$$= \frac{1}{[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,] - b^2}$$

$$\cdot \left\{ c_A \left[\left(2\,b + k_N + k^R\right)(2\,b + k) - b^2 \right] + [\,a - c_N\,]\,[\,2\,b + k\,]\left[\,b + k^R\,\right] - b\left[\,b + k^R\,\right][\,a - c\,] \right\}. \tag{105}$$

Recall from (15) that when $q_A = 0$ and the price cap binds, the equilibrium price is:

$$P(Q) = a - b\,[\,q + q_N\,] = a - b\,\frac{[\,a - c\,]\left[\,b + k_N + k^R\,\right] + [\,a - c_N\,]\,[\,b + k\,]}{[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,] - b^2}$$

$$= \frac{1}{[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,] - b^2}$$

$$\cdot \left\{ a \left[\left(2\,b + k_N + k^R\right)(2\,b + k) - b^2 \right] - b\,[\,a - c\,]\left[\,b + k_N + k^R\,\right] - b\,[\,a - c_N\,]\,[\,b + k\,] \right\}. \tag{106}$$

(105) and (106) imply that $\overline{p}_0 < P(Q)$ if:

$$a \left[\left(2\,b + k_N + k^R\right)(2\,b + k) - b^2 \right] - b\,[\,a - c\,]\left[\,b + k_N + k^R\,\right] - b\,[\,a - c_N\,]\,[\,b + k\,]$$

$$> c_A \left[\left(2\,b + k_N + k^R\right)(2\,b + k) - b^2 \right] + [\,a - c_N\,]\,[\,2\,b + k\,]\left[\,b + k^R\,\right] - b\left[\,b + k^R\,\right][\,a - c\,]$$

$$\Leftrightarrow \quad [\,a - c_A\,]\left[\left(2\,b + k_N + k^R\right)(2\,b + k) - b^2 \right] - b\,[\,a - c\,]\,k_N - [\,a - c_N\,]\left[(b + k)\,b + (2\,b + k)\left(b + k^R\right) \right] > 0$$

$$\Leftrightarrow \quad [\,a - c_A\,]\left[\,2\,b\,k + 2\,b\,k_N + 2\,b\,k^R + k\,k_N + k\,k^R + 3\,b^2\,\right] - b\,k_N\,[\,a - c\,] - [\,a - c_N\,]\left[\,2\,b\,k + 2\,b\,k^R + k\,k^R + 3\,b^2\,\right] > 0$$

$$\Leftrightarrow \quad [\,c_N - c_A\,]\left[\,2\,b\,k + 2\,b\,k^R + k\,k^R + 3\,b^2\,\right] + k_N\,[\,(a - c_A)\,(2\,b + k) - b\,(a - c)\,] > 0\,.$$

The inequality here holds because $c_N \geq c_A$, by assumption and because (3) implies:

$$k_A\,[\,(a - c_N)\,(2\,b + k) - b\,(a - c)\,] > [\,c_N - c_A\,]\left[\,2\,b\,k + 2\,b\,k^R + k\,k^R + 3\,b^2\,\right]$$

$$\Rightarrow \quad [\,a - c_N\,]\,[\,2\,b + k\,] - b\,[\,a - c\,] > 0 \quad \Rightarrow \quad [\,a - c_A\,]\,[\,2\,b + k\,] - b\,[\,a - c\,] > 0. \tag{107}$$

The last two inequalities in (107) hold because $c_N \geq c_A$, by assumption. □ ■

Proposition 2. *In equilibrium: (i)* $\frac{dq_A}{d\overline{p}} < 0\,,\ \frac{dq_N}{d\overline{p}} < 0\,,\ \frac{dq}{d\overline{p}} > 0\,,\ \frac{dQ}{d\overline{p}} < 0\,,$ *and* $\frac{dP(Q)}{d\overline{p}} = 1$ *for* $\overline{p} \in (\overline{p}_d, \overline{p}_b)$*; and (ii)* $\frac{dq_A}{d\overline{p}} > 0,\ \frac{dq_N}{d\overline{p}} < 0\,,\ \frac{dq}{d\overline{p}} < 0\,,\ \frac{dQ^R}{d\overline{p}} > 0\,,\ \frac{dQ}{d\overline{p}} > 0\,,$ *and* $\frac{dP(Q)}{d\overline{p}} < 0$ *for* $\overline{p} \in (\overline{p}_0, \overline{p}_d)\,.$

Proof. Lemma A3 implies that for $\overline{p} \in (\overline{p}_d, \overline{p}_b)$:

$$\frac{dq_A}{d\overline{p}} = -\frac{k_N[b+k]+b\,k_N}{b[b+k][k_N+k_A]} < 0\,;$$

$$\frac{dq_N}{d\overline{p}} = -\frac{k_A[b+k]+b\,k_A}{b[b+k][k_N+k_A]} < 0\,; \quad \frac{dq}{d\overline{p}} = \frac{1}{b+k} > 0\,;$$

$$\frac{dQ}{d\overline{p}} = -\frac{1}{b} < 0 \;\Rightarrow\; \frac{dP(Q)}{d\overline{p}} = -b\left[-\frac{1}{b}\right] = 1\,.$$

Lemma A2 implies that for $\overline{p} \in (\overline{p}_0, \overline{p}_d)$:

$$\frac{dq_A}{d\overline{p}} = \frac{3b^2+2b\left[k+k_N+k^R\right]+k\left[k_N+k^R\right]}{D} > 0\,;$$

$$\frac{dq_N}{d\overline{p}} = -\frac{b\left[b+2k^R\right]+k\left[b+k^R\right]}{D} < 0\,; \quad \frac{dQ^R}{d\overline{p}} = \frac{[2b+k][b+k_N]}{D} > 0\,;$$

$$\frac{dq}{d\overline{p}} = -\frac{b[b+k_N]}{D} < 0\,; \quad \text{and} \quad \frac{dQ}{d\overline{p}} = \frac{[b+k][b+k_N]}{D} > 0\,. \;\blacksquare \tag{108}$$

Proposition 3. *For $\overline{p} \in (\overline{p}_d, \overline{p}_b)$: (i) $V(\overline{p})$ is a strictly concave function of $\overline{p}$; (ii) $\frac{\partial V(\overline{p})}{\partial \overline{p}} \lesseqgtr 0 \Leftrightarrow \overline{p} \gtreqless \overline{p}_{VM}$ where $\overline{p}_{VM} \in [\,\overline{p}_d, \overline{p}_b)$; and (iii) $\overline{p}_{VM} = \overline{p}_d$ if $\Phi_1 \geq 0$, whereas $\overline{p}_{VM} > \overline{p}_d$ if $\Phi_1 < 0$, where*

$$\Phi_1 \equiv \left[k^R + \frac{b^2}{2b+k}\right][k_A+k_N]A + 2b[b+k]c_A[k_N+b]$$
$$+\,[2b(b+k)c_N + A\,k_N][k_A-b] \quad \textit{where} \;\; A \equiv a[b+k]+b\,c\,. \tag{109}$$

Proof. (62) implies that for $\overline{p} \in (\overline{p}_d, \overline{p}_b)$, R's revenue is:

$$V(\overline{p}) = \overline{p}\left[\frac{a(b+k)+b\,c-\overline{p}\,(2b+k)}{b[b+k]}\right] = \frac{[a(b+k)+b\,c]\,\overline{p}-[2b+k]\,\overline{p}^2}{b[b+k]}\,. \tag{110}$$

The value of $\overline{p}$ at which $V(\overline{p})$ in (110) is maximized is determined by:

$$a[b+k]+b\,c-2[2b+k]\,\overline{p} = 0 \;\Rightarrow\; \overline{p} = \frac{a[b+k]+b\,c}{2[2b+k]} \equiv \overline{p}_{VM}\,. \tag{111}$$

From (8):

$$\overline{p}_b = \frac{[a(b+k)+b\,c]\left[\left(b+k^R\right)(k_N+k_A)+k_N\,k_A\right]+b\,c_N[b+k]k_A+b\,k_N[b+k]c_A}{b[b+k][k_N+k_A]+k_N\,k_A[2b+k]+[k_N+k_A][2b+k][b+k^R]}$$

$$= \frac{\left[\left(b+k^R\right)(k_N+k_A)+k_N\,k_A\right]\frac{a\,[b+k]+b\,c}{b\,[b+k]}+c_N\;k_A+k_N\,c_A}{k_N+k_A+\left[(b+k^R)(k_N+k_A)+k_N\,k_A\right]\frac{2\,b+k}{b\,[b+k]}}\,. \tag{112}$$

(111) and (112) imply that $\overline{p}_{VM} < \overline{p}_b$ if:

$$\frac{a\,[b+k]+b\,c}{2\,[2\,b+k]} < \frac{\left[\left(b+k^R\right)(k_N+k_A)+k_N\,k_A\right]\frac{a\,[b+k]+b\,c}{b\,[b+k]}+c_N\;k_A+k_N\,c_A}{k_N+k_A+\left[(b+k^R)(k_N+k_A)+k_N\,k_A\right]\frac{2\,b+k}{b\,[b+k]}}\,. \tag{113}$$

The inequality in (113) holds if:

$$\frac{\left[\left(b+k^R\right)(k_N+k_A)+k_N\,k_A\right]\frac{a\,[b+k]+b\,c}{b\,[b+k]}}{k_N+k_A+\left[(b+k^R)(k_N+k_A)+k_N\,k_A\right]\frac{2\,b+k}{b\,[b+k]}} > \frac{a\,[b+k]+b\,c}{2\,[2\,b+k]}\,. \tag{114}$$

Define $z \equiv \left[\left(b+k^R\right)(k_N+k_A)+k_N\,k_A\right]\frac{1}{b\,[b+k]}$. Then the inequality in (114) holds if:

$$\begin{aligned}
&\frac{z\,[a\,(b+k)+b\,c\,]}{k_N+k_A+z\,[2\,b+k]} > \frac{a\,[b+k]+b\,c}{2\,[2\,b+k]} \\
\Leftrightarrow\quad &\frac{z}{k_N+k_A+z\,[2\,b+k]} > \frac{1}{2\,[2\,b+k]} \\
\Leftrightarrow\quad &2\,z\,[2\,b+k] > k_N+k_A+z\,[2\,b+k] \;\Leftrightarrow\; [2\,b+k]\,z > k_N+k_A \\
\Leftrightarrow\quad &\left[\left(b+k^R\right)(k_N+k_A)+k_N\,k_A\right]\frac{2\,b+k}{b\,[b+k]} > k_N+k_A \\
\Leftrightarrow\quad &\frac{[2\,b+k]\left[b+k^R\right]}{b\,[b+k]}\,[k_N+k_A]+k_N\,k_A\left[\frac{2\,b+k}{b\,(b+k)}\right] > k_N+k_A \\
\Leftrightarrow\quad &\frac{[2\,b+k]\left[b+k^R\right]-b\,[b+k]}{b\,[b+k]}\,[k_N+k_A]+k_N\,k_A\left[\frac{2\,b+k}{b\,(b+k)}\right] > 0\,.
\end{aligned} \tag{115}$$

The inequality in (115) always holds because:

$$\begin{aligned}
[2\,b+k]\left[b+k^R\right]-b\,[b+k] &= 2\,b^2+2\,b\,k^R+b\,k+k\,k^R-b^2-b\,k \\
&= b^2+2\,b\,k^R+k\,k^R > 0\,.
\end{aligned}$$

(115) implies that $\overline{p}_{VM} < \overline{p}_b$.

(110) and (111) imply that for $\overline{p} \in (\overline{p}_d, \overline{p}_b)$, $V(\overline{p})$ is a strictly concave function that attains its maximum at $\overline{p}_{VM}$. Therefore, $\frac{\partial V(\overline{p})}{\partial \overline{p}} < 0$ for $\overline{p} \in (\overline{p}_{VM}, \overline{p}_b)$.

(7) and (111) imply that $\overline{p}_d \geq \overline{p}_{VM}$ if and only if:

$$\frac{1}{b\,[b+k]\,[k_N+k_A]+[k_A\,k_N-k_N\,b]\,[2\,b+k]+[k_N+k_A]\,[2\,b+k]\,[b+k^R]}$$

$$\cdot\,\left\{\,[\,(b+k)\,a+b\,c\,]\left[\left(b+k^R\right)(k_N+k_A)+k_N\,k_A-b\,k_N\right]\right.$$

$$+\ b\,[\,b+k\,]\,[\,k_A-b\,]\,c_N+b\,[\,k_N+b\,]\,[\,b+k\,]\,c_A\,\}$$

$$\geq\ \frac{a\,[\,b+k\,]+b\,c}{2\,[\,b+k\,]}$$

$$\Leftrightarrow\ \frac{1}{\frac{b\,[\,b+k\,]}{2\,b+k}\,[\,k_N+k_A\,]+[\,k_A\,k_N-k_N\,b\,]+[\,k_N+k_A\,]\,[\,b+k^R\,]}$$

$$\cdot\,\{\,2\,[\,(b+k)\,a+b\,c\,]\,\Big[\,\Big(b+k^R\Big)\,(k_N+k_A)+k_N\,k_A-b\,k_N\,\Big]$$

$$+\ 2\,b\,[\,b+k\,]\,[\,k_A-b\,]\,c_N+2\,b\,[\,k_N+b\,]\,[\,b+k\,]\,c_A\,\}$$

$$\geq\ a\,[\,b+k\,]+b\,c$$

$$\Leftrightarrow\ 2\,[\,(b+k)\,a+b\,c\,]\,\Big[\,\Big(b+k^R\Big)\,(k_N+k_A)+k_N\,k_A-b\,k_N\,\Big]+2\,b\,[\,b+k\,]\,[\,k_A-b\,]\,c_N$$

$$+\ 2\,b\,[\,k_N+b\,]\,[\,b+k\,]\,c_A$$

$$\geq\ [\,(b+k)\,a+b\,c\,]\,\frac{b\,[\,b+k\,]}{2\,b+k}\,[\,k_N+k_A\,]+[\,a\,(b+k)+b\,c\,]\,[\,k_A\,k_N-b\,k_N\,]$$

$$+\ [\,a\,(b+k)+b\,c\,]\,[\,k_N+k_A\,]\,\Big[\,b+k^R\,\Big]$$

$$\Leftrightarrow\ [\,(b+k)\,a+b\,c\,]\,\Big[\,\Big(b+k^R\Big)\,(k_N+k_A)+k_N\,k_A-b\,k_N\,\Big]+2\,b\,[\,b+k\,]\,[\,k_A-b\,]\,c_N$$

$$+\ 2\,b\,[\,k_N+b\,]\,[\,b+k\,]\,c_A\ \geq\ [\,a\,(b+k)+b\,c\,]\,\frac{b\,[\,b+k\,]}{2\,b+k}\,[\,k_N+k_A\,]$$

$$\Leftrightarrow\ [\,(b+k)\,a+b\,c\,]\left[\left(b-\frac{b\,(b+k)}{2\,b+k}+k^R\right)(k_N+k_A)+k_N\,k_A-b\,k_N\right]$$

$$+\ 2\,b\,[\,b+k\,]\,[\,k_A-b\,]\,c_N+2\,b\,[\,k_N+b\,]\,[\,b+k\,]\,c_A\ \geq\ 0$$

$$\Leftrightarrow\ [\,(b+k)\,a+b\,c\,]\left[\left(\frac{2\,b^2+k\,b}{2\,b+k}-\frac{b\,(b+k)}{2\,b+k}+k^R\right)(k_N+k_A)+k_N\,k_A-b\,k_N\right]$$

$$+\ 2\,b\,[\,b+k\,]\,[\,k_A-b\,]\,c_N+2\,b\,[\,k_N+b\,]\,[\,b+k\,]\,c_A\ \geq\ 0$$

$$\Leftrightarrow\ [\,(b+k)\,a+b\,c\,]\left[\left(\frac{b^2}{2\,b+k}+k^R\right)(k_N+k_A)+k_N\,k_A-b\,k_N\right]$$

$$+\ 2\,b\,[\,b+k\,]\,[\,k_A-b\,]\,c_N+2\,b\,[\,k_N+b\,]\,[\,b+k\,]\,c_A\ \geq\ 0$$

$$\Leftrightarrow\ [\,(b+k)\,a+b\,c\,]\left[\left(\frac{b^2}{2\,b+k}+k^R\right)(k_N+k_A)+k_N\,k_A-b\,k_N\right]$$

$$+\ 2\,b\,[\,b+k\,]\,[\,k_A\,c_N+k_N\,c_A-b\,(c_N-c_A)\,]\ \geq\ 0$$

$$\Leftrightarrow \quad \left[\frac{(b+k)\,a + b\,c}{2\,b+k} \right] \left[\left(b^2 + k^R\,[\,2\,b+k\,] \right) (k_N + k_A) + k_N\,k_A\,(2\,b+k) - b\,k_N\,(2\,b+k) \right]$$

$$+ \, 2\,b\,[\,b+k\,]\,[\,k_A\,c_N + k_N\,c_A - b\,(c_N - c_A)\,] \;\geq\; 0\,. \tag{116}$$

Observe that:

$$\left[\,b^2 + k^R\,(2\,b+k)\,\right][\,k_N + k_A\,] + k_N\,k_A\,[\,2\,b+k\,] - b\,k_N\,[\,2\,b+k\,]$$

$$= \; b^2\,[\,k_N + k_A\,] + k^R\,[\,2\,b+k\,]\,[\,k_N + k_A\,] + k_N\,[\,2\,b+k\,]\,[\,k_A - b\,]$$

$$= \; b^2\,[\,k_N + k_A\,] - b\,[\,2\,b+k\,]\,[\,k_N + k_A\,] + b\,[\,2\,b+k\,]\,[\,k_N + k_A\,]$$

$$+ \;\; k^R\,[\,2\,b+k\,]\,[\,k_N + k_A\,] + k_N\,[\,2\,b+k\,]\,[\,k_A - b\,]$$

$$= \; -\,b\,[\,b+k\,]\,[\,k_N + k_A\,] + \left[\,b + k^R\,\right][\,2\,b+k\,]\,[\,k_N + k_A\,] + k_N\,[\,2\,b+k\,]\,[\,k_A - b\,]$$

$$= \; -\,2\,b\,[\,b+k\,]\,[\,k_N + k_A\,] + D_2\,. \tag{117}$$

The last equality in (117) reflects (7). (116) and (117) imply:

$$\overline{p}_d \;\geq\; \overline{p}_{VM} \quad \Leftrightarrow \quad \widetilde{\Phi}_1 \;\geq\; 0\,,$$

$$\text{where} \;\; \widetilde{\Phi}_1 \;\equiv\; \left[\frac{(b+k)\,a + b\,c}{2\,b+k} \right] \{\, D_2 - 2\,b\,[\,b+k\,]\,[\,k_N + k_A\,]\,\}$$

$$+ \;\; 2\,b\,[\,b+k\,]\,[\,k_A\,c_N + k_N\,c_A - b\,(c_N - c_A)\,]\,. \tag{118}$$

(7) implies:

$$D_2 - 2\,b\,[\,b+k\,]\,[\,k_N + k_A\,] \;\; = \;\; -\,b\,[\,b+k\,]\,[\,k_N + k_A\,] + k_N\,[\,k_A - b\,]\,[\,2\,b+k\,]$$

$$+ \;[\,2\,b+k\,]\left[\,b + k^R\,\right][\,k_N + k_A\,]$$

$$= \; [\,k_A + k_N\,]\left[\,2\,b^2 + 2\,b\,k^R + b\,k + k\,k^R - b^2 - b\,k\,\right] + k_N\,[\,k_A - b\,]\,[\,2\,b+k\,]$$

$$= \; [\,k_A + k_N\,]\left[\,b^2 + 2\,b\,k^R + k\,k^R\,\right] + k_N\,[\,k_A - b\,]\,[\,2\,b+k\,]\,. \tag{119}$$

(119) implies:

$$\left[\frac{(b+k)\,a + b\,c}{2\,b+k} \right] [\,D_2 - 2\,b\,(b+k)\,(k_N + k_A)\,]$$

$$= \; \left[\frac{(b+k)\,a + b\,c}{2\,b+k} \right] [\,k_A + k_N\,]\left[\,b^2 + k^R\,(2\,b+k)\,\right] + [\,(b+k)\,a + b\,c\,]\,k_N\,[\,k_A - b\,]\,. \tag{120}$$

(109) and (120) imply:

$$\widetilde{\Phi}_1 \;\; = \;\; \frac{b^2}{2\,b+k}\,[\,k_A + k_N\,]\,[\,(b+k)\,a + b\,c\,] + k^R\,[\,k_A + k_N\,]\,[\,(b+k)\,a + b\,c\,]$$

$$+ [\,(b+k)\,a + b\,c\,]\,k_N\,[\,k_A - b\,] + 2\,b\,[\,b+k\,]\,c_N\,[\,k_A - b\,]$$

$$+ 2\,b\,[\,b+k\,]\,c_A\,[\,k_N + b\,]$$

$$= \left[k^R + \frac{b^2}{2\,b+k} \right] [\,k_A + k_N\,]\,[\,(b+k)\,a + b\,c\,] + 2\,b\,[\,b+k\,]\,c_A\,[\,k_N + b\,]$$

$$+ \{\,2\,b\,[\,b+k\,]\,c_N + [\,(b+k)\,a + b\,c\,]\,k_N\,\}\,[\,k_A - b\,] \;\equiv\; \Phi_1\,. \quad \blacksquare$$

Proposition 4. *$\overline{p}_b - \overline{p}_d$ increases as: (i) c_A, k_A, or k^R declines; (ii) c or c_N increases; or (iii) k_N increases if $k_A - b$ is sufficiently small.*

Proof. (7) and (8) imply:

$$\overline{p}_d \;=\; \frac{N_2}{D_2} \quad \text{and} \quad \overline{p}_b \;=\; \frac{N_3}{D_2 + b\,[\,2\,b+k\,]\,k_N}$$

$$\text{where} \quad N_3 \;\equiv\; [\,a\,(b+k) + b\,c\,]\,\left[\,\left(b + k^R\right)(k_N + k_A) + k_N\,k_A\,\right]$$

$$+ \,b\,c_N\,[\,b+k\,]\,k_A + b\,k_N\,[\,b+k\,]\,c_A \quad \text{and}$$

$$N_2 \;\equiv\; [\,a\,(b+k) + b\,c\,]\,\left[\,\left(b + k^R\right)(k_N + k_A) + k_N\,k_A\,\right]$$

$$+ \,b\,c_N\,[\,b+k\,]\,k_A + b\,k_N\,[\,b+k\,]\,c_A$$

$$- \;b\,k_N\,[\,a\,(b+k) + b\,c\,] - b^2\,[\,b+k\,]\,c_N + b^2\,[\,b+k\,]\,c_A$$

$$= \; N_3 - b\,k_N\,[\,a\,(b+k) + b\,c\,] - b^2\,[\,b+k\,]\,[\,c_N - c_A\,]\,. \tag{121}$$

To prove that $\frac{\partial(\overline{p}_b - \overline{p}_d)}{\partial k^R} < 0$, let $q_A(\overline{p})$ denote R's equilibrium output using A's input when the price cap is $\overline{p} \in [\,\overline{p}_d, \overline{p}_b\,]$. Let $q_N(\overline{p})$ denote R's corresponding output when R does not employ A's input. Also let $Q^R(\overline{p}) = q_A(\overline{p}) + q_N(\overline{p})$. (73) implies:

$$\overline{p}_b \;=\; \left[\,b + k^R\,\right] Q^R(\overline{p}_b) + c_N + k_N\,q_N(\overline{p}_b)$$

where, from (57):

$$q_N(\overline{p}_b) \;=\; \frac{k_A\,[\,b+k\,]\,[\,a - \overline{p}_b\,] - b\,k_A\,[\,\overline{p}_b - c\,] - b\,[\,b+k\,]\,[\,c_N - c_A\,]}{b\,[\,b+k\,]\,[\,k_N + k_A\,]} \quad \text{and}$$

$$Q^R(\overline{p}_b) \;=\; \frac{[\,b+k\,]\,[\,a - \overline{p}_b\,] - b\,[\,\overline{p}_b - c\,]}{b\,[\,b+k\,]}\,. \tag{122}$$

(122) implies that $q_N(\overline{p}_b)$ and $Q^R(\overline{p}_b)$ vary with k^R only through $\overline{p}_b$. Therefore, (122) implies:

$$\frac{\partial \overline{p}_b}{\partial k^R} \;=\; Q^R(\overline{p}_b) + \left[\,b + k^R\,\right] \frac{\partial Q^R(\overline{p}_b)}{\partial \overline{p}_b}\,\frac{\partial \overline{p}_b}{\partial k^R} + k_N\,\frac{\partial q_N(\overline{p}_b)}{\partial \overline{p}_b}\,\frac{\partial \overline{p}_b}{\partial k^R}\,. \tag{123}$$

(122) also implies:

$$\frac{\partial q_N(\overline{p}_b)}{\partial \overline{p}_b} \;=\; -\,\frac{k_A\,[\,b+k\,]+b\,k_A}{b\,[\,b+k\,]\,[\,k_N+k_A\,]} \;\equiv\; D_N \;<\; 0\,;$$

$$\frac{\partial Q^R(\overline{p}_b)}{\partial \overline{p}_b} \;=\; -\,\frac{2\,b+k}{b\,[\,b+k\,]} \;\equiv\; D_R \;<\; 0\,. \tag{124}$$

(123) and (124) imply:

$$\frac{\partial \overline{p}_b}{\partial k^R} \;=\; Q^R(\overline{p}_b) + \left[\,b+k^R\,\right] D_R\,\frac{\partial \overline{p}_b}{\partial k^R} + k_N\,D_N\,\frac{\partial \overline{p}_b}{\partial k^R}$$

$$\Rightarrow\;\; \frac{\partial \overline{p}_b}{\partial k^R}\left[\,1-\left(b+k^R\right) D_R - k_N\,D_N\,\right] \;=\; Q^R(\overline{p}_b)$$

$$\Rightarrow\;\; \frac{\partial \overline{p}_b}{\partial k^R} \;=\; \frac{Q^R(\overline{p}_b)}{1-[\,b+k^R\,]\,D_R - k_N\,D_N} \;>\; 0\,. \tag{125}$$

The inequality in (125) holds because $D_R < 0$ and $D_N < 0$, from (124).

(73) implies:

$$\overline{p}_d \;=\; \left[\,b+k^R\,\right] Q^R(\overline{p}_d) + c_N + k_N\,q_N(\overline{p}_d) - b\,q_A(\overline{p}_d)$$

where, from (57):

$$q_A(\overline{p}_d) \;=\; \frac{b\,[\,b+k\,]\,[\,c_N-c_A\,]+k_N\,[\,a-\overline{p}\,]\,[\,b+k\,]-b\,k_N\,[\,\overline{p}-c\,]}{b\,[\,b+k\,]\,[\,k_N+k_A\,]}\,;$$

$$q_N(\overline{p}_d) \;=\; \frac{k_A\,[\,b+k\,]\,[\,a-\,\overline{p}_d\,]-b\,k_A\,[\,\overline{p}_d-c\,]-b\,[\,b+k\,]\,[\,c_N-c_A\,]}{b\,[\,b+k\,]\,[\,k_N+k_A\,]}\,;\;\text{ and}$$

$$Q^R(\overline{p}_d) \;=\; \frac{[\,b+k\,]\,[\,a-\,\overline{p}_d\,]-b\,[\,\overline{p}_d-c\,]}{b\,[\,b+k\,]}\,. \tag{126}$$

(126) implies that $q_A(\overline{p}_d)$, $q_N(\overline{p}_d)$, and $Q^R(\overline{p}_d)$ vary with k^R only through $\overline{p}_d$. Therefore, (126) implies:

$$\frac{\partial \overline{p}_d}{\partial k^R} \;=\; Q^R(\overline{p}_d) + \left[\,b+k^R\,\right]\frac{\partial Q^R(\overline{p}_d)}{\partial \overline{p}_d}\,\frac{\partial \overline{p}_d}{\partial k^R}$$

$$+\;\; k_N\,\frac{\partial q_N(\overline{p}_d)}{\partial \overline{p}_d}\,\frac{\partial \overline{p}_d}{\partial k^R} - b\,\frac{\partial q_A(\overline{p}_d)}{\partial \overline{p}_d}\,\frac{\partial \overline{p}_d}{\partial k^R}\,. \tag{127}$$

(126) also implies:

$$\frac{\partial q_A(\overline{p}_d)}{\partial \overline{p}_d} \;=\; -\,\frac{k_N\,[\,b+k\,]+b\,k_N}{b\,[\,b+k\,]\,[\,k_N+k_A\,]} \;\equiv\; D_A \;<\; 0\,;$$

$$\frac{\partial q_N(\overline{p}_d)}{\partial \overline{p}_d} \;=\; -\,\frac{k_A\,[\,b+k\,]+b\,k_A}{b\,[\,b+k\,]\,[\,k_N+k_A\,]} \;\equiv\; D_N \;<\; 0\,;$$

$$\frac{\partial Q^R(\overline{p}_d)}{\partial \overline{p}_d} \;=\; -\,\frac{2\,b+k}{b\,[\,b+k\,]} \;\equiv\; D_R \;<\; 0\,. \tag{128}$$

(127) and (128) imply:

$$\frac{\partial \overline{p}_d}{\partial k^R} \;=\; Q^R(\overline{p}_d) + \left[\,b+k^R\,\right] D_R\,\frac{\partial \overline{p}_d}{\partial k^R} + k_N\,D_N\,\frac{\partial \overline{p}_d}{\partial k^R} - b\,D_A\,\frac{\partial \overline{p}_d}{\partial k^R}$$

$$\Rightarrow \;\; \frac{\partial \overline{p}_d}{\partial k^R}\,\left[\,1-\left(b+k^R\right)D_R - k_N D_N + b\,D_A\,\right] \;=\; Q^R(\overline{p}_d)$$

$$\Rightarrow \;\; \frac{\partial \overline{p}_d}{\partial k^R} \;=\; \frac{Q^R(\overline{p}_d)}{1-[\,b+k^R\,]\,D_R - k_N\,D_N + b\,D_A}\,. \tag{129}$$

(128) implies:

$$-\,b\,D_R + b\,D_A \;=\; b\,[\,-\,D_R + D_A\,] \;=\; b\left[\frac{2\,b+k}{b\,(b+k)} - \frac{k_N\,(b+k)+b\,k_N}{b\,(b+k)\,(k_N+k_A)}\right]$$

$$=\; b\left[\frac{2\,b+k}{b\,(b+k)} - \left(\frac{k_N}{k_N+k_A}\right)\frac{2\,b+k}{b\,(b+k)}\right]$$

$$=\; b\left[\frac{2\,b+k}{b\,(b+k)}\right]\left[1-\frac{k_N}{k_N+k_A}\right] \;>\; 0 \tag{130}$$

Because $D_N \;<\; 0$ and $D_R \;<\; 0$ from (128), (130) implies:

$$1-\left[\,b+k^R\,\right]D_R - k_N\,D_N + b\,D_A \;=\; 1-k^R\,D_R - k_N\,D_N - b\,D_R + b\,D_A$$

$$>\; 1-k^R\,D_R - k_N\,D_N \;>\; 0\,. \tag{131}$$

Because $D_A < 0$ from (128), (131) implies:

$$1-\left[\,b+k^R\,\right]D_R - k_N\,D_N \;>\; 0\,. \tag{132}$$

(129) and (131) imply:

$$\frac{\partial\,\overline{p}_d}{\partial k^R} \;=\; \frac{Q^R(\overline{p}_d)}{1-[\,b+k^R\,]\,D_R - k_N\,D_N + b\,D_A} \;>\; 0\,. \tag{133}$$

(125) and (131) – (133) imply:

$$\frac{\partial\,\overline{p}_b}{\partial k^R} - \frac{\partial\,\overline{p}_d}{\partial k^R} \;=\; \frac{Q^R(\overline{p}_b)}{1-[\,b+k^R\,]\,D_R - k_N\,D_N} - \frac{Q^R(\overline{p}_d)}{1-[\,b+k^R\,]\,D_R - k_N\,D_N + b\,D_A} \;<\; 0$$

$$\Leftrightarrow \quad \frac{Q^R(\overline{p}_b)}{1-[\,b+k^R\,]\,D_R-k_N\,D_N} \;<\; \frac{Q^R(\overline{p}_d)}{1-[\,b+k^R\,]\,D_R-k_N\,D_N+b\,D_A}$$

$$\Leftrightarrow \quad \frac{Q^R(\overline{p}_b)}{Q^R(\overline{p}_d)} \;<\; \frac{1-\left[\,b+k^R\,\right]D_R-k_N\,D_N}{1-[\,b+k^R\,]\,D_R-k_N\,D_N+b\,D_A}\,. \tag{134}$$

(62) implies that $Q^R(\overline{p}_b) \,<\, Q^R(\overline{p}_d)$. Therefore:

$$\frac{Q^R(\overline{p}_b)}{Q^R(\overline{p}_d)} \;<\; 1\,. \tag{135}$$

Furthermore, because $1-\left[\,b+k^R\,\right]D_R-k_N D_N+bD_A \,>\, 0$ from (131):

$$\frac{1-\left[\,b+k^R\,\right]D_R-k_N\,D_N}{1-[\,b+k^R\,]\,D_R-k_N\,D_N+b\,D_A} \;>\; 1$$

$$\Leftrightarrow \quad 1-\left[\,b+k^R\,\right]D_R-k_N\,D_N \;>\; 1-\left[\,b+k^R\,\right]D_R-k_N\,D_N+b\,D_A$$

$$\Leftrightarrow \quad D_A \;<\; 0\,. \tag{136}$$

(128) implies that the last inequality in (136) holds. (135) and (136) imply that (134) holds. Therefore, because $\overline{p}_b \,>\, \overline{p}_d \,>\, 0$ from Proposition 1, (125) and (134) imply that $\frac{\partial(\overline{p}_b-\overline{p}_d)}{\partial k^R} < 0\,.$

To prove that $\frac{\partial(\,\overline{p}_b\,-\,\overline{p}_d)}{\partial c_N} \,>\, 0$, observe that (7) and (8) imply:

$$\frac{\partial\,\overline{p}_d}{\partial c_N} \;=\; \frac{b\,[\,b+k\,]\,[\,k_A-b\,]}{D_2} \quad \text{and} \quad \frac{\partial\,\overline{p}_b}{\partial c_N} \;=\; \frac{b\,[\,b+k\,]\,k_A}{D_2+b\,k_N[\,2\,b+k\,]}\,. \tag{137}$$

(137) implies:

$$\frac{\partial\,\overline{p}_b}{\partial c_N}-\frac{\partial\,\overline{p}_d}{\partial c_N} \;=\; \frac{b\,[\,b+k\,]\,k_A}{D_2+b\,k_N\,[\,2\,b+k\,]}-\frac{b\,[\,b+k\,]\,[\,k_A-b\,]}{D_2} \;>\; 0$$

$$\Leftrightarrow \quad \frac{k_A}{D_2+b\,k_N\,[\,2\,b+k\,]} \;>\; \frac{k_A-b}{D_2}$$

$$\Leftrightarrow \quad D_2\,k_A \;>\; [\,D_2+b\,k_N\,(2\,b+k)\,]\,[\,k_A-b\,]$$

$$\Leftrightarrow \quad D_2\,k_A \;>\; D_2\,k_A-b\,D_2+b\,k_N\,[\,2\,b+k\,]\,[\,k_A-b\,]$$

$$\Leftrightarrow \quad D_2-k_N\,[\,2\,b+k\,]\,[\,k_A-b\,] \;>\; 0\,. \tag{138}$$

The inequality in (138) holds because, from (7):

$$D_2-k_N\,[\,2\,b+k\,]\,[\,k_A-b\,]$$

$$=\; b\,[\,b+k\,]\,[\,k_N+k_A\,]+[\,2\,b+k\,]\,k_N\,[\,k_A-b\,]$$

$$+ [\,2\,b+k\,]\,[\,k_N+k_A\,]\left[\,b+k^R\,\right] - k_N\,[\,2\,b+k\,]\,[\,k_A-b\,]$$

$$= \; b\,[\,b+k\,]\,[\,k_N+k_A\,] + [\,2\,b+k\,]\,[\,k_N+k_A\,]\left[\,b+k^R\,\right] \; > \; 0\,.$$

To prove that $\frac{\partial(\,\overline{p}_b\,-\,\overline{p}_d)}{\partial c_A} \, < \, 0$, observe that (7) and (8) imply:

$$\frac{\partial\,\overline{p}_d}{\partial c_A} \;=\; \frac{b\,[\,b+k\,]\,[\,k_N+b\,]}{D_2} \quad \text{and} \quad \frac{\partial\,\overline{p}_b}{\partial c_A} \;=\; \frac{b\,[\,b+k\,]\,k_N}{D_2+b\,k_N[\,2\,b+k\,]}\,. \tag{139}$$

(139) implies:

$$\frac{\partial\,\overline{p}_b}{\partial c_A} - \frac{\partial\,\overline{p}_d}{\partial c_A} \;=\; \frac{b\,[\,b+k\,]\,k_N}{D_2+b\,k_N\,[\,2\,b+k\,]} - \frac{b\,[\,b+k\,]\,[\,k_N+b\,]}{D_2} \;<\; 0$$

$$\Leftrightarrow \quad \frac{b\,[\,b+k\,]\,k_N}{D_2+b\,k_N\,[\,2\,b+k\,]} \;<\; \frac{b\,[\,b+k\,]\,[\,k_N+b\,]}{D_2}$$

$$\Leftrightarrow \quad b\,D_2\,[\,b+k\,]\,k_N \;<\; [\,D_2+b\,k_N\,(2\,b+k)\,]\,b\,[\,b+k\,]\,[\,k_N+b\,]$$

$$\Leftrightarrow \quad D_2\,k_N \;<\; [\,D_2+b\,k_N\,(2\,b+k)\,]\,[\,k_N+b\,]$$

$$\Leftrightarrow \quad D_2\,k_N \;<\; D_2\,k_N+b\,D_2+b\,k_N\,[\,2\,b+k\,]\,[\,k_N+b\,]$$

$$\Leftrightarrow \quad D_2+k_N\,[\,2\,b+k\,]\,[\,k_N+b\,] \;>\; 0\,. \tag{140}$$

The inequality in (140) holds because, from (7):

$$D_2 \;=\; b\,[\,b+k\,]\,[\,k_N+k_A\,]+[\,2\,b+k\,]\left\{\,k_N\,[\,k_A-b\,]+[\,k_N+k_A\,]\left[\,b+k^R\,\right]\,\right\}$$

$$=\; b\,[\,b+k\,]\,[\,k_N+k_A\,]+[\,2\,b+k\,]\left\{\,k_N\left[\,k_A+k^R\,\right]+k_A\left[\,b+k^R\,\right]\,\right\} \;>\; 0\,.$$

To prove that $\frac{\partial(\overline{p}_b-\overline{p}_d)}{\partial k_A} < 0$, we introduce the following:

Definition. $\quad Y_1 \;\equiv\; b\,[\,b+k\,]\;\{\,k_N\,[\,a\,(b+k)+b\,c-(2\,b+k)\,c_A\,]$

$$+\;[\,c_N-c_A\,]\left[\,b\,(b+k)+(2\,b+k)\left(b+k^R\right)\,\right]\,\}\,. \tag{141}$$

Observe that:

$$Y_1 \;>\; 0\,. \tag{142}$$

(142) holds because $c_N \;\geq\; c_A$ by assumption, and (3) implies:

$$[\,a-c_A\,]\,[\,2\,b+k\,]-b\,[\,a-c\,] \;>\; 0 \quad \Rightarrow \quad a\,[\,b+k\,]+b\,c-[\,2\,b+k\,]\,c_A \;>\; 0\,.$$

(8) implies:

$$(D_3)^2\,\frac{\partial\,\overline{p}_b}{\partial k_A} \;=\; D_3\left\{\,[\,a\,(b+k)+b\,c\,]\left[\,b+k^R+k_N\,\right]+b\,[\,b+k\,]\,c_N\,\right\}$$

$$- \Big\{ b\,[\,b+k\,] + [\,2\,b+k\,]\,\Big[\,k_N + b + k^R\,\Big]\,\Big\}$$

$$\cdot\ \Big\{\,[\,a\,(b+k) + b\,c\,]\,\Big[\,\Big(b+k^R\Big)\,(k_A+k_N) + k_A\,k_N\,\Big]$$

$$+\ \ b\,[\,b+k\,]\,[\,c_N\,k_A + c_A\,k_N\,]\,\Big\}$$

$$= \ \Big\{\,[\,a\,(b+k) + b\,c\,]\,\Big[\,b+k^R+k_N\,\Big] + b\,[\,b+k\,]\,c_N\,\Big\}$$

$$\cdot \Big\{\,b\,[\,b+k\,]\,[\,k_A+k_N\,] + [\,2\,b+k\,]\,\Big[\,k_A\,k_N + (k_A+k_N)\,\Big(b+k^R\Big)\,\Big]\,\Big\}$$

$$- \Big\{\,b\,[\,b+k\,] + [\,2\,b+k\,]\,\Big[\,k_N+b+k^R\,\Big]\,\Big\}$$

$$\cdot\,\Big\{\,[\,a\,(b+k) + b\,c\,]\,\Big[\,\Big(b+k^R\Big)\,(k_A+k_N) + k_A\,k_N\,\Big]$$

$$+\ \ b\,[\,b+k\,]\,[\,c_N\,k_A + c_A\,k_N\,]\,\Big\}$$

$$= \ [\,a\,(b+k) + b\,c\,]\,\Big[\,b+k^R+k_N\,\Big]\,b\,[\,b+k\,]\,[\,k_A+k_N\,]$$

$$+\ [\,a\,(b+k) + b\,c\,]\,\Big[\,b+k^R+k_N\,\Big]\,[\,2\,b+k\,]\,\Big[\,\Big(b+k^R\Big)\,(k_A+k_N) + k_A\,k_N\,\Big]$$

$$+\ b\,[\,b+k\,]\,c_N\,b\,[\,b+k\,]\,[\,k_A+k_N\,]$$

$$+\ b\,[\,b+k\,]\,c_N\,[\,2\,b+k\,]\,\Big[\,\Big(b+k^R\Big)\,(k_A+k_N) + k_A\,k_N\,\Big]$$

$$-\ b\,[\,b+k\,]\,[\,a\,(b+k) + b\,c\,]\,\Big[\,\Big(b+k^R\Big)\,(k_A+k_N) + k_A\,k_N\,\Big]$$

$$-\ b\,[\,b+k\,]\,b\,\,[\,b+k\,]\,[\,c_N\,k_A + c_A\,k_N\,]$$

$$-\ [\,2\,b+k\,]\,\Big[\,k_N+b+k^R\,\Big]\,[\,a\,(b+k) + b\,c\,]\,\Big[\,\Big(b+k^R\Big)\,(k_A+k_N) + k_A\,k_N\,\Big]$$

$$-\ \,[\,2\,b+k\,]\,\Big[\,k_N+b+k^R\,\Big]\,b\,\,[\,b+k\,]\,[\,c_N\,k_A + c_A\,k_N\,]\ \ \equiv\ \ \Phi\,. \tag{143}$$

(143) implies:

$$\Phi \ \ = \ \ [\,a\,(b+k) + b\,c\,]\,\Phi_A + b\,[\,b+k\,]\,\Phi_B \tag{144}$$

where

$$\Phi_A \ \equiv\ b\,[\,b+k\,]\,\Big[\,b+k^R+k_N\,\Big]\,[\,k_A+k_N\,]$$

$$+\ [\,2\,b+k\,]\,\Big[\,b+k^R+k_N\,\Big]\,\Big[\,\Big(b+k^R\Big)\,(k_A+k_N) + k_A\,k_N\,\Big]$$

$$-\ b\,[\,b+k\,]\,\Big[\,\Big(b+k^R\Big)\,(k_A+k_N) + k_A\,k_N\,\Big]$$

$$-\ [\,2\,b+k\,]\,\Big[\,b+k^R+k_N\,\Big]\,\Big[\,\Big(b+k^R\Big)\,(k_A+k_N) + k_A\,k_N\,\Big]$$

$$= \ b\,[\,b+k\,]\,\Big\{\,\Big[\,b+k^R+k_N\,\Big]\,[\,k_A+k_N\,] - \Big[\,\Big(b+k^R\Big)\,(k_A+k_N) + k_A\,k_N\,\Big]\,\Big\}$$

$$= \ b\,[\,b+k\,]\,\{\,k_N\,[\,k_A+k_N\,] - k_A\,k_N\,\}\ \ = \ \ b\,[\,b+k\,]\,(k_N)^2 \quad \text{and} \tag{145}$$

$$\Phi_B \equiv c_N\, b\, [\, b + k\,]\, [\, k_A + k_N\,] + c_N\, [\, 2\, b + k\,] \left[\left(b + k^R \right) (k_A + k_N) + k_A\, k_N \,\right]$$

$$- \; b\, [\, b + k\,]\, [\, c_N\, k_A + c_A\, k_N\,] - [\, 2\, b + k\,] \left[k_N + b + k^R \right] [\, c_N\, k_A + c_A\, k_N\,]$$

$$= b\, [\, b + k\,]\, k_N\, [\, c_N - c_A\,] + [\, 2\, b + k\,]\, \Phi_C \tag{146}$$

where

$$\Phi_C \equiv c_N \left[\left(b + k^R \right) (k_A + k_N) + k_A\, k_N \,\right] - \left[k_N + b + k^R \right] [\, c_N\, k_A + c_A\, k_N\,]$$

$$= c_N \left[\left(b + k^R \right) (k_A + k_N) + k_A\, k_N - k_A \left(k_N + b + k^R \right) \right] - c_A\, k_N \left[k_N + b + k^R \right]$$

$$= c_N\, k_N \left[b + k^R \right] - c_A\, k_N \left[k_N + b + k^R \right] = \left[b + k^R \right] k_N\, [\, c_N - c_A\,] - c_A\, (k_N)^2 . \tag{147}$$

(146) and (147) imply:

$$\Phi_B = b\, [\, b + k\,]\, k_N\, [\, c_N - c_A\,] + [\, 2\, b + k\,] \left\{ \left[b + k^R \right] k_N\, [\, c_N - c_A\,] - c_A\, (k_N)^2 \right\}$$

$$= k_N\, [\, c_N - c_A\,] \left\{ b\, [\, b + k\,] + [\, 2\, b + k\,] \left[b + k^R \right] \right\} - [\, 2\, b + k\,]\, c_A\, (k_N)^2 . \tag{148}$$

(141), (144), (145), and (148) imply:

$$\Phi = [\, a\, (b + k) + b\, c\,]\, b\, [\, b + k\,]\, (k_N)^2$$

$$+ \; b\, [\, b + k\,]\, k_N\, [\, c_N - c_A\,] \left\{ b\, [\, b + k\,] + [\, 2\, b + k\,] \left[b + k^R \right] \right\}$$

$$- \; b\, [\, b + k\,]\, [\, 2\, b + k\,]\, c_A\, (k_N)^2$$

$$= b\, [\, b + k\,]\, k_N\, \{\, [\, a\, (b + k) + b\, c\,]\, k_N - [\, 2\, b + k\,]\, c_A\, k_N$$

$$+ \; [\, c_N - c_A\,] \left[b\, (b + k) + (2\, b + k) \left(b + k^R \right) \right] \}$$

$$= b\, [\, b + k\,]\, k_N\, \{\, k_N\, [\, a\, (b + k) + b\, c - (2\, b + k)\, c_A\,]$$

$$+ \; [\, c_N - c_A\,] \left[b\, (b + k) + (2\, b + k) \left(b + k^R \right) \right] \} = k_N\, Y_1 . \tag{149}$$

(142), (143), and (149) imply that $\frac{\partial \overline{p}_b}{\partial k_A} = \frac{k_N\, Y_1}{(D_3)^2} > 0$.

(7) implies:

$$(D_2)^2\, \frac{\partial \overline{p}_d}{\partial k_A} = D_2 \left\{ [\, a\, (b + k) + b\, c\,] \left[b + k^R + k_N \right] + b\, [\, b + k\,]\, c_N \right\}$$

$$- \left\{ b\, [\, b + k\,] + [\, 2\, b + k\,] \left[k_N + b + k^R \right] \right\}$$

$$\cdot \; \left\{ [\, a\, (b + k) + b\, c\,] \left[\left(b + k^R \right) (k_A + k_N) + k_A\, k_N - b\, k_N \right] \right.$$

$$\left. + \; b\, [\, b + k\,]\, [\, c_N\, (k_A - b) + c_A\, (k_N + b)\,] \right\}$$

$$= \Big\{ [\, a\,(b+k) + b\,c\,] \big[\, b + k^R + k_N \big] + b\,[\, b+k\,]\, c_N \Big\}$$

$$\cdot \{\, b\,[\, b+k\,]\,[\, k_A + k_N\,] + [\, 2\,b + k\,] \big[\, k_N \big(k_A + k^R\big) + k_A \big(b + k^R\big) \big] \}$$

$$- \Big\{ b\,[\, b+k\,] + [\, 2\,b+k\,] \big[\, k_N + b + k^R \big] \Big\}$$

$$\cdot \{ [\, a\,(b+k) + b\,c\,] \big[\, k_A\,(b + k_N) + k^R\,(k_A + k_N)\,\big]$$

$$+ \, b\,[\, b+k\,]\,[\, c_N\,(k_A - b) + c_A\,(k_N + b)\,] \}$$

$$= [\, a\,(b+k) + b\,c\,] \big[\, b + k^R + k_N \big]\, b\,[\, b+k\,]\,[\, k_N + k_A\,]$$

$$+ [\, a\,(b+k) + b\,c\,] \big[\, b + k^R + k_N \big]\,[\, 2\,b+k\,] \big[\, k_N \big(k_A + k^R\big) + k_A \big(b + k^R\big) \big]$$

$$+ \, b\,[\, b+k\,]\, c_N\, b\,[\, b+k\,]\,[\, k_N + k_A\,]$$

$$+ \, b\,[\, b+k\,]\, c_N \; [\, 2\,b+k\,] \big[\, k_N \big(k_A + k^R\big) + k_A \big(b + k^R\big) \big]$$

$$- \, b\,[\, b+k\,]\,[\, a\,(b+k) + b\,c\,] \big[\, k_A\,(b + k_N) + k^R\,(k_A + k_N)\,\big]$$

$$- \, b\,[\, b+k\,]\; b\,[\, b+k\,]\,[\, c_N\,(k_A - b) + c_A\,(k_N + b)\,]$$

$$- [\, 2\,b+k\,] \big[\, k_N + b + k^R \big]\,[\, a\,(b+k) + b\,c\,] \big[\, k_A\,(b + k_N) + k^R\,(k_A + k_N)\,\big]$$

$$- [\, 2\,b+k\,] \big[\, k_N + b + k^R \big]\; b\,[\, b+k\,]\,[\, c_N\,(k_A - b) + c_A\,(k_N + b)\,] \;\equiv\; \mathcal{F} \,. \qquad (150)$$

(150) implies:

$$\mathcal{F} \;=\; [\, a\,(b+k) + b\,c\,]\,\mathcal{F}_1 + b\,[\, b+k\,]\,\mathcal{F}_2 \qquad (151)$$

where

$$\mathcal{F}_1 \;\equiv\; \big[\, b + k^R + k_N \big]\, b\,[\, b+k\,]\,[\, k_N + k_A\,]$$

$$+ [\, 2\,b+k\,] \big[\, b + k^R + k_N \big] \big[\, k_N \big(k_A + k^R\big) + k_A \big(b + k^R\big) \big]$$

$$- \, b\,[\, b+k\,] \big[\, k_A\,(b + k_N) + k^R\,(k_A + k_N)\,\big]$$

$$- [\, 2\,b+k\,] \big[\, k_N + b + k^R \big] \big[\, k_A\,(b + k_N) + k^R\,(k_A + k_N)\,\big]$$

$$= \; b\,[\, b+k\,] \Big\{ \big[\, b + k^R + k_N \big]\,[\, k_N + k_A\,] - \big[\, k_A\,(b + k_N) + k^R\,(k_A + k_N)\,\big] \Big\}$$

$$= \; b\,[\, b+k\,]\,\{ [\, b + k_N\,]\,[\, k_N + k_A\,] - k_A\,[\, b + k_N\,] \} \;=\; b\,[\, b+k\,]\,[\, b + k_N\,]\, k_N \qquad (152)$$

and

$$\mathcal{F}_2 \;\equiv\; b\,[\, b+k\,]\,[\, k_N + k_A\,]\, c_N + c_N\,[\, 2\,b+k\,] \big[\, k_N \big(k_A + k^R\big) + k_A \big(b + k^R\big) \big]$$

$$- \, b\,[\, b+k\,]\,[\, c_N\,(k_A - b) + c_A\,(k_N + b)\,]$$

$$
\begin{aligned}
& - [\,2\,b+k\,] \left[\,k_N+b+k^R\right] [\,c_N\,(k_A-b)+c_A\,(k_N+b)\,] \\
= {} & b\,[\,b+k\,]\,[\,c_N\,(k_A+k_N)-c_N\,(k_A-b)-c_A\,(k_N+b)\,] \\
& + [\,2\,b+k\,] \,\{\, c_N \left[\,k_N \left(k_A+k^R\right)+k_A\left(b+k^R\right)\right] \\
& \qquad - \left[\,k_N+b+k^R\right] [\,c_N\,(k_A-b)+c_A\,(k_N+b)\,]\,\} \\
= {} & b\,[\,b+k\,]\,[\,c_N\,k_N+b\,c_N-c_A\,(k_N+b)\,] \\
& + [\,2\,b+k\,] \,\{\, c_N\,[\,k_N\left(k_A+k^R\right)-k_N\,(k_A-b)+k_A\left(b+k^R\right) \\
& \qquad - (k_A-b)\left(b+k^R\right)]-c_A\,[\,k_N+b\,]\left[\,k_N+b+k^R\right]\} \\
= {} & b\,[\,b+k\,]\,[\,k_N\,(c_N-c_A)+b\,(c_N-c_A)\,] \\
& + [\,2\,b+k\,] \,\{\, c_N \left[\,k_N\left(b+k^R\right)+b\left(b+k^R\right)\right]-c_A\,[\,k_N+b\,]\,k_N \\
& \qquad - c_A\,[\,k_N+b\,]\left[\,b+k^R\right]\} \\
= {} & b\,[\,b+k\,]\,[\,c_N-c_A\,]\,[\,k_N+b\,] \\
& + [\,2\,b+k\,] \left\{ [\,c_N-c_A\,]\,[\,k_N+b\,]\left[\,b+k^R\right]-c_A\,[\,k_N+b\,]\,k_N \right\} \\
= {} & b\,[\,b+k\,]\,[\,c_N-c_A\,]\,[\,k_N+b\,] \\
& + [\,2\,b+k\,]\,[\,k_N+b\,]\left[\left(b+k^R\right)(c_N-c_A)-c_A\,k_N\right]. \qquad (153)
\end{aligned}
$$

(141), (151), (152), and (153) imply:

$$
\begin{aligned}
F = {} & [\,a\,(b+k)+b\,c\,]\,b\,[\,b+k\,]\,[\,b+k_N\,]\,k_N \\
& + b\,[\,b+k\,] \,\{\, b\,[\,b+k\,]\,[\,c_N-c_A\,]\,[\,k_N+b\,] \\
& \qquad + [\,2\,b+k\,]\,[\,k_N+b\,]\left[\left(b+k^R\right)(c_N-c_A)-c_A\,k_N\right]\} \\
= {} & b\,[\,b+k\,]\,[\,b+k_N\,]\,\{\,k_N\,[\,a\,(b+k)+b\,c-(2\,b+k)\,c_A\,] \\
& \qquad + [\,c_N-c_A\,]\left[\,b\,(b+k)+(2\,b+k)\left(b+k^R\right)\right]\} \\
= {} & [\,b+k_N\,]\,Y_1\,. \qquad (154)
\end{aligned}
$$

(142), (150), and (154) imply that $\frac{\partial\,\overline{p}_d}{\partial k_A} = \frac{[\,k_N+b\,]\,Y_1}{(D_2)^2} > 0$.

(141), (143), (149), (150), and (154) imply:

$$
\frac{\partial\,(\overline{p}_b-\overline{p}_d)}{\partial k_A} = Y_1\left[\frac{k_N}{(D_3)^2}-\frac{k_N+b}{(D_2)^2}\right] < 0\,.
$$

The inequality here holds because: (i) $Y_1 > 0$, from (142); (ii) $k_N < k_N + b$; and (iii) $D_3 > D_2 > 0$, from (7) and (8).

To prove that $\frac{\partial(\overline{p}_b - \overline{p}_d)}{\partial k_N} > 0$ if $k_A - b$ is sufficiently small, we introduce the following:

Definition. $$Y_2 \equiv b\,[\,b+k\,]\,\{\,k_A\,[\,a\,(b+k) + b\,c - (2\,b+k)\,c_N\,] - [\,c_N - c_A\,]\left[\,b\,(b+k) + (2\,b+k)\left(b+k^R\right)\right]\}\,. \tag{155}$$

Observe that:

$$Y_2 > 0. \tag{156}$$

(156) follows from (3) and (155) because:

$$a\,[\,b+k\,] + b\,c - [\,2\,b+k\,]\,c_N = a\,[\,2\,b+k\,] - a\,b + b\,c - [\,2\,b+k\,]\,c_N$$
$$= [\,a - c_N\,]\,[\,2\,b+k\,] - b\,[\,a - c\,] \quad \text{and}$$

$$b\,[\,b+k\,] + [\,2\,b+k\,]\left[\,b+k^R\right] = b^2 + b\,k + 2\,b^2 + 2\,b\,k^R + b\,k + k\,k^R$$
$$= 3\,b^2 + 2\,b\,k + 2\,b\,k^R + k\,k^R = 3\,b^2 + 2\,b\left[\,k+k^R\right] + k\,k^R\,.$$

(8) implies:

$$\begin{aligned}
(D_3)^2\,\frac{\partial \overline{p}_b}{\partial k_N} = {} & D_3\left\{\,[\,a\,(b+k) + b\,c\,]\left[\,b+k^R+k_A\right] + b\,[\,b+k\,]\,c_A\right\} \\
& - \left\{\,b\,[\,b+k\,] + [\,2\,b+k\,]\left[\,k_A + b + k^R\right]\right\} \\
& \cdot \left\{\,[\,a\,(b+k) + b\,c\,]\left[\left(b+k^R\right)(k_A+k_N) + k_A\,k_N\,\right] \right. \\
& \left. + \; b\,[\,b+k\,]\,[\,c_N\,k_A + c_A\,k_N\,]\right\} \\
= {} & \left\{\,[\,a\,(b+k) + b\,c\,]\left[\,b+k^R+k_A\right] + b\,[\,b+k\,]\,c_A\right\} \\
& \cdot \left\{\,b\,[\,b+k\,]\,[\,k_A+k_N\,] + [\,2\,b+k\,]\left[\,k_A\,k_N + (k_A+k_N)\left(b+k^R\right)\right]\right\} \\
& - \left\{\,b\,[\,b+k\,] + [\,2\,b+k\,]\left[\,k_A + b + k^R\right]\right\} \\
& \cdot \left\{\,[\,a\,(b+k) + b\,c\,]\left[\left(b+k^R\right)(k_A+k_N) + k_A\,k_N\,\right] \right. \\
& \left. + \; b\,[\,b+k\,]\,[\,c_N\,k_A + c_A\,k_N\,]\right\} \\
= {} & [\,a\,(b+k) + b\,c\,]\left[\,b+k^R+k_A\right] b\,[\,b+k\,]\,[\,k_A+k_N\,] \\
& + [\,a\,(b+k) + b\,c\,]\left[\,b+k^R+k_A\right][\,2\,b+k\,]\left[\left(b+k^R\right)(k_A+k_N) + k_A\,k_N\,\right] \\
& + b\,[\,b+k\,]\,c_A\,b\,[\,b+k\,]\,[\,k_A+k_N\,]
\end{aligned}$$

$$+\ b\,[\,b+k\,]\,c_A\,[\,2\,b+k\,]\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big]$$

$$-\ b\,[\,b+k\,]\,[\,a\,(b+k)+b\,c\,]\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big]$$

$$-\ b\,[\,b+k\,]\,b\,[\,b+k\,]\,[\,c_N\,k_A+c_A\,k_N\,]$$

$$-\ [\,2\,b+k\,]\,\Big[\,k_A+b+k^R\,\Big]\,[\,a\,(b+k)+b\,c\,]\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big]$$

$$-\ [\,2\,b+k\,]\,\Big[\,k_A+b+k^R\,\Big]\,b\,[\,b+k\,]\,[\,c_N\,k_A+c_A\,k_N\,]\ \equiv\ \Lambda\,. \tag{157}$$

(157) implies:

$$\Lambda\ =\ [\,a\,(b+k)+b\,c\,]\,\Lambda_1+b\,[\,b+k\,]\,\Lambda_2 \tag{158}$$

where

$$\begin{aligned}
\Lambda_1\ \equiv\ & b\,[\,b+k\,]\,\Big[\,b+k^R+k_A\,\Big]\,[\,k_A+k_N\,] \\
& +\ [\,2\,b+k\,]\,\Big[\,b+k^R+k_A\,\Big]\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big] \\
& -\ b\,[\,b+k\,]\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big] \\
& -\ [\,2\,b+k\,]\,\Big[\,b+k^R+k_A\,\Big]\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big] \\
=\ & b\,[\,b+k\,]\,\Big\{\,\Big[\,b+k^R+k_A\,\Big]\,[\,k_A+k_N\,]-\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big]\,\Big\} \\
=\ & b\,[\,b+k\,]\,\{\,k_A\,[\,k_A+k_N\,]-k_A\,k_N\,\}\ =\ b\,[\,b+k\,]\,(k_A)^2 \quad \text{and}
\end{aligned} \tag{159}$$

$$\begin{aligned}
\Lambda_2\ \equiv\ & c_A\,b\,[\,b+k\,]\,[\,k_A+k_N\,]+c_A\,[\,2\,b+k\,]\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big] \\
& -\ b\,[\,b+k\,]\,[\,c_N\,k_A+c_A\,k_N\,]-[\,2\,b+k\,]\,\Big[\,k_A+b+k^R\,\Big]\,[\,c_N\,k_A+c_A\,k_N\,] \\
=\ & -b\,[\,b+k\,]\,k_A\,[\,c_N-c_A\,]+[\,2\,b+k\,]\,\Lambda_3
\end{aligned} \tag{160}$$

where

$$\begin{aligned}
\Lambda_3\ \equiv\ & c_A\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big]-\Big[\,k_A+b+k^R\,\Big]\,[\,c_N\,k_A+c_A\,k_N\,] \\
=\ & c_A\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N-k_N\,\big(k_A+b+k^R\big)\,\Big]-c_N\,k_A\,\Big[\,k_A+b+k^R\,\Big] \\
=\ & c_A\,k_A\,\Big[\,b+k^R\,\Big]-c_N\,k_A\,\Big[\,k_A+b+k^R\,\Big]\ =\ -\ \Big[\,b+k^R\,\Big]\,k_A\,[\,c_N-c_A\,]-c_N\,(k_A)^2\,.
\end{aligned} \tag{161}$$

(160) and (161) imply:

$$\begin{aligned}
\Lambda_2\ =\ & -b\,[\,b+k\,]\,k_A\,[\,c_N-c_A\,]-[\,2\,b+k\,]\,\Big\{\,\Big[\,b+k^R\,\Big]\,k_A\,[\,c_N-c_A\,]-c_N\,(k_A)^2\,\Big\} \\
=\ & -k_A\,[\,c_N-c_A\,]\,\Big\{\,b\,[\,b+k\,]+[\,2\,b+k\,]\,\Big[\,b+k^R\,\Big]\,\Big\}-[\,2\,b+k\,]\,c_N\,(k_A)^2\ .
\end{aligned} \tag{162}$$

(155), (158), (159), and (162) imply:

$$\Lambda \;=\; [\, a\,(b+k) + b\,c\,]\, b\,[\, b+k\,]\,(k_A)^2$$

$$- \; b\,[\, b+k\,]\, k_A\,[\, c_N - c_A\,] \left\{ b\,[\, b+k\,] + [\, 2\,b + k\,] \left[\, b + k^R \,\right] \right\}$$

$$- \; b\,[\, b+k\,]\,[\, 2\,b+k\,]\, c_N\,(k_A)^2$$

$$= \; b\,[\, b+k\,]\, k_A \,\{\, [\, a\,(b+k) + b\,c\,]\, k_A - [\, 2\,b+k\,]\, c_N\, k_A$$

$$- \; [\, c_N - c_A\,] \left[\, b\,(b+k) + (2\,b+k) \left(b + k^R \right) \right] \}$$

$$= \; b\,[\, b+k\,]\, k_A \,\{\, k_A\,[\, a\,(b+k) + b\,c - (2\,b+k)\, c_N\,]$$

$$- \; [\, c_N - c_A\,] \left[\, b\,(b+k) + (2\,b+k) \left(b + k^R \right) \right] \} \; = \; k_A\, Y_2\,. \qquad (163)$$

(156), (157), and (163) imply that $\frac{\partial \overline{p}_b}{\partial k_N} \;=\; \frac{k_A\, Y_2}{(D_3)^2} \;>\; 0$.

(7) implies:

$$(D_2)^2\, \frac{\partial \overline{p}_d}{\partial k_N} \;=\; D_2 \left\{ [\, a\,(b+k) + b\,c\,] \left[\, b + k^R + k_A - b \,\right] + b\,[\, b+k\,]\, c_A \right\}$$

$$- \left\{ b\,[\, b+k\,] + [\, 2\,b+k\,] \left[\, k_A - b + b + k^R \,\right] \right\}$$

$$\cdot \; \{\, [\, a\,(b+k) + b\,c\,] \left[\left(b + k^R \right) (k_A + k_N) + k_A\, k_N - b\, k_N \,\right]$$

$$+ \; b\,[\, b+k\,]\,[\, c_N\,(k_A - b) + c_A\,(k_N + b)\,]\,\}$$

$$= \; \left\{ [\, a\,(b+k) + b\,c\,] \left[\, k^R + k_A \,\right] + b\,[\, b+k\,]\, c_A \right\}$$

$$\cdot \{\, b\,[\, b+k\,]\,[\, k_A + k_N\,] + [\, 2\,b+k\,] \left[\, k_N \left(k_A + k^R \right) + k_A \left(b + k^R \right) \right] \}$$

$$- \left\{ b\,[\, b+k\,] + [\, 2\,b+k\,] \left[\, k_A + k^R \,\right] \right\}$$

$$\cdot \{\, [\, a\,(b+k) + b\,c\,] \left[\, k_A\,(b + k_N) + k^R\,(k_A + k_N) \right]$$

$$+ \; b\,[\, b+k\,]\,[\, c_N\,(k_A - b) + c_A\,(k_N + b)\,]\,\}$$

$$= \; [\, a\,(b+k) + b\,c\,] \left[\, k^R + k_A \,\right] b\,[\, b+k\,]\,[\, k_A + k_N\,]$$

$$+ \; [\, a\,(b+k) + b\,c\,] \left[\, k^R + k_A \,\right] [\, 2\,b+k\,] \left[\, k_N \left(k_A + k^R \right) + k_A \left(b + k^R \right) \right]$$

$$+ \; b\,[\, b+k\,]\, c_A\, b\,[\, b+k\,]\,[\, k_A + k_N\,]$$

$$+ \; b\,[\, b+k\,]\, c_A\; [\, 2\,b+k\,] \left[\, k_N \left(k_A + k^R \right) + k_A \left(b + k^R \right) \right]$$

$$
\begin{aligned}
&- b\,[\,b+k\,]\,[\,a\,(b+k)+b\,c\,]\,\left[\,k_A\,(b+k_N)+k^R\,(k_A+k_N)\,\right] \\
&- b\,[\,b+k\,]\;b\,[\,b+k\,]\,[\,c_N\,(k_A-b)+c_A\,(k_N+b)\,] \\
&- [\,2\,b+k\,]\,\left[\,k_A+k^R\,\right]\,[\,a\,(b+k)+b\,c\,]\,\left[\,k_A\,(b+k_N)+k^R\,(k_A+k_N)\,\right] \\
&- [\,2\,b+k\,]\,\left[\,k_A+k^R\,\right]\;b\,[\,b+k\,]\,[\,c_N\,(k_A-b)+c_A\,(k_N+b)\,] \;\equiv\; \Gamma\,. \qquad (164)
\end{aligned}
$$

(164) implies:

$$
\Gamma \;=\; [\,a\,(b+k)+b\,c\,]\,\Gamma_1 + b\,[\,b+k\,]\,\Gamma_2 \qquad (165)
$$

where

$$
\begin{aligned}
\Gamma_1 \;\equiv\;& b\,[\,b+k\,]\,\left[\,k^R+k_A\,\right]\,[\,k_A+k_N\,] \\
&+ [\,2\,b+k\,]\,\left[\,k^R+k_A\,\right]\,\left[\,k_N\left(k_A+k^R\right)+k_A\left(b+k^R\right)\,\right] \\
&- b\,[\,b+k\,]\,\left[\,k_A\,(b+k_N)+k^R\,(k_A+k_N)\,\right] \\
&- [\,2\,b+k\,]\,\left[\,k_A+k^R\,\right]\,\left[\,k_A\,(b+k_N)+k^R\,(k_A+k_N)\,\right] \\
=\;& b\,[\,b+k\,]\left\{\,\left[\,k^R+k_A\,\right][\,k_A+k_N\,]-\left[\,k_A\,(b+k_N)+k^R\,(k_A+k_N)\,\right]\,\right\} \\
=\;& b\,[\,b+k\,]\,\{\,k_A\,[\,k_A+k_N\,]-k_A\,[\,b+k_N\,]\,\} \;=\; b\,[\,b+k\,]\,[\,k_A-b\,]\,k_A \qquad (166)
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_2 \;\equiv\;& b\,[\,b+k\,]\,[\,k_A+k_N\,]\,c_A + c_A\,[\,2\,b+k\,]\,\left[\,k_N\left(k_A+k^R\right)+k_A\left(b+k^R\right)\,\right] \\
&- b\,[\,b+k\,]\,[\,c_N\,(k_A-b)+c_A\,(k_N+b)\,] \\
&- [\,2\,b+k\,]\,\left[\,k_A+k^R\,\right]\,[\,c_N\,(k_A-b)+c_A\,(k_N+b)\,] \\
=\;& b\,[\,b+k\,]\,[\,c_A\,(k_A+k_N)-c_N\,(k_A-b)-c_A\,(k_N+b)\,] \\
&+ [\,2\,b+k\,]\;\{\,c_A\left[\,k_N\left(k_A+k^R\right)+k_A\left(b+k^R\right)\,\right] \\
&\qquad\qquad - \left[\,k_A+k^R\,\right][\,c_N\,(k_A-b)+c_A\,(k_N+b)\,]\,\} \\
=\;& b\,[\,b+k\,]\,[\,c_A\,(k_A-b)-c_N\,(k_A-b)\,] \\
&+ [\,2\,b+k\,]\;\{\,c_A\left[\,k_N\left(k_A+k^R\right)+k_A\left(b+k^R\right)-\left(k_A+k^R\right)(k_N+b)\,\right] \\
&\qquad\qquad - c_N\left[\,k_A+k^R\,\right][\,k_A-b\,]\,\} \\
=\;& -b\,[\,b+k\,]\,[\,k_A-b\,]\,[\,c_N-c_A\,] \\
&+ [\,2\,b+k\,]\;\{\,c_A\left[\,k_A\left(b+k^R\right)-b\left(k_A+k^R\right)\,\right]-c_N\left[\,k_A+k^R\,\right][\,k_A-b\,]\,\}
\end{aligned}
$$

$$= \; -b\,[\,b+k\,]\,[\,k_A - b\,]\,[\,c_N - c_A\,]$$

$$+ \; [\,2\,b+k\,]\,\big\{\,c_A\,k^R\,[\,k_A-b\,] - c_N\,k^R\,[\,k_A-b\,] - c_N\,k_A\,[\,k_A-b\,]\,\big\}$$

$$= \; [\,k_A-b\,]\,\big\{-b\,[\,b+k\,]\,[\,c_N-c_A\,] - [\,2\,b+k\,]\,k^R\,[\,c_N-c_A\,] - \;[\,2\,b+k\,]\,k_A\,c_N\,\big\}$$

$$= \; -\;[\,k_A-b\,]\,\big\{\,[\,c_N-c_A\,]\,\big[\,b\,(b+k)+(2\,b+k)\,k^R\,\big] + [\,2\,b+k\,]\,k_A\,c_N\,\big\}\,. \tag{167}$$

(155), (165), (166), and (167) imply:

$$\begin{aligned}
\Gamma \; &= \; [\,a\,(b+k)+b\,c\,]\,b\,[\,b+k\,]\,[\,k_A-b\,]\,k_A \\
&\qquad -\;b\,[\,b+k\,]\,[\,k_A-b\,]\;\{\,[\,c_N-c_A\,]\,\big[\,b\,(b+k)+(2\,b+k)\,k^R\,\big] \\
&\qquad\qquad\qquad\qquad\qquad + [\,2\,b+k\,]\,k_A\,c_N\,\} \\
&= \; b\,[\,b+k\,]\,[\,k_A-b\,]\;\{\,[\,a\,(b+k)+b\,c\,]\,k_A - [\,c_N-c_A\,]\,\big[\,b\,(b+k)+(2\,b+k)\,k^R\,\big] \\
&\qquad\qquad\qquad\qquad\qquad -\;[\,2\,b+k\,]\,k_A\,c_N\,\} \\
&= \; [\,k_A-b\,]\,b\,[\,b+k\,]\;\{\,k_A\,[\,a\,(b+k)+b\,c-(2\,b+k)\,c_N\,] \\
&\qquad\qquad\qquad\qquad -\;[\,c_N-c_A\,]\,\big[\,b\,(b+k)+(2\,b+k)\,k^R\,\big]\,\} \\
&= \; [\,k_A-b\,]\;Y_2\,.
\end{aligned} \tag{168}$$

(164) and (168) imply that $\frac{\partial \overline{p}_d}{\partial k_N} \; = \; \frac{[\,k_A-b\,]\,Y_2}{(D_2)^2} \gtreqless 0 \;\Leftrightarrow\; k_A \gtreqless b$.

(155), (156), (157), (163), (164), and (168) imply:

$$\frac{\partial\,(\overline{p}_b - \overline{p}_d)}{\partial k_N} \; = \; Y_2 \left[\, \frac{k_A}{(D_3)^2} - \frac{k_A - b}{(D_2)^2} \,\right] \; > \; 0 \;\; \text{if } k_A - b \text{ is sufficiently small.} \tag{169}$$

To prove that $\frac{\partial(\overline{p}_b - \overline{p}_d)}{\partial c} > 0$, observe that (7) implies:

$$\begin{aligned}
\frac{\partial \overline{p}_d}{\partial c} \; &= \; \frac{b}{D_2}\,\big[\,\big(b+k^R\big)\,(k_A+k_N) + k_N\,(k_A-b)\,\big] \\
&= \; \frac{b}{D_2}\,\big[\,k_A\,\big(b+k^R\big) + k_N\,\big(b+k^R+k_A-b\big)\,\big] \\
&= \; \frac{b}{D_2}\,\big[\,k_A\,\big(b+k^R\big) + k_N\,\big(k_A+k^R\big)\,\big] \; > \; 0\,.
\end{aligned} \tag{170}$$

Furthermore, (8) implies:

$$\frac{\partial \overline{p}_b}{\partial c} \; = \; \frac{1}{D_3}\;\big\{\,b\,\big[\,\big(b+k^R\big)\,(k_A+k_N) + k_N\,k_A\,\big]\,\big\}$$

$$= \frac{b}{D_3} \left[\left(b + k^R\right) (k_A + k_N) + k_N \, k_A \right]$$

$$= \frac{b}{D_3} \left[k_A \left(b + k^R\right) + k_N \left(k_A + k^R + b\right) \right] \; > \; 0 \, . \tag{171}$$

(170) and (171) imply:

$$\frac{\partial \left(\overline{p}_b \, - \, \overline{p}_d \right)}{\partial c} \; \stackrel{s}{=} \; \frac{k_A \left[b + k^R \right] + k_N \left[k_A + k^R + b \right]}{D_3} - \frac{k_A \left[b + k^R \right] + k_N \left[k_A + k^R \right]}{D_2} \; > \; 0$$

$$\Leftrightarrow \; \frac{k_A \left[b + k^R \right] + k_N \left[k^R + b + k_A \right]}{D_3} \; > \; \frac{k_A \left[b + k^R \right] + k_N \left[k_A + k^R \right]}{D_2}$$

$$\Leftrightarrow \; \frac{k_A \left[b + k^R \right] + k_N \left[k^R + b + k_A \right]}{D_2 + b \, k_N \; [\, 2 \, b + k \,]} \; > \; \frac{k_A \left[b + k^R \right] + k_N \left[k_A + k^R \right]}{D_2}$$

$$\Leftrightarrow \; \frac{Z + b \, k_N}{D_2 + b \, k_N \; [\, 2 \, b + k \,]} \; > \; \frac{Z}{D_2} \; \text{ where } \; Z \; \equiv \; k_A \left[b + k^R \right] + k_N \left[k_A + k^R \right] . \tag{172}$$

(172) implies:

$$\frac{\partial \left(\overline{p}_b \, - \, \overline{p}_d \right)}{\partial c} \; > \; 0 \quad \Leftrightarrow \quad Z \, D_2 + b \, k_N \, D_2 \; > \; Z \, D_2 + Z \, b \, k_N \; [\, 2 \, b + k \,]$$

$$\Leftrightarrow \quad b \, k_N \, D_2 \; > \; Z \, b \, k_N \; [\, 2 \, b + k \,] \quad \Leftrightarrow \quad D_2 \; > \; Z \; [\, 2 \, b + k \,]$$

$$\Leftrightarrow \quad D_2 > \left[k_A \left(b + k^R \right) + k_N \left(k_A + k^R \right) \right] [\, 2 \, b + k \,]$$

$$\Leftrightarrow \quad b \, [\, b + k \,] \, [\, k_N + k_A \,] + k_N \, [\, k_A - b \,] \, [\, 2 \, b + k \,] + \; [\, k_N + k_A \,] \, [\, 2 \, b + k \,] \left[\, b + k^R \, \right]$$

$$> \; \left[\, k_A \left(b + k^R \right) + k_N \left(k_A + k^R \right) \, \right] [\, 2 \, b + k \,]$$

$$\Leftrightarrow \quad b \, [\, b + k \,] \, [\, k_N + k_A \,] + k_N \, [\, k_A - b \,] \, [\, 2 \, b + k \,] + \; [\, k_N + k_A \,] \, [\, 2 \, b + k \,] \left[\, b + k^R \, \right]$$

$$> \; \left[\, \left(b + k^R \right) (k_A + k_N) + k_N \, (k_A - b) \, \right] [\, 2 \, b + k \,]$$

$$\Leftrightarrow \quad b \, [\, b + k \,] \, [\, k_N + k_A \,] \; > \; 0 \, . \; \blacksquare$$

Recall that welfare is:

$$W(\overline{p}) \; \equiv \; S(\overline{p}) - r \; [\; \overline{p} \; q_A + (\, a - b \, [\, q_A + q_N + q \,] \,) \, q_N \,] \; = \; S(\overline{p}) - r \; V(\overline{p}) \tag{173}$$

where $r > 0$ is a parameter and $S(\cdot)$ denotes consumer surplus. The gross value that consumers derive from Q units of output is:

$$\frac{1}{2} \, [\, a - P(Q) \,] \, Q + P(Q) \, Q \; = \; \frac{1}{2} \, [\, a + P(Q) \,] \, Q \; = \; \frac{1}{2} \, [\, a + a - b \, Q \,] \, Q \; = \; a \, Q - \frac{b}{2} \, Q^2 .$$

Therefore, consumer surplus when the price cap is $\overline{p}$ is:

$$S(\overline{p}) \; = \; a\,Q - \frac{b}{2}\,Q^2 - \overline{p}\,q_A - P(Q)\,[\,q_N + q\,]\,. \tag{174}$$

Lemma 1. *Equilibrium consumer surplus,* $S(\overline{p})$*, is a strictly decreasing, strictly convex function of* $\overline{p}$ *for* $\overline{p} \in (\overline{p}_d, \overline{p}_b)$*.*

Proof. (57) implies that when $\overline{p} \in (\overline{p}_d, \overline{p}_b)$, so $P(Q) = \overline{p}$:

$$Q \; = \; \frac{a - \overline{p}}{b} \quad \Rightarrow \quad \frac{\partial Q}{\partial \overline{p}} \; = \; -\,\frac{1}{b}\,. \tag{175}$$

(174) and (175) imply that for $\overline{p} \in (\overline{p}_d, \overline{p}_b)$, where $P(Q) = \overline{p}$:

$$\begin{aligned} \frac{\partial S(\overline{p})}{\partial \overline{p}} \; &= \; a\,\frac{\partial Q}{\partial \overline{p}} - b\,Q\,\frac{\partial Q}{\partial \overline{p}} - Q - \overline{p}\,\frac{\partial Q}{\partial \overline{p}} \; = \; -\,\frac{a}{b} + Q - Q + \frac{\overline{p}}{b} \\ &= \; -\,\frac{a - \overline{p}}{b} \; < \; 0 \quad \Rightarrow \quad \frac{\partial^2 S(\overline{p})}{\partial\,(\overline{p})^2} \; = \; \frac{1}{b} \; > \; 0\,. \; \blacksquare \end{aligned} \tag{176}$$

Lemma A7. $V(\overline{p}_0) \, < \, V(\overline{p}_b)$.

Proof. Lemmas A1 and A3 imply that because $q_A(\overline{p}_0) = 0$ and $P(Q(\overline{p}_b)) = \overline{p}_b$:

$$\begin{aligned} V(\overline{p}_0) \; &= \; \overline{p}_0\,q_N(\overline{p}_0) \; = \; \overline{p}_0\,\frac{[\,a - c_N\,]\,[\,2\,b + k\,] - b\,[\,a - c\,]}{[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,] - b^2}\,; \\ V(\overline{p}_b) \; &= \; \overline{p}_b\,Q^R(\overline{p}_b) \; = \; \overline{p}_b\,\frac{[\,b + k\,]\,[\,a - \overline{p}_b\,] - b\,[\,\overline{p}_b - c\,]}{b\,[\,b + k\,]}\,. \end{aligned} \tag{177}$$

Definition. $D_N \; \equiv \; \left[\,2\,b + k_N + k^R\,\right][\,2\,b + k\,] - b^2\,.$ (178)

Because $\overline{p}_0 \, < \, \overline{p}_b$, (177) and (178) imply that $V(\overline{p}_0) \, < \, V(\overline{p}_b)$ if:

$$\begin{aligned} q_N(\overline{p}_0) \; &= \; \frac{[\,a - c_N\,]\,[\,2\,b + k\,] - b\,[\,a - c\,]}{D_N} \; < \; \frac{[\,b + k\,]\,[\,a - \overline{p}_b\,] - b\,[\,\overline{p}_b - c\,]}{b\,[\,b + k\,]} \; = \; Q^R(\overline{p}_b) \\ \Leftrightarrow \quad & \frac{a\,[\,b + k\,] + a\,b - c_N\,[\,2\,b + k\,] - b\,a + b\,c}{D_N} \; < \; \frac{[\,b + k\,]\,a - [\,b + k\,]\,\overline{p}_b - b\,\overline{p}_b + b\,c}{b\,[\,b + k\,]} \\ \Leftrightarrow \quad & \frac{a\,[\,b + k\,] + b\,c - c_N\,[\,2\,b + k\,]}{D_N} \; < \; \frac{[\,b + k\,]\,a + b\,c - [\,2\,b + k\,]\,\overline{p}_b}{b\,[\,b + k\,]} \\ \Leftrightarrow \quad & \frac{a\,[\,b + k\,] + b\,c - c_N\,[\,2\,b + k\,]}{D_N}\,b\,[\,b + k\,] - [\,b + k\,]\,a - b\,c \; < \; -\,[\,2\,b + k\,]\,\overline{p}_b \end{aligned}$$

$$\Leftrightarrow \frac{a\,[\,b+k\,]+b\,c}{2\,b+k} - \frac{a\,[\,b+k\,]+b\,c-c_N\,[\,2\,b+k\,]}{[\,2\,b+k\,]\,D_N}\;b\,[\,b+k\,] \;>\; \overline{p}_b$$

$$\Leftrightarrow \frac{a\,[\,b+k\,]+b\,c}{2\,b+k} - \frac{[\,a\,(b+k)+b\,c\,]\,b\,[\,b+k\,]-c_N\,[\,2\,b+k\,]\,b\,[\,b+k\,]}{[\,2\,b+k\,]\,D_N} \;>\; \overline{p}_b$$

$$\Leftrightarrow \frac{1}{[\,2\,b+k\,]\,D_N}\,\{\,[\,a\,(b+k)+b\,c\,]\,\left[\,\left(2\,b+k_N+k^R\right)(2\,b+k)-b^2-b\,(b+k)\,\right] + c_N\,[\,2\,b+k\,]\,b\,[\,b+k\,]\,\} \;>\; \overline{p}_b$$

$$\Leftrightarrow \frac{1}{[\,2\,b+k\,]\,D_N}\,\{\,[\,a\,(b+k)+b\,c\,]\,\left[\,\left(2\,b+k_N+k^R\right)(2\,b+k)-b\,(2\,b+k)\,\right] + c_N\,[\,2\,b+k\,]\,b\,[\,b+k\,]\,\} \;>\; \overline{p}_b$$

$$\Leftrightarrow \frac{[\,a\,(b+k)+b\,c\,]\,\left[\,2\,b+k_N+k^R-b\,\right]+c_N\,b\,[\,b+k\,]}{D_N} \;>\; \overline{p}_b$$

$$\Leftrightarrow \frac{[\,a\,(b+k)+b\,c\,]\,\left[\,b+k_N+k^R\,\right]+c_N\,b\,[\,b+k\,]}{D_N} \;>\; \overline{p}_b\,. \tag{179}$$

(8) implies:

$$\overline{p}_b \;=\; \frac{[\,a\,(b+k)+b\,c\,]\,\left[\,\left(b+k^R\right)(k_N+k_A)+k_N\,k_A\,\right]+b\,c_N\,[\,b+k\,]\,k_A+b\,k_N\,[\,b+k\,]\,c_A}{b\,[\,b+k\,]\,[\,k_N+k_A\,]+k_N\,k_A\,[\,2\,b+k\,]+[\,k_N+k_A\,]\,[\,2\,b+k\,]\,[\,b+k^R\,]}\,. \tag{180}$$

As established in the proof of Proposition 4 (just below (149)), $\overline{p}_b$ is increasing in k_A. Therefore, (180) implies that because $k_A \leq k_N$ by assumption:

$$\overline{p}_b \;\leq\; \frac{[\,a\,(b+k)+b\,c\,]\,\left[\,2\,k_N\left(b+k^R\right)+(k_N)^2\,\right]+b\,c_N\,[\,b+k\,]\,k_N+b\,k_N\,[\,b+k\,]\,c_A}{2\,b\,[\,b+k\,]\,k_N+(k_N)^2\,[\,2\,b+k\,]+2\,k_N\,[\,2\,b+k\,]\,[\,b+k^R\,]}\,. \tag{181}$$

(8) implies that $\overline{p}_b$ is increasing in c_A. Therefore, because $c_A \leq c_N$ by assumption, (181) implies:

$$\begin{aligned}
\overline{p}_b \;\leq\; & \frac{[\,a\,(b+k)+b\,c\,]\,\left[\,2\,k_N\left(b+k^R\right)+(k_N)^2\,\right]+2\,b\,c_N\,[\,b+k\,]\,k_N}{2\,b\,[\,b+k\,]\,k_N+(k_N)^2\,[\,2\,b+k\,]+2\,k_N\,[\,2\,b+k\,]\,[\,b+k^R\,]} \\
=\; & \frac{[\,a\,(b+k)+b\,c\,]\,\left[\,2\left(b+k^R\right)+k_N\,\right]+2\,b\,c_N\,[\,b+k\,]}{2\,b\,[\,b+k\,]+k_N\,[\,2\,b+k\,]+2\,[\,2\,b+k\,]\,[\,b+k^R\,]} \\
=\; & \frac{[\,a\,(b+k)+b\,c\,]\,\left[\,b+k^R+\frac{k_N}{2}\,\right]+b\,c_N\,[\,b+k\,]}{b\,[\,b+k\,]+\frac{k_N}{2}\,[\,2\,b+k\,]+[\,2\,b+k\,]\,[\,b+k^R\,]} \\
=\; & \frac{[\,a\,(b+k)+b\,c\,]\,\left[\,b+k^R+\frac{k_N}{2}\,\right]+b\,c_N\,[\,b+k\,]}{[\,2\,b+k\,]\left[\,b+k^R+\frac{k_N}{2}\,\right]+b\,[\,b+k\,]}
\end{aligned}$$

$$= \frac{\left[\, a\,(b+k) + b\,c \,\right] \left[\, b + k^R + \frac{k_N}{2} \,\right] + b\,c_N\,[\, b+k \,]}{[\, 2\,b + k \,] \left[\, 2\,b + k^R + \frac{k_N}{2} \,\right] - b^2} \,. \tag{182}$$

The last equality in (182) holds because:

$$\begin{aligned} & [\, 2\,b + k \,] \left[b + k^R + \frac{k_N}{2} \right] + b\,[\, b+k \,] \\ & \quad = [\, 2\,b + k \,] \left[2\,b + k^R + \frac{k_N}{2} \right] - b\,[\, 2\,b+k \,] + b\,[\, b+k \,] \\ & \quad = [\, 2\,b + k \,] \left[2\,b + k^R + \frac{k_N}{2} \right] - 2\,b^2 - b\,k + b^2 + b\,k \\ & \quad = [\, 2\,b + k \,] \left[2\,b + k^R + \frac{k_N}{2} \right] - b^2. \end{aligned}$$

(178), (179), and (182) imply that the Lemma holds if:

$$\begin{aligned} & \frac{\left[\, a\,(b+k) + b\,c \,\right] \left[\, b + k^R + \frac{k_N}{2} \,\right] + b\,c_N\,[\, b+k \,]}{[\, 2\,b + k \,] \left[\, 2\,b + k^R + \frac{k_N}{2} \,\right] - b^2} \\ & \qquad < \frac{\left[\, a\,(b+k) + b\,c \,\right] \left[\, b + k^R + k_N \,\right] + b\,c_N\,[\, b+k \,]}{[\, 2\,b + k \,]\,[\, 2\,b + k^R + k_N \,] - b^2} \,. \end{aligned} \tag{183}$$

<u>Definition</u>. $f(x) \;\equiv\; \dfrac{A \left[\, b + k^R + x \,\right] + b\,c_N\,[\, b+k \,]}{[\, 2\,b + k \,]\,[\, 2\,b + k^R + x \,] - b^2}$ where $A \;\equiv\; a\,[\, b+k \,] + b\,c\,.$ (184)

(184) implies that (183) holds if $\frac{\partial f}{\partial x} > 0$. (178) and (184) imply:

$$\begin{aligned} \frac{\partial f(\cdot)}{\partial x} \;&\overset{s}{=}\; \left\{ [\, 2\,b + k \,] \left[\, 2\,b + k^R + x \,\right] - b^2 \right\} A \\ & \quad - [\, 2\,b + k \,] \left\{ A \left[\, b + k^R + x \,\right] + b\,c_N\,[\, b+k \,] \right\} \\ &= A \left\{ [\, 2\,b + k \,] \left[\, 2\,b + k^R + x - \left(b + k^R + x \right) \right] - b^2 \right\} - b\,[\, b+k \,]\,[\, 2\,b+k \,]\,c_N \\ &= A \left\{ b\,[\, 2\,b + k \,] - b^2 \right\} - b\,[\, b+k \,]\,[\, 2\,b+k \,]\,c_N \\ &= A\,b\,[\, b+k \,] - b\,[\, b+k \,]\,[\, 2\,b+k \,]\,c_N \;\overset{s}{=}\; A - [\, 2\,b+k \,]\,c_N \\ &= a\,[\, b+k \,] + b\,c - [\, 2\,b+k \,]\,c_N \;>\; 0\,. \end{aligned}$$

The inequality here holds because (3) implies:

$$[\, a - c_N \,]\,[\, 2\,b + k \,] - b\,[\, a - c \,] \;>\; 0$$

$$\Rightarrow \; [\,a - c_N\,]\,[\,b + k\,] + b\,[\,a - c_N\,] - b\,[\,a - c\,] \; > \; 0$$

$$\Rightarrow \; [\,a - c_N\,]\,[\,b + k\,] + b\,[\,c - c_N\,] \; > \; 0$$

$$\Rightarrow \; a\,[\,b + k\,] - c_N\,[\,b + k\,] + b\,[\,c - c_N\,] \; > \; 0$$

$$\Rightarrow \; a\,[\,b + k\,] + b\,c - [\,2\,b + k\,]\,c_N \; > \; 0\,. \; \blacksquare$$

Proposition A1. $\overline{p}^* \in [\,\overline{p}_0, \overline{p}_d\,]$.

Proof. Proposition 3 and Lemma 1 imply that $W(\cdot)$ is a strictly convex function of $\overline{p}$ for $\overline{p} \in (\overline{p}_d, \overline{p}_b)$. Therefore, $\overline{p}^* \notin (\overline{p}_d, \overline{p}_b)$. Lemma A1 implies that $W(\overline{p}\,) = W(\overline{p}_0)$ for all $\overline{p} < \overline{p}_0$. Lemma A4 implies that $W(\overline{p}\,) = W(\overline{p}_b)$ for all $\overline{p} > \overline{p}_b$. Therefore, $\overline{p}^* \in [\,\overline{p}_0, \overline{p}_d\,] \bigcup \overline{p}_b$.

It remains to show that $\overline{p}^* \neq \overline{p}_b$. The proof of Lemma A7 establishes that:

$$Q^R(\overline{p}_0) \; < \; Q^R(\overline{p}_b) \tag{185}$$

where $Q^R(\overline{p})$ is R's total output when the price cap is $\overline{p}$. Lemma A6 and Proposition 2 imply:

$$Q^R(\overline{p}_b) \; < \; Q^R(\overline{p}_d)\,. \tag{186}$$

(185) and (186) imply that $Q^R(\overline{p}_0) \, < \, Q^R(\overline{p}_b) \, < \, Q^R(\overline{p}_d)$. $Q^R(\overline{p}\,)$ is continuous and monotonically increasing in $\overline{p}$ for $\overline{p} \in (\overline{p}_0, \overline{p}_d)$ (from Lemma A2). Therefore, the intermediate value theorem implies that there exists a $\overline{p}_E \in (\overline{p}_0, \overline{p}_d)$ such that:

$$Q^R(\overline{p}_E) \; = \; Q^R(\overline{p}_b)\,. \tag{187}$$

(12) implies that the rival's output q is determined by:

$$a - b\left[\,Q^R(\overline{p}\,) + q(\overline{p}\,)\,\right] - c - b\,q(\overline{p}\,) - k\,q(\overline{p}\,) \; = \; 0\,. \tag{188}$$

(187) and (188) imply:

$$q(\overline{p}_E) \; = \; q(\overline{p}_b)\,. \tag{189}$$

(187) and (189) imply:

$$Q(\overline{p}_E) \; = \; Q(\overline{p}_b) \;\; \text{and} \;\; P(Q(\overline{p}_E)\,) \; = \; P(Q(\overline{p}_b)\,)\,. \tag{190}$$

R's revenue is:

$$\begin{aligned} V_2(\overline{p}_E) \; &= \; \overline{p}_E\; q_A(\overline{p}_E) + P(Q(\overline{p}_E)\,)\; q_N(\overline{p}_E) \\ &< \; P(Q(\overline{p}_E)\,)\; q_A(\overline{p}_E) + P(Q(\overline{p}_E)\,)\; q_N(\overline{p}_E) \\ &= \; P(Q(\overline{p}_E)\,)\; Q^R(\overline{p}_E) \; = \; P(Q(\overline{p}_b)\,)\; Q^R(\overline{p}_b) \; = \; V_3(\overline{p}_b)\,. \end{aligned} \tag{191}$$

The inequality in (191) holds because $\overline{p}_E \, < \, P(Q(\overline{p}_E)\,)$, since $\overline{p}_E \in (\overline{p}_0, \overline{p}_d)$. The penultimate equality in (191) reflects (190). The last equality in (191) holds because $P(Q(\overline{p}_b)\,) = \overline{p}_b$.

(174) and (190) imply:

$$
\begin{aligned}
S(\overline{p}_E) \;&=\; a\,Q(\overline{p}_E) - \frac{b}{2}\,Q(\overline{p}_E)^2 - P(Q(\overline{p}_E))\,\left[\,q(\overline{p}_E) + q_N(\overline{p}_E)\,\right] - \overline{p}_E\,q_A(\overline{p}_E) \\
&>\; a\,Q(\overline{p}_E) - \frac{b}{2}\,Q(\overline{p}_E)^2 - P(Q(\overline{p}_E))\,\left[\,q(\overline{p}_E) + q_N(\overline{p}_E) + q_A(\overline{p}_E)\,\right] \\
&=\; a\,Q(\overline{p}_E) - \frac{b}{2}\,Q(\overline{p}_b)^2 - P(Q(\overline{p}_E))\,Q(\overline{p}_E) \\
&=\; a\,Q(\overline{p}_E) - \frac{b}{2}\,Q(\overline{p}_b)^2 - P\left(Q(\overline{p}_b)\right)\,Q(\overline{p}_b) \;=\; S(\overline{p}_b)\,. \qquad (192)
\end{aligned}
$$

The inequality in (192) holds because $\overline{p}_E < P\left(Q(\overline{p}_E)\right)$, since $\overline{p}_E \in (\overline{p}_0, \overline{p}_d)$. (191) and (192) imply that consumer surplus is higher and R's revenue is lower when $\overline{p} = \overline{p}_E$ than when $\overline{p} = \overline{p}_b$. Therefore, $W(\overline{p}_E) > W(\overline{p}_b)$, so $\overline{p}^* \neq \overline{p}_b$. ■

Lemma 2. *For $\overline{p} \in (\overline{p}_0, \overline{p}_d)$: (i) $V(\overline{p})$ is a strictly convex function of $\overline{p}$; (ii) $\frac{\partial V(\overline{p})}{\partial \overline{p}} \lesseqgtr 0 \Leftrightarrow \overline{p} \lesseqgtr \overline{p}_{Vm}$ where $\overline{p}_{Vm} \in [\,\overline{p}_0, \overline{p}_d)$; and (iii) $\overline{p}_{Vm} > \overline{p}_0$ if $\Phi_2 > 0$, where*

$$
\begin{aligned}
\Phi_2 \;\equiv\; &\left\{\, k^R\,[\,2\,b + k\,]\left[\,k^R\,(2\,b+k) + 2\,b\,(3\,b + 2\,k)\,\right]\right. \\
&\left.+\; k_N\,[\,2\,b + k\,]\left[\,k^R\,(2\,b+k) + b^2\,\right] + b^2\left[\,5\,b^2 + 6\,b\,k + 2\,k^2\,\right]\right\} c_N \\
&- \left\{\, b\,[\,3\,b + 2\,k\,] + [\,2\,b + k\,]\left[\,k_N + k^R\,\right]\right\}^2 c_A \\
&- b\left[\,b^2 - k\,k_N + (2\,b + k)\,k^R\,\right][\,a\,(b + k) + b\,c\,]\,. \qquad (193)
\end{aligned}
$$

Corollary to Lemma 2. $\left.\frac{\partial V(\overline{p})}{\partial \overline{p}}\right|_{\overline{p} = \overline{p}_0} < 0$ *if* $\Phi_2 > 0$.

<u>Proof of Lemma 2 and its Corollary.</u>

Define:

$$
\widetilde{V}_2(\overline{p}) \;\equiv\; q_{A2}(\overline{p})\,\overline{p} + q_{N2}(\overline{p})\,P(Q_2(\overline{p})) \qquad (194)
$$

where $q_{A2}(\overline{p})$ and $q_{N2}(\overline{p})$ are as defined in (20) and (21), respectively. Observe that $\widetilde{V}_2(\overline{p}) = V(\overline{p})$ for $\overline{p} \in [\overline{p}_0, \overline{p}_d]$.

Because $P(Q_2) = a - b\,Q_2$, (194) implies:

$$
\frac{\partial \widetilde{V}_2(\overline{p})}{\partial \overline{p}} \;=\; q_{A2} + \overline{p}\,\frac{\partial q_{A2}}{\partial \overline{p}} + P(Q_2)\,\frac{\partial q_{N2}}{\partial \overline{p}} - b\,q_{N2}\,\frac{\partial Q_2}{\partial \overline{p}}\,. \qquad (195)
$$

(2) and Lemma A2 imply:

$$
\frac{\partial^2 q_{A2}}{\partial\,(\overline{p})^2} \;=\; \frac{\partial^2 q_{N2}}{\partial\,(\overline{p})^2} \;=\; \frac{\partial^2 q_2}{\partial\,(\overline{p})^2} \;=\; \frac{\partial^2 Q_2}{\partial\,(\overline{p})^2} \;=\; 0\,. \qquad (196)
$$

(195) and (196) imply:

$$\frac{\partial^2 \widetilde{V}_2(\overline{p})}{\partial(\overline{p})^2} = \frac{\partial q_{A2}}{\partial \overline{p}} + \frac{\partial q_{A2}}{\partial \overline{p}} - b\,\frac{\partial Q_2}{\partial \overline{p}}\,\frac{\partial q_{N2}}{\partial \overline{p}} - b\,\frac{\partial q_{N2}}{\partial \overline{p}}\,\frac{\partial Q_2}{\partial \overline{p}}$$

$$= 2\,\frac{\partial q_{A2}}{\partial \overline{p}} - 2\,b\,\frac{\partial Q_2}{\partial \overline{p}}\,\frac{\partial q_{N2}}{\partial \overline{p}} > 0\,. \tag{197}$$

The inequality in (197) holds because $D > 0$ by assumption, so $\frac{\partial q_{A2}}{\partial \overline{p}} > 0$ from (20), $\frac{\partial Q_2}{\partial \overline{p}} > 0$ from (24), and $\frac{\partial q_{N2}}{\partial \overline{p}} < 0$ from (21).

$\overline{p}_{Vm} \equiv \arg\min_{\overline{p}} \{\widetilde{V}_2(\overline{p})\}$ is unique and is determined by:

$$\frac{\partial \widetilde{V}_2(\overline{p}_{V_2m})}{\partial \overline{p}} \equiv \left.\frac{\partial \widetilde{V}_2(\overline{p})}{\partial \overline{p}}\right|_{\overline{p}=\overline{p}_{V_2m}} = 0. \tag{198}$$

This is the case because (2), (20) – (24), and (195) imply that $\frac{\partial \widetilde{V}_2(\overline{p})}{\partial \overline{p}}$ is a linear function of $\overline{p}$. Therefore, $\widetilde{V}_2(\overline{p})$ is a quadratic function of $\overline{p}$. Consequently, (197) implies that $\widetilde{V}_2(\overline{p})$ has a unique minimum that is determined by (198).

To prove the Corollary to Lemma 2 and thereby establish that $\overline{p}_{Vm} > \overline{p}_0$ when $\Phi_2 > 0$, observe that R's revenue is:

$$V(\overline{p}) = \overline{p}\,q_A + P(Q)\,q_N = \overline{p}\,q_A + [\,a - b\,Q\,]\,q_N\,. \tag{199}$$

(199) implies that the Corollary to Lemma 2 holds if:

$$\frac{\partial^+ V(\overline{p}_0)}{\partial \overline{p}} = q_A + \overline{p}_0\,\frac{\partial q_A}{\partial \overline{p}} - b\,\frac{\partial Q}{\partial \overline{p}}\,q_N + P(Q)\,\frac{\partial q_N}{\partial \overline{p}} < 0\,, \tag{200}$$

where: (i) $\frac{\partial^+ V(\overline{p}_0)}{\partial \overline{p}} = \left.\frac{\partial^+ V(\overline{p})}{\partial \overline{p}}\right|_{\overline{p}=\overline{p}_0}$ denotes the right-sided derivative of $V(\cdot)$; (ii) $\frac{\partial q_A}{\partial \overline{p}}$, $\frac{\partial q_N}{\partial \overline{p}}$, and $\frac{\partial Q}{\partial \overline{p}}$ pertain to the quantities identified in Lemma A2 (which prevail when $\overline{p} \in (\overline{p}_0, \overline{p}_d)$); and (iii) q_A, q_N, and Q are as defined in Lemma A1.

Define:

$$E = 2\,b\,[\,2\,b + k\,] + \left[\,k_N + k^R\,\right][\,2\,b + k\,] - b^2$$

$$= 3\,b^2 + 2\,b\,k + \left[\,k_N + k^R\,\right][\,2\,b + k\,]$$

$$= b\,[\,3\,b + 2\,k\,] + [\,2\,b + k\,]\left[\,k_N + k^R\,\right]. \tag{201}$$

(201) and Lemma A2 imply that when $\overline{p} \in (\overline{p}_0, \overline{p}_d)$:

$$\frac{\partial q_N}{\partial \overline{p}} = -\,\frac{b\,k + 2\,b\,k^R + k\,k^R + b^2}{D}\,;$$

$$\frac{\partial q_A}{\partial \overline{p}} = \frac{2\,b\,k + 2\,b\,k_N + 2\,b\,k^R + k\,k_N + k\,k^R + 3\,b^2}{D}$$

$$= \frac{\left[\,2\,b + k_N + k^R\,\right]\,[\,2\,b + k\,] - b^2}{D} \;=\; \frac{E}{D}\,;$$

$$\frac{\partial Q}{\partial \overline{p}} \;=\; \frac{1}{D}\,\{\,2\,b\,k + 2\,b\,k_N + 2\,b\,k^R + k\,k_N + k\,k^R + 3\,b^2$$

$$-\;\left[\,b\,k + 2\,b\,k^R + k\,k^R + b^2\,\right] - \left[\,b^2 + k_N\,b\,\right]\,\}$$

$$= \frac{b\,k + b\,k_N + k\,k_N + b^2}{D} \;=\; \frac{[\,b+k\,]\,[\,b+k_N\,]}{D}\,. \tag{202}$$

Lemma A1 implies that when $\overline{p} \,\leq\, \overline{p}_0$:

$$q_N \;=\; \frac{[\,a - c_N\,]\,[\,2\,b + k\,] - b\,[\,a - c\,]}{E}\,,\;\; q \;=\; \frac{[\,a - c\,]\left[\,2\,b + k_N + k^R\,\right] - b\,[\,a - c_N\,]}{E}, \text{ and}$$

$$P(Q) \;=\; a - b\,[\,q_N + q\,] \;=\; a - b\,\frac{[\,a - c_N\,]\,[\,b + k\,] + \left[\,b + k_N + k^R\,\right]\,[\,a - c\,]}{E}$$

$$= \frac{a\,E - b\,[\,a - c_N\,]\,[\,b + k\,] - b\,\left[\,b + k_N + k^R\,\right]\,[\,a - c\,]}{E}\,. \tag{203}$$

(200) – (203) imply that because $q_A = 0$ when $\overline{p} \,=\, \overline{p}_0$ (from Lemma A1):

$$\frac{\partial^+ V(\overline{p}_0)}{\partial \overline{p}} \;=\; \overline{p}_0\,\frac{E}{D} - b\left[\,\frac{(b+k)\,(b+k_N)}{D}\,\right]\left[\,\frac{(a - c_N)\,(2\,b + k) - b\,(a - c)}{E}\,\right] \tag{204}$$

$$-\left[\,\frac{a\,E - b\,(a - c_N)\,(b + k) - b\,\left(b + k_N + k^R\right)\,(a - c)}{E}\,\right]$$

$$\cdot\left[\,\frac{b\,k + 2\,b\,k^R + k\,k^R + b^2}{D}\,\right]$$

$$= \frac{1}{D\,E}\,\{\,\overline{p}_0\,E^2 - b\,[\,b + k\,]\,[\,b + k_N\,]\,[\,(a - c_N)\,(2\,b + k) - b\,(a - c)\,]$$

$$-\;\left[\,a\,E - b\,(a - c_N)\,(b + k) - b\,\left(b + k_N + k^R\right)\,(a - c)\,\right]$$

$$\cdot\,\left[\,b\,k + 2\,b\,k^R + k\,k^R + b^2\,\right]\,\}\,. \tag{205}$$

(6) and (201) imply:

$$E\;\overline{p}_0 \;=\; c_A\,E + [\,a - c_N\,]\,[\,2\,b + k\,]\left[\,b + k^R\,\right] - b\,[\,a - c\,]\left[\,b + k^R\,\right]. \tag{206}$$

(201), (205), and (206) imply:

$$\frac{\partial^+ V(\overline{p}_0)}{\partial \overline{p}} \;=\; \frac{1}{D\,E}\,\{\,c_A\,E^2 + E\left[\,(a - c_N)\,(2\,b + k)\left(b + k^R\right) - b\,(a - c)\left(b + k^R\right)\,\right]$$

$$-\;b\,[\,b + k\,]\,[\,b + k_N\,]\,[\,(a - c_N)\,(2\,b + k) - b\,(a - c)\,]$$

$$- \left[a\,E - b\,(a - c_N)\,(b+k) - b\,\left(b + k_N + k^R\right)(a-c) \right] \left[b\,k + 2\,b\,k^R + k\,k^R + b^2 \right] \}$$

$$= \frac{1}{D\,E} \{ \, c_A\,E^2 + \left[\,E\,\left(b + k^R\right) - b\,(b+k)\,(b+k_N) \right] \left[(a-c_N)\,(2\,b+k) - b\,(a-c) \right]$$

$$- \left[a\,E - b\,(a - c_N)\,(b+k) - b\,\left(b + k_N + k^R\right)(a-c) \right] \left[b\,k + 2\,b\,k^R + k\,k^R + b^2 \right] \}$$

$$= \frac{1}{D\,E} \left[c_A\,E^2 - \tilde{E} \right]. \tag{207}$$

where

$$\tilde{E} = \left[a\,E - b\,(a - c_N)\,(b+k) - b\,\left(b + k_N + k^R\right)(a-c) \right] \left[b\,k + 2\,b\,k^R + k\,k^R + b^2 \right]$$

$$- \left[E\left(b + k^R\right) - b\,(b+k)\,(b+k_N) \right] \left[(a-c_N)\,(2\,b+k) - b\,(a-c) \right]$$

$$= \left[a\,E - b\,(a - c_N)\,(b+k) - b\,\left(b + k_N + k^R\right)(a-c) \right] \left[b\,k + 2\,b\,k^R + k\,k^R + b^2 \right]$$

$$+ \,b\,[\,b+k\,]\,[\,b+k_N\,]\,[\,(a-c_N)\,(2\,b+k) - b\,(a-c)\,]$$

$$- \,E\,\left[b + k^R\right][\,a - c_N\,][\,2\,b+k\,] + E\,\left[b + k^R\right]\,b\,[\,a-c\,]$$

$$= \left[a\,E - b\,(a - c_N)\,(b+k) - b\,\left(b + k_N + k^R\right)(a-c) \right] \left[b\,k + 2\,b\,k^R + k\,k^R + b^2 \right]$$

$$+ \,b\,[\,b+k\,]\,[\,b+k_N\,]\,[\,(a-c_N)\,(2\,b+k) - b\,(a-c)\,]$$

$$- \,E\left[b + k^R\right]\,a\,[\,2\,b+k\,] + E\,\left[b + k^R\right] b\,[\,a-c\,] + c_N\,E\,\left[b + k^R\right]\,[\,2\,b+k\,]. \tag{208}$$

(201) implies:

$$\left[b + k^R\right][\,2\,b+k\,] = \left[2\,b + k_N + k^R\right][\,2\,b+k\,] - [\,b+k_N\,]\,[\,2\,b+k\,]$$

$$= E + b^2 - [\,b+k_N\,]\,[\,2\,b+k\,] = E - \left[(b+k_N)\,(2\,b+k) - b^2 \right]. \tag{209}$$

(201), (208), and (209) imply:

$\tilde{E} = \hat{E} + c_N\,E^2$, where

$$\hat{E} \equiv \left[a\,E - b\,(a - c_N)\,(b+k) - b\left(b + k_N + k^R\right)(a-c) \right] \left[b\,k + 2\,b\,k^R + k\,k^R + b^2 \right]$$

$$+ \,b\,[\,b+k\,]\,[\,b+k_N\,]\,[\,(a-c_N)\,(2\,b+k) - b\,(a-c)\,]$$

$$- \,E\left[b + k^R\right]\,a\,[\,2\,b+k\,] + E\,\left[b + k^R\right] b\,[\,a-c\,]$$

$$- \,E\left[(b+k_N)\,(2\,b+k) - b^2 \right] c_N\,. \tag{210}$$

(207) and (210) imply:

$$\frac{\partial^+ V(\overline{p}_0)}{\partial \overline{p}} = \frac{1}{D\,E} \left[c_A\,E^2 - \left(\hat{E} + c_N\,E^2 \right) \right] = -\frac{1}{D\,E} \left[(c_N - c_A)\,E^2 + \hat{E} \right]$$

$$< \; 0 \;\; \text{if} \;\; c_N - c_A \; > \; - \frac{\hat{E}}{E^2} \quad \Leftrightarrow \quad \Phi_2 \; \equiv \; E^2 \, [\, c_N - c_A \,] + \hat{E} \; > \; 0 \, . \tag{211}$$

(211) reflects the facts that $E > 0$ (from (201)) and $D > 0$ (by assumption).

It remains to demonstrate that Φ_2 is as specified in (193). (201) and (210) imply:

$$\hat{E} \; = \; \psi_1 \, E + \psi_2 \, , \;\; \text{where} \tag{212}$$

$$\begin{aligned} \psi_1 \; \equiv \; & a \left[\, b \, k + 2 \, b \, k^R + k \, k^R + b^2 \, \right] - a \left[\, b + k^R \, \right] [\, 2 \, b + k \,] + b \left[\, b + k^R \, \right] [\, a - c \,] \\ & - \left[\, (b + k_N) \, (2 \, b + k) - b^2 \, \right] c_N , \;\; \text{and} \end{aligned} \tag{213}$$

$$\begin{aligned} \psi_2 \; \equiv \; & - \left[\, b \, (a - c_N) \, (b + k) + b \left(b + k_N + k^R \right) (a - c) \, \right] \left[\, b \, k + 2 \, b \, k^R + k \, k^R + b^2 \, \right] \\ & + \, b \, [\, b + k \,] \, [\, b + k_N \,] \, [\, (a - c_N) \, (2 \, b + k) - b \, (a - c) \,] \, . \end{aligned} \tag{214}$$

(213) implies:

$$\begin{aligned} \psi_1 \; = \; & a \left[\, b \, k + 2 \, b \, k^R + k \, k^R + b^2 - \left(b + k^R \right) (2 \, b + k) + b \left(b + k^R \right) \, \right] \\ & - b \left[\, b + k^R \, \right] c - \left[\, (b + k_N) \, (2 \, b + k) - b^2 \, \right] c_N \\ = \; & a \left[\, b \, k + 2 \, b \, k^R + k \, k^R + b^2 - 2 \, b^2 - b \, k - 2 \, b \, k^R - k \, k^R + b^2 + b \, k^R \, \right] \\ & - b \left[\, b + k^R \, \right] c - \left[\, (b + k_N) \, (2 \, b + k) - b^2 \, \right] c_N \\ = \; & a \, b \, k^R - b \left[\, b + k^R \, \right] c - \left[\, (b + k_N) \, (2 \, b + k) - b^2 \, \right] c_N \, . \end{aligned} \tag{215}$$

(214) implies:

$$\begin{aligned} \psi_2 \; = \; & - \left\{ \, b \, [\, b + k \,] \, [\, a - c_N \,] + b \left[\, b + k_N + k^R \, \right] [\, a - c \,] \, \right\} \left[\, b \, k + 2 \, b \, k^R + k \, k^R + b^2 \, \right] \\ & + \; b \, [\, b + k \,] \, [\, b + k_N \,] \, [\, 2 \, b + k \,] \, [\, a - c_N \,] - b^2 \, [\, b + k \,] \, [\, b + k_N \,] \, [\, a - c \,] \\ = \; & [\, a - c_N \,] \left\{ \, b \, [\, b + k \,] \, [\, b + k_N \,] \, [\, 2 \, b + k \,] - b \, [\, b + k \,] \left[\, b \, k + 2 \, b \, k^R + k \, k^R + b^2 \, \right] \right\} \\ & - [\, a - c \,] \left\{ \, b^2 \, [\, b + k \,] \, [\, b + k_N \,] + b \left[\, b + k_N + k^R \, \right] \left[\, b \, k + 2 \, b \, k^R + k \, k^R + b^2 \, \right] \right\} \\ = \; & [\, a - c_N \,] \, b \, [\, b + k \,] \left\{ \, [\, b + k_N \,] \, [\, 2 \, b + k \,] - \left[\, b \, k + 2 \, b \, k^R + k \, k^R + b^2 \, \right] \right\} \\ & - \; b \, [\, a - c \,] \left\{ \, b \, [\, b + k \,] \, [\, b + k_N \,] + \left[\, b + k_N + k^R \, \right] \left[\, b \, k + 2 \, b \, k^R + k \, k^R + b^2 \, \right] \right\} . \end{aligned} \tag{216}$$

The coefficient on $[\, a - c_N \,] \, b \, [\, b + k \,]$ in (216) is:

$$\begin{aligned} & 2 \, b^2 + b \, k + 2 \, b \, k_N + k \, k_N - b \, k - 2 \, b \, k^R - k \, k^R - b^2 \\ & \quad = \; b^2 + 2 \, b \, k_N + k \, k_N - 2 \, b \, k^R - k \, k^R \; = \; b^2 + [\, 2 \, b + k \,] \left[\, k_N - k^R \, \right] . \end{aligned} \tag{217}$$

The coefficient on $-b[a-c]$ in (216) is:

$$\begin{aligned}
& b[b+k][b+k_N]+[b+k_N]\left[bk+2bk^R+kk^R+b^2\right] \\
& \qquad + k^R\left[bk+2bk^R+kk^R+b^2\right] \\
& = [b+k_N]\left[b^2+bk+bk+2bk^R+kk^R+b^2\right]+k^R\left[bk+2bk^R+kk^R+b^2\right] \\
& = [b+k_N]\left[2b^2+2bk+2bk^R+kk^R\right]+k^R\left[bk+2bk^R+kk^R+b^2\right] \\
& = [b+k_N]\left[2b^2+2bk\right]+k^R\left[(b+k_N)(2b+k)+bk+2bk^R+kk^R+b^2\right] \\
& = 2b[b+k][b+k_N]+k^R\left[2b^2+bk+2bk_N+kk_N+bk+2bk^R+kk^R+b^2\right] \\
& = 2b[b+k][b+k_N]+k^R\left[3b^2+2bk+2bk_N+kk_N+2bk^R+kk^R\right] \\
& = 2b[b+k][b+k_N]+k^R\left[3b^2+2bk+\left(k_N+k^R\right)(2b+k)\right] \\
& = 2b[b+k][b+k_N]+k^R E\,. \qquad (218)
\end{aligned}$$

The last equality in (218) reflects (201).

(212) and (215) – (218) imply:

$$\begin{aligned}
\hat{E} \; = \; & E\left\{[a-c]\,b\,k^R-b^2c-\left[(b+k_N)(2b+k)-b^2\right]c_N\right\} \\
& + \left\{b^2+[2b+k]\left[k_N-k^R\right]\right\} b[b+k][a-c_N] \\
& - \left\{2b[b+k][b+k_N]+k^R E\right\} b[a-c] \\
= \; & -E\left\{b^2c+\left[(b+k_N)(2b+k)-b^2\right]c_N\right\} \\
& + b[b+k]\left\{b^2+[2b+k]\left[k_N-k^R\right]\right\}[a-c_N] \\
& - 2b^2[b+k][b+k_N][a-c]\,. \qquad (219)
\end{aligned}$$

(201) and (219) imply:

$$\begin{aligned}
\Phi_2 \; \equiv \; & E^2[c_N-c_A]+\widehat{E} \\
= \; & \left\{b[3b+2k]+[2b+k]\left[k_N+k^R\right]\right\}^2[c_N-c_A] \\
& - \left\{b[3b+2k]+[2b+k]\left[k_N+k^R\right]\right\} \\
& \quad \cdot\left\{b^2c+\left[(b+k_N)(2b+k)-b^2\right]c_N\right\} \\
& + b[b+k]\left\{b^2+[2b+k]\left[k_N-k^R\right]\right\}[a-c_N]-2b^2[b+k][b+k_N][a-c]\,. \qquad (220)
\end{aligned}$$

Observe that:

$$[\,b+k_N\,]\,[\,2\,b+k\,]-b^2 \;=\; b^2+b\,k+k_N\,[\,2\,b+k\,] \;=\; b\,[\,b+k\,]+[\,2\,b+k\,]\,k_N. \tag{221}$$

(220) and (221) imply:

$$\begin{aligned}
\Phi_2 \;=\; & \Big\{\,b\,[\,3\,b+2\,k\,]+[\,2\,b+k\,]\,\big[\,k_N+k^R\,\big]\,\Big\}^2\,[\,c_N-c_A\,] \\
& - \Big\{\,b\,[\,3\,b+2\,k\,]+[\,2\,b+k\,]\,\big[\,k_N+k^R\,\big]\,\Big\} \\
& \quad \cdot \Big\{\,b^2\,c+[\,b\,(b+k)+(2\,b+k)\,k_N\,]\,c_N\,\Big\} \\
& + \; b\,[\,b+k\,]\Big\{\,b^2+[\,2\,b+k\,]\,\big[\,k_N-k^R\,\big]\,\Big\}\,[\,a-c_N\,] \\
& - \,2\,b^2\,[\,b+k\,]\,[\,b+k_N\,]\,[\,a-c\,] \\
= \; & \Big\{\,b\,[\,3\,b+2\,k\,]+[\,2\,b+k\,]\,\big[\,k_N+k^R\,\big]\,\Big\} \\
& \cdot \bigg[\,\Big\{\,b\,[\,3\,b+2\,k\,]+[\,2\,b+k\,]\,\big[\,k_N+k^R\,\big]\,\Big\}\,[\,c_N-c_A\,] \\
& \qquad - \; b^2\,c-b\,[\,b+k\,]\,c_N-[\,2\,b+k\,]\,k_N\,c_N\,\bigg] \\
& + b\,[\,b+k\,]\;\{\,b^2\,[\,a-c_N\,]+[\,2\,b+k\,]\,\big[\,k_N-k^R\,\big]\,[\,a-c_N\,] \\
& \qquad - \; 2\,b\,[\,b+k_N\,]\,[\,a-c\,]\,\} \\
= \; & \Big\{\,b\,[\,3\,b+2\,k\,]+[\,2\,b+k\,]\,\big[\,k_N+k^R\,\big]\,\Big\} \\
& \cdot \bigg[\,\Big\{\,b\,[\,3\,b+2\,k\,]+[\,2\,b+k\,]\,\big[\,k_N+k^R\,\big]-b\,[\,b+k\,]-[\,2\,b+k\,]\,k_N\,\Big\}\;c_N \\
& \qquad - b^2\,c-\Big\{\,b\,[\,3\,b+2\,k\,]+[\,2\,b+k\,]\,\big[\,k_N+k^R\,\big]\,\Big\}\;c_A\,\bigg] \\
& + b\,[\,b+k\,]\,\bigg[\,\Big\{\,b^2+[\,2\,b+k\,]\,\big[\,k_N-k^R\,\big]-2\,b\,[\,b+k_N\,]\,\Big\}\;a \\
& \qquad - \Big\{\,b^2+[\,2\,b+k\,]\,\big[\,k_N-k^R\,\big]\,\Big\}\;c_N+2\,b\,[\,b+k_N\,]\,c\,\bigg].
\end{aligned} \tag{222}$$

Observe that:

$$\begin{aligned}
& b\,[\,3\,b+2\,k\,]+[\,2\,b+k\,]\,\big[\,k_N+k^R\,\big]-b\,[\,b+k\,]-[\,2\,b+k\,]\,k_N \\
& \quad = \; b\,[\,3\,b+2\,k-b-k\,]+[\,2\,b+k\,]\,k^R \;=\; [\,2\,b+k\,]\,\big[\,b+k^R\,\big]\,.
\end{aligned} \tag{223}$$

Further observe that:

$$\begin{aligned}
& b^2+[\,2\,b+k\,]\,\big[\,k_N-k^R\,\big]-2\,b\,[\,b+k_N\,] \\
& \quad = \; b^2+[\,2\,b+k-2\,b\,]\,k_N-[\,2\,b+k\,]\,k^R-2\,b^2
\end{aligned}$$

$$= \; - b^2 + k\, k_N - [\, 2\, b + k\,]\, k^R \; = \; - \left[b^2 - k\, k_N + (2\, b + k)\, k^R \right]. \tag{224}$$

(222) – (224) imply:

$$\begin{aligned}
& E^2\, [\, c_N - c_A\,] + \hat{E} \\
& \quad = \left\{ b\, [\, 3\, b + 2\, k\,] + [\, 2\, b + k\,] \left[k_N + k^R \right] \right\} \\
& \qquad \cdot \Bigg\{ [\, 2\, b + k\,] \left[b + k^R \right] c_N - b^2\, c \\
& \qquad\qquad - \left\{ b\, [\, 3\, b + 2\, k\,] + [\, 2\, b + k\,] \left[k_N + k^R \right] \right\} c_A \Bigg\} \\
& \qquad - b\, [\, b + k\,] \Bigg\{ \left[b^2 - k\, k_N + (2\, b + k)\, k^R \right] a \\
& \qquad\qquad + \left[b^2 + (2\, b + k) \left(k_N - k^R \right) \right] c_N - 2\, b\, [\, b + k_N\,]\, c \Bigg\}.
\end{aligned} \tag{225}$$

The coefficient on c_N in (225) is:

$$\begin{aligned}
& b\, [\, 2\, b + k\,]\, [\, 3\, b + 2\, k\,] \left[b + k^R \right] + [\, 2\, b + k\,]^2 \left[b + k^R \right] \left[k_N + k^R \right] \\
& \quad - b^3\, [\, b + k\,] - b\, [\, b + k\,]\, [\, 2\, b + k\,] \left[k_N - k^R \right] \\
& \quad = \; k^R \left\{ b\, [\, 2\, b + k\,]\, [\, 3\, b + 2\, k\,] + [\, 2\, b + k\,]^2 \left[b + k^R \right] + b\, [\, b + k\,]\, [\, 2\, b + k\,] \right\} \\
& \qquad\quad + k_N \left\{ [\, 2\, b + k\,]^2 \left[b + k^R \right] - b\, [\, b + k\,]\, [\, 2\, b + k\,] \right\} \\
& \qquad\quad + b^2\, [\, 2\, b + k\,]\, [\, 3\, b + 2\, k\,] - b^3\, [\, b + k\,] \\
& \quad = \; k^R\, [\, 2\, b + k\,] \left\{ b\, [\, 3\, b + 2\, k\,] + [\, 2\, b + k\,] \left[b + k^R \right] + b\, [\, b + k\,] \right\} \\
& \qquad\quad + k_N\, [\, 2\, b + k\,] \left\{ [\, 2\, b + k\,] \left[b + k^R \right] - b\, [\, b + k\,] \right\} \\
& \qquad\quad + b^2 \left\{ [\, 2\, b + k\,]\, [\, 3\, b + 2\, k\,] - b\, [\, b + k\,] \right\} \\
& \quad = \; k^R\, [\, 2\, b + k\,] \left\{ k^R\, [\, 2\, b + k\,] + b\, [\, 3\, b + 2\, k + 2\, b + k + b + k\,] \right\} \\
& \qquad + k_N\, [\, 2\, b + k\,] \left\{ k^R\, [\, 2\, b + k\,] + b\, [\, 2\, b + k - b - k\,] \right\} \\
& \qquad + b^2 \left[6\, b^2 + 7\, b\, k + 2\, k^2 - b^2 - b\, k \right] \\
& \quad = \; k^R\, [\, 2\, b + k\,] \left[k^R\, (2\, b + k) + 2\, b\, (3\, b + 2\, k) \right] \\
& \qquad\quad + \; k_N\, [\, 2\, b + k\,] \left[k^R\, (2\, b + k) + b^2 \right] + \; b^2 \left[5\, b^2 + 6\, b\, k + 2\, k^2 \right].
\end{aligned} \tag{226}$$

The coefficient on c in (225) is:

$$2\,b^2\,[\,b+k\,]\,[\,b+k_N\,] - b^2\left\{\,b\,[\,3\,b+2\,k\,] + [\,2\,b+k\,]\left[\,k_N + k^R\,\right]\,\right\}$$

$$= \; b^2\left\{\,2\,[\,b+k\,]\,[\,b+k_N\,] - b\,[\,3\,b+2\,k\,] - [\,2\,b+k\,]\left[\,k_N+k^R\,\right]\,\right\}$$

$$= \; b^2\left\{\,2\left[\,b^2+b\,k+b\,k_N+k\,k_N\,\right] - 3\,b^2 - 2\,b\,k - 2\,b\,k_N - k\,k_N - [\,2\,b+k\,]\,k^R\,\right\}$$

$$= \; b^2\left\{-\,b^2 + k\,k_N - [\,2\,b+k\,]\,k^R\right\} \; = \; -\,b^2\left[\,b^2 - k\,k_N + (2\,b+k)\,k^R\,\right]. \tag{227}$$

(201) implies that the coefficient on c_A in (225) is $-\,E^2$. Therefore, (201) and (225) – (227) imply:

$$\begin{aligned}
\Phi_2 \; = \; & \Big\{\,k^R\,[\,2\,b+k\,]\left[\,k^R\,(2\,b+k) + 2\,b\,(3\,b+2\,k)\,\right] \\
& \quad + \; k_N\,[\,2\,b+k\,]\left[k^R\,(2\,b+k)+b^2\,\right] + b^2\left[\,5\,b^2+6\,b\,k+2\,k^2\,\right]\Big\}\; c_N \\
& - \; b^2\left[\,b^2 - k\,k_N + (2\,b+k)\,k^R\,\right]\; c - \; E^2\,c_A \\
& - \; b\,[\,b+k\,]\left[\,b^2 - k\,k_N + (2\,b+k)\,k^R\,\right]\; a \\
= \; & \Big\{\,k^R\,[\,2\,b+k\,]\left[\,k^R\,(2\,b+k) + 2\,b\,(3\,b+2\,k)\,\right] \\
& \quad + \; k_N\,[\,2\,b+k\,]\left[k^R\,(2\,b+k)+b^2\,\right] + b^2\left[\,5\,b^2+6\,b\,k+2\,k^2\,\right]\Big\}\; c_N \\
& - \; b\left[\,b^2 - k\,k_N + (2\,b+k)\,k^R\,\right][\,a\,(b+k) + b\,c\,] \\
& - \; \left\{\,b\,[\,3\,b+2\,k\,] + [\,2\,b+k\,]\left[\,k_N+k^R\,\right]\right\}^2 c_A\,.
\end{aligned} \tag{228}$$

It remains to prove that $\overline{p}_{Vm} < \overline{p}_d$, which is established by demonstrating that $\left.\frac{\partial^- V(\overline{p})}{\partial\,\overline{p}}\right|_{\overline{p}=\overline{p}_d} > 0$. Define $V_2(\overline{p}) \equiv \overline{p}\,q_A(\cdot) + P(Q(\cdot))\,q_N(\cdot)$ for $\overline{p} \in (\overline{p}_0, \overline{p}_d)$. Because $P(Q) = a - b\,Q$:

$$\frac{\partial^- V_2(\overline{p}_d)}{\partial\,\overline{p}} \; = \; q_A + \overline{p}_d\,\frac{\partial q_A}{\partial\,\overline{p}} + P(Q)\,\frac{\partial q_N}{\partial\,\overline{p}} - b\,q_N\,\frac{\partial Q}{\partial\,\overline{p}} \tag{229}$$

where q_A, q_N, and Q are as specified in Lemma A2, evaluated at $\overline{p} = \overline{p}_d$. Because $\overline{p}_d = P(Q)$, (229) implies:

$$\frac{\partial^- V_2(\overline{p}_d)}{\partial\,\overline{p}} \; = \; q_A + \overline{p}_d\left[\frac{\partial q_A}{\partial\,\overline{p}} + \frac{\partial q_N}{\partial\,\overline{p}}\right] - b\,q_N\,\frac{\partial Q}{\partial\,\overline{p}}\,. \tag{230}$$

(68) implies:

$$\begin{aligned}
\overline{p}_d \; &= \; \left[\,b+k^R\,\right]Q^R + c_N + k_N\,q_N - b\,q_A \\
&= \; \left[\,b+k^R\,\right]q_A + \left[\,b+k^R\,\right]q_N + c_N + k_N\,q_N - b\,q_A \\
&= \; k^R\,q_A + \left[\,b+k_N+k^R\,\right]q_N + c_N\,.
\end{aligned} \tag{231}$$

(230) and (231) imply:

$$
\begin{aligned}
\frac{\partial^- V_2(\overline{p}_d)}{\partial \overline{p}} &= q_A - b\, q_N \, \frac{\partial Q}{\partial \overline{p}} + \left[\, k^R q_A + \left(b + k_N + k^R\right) \, q_N + c_N \,\right] \left[\frac{\partial q_A}{\partial \overline{p}} + \frac{\partial q_N}{\partial \overline{p}} \right] \\
&= q_A + \left[\, k^R q_A + \left(k_N + k^R\right) \, q_N + c_N \,\right] \left[\frac{\partial q_A}{\partial \overline{p}} + \frac{\partial q_N}{\partial \overline{p}} \right] \\
&\qquad + b\, q_N \left[\frac{\partial q_A}{\partial \overline{p}} + \frac{\partial q_N}{\partial \overline{p}} \right] - b\, q_N \, \frac{\partial Q}{\partial \overline{p}} \\
&= q_A + \left[\, k^R q_A + \left(k_N + k^R\right) \, q_N + c_N \,\right] \left[\frac{\partial q_A}{\partial \overline{p}} + \frac{\partial q_N}{\partial \overline{p}} \right] \\
&\qquad + b\, q_N \left[\frac{\partial q_A}{\partial \overline{p}} + \frac{\partial q_N}{\partial \overline{p}} - \frac{\partial Q}{\partial \overline{p}} \right] \\
&= q_A + \left[\, k^R q_A + \left(k_N + k^R\right) \, q_N + c_N \,\right] \left[\frac{\partial q_A}{\partial \overline{p}} + \frac{\partial q_N}{\partial \overline{p}} \right] - b\, q_N \, \frac{\partial q}{\partial \overline{p}} \; > \; 0 \, . \qquad (232)
\end{aligned}
$$

The inequality holds here because $\frac{\partial q_A}{\partial \overline{p}} + \frac{\partial q_N}{\partial \overline{p}} = \frac{\partial Q^R}{\partial \overline{p}} > 0$ (from (22)) and $\frac{\partial q}{\partial \overline{p}} < 0$ (from (23)). ■

Lemma 3. *For $\overline{p} \in (\overline{p}_0, \overline{p}_d)$: (i) $S(\overline{p})$ is a strictly concave function of $\overline{p}$; (ii) $\frac{\partial S(\overline{p})}{\partial \overline{p}} \gtreqless 0 \Leftrightarrow \overline{p} \lesseqgtr \overline{p}_{SM}$ where $\overline{p}_{SM} \in (\overline{p}_0, \overline{p}_d]$; and (iii) $\overline{p}_{S_2 M} > \overline{p}_{Vm}$.*

Proof. As in (174), define:

$$
\widetilde{S}_2(\overline{p}) \;\equiv\; a\, Q_2(\overline{p}) - \frac{b}{2}\, Q_2(\overline{p})^2 - q_{A2}(\overline{p})\, \overline{p} - \left[\, q_2(\overline{p}) + q_{N2}(\overline{p}) \,\right] P(Q_2(\overline{p})) \qquad (233)
$$

where $q_{A2}(\overline{p})$, $q_{N2}(\overline{p})$, $q_2(\overline{p})$, and $Q_2(\overline{p})$ are as defined in (20), (21), (23), and (24), respectively. Observe that $\widetilde{S}_2(\overline{p}) = S(\overline{p})$ for $\overline{p} \in [\overline{p}_0, \overline{p}_d]$.

(233) implies that because $P(Q_2) = a - b\, Q_2$ and $Q_2 = q_{A2} + q_{N2} + q_2$:

$$
\begin{aligned}
\frac{\partial \widetilde{S}_2(\overline{p})}{\partial \overline{p}} &= a\, \frac{\partial Q_2}{\partial \overline{p}} - b\, Q_2\, \frac{\partial Q_2}{\partial \overline{p}} - q_{A2} - \overline{p}\, \frac{\partial q_{A2}}{\partial \overline{p}} - P(Q_2) \left[\frac{\partial q_{N2}}{\partial \overline{p}} + \frac{\partial q_2}{\partial \overline{p}} \right] + b\, \frac{\partial Q_2}{\partial \overline{p}} \left[\, q_{N2} + q_2 \,\right] \\
&= P(Q_2) \left[\frac{\partial Q_2}{\partial \overline{p}} - \frac{\partial q_{N2}}{\partial \overline{p}} - \frac{\partial q_2}{\partial \overline{p}} \right] - \overline{p}\, \frac{\partial q_{A2}}{\partial \overline{p}} + b\, \frac{\partial Q_2}{\partial \overline{p}} \left[\, q_{N2} + q_2 \,\right] - q_{A2} \\
&= \left[\, P(Q_2) - \overline{p} \,\right] \frac{\partial q_{A2}}{\partial \overline{p}} + b\, \frac{\partial Q_2}{\partial \overline{p}} \left[\, q_{N2} + q_2 \,\right] - q_{A2} \, . \qquad (234)
\end{aligned}
$$

(196) and (234) imply that because $P(Q_2) = a - b\, Q_2$:

$$\frac{\partial^2 \widetilde{S}_2(\overline{p})}{\partial\,(\overline{p})^2} \;=\; \left[-\, b\, \frac{\partial Q_2}{\partial \overline{p}} - 1 \right] \frac{\partial q_{A2}}{\partial \overline{p}} + b\, \frac{\partial Q_2}{\partial \overline{p}} \left[\frac{\partial q_{N2}}{\partial \overline{p}} + \frac{\partial q_2}{\partial \overline{p}} \right] - \frac{\partial q_{A2}}{\partial \overline{p}} \;<\; 0\,. \tag{235}$$

The inequality in (235) holds because $D > 0$ by assumption, so $\frac{\partial q_{A2}}{\partial \overline{p}} > 0$ from (20), $\frac{\partial Q_2}{\partial \overline{p}} > 0$ from (24), $\frac{\partial q_{N2}}{\partial \overline{p}} < 0$ from (21), and $\frac{\partial q_2}{\partial \overline{p}} < 0$ from (23).

$\overline{p}_{SM} \equiv \arg\max_{\overline{p}} \{ \widetilde{S}_2(\overline{p}) \}$ is unique and is determined by:

$$\frac{\partial \widetilde{S}_2(\overline{p}_{SM})}{\partial \overline{p}} \;\equiv\; \left. \frac{\partial \widetilde{S}_2(\overline{p})}{\partial \overline{p}} \right|_{\overline{p} = \overline{p}_{S_2 M}} \;=\; 0\,. \tag{236}$$

This is the case because (2), (20) – (24), and (234) imply that $\frac{\partial \widetilde{S}_2(\overline{p})}{\partial \overline{p}}$ is a linear function of $\overline{p}$. Therefore, $\widetilde{S}_2(\overline{p})$ is a quadratic function of $\overline{p}$. Consequently, (235) implies that $\widetilde{S}_2(\overline{p})$ has a unique maximum that is determined by (236).

To prove that $\overline{p}_{SM} > \overline{p}_{Vm}$, let:

$$H(\overline{p}) \;\equiv\; a\, Q_2 - \frac{b}{2}\, Q_2^2 - [\, a - b\, Q_2\,]\; q_2 \tag{237}$$

$$\Rightarrow \quad \frac{\partial H(\overline{p})}{\partial \overline{p}} \;\equiv\; [\, a - b\, Q_2\,]\, \frac{\partial Q_2}{\partial \overline{p}} - [\, a - b\, Q_2\,]\, \frac{\partial q_2}{\partial \overline{p}} + b\, \frac{\partial Q_2}{\partial \overline{p}}\; q_2 \tag{238}$$

where q_2 and Q_2 are defined in (23) and (24). Differentiating (238) provides:

$$\frac{\partial^2 H(\overline{p})}{(\partial \overline{p})^2} \;\equiv\; -\, b \left(\frac{\partial Q_2}{\partial \overline{p}} \right)^2 + 2\, b\, \frac{\partial Q_2}{\partial \overline{p}}\, \frac{\partial q_2}{\partial \overline{p}} \;<\; 0\,. \tag{239}$$

The inequality in (239) holds because $\frac{\partial Q_2}{\partial \overline{p}} > 0$ and $\frac{\partial q_2}{\partial \overline{p}} < 0$, from (23) and (24). (238) implies:

$$\frac{\partial H(\overline{p}_d)}{\partial \overline{p}} \;\equiv\; \left. \frac{\partial H(\overline{p})}{\partial \overline{p}} \right|_{\overline{p} = \overline{p}_d} \;=\; \overline{p}_d\, \frac{\partial Q_2}{\partial \overline{p}} - \overline{p}_d\, \frac{\partial q_2}{\partial \overline{p}} + b\, \frac{\partial Q_2}{\partial \overline{p}}\; q_2(\overline{p}_d) \;>\; 0\,. \tag{240}$$

The inequality in (240) holds because $\frac{\partial Q_2}{\partial \overline{p}} > 0$ and $\frac{\partial q_2}{\partial \overline{p}} < 0$, from (23) and (24). The concavity of $H(\overline{p})$ established in (239), along with (240), imply:

$$\frac{\partial H(\overline{p})}{\partial \overline{p}} \;>\; 0 \;\text{ for all }\; \overline{p} \;<\; \overline{p}_d \quad \Rightarrow \quad \frac{\partial H(\overline{p}_{Vm})}{\partial \overline{p}} \;>\; 0\,. \tag{241}$$

The implication in (241) holds because $\overline{p}_{V_2 m} < \overline{p}_d$, from Lemma 2.

(195) and (236) imply:

$$\frac{\partial \widetilde{V}_2(\overline{p}_{Vm})}{\partial \overline{p}} \;=\; [\, a - b\, Q_2(\cdot)\,]\, \frac{\partial q_{N2}(\cdot)}{\partial \overline{p}} - b\, \frac{\partial Q_2(\cdot)}{\partial \overline{p}}\; q_{N2}(\cdot)$$

$$+ \, q_{A2}(\cdot) + \overline{p}_{Vm} \, \frac{\partial q_{A2}(\cdot)}{\partial \, \overline{p}} \;\; = \;\; 0 \tag{242}$$

where $q_{A2}(\cdot)$, $q_{N2}(\cdot)$, and $Q_2(\cdot)$ are defined in (20), (21), and (24), and evaluated at $\overline{p}_{Vm}$.

(234) implies:

$$\begin{aligned} \frac{\partial \widetilde{S}_2(\overline{p})}{\partial \overline{p}} \;&=\; [\, a - b\, Q_2 \,] \, \frac{\partial Q_2}{\partial \overline{p}} - [\, a - b\, Q_2 \,] \, \frac{\partial q_2}{\partial \overline{p}} + b \, \frac{\partial Q_2}{\partial \overline{p}} \, q_2 \\ &\quad - \, [\, a - b\, Q_2 \,] \, \frac{\partial q_{N2}}{\partial \overline{p}} + b \, \frac{\partial Q_2}{\partial \overline{p}} \, q_{N2} - q_{A2} - \overline{p} \, \frac{\partial q_{A2}}{\partial \overline{p}} \end{aligned} \tag{243}$$

where q_{A2}, q_{N2}, q_2, and Q_2 are defined in (20), (21), (23), and (24). (243) implies:

$$\begin{aligned} \frac{\partial \widetilde{S}_2(\overline{p}_{Vm})}{\partial \overline{p}} \;&=\; \big[\, a - b\, Q_2(\overline{p}_{V_2 m}) \,\big] \, \frac{\partial Q_2}{\partial \overline{p}} - [\, a - b\, Q_2(\overline{p}_{Vm}) \,] \, \frac{\partial q_2}{\partial \overline{p}} + b \, \frac{\partial Q_2}{\partial \overline{p}} \, q_2(\overline{p}_{Vm}) \\ &\quad - \, [\, a - b\, Q_2(\overline{p}_{Vm}) \,] \, \frac{\partial q_{N2}}{\partial \overline{p}} + b \, \frac{\partial Q_2}{\partial \overline{p}} \, q_{N2}(\overline{p}_{Vm}) - q_{A2}(\overline{p}_{Vm}) - \overline{p}_{V_2 m} \, \frac{\partial q_{A2}}{\partial \overline{p}} \\ &=\; [\, a - b\, Q_2(\overline{p}_{Vm}) \,] \, \frac{\partial Q_2}{\partial \overline{p}} - [\, a - b\, Q_2(\overline{p}_{Vm}) \,] \, \frac{\partial q_2}{\partial \overline{p}} + b \, \frac{\partial Q_2}{\partial \overline{p}} \, q_2(\overline{p}_{Vm}) \\ &=\; \frac{\partial H(\overline{p}_{V_2 m})}{\partial \overline{p}} \;>\; 0 \, . \end{aligned} \tag{244}$$

The last equality in (244) reflects (242). The inequality in (244) reflects (241).

(235) implies that $\widetilde{S}_2(\overline{p})$ is a strictly concave function of $\overline{p}$. Therefore, $\overline{p}_{V_2 m} < \overline{p}_{SM}$ because: (i) $\frac{\partial \widetilde{S}_2(\overline{p}_{SM})}{\partial \overline{p}} = 0$ from (236); and (ii) $\frac{\partial \widetilde{S}_2(\overline{p}_{Vm})}{\partial \overline{p}} > 0$, from (244).

To prove that $\overline{p}_{SM} > \overline{p}_0$, it suffices to establish that $\frac{\partial^+ S_2(\overline{p}_0)}{\partial \overline{p}} \equiv \left. \frac{\partial^+ S_2(\overline{p}_0)}{\partial \overline{p}} \right|_{\overline{p} = \overline{p}_0} > 0$. Lemma A1 implies that $q_A = 0$ when $\overline{p} = \overline{p}_0$. Therefore, (174) implies:

$$\begin{aligned} \frac{\partial^+ \widetilde{S}_2(\overline{p}_0)}{\partial \overline{p}} \;&=\; [\, a - b\, Q \,] \, \frac{\partial Q}{\partial \overline{p}} - \overline{p}_0 \, \frac{\partial q_A}{\partial \overline{p}} - P(Q) \left[\frac{\partial q_N}{\partial \overline{p}} + \frac{\partial q}{\partial \overline{p}} \right] + b \, [\, q_N + q \,] \, \frac{\partial Q}{\partial \overline{p}} \\ &=\; P(Q) \, \frac{\partial Q}{\partial \overline{p}} - \overline{p}_0 \, \frac{\partial q_A}{\partial \overline{p}} - P(Q) \left[\frac{\partial q_N}{\partial \overline{p}} + \frac{\partial q}{\partial \overline{p}} \right] + b \, [\, q_N + q \,] \, \frac{\partial Q}{\partial \overline{p}} \\ &=\; [\, P(Q) - \overline{p}_0 \,] \, \frac{\partial q_A}{\partial \overline{p}} + b \, [\, q_N + q \,] \, \frac{\partial Q}{\partial \overline{p}} \;>\; 0 \, . \end{aligned} \tag{245}$$

The inequality in (245) holds because $D > 0$ by assumption, so $\frac{\partial q_A}{\partial \overline{p}} > 0$ from (20), $\frac{\partial Q}{\partial \overline{p}} > 0$ from (24), and $P(Q) > \overline{p}_0$ when $\overline{p} \in (\overline{p}_0, \overline{p}_d)$. ■

Proposition 5. *$\overline{p}^* \in (\overline{p}_0, \overline{p}_d\,]$ if $\Phi_2 \geq 0$. $\overline{p}^* = \overline{p}_0$ if $\Phi_2 < 0$ and r is sufficiently large.*

Proof. The first conclusion in the Proposition follows from Proposition A1 because (173) implies that when if $\Phi_2 \geq 0$:

$$\frac{\partial^+ W_2(\overline{p}_0)}{\partial \overline{p}} \;\equiv\; \left.\frac{\partial^+ W_2(\overline{p})}{\partial \overline{p}}\right|_{\overline{p}=\overline{p}_0} \;=\; \frac{\partial^+ S_2(\overline{p}_0)}{\partial \overline{p}} - r\,\frac{\partial^+ V_2(\overline{p}_0)}{\partial \overline{p}} \;>\; 0\,. \tag{246}$$

The inequality in (246) holds because when $\Phi_2 \geq 0$: (i) $\frac{\partial^+ V_2(\overline{p}_0)}{\partial \overline{p}} \leq 0$ from (211); and (ii) $\frac{\partial^+ S_2(\overline{p}_0)}{\partial \overline{p}} > 0$ from (245).

The second conclusion in the Proposition holds if $V(\overline{p}_0) < V(\overline{p})$ for all $\overline{p} > \overline{p}_0$ when r is sufficiently large and $\Phi_2 < 0$. (211) and (220) imply:

$$\left.\frac{\partial^+ V(\overline{p})}{\partial \overline{p}}\right|_{\overline{p}=\overline{p}_0} \;>\; 0 \text{ when } \Phi_2 \,<\, 0\,. \tag{247}$$

$V(\overline{p})$ is a strictly convex function of $\overline{p}$ for $\overline{p} \in (\overline{p}_0, \overline{p}_d)$, from Lemma 2. Therefore, (247) implies that $V(\overline{p})$ is a strictly increasing function of $\overline{p}$ for $\overline{p} \in [\,\overline{p}_0, \overline{p}_d\,]$ under the maintained conditions. Consequently:

$$V(\overline{p}_0) \;<\; V(\overline{p}) \text{ for all } \overline{p} \in (\overline{p}_0, \overline{p}_d\,]\,. \tag{248}$$

Lemma A7 implies that under the maintained conditions:

$$V(\overline{p}_0) \;<\; V(\overline{p}_b)\,. \tag{249}$$

(110) implies that $V(\overline{p})$ is a strictly concave function of $\overline{p}$ for $\overline{p} \in (\overline{p}_d, \overline{p}_b)$. Therefore, (248) and (249) imply:

$$V(\overline{p}) \;>\; V(\overline{p}_0) \text{ for all } \overline{p} \in (\overline{p}_d, \overline{p}_b\,]\,. \tag{250}$$

The conclusion follows from (248), (250), and Proposition A1. ■

Proposition A2. $\overline{p}^* \in [\,\overline{p}_{Vm}, \overline{p}_{SM}\,]$. Furthermore: (i) $\overline{p}^* < \overline{p}_{SM}$ when $\overline{p}_{SM} < \overline{p}_d$ and $d > 0$; (ii) $\overline{p}^* > \overline{p}_{Vm}$ when $\overline{p}_{Vm} > \overline{p}_0$; (iii) $\overline{p}^* \to \overline{p}_{SM}$ as $r \to 0$; and (iv) $\overline{p}^* \to \overline{p}_{Vm}$ as $r \to \infty$.

Proof. To prove that $\overline{p}^* \leq \overline{p}_{SM}$, suppose that $\overline{p}^* > \overline{p}_{SM}$. $\widetilde{S}_2(\overline{p})$ is a strictly concave function of $\overline{p}$, from Lemma 3. Therefore, because $\overline{p}^* > \overline{p}_{SM}$, (236) implies:

$$\frac{\partial \widetilde{S}_2(\overline{p}^*)}{\partial \overline{p}} \;<\; \frac{\partial \widetilde{S}_2(\overline{p}_{SM})}{\partial \overline{p}} \;=\; 0\,. \tag{251}$$

$\widetilde{V}_2(\overline{p})$ is a strictly convex function of $\overline{p}$, from Lemma 2. Therefore, because $\overline{p}_{Vm} < \overline{p}_{SM}$ from Lemma 3 and because $\overline{p}^* > \overline{p}_{SM}$ by assumption, (198) implies:

$$\frac{\partial \widetilde{V}_2(\overline{p}^*)}{\partial \overline{p}} \;>\; \frac{\partial \widetilde{V}_2(\overline{p}_{SM})}{\partial \overline{p}} \;>\; \frac{\partial \widetilde{V}_2(\overline{p}_{Vm})}{\partial \overline{p}} \;=\; 0\,. \tag{252}$$

(251) and (252) imply that R's revenue declines and consumer surplus increases as $\overline{p}$ declines below $\overline{p}^*$. Therefore, $\overline{p}^*$ is not the welfare-maximizing value of $\overline{p}$. Hence, by contradiction, $\overline{p}^* \leq \overline{p}_{SM}$.

To prove that $\overline{p}^* \geq \overline{p}_{Vm}$, suppose that $\overline{p}^* < \overline{p}_{Vm}$. $\widetilde{V}_2(\overline{p})$ is a strictly convex function of $\overline{p}$, from Lemma 2. Therefore, because $\overline{p}_{Vm} < \overline{p}_{SM}$ from Lemma 3, (198) implies:

$$\frac{\partial \widetilde{V}_2(\overline{p}^*)}{\partial \overline{p}} < \frac{\partial \widetilde{V}_2(\overline{p}_{Vm})}{\partial \overline{p}} = 0\,. \tag{253}$$

$\widetilde{S}_2(\overline{p})$ is a strictly concave function of $\overline{p}$, from Lemma 3. Therefore, because $\overline{p}_{Vm} < \overline{p}_{SM}$ from Lemma 3 and because $\overline{p}^* < \overline{p}_{Vm}$ by assumption, (236) implies:

$$\frac{\partial \widetilde{S}_2(\overline{p}^*)}{\partial \overline{p}} > \frac{\partial \widetilde{S}_2(\overline{p}_{Vm})}{\partial \overline{p}} > \frac{\partial \widetilde{S}_2(\overline{p}_{SM})}{\partial \overline{p}} = 0\,. \tag{254}$$

(253) and (254) imply that R's revenue declines and consumer surplus increases as $\overline{p}$ increases above $\overline{p}^*$. Therefore, $\overline{p}^*$ is not the welfare-maximizing value of $\overline{p}$. Hence, by contradiction, $\overline{p}^* \geq \overline{p}_{Vm}$.

To prove conclusion (i) in the Proposition, define $\widetilde{W}_2(\cdot) \equiv \widetilde{S}_2(\cdot) - r\,\widetilde{V}_2(\cdot)$ and observe that when $\overline{p}_{SM} < \overline{p}_d$ and $r > 0$:

$$\begin{aligned} \left.\frac{\partial \widetilde{W}_2(\overline{p})}{\partial \overline{p}}\right|_{\overline{p}=\overline{p}_{SM}} &= \frac{\partial \widetilde{S}_2(\overline{p}_{SM})}{\partial \overline{p}} - r\,\frac{\partial \widetilde{V}_2(\overline{p}_{SM})}{\partial \overline{p}} \\ &= -r\,\frac{\partial \widetilde{V}_2(\overline{p}_{S_2M})}{\partial \overline{p}} < -r\,\frac{\partial \widetilde{V}_2(\overline{p}_{Vm})}{\partial \overline{p}} = 0\,. \end{aligned} \tag{255}$$

The inequality in (255) holds because: (i) $\overline{p}_{S_2M} > \overline{p}_{Vm}$, from Lemma 3; and (ii) $\widetilde{V}_2(\cdot)$ is a strictly convex function of $\overline{p}$, from Lemma 2. (255) implies that $\overline{p}_{SM} > \overline{p}^*$ because $\widetilde{W}_2(\cdot)$ is a strictly concave function of $\overline{p}$ (because $\widetilde{S}_2(\cdot)$ is a strictly concave function of $\overline{p}$ and $\widetilde{V}_2(\cdot)$ is a strictly convex function of $\overline{p}$).

To prove conclusion (ii) in the Proposition, observe that when $\overline{p}_{V_2m} > \overline{p}_0$:

$$\begin{aligned} \left.\frac{\partial \widetilde{W}_2(\overline{p})}{\partial \overline{p}}\right|_{\overline{p}=\overline{p}_{Vm}} &= \frac{\partial \widetilde{S}_2(\overline{p}_{Vm})}{\partial \overline{p}} - r\,\frac{\partial \widetilde{V}_2(\overline{p}_{Vm})}{\partial \overline{p}} \\ &= \frac{\partial \widetilde{S}_2(\overline{p}_{V_2m})}{\partial \overline{p}} > \frac{\partial \widetilde{S}_2(\overline{p}_{SM})}{\partial \overline{p}} = 0\,. \end{aligned} \tag{256}$$

The inequality in (255) holds because: (i) $\overline{p}_{S_2M} > \overline{p}_{Vm}$, from Lemma 3; and (ii) $\widetilde{S}_2(\cdot)$ is a strictly concave function of $\overline{p}$, from Lemma 3. (256) implies that $\overline{p}^* > \overline{p}_{Vm}$ because $\widetilde{W}_2(\cdot)$ is a strictly concave function of $\overline{p}$.

Conclusions (iii) and (iv) in the Proposition follow immediately from (173) because $\overline{p}^* \in (\overline{p}_0, \overline{p}_d)$ is a non-increasing function of r. This is the case because (173) implies that when $\overline{p}^* \in (\overline{p}_0, \overline{p}_d)$:

$$\frac{\partial S(\overline{p}^*)}{\partial \overline{p}} - r\,\frac{\partial \widetilde{V}(\overline{p}^*)}{\partial \overline{p}} \;=\; 0 \;\;\Rightarrow\;\; \frac{\partial^2 \widetilde{S}(\overline{p}^*)}{\partial (\overline{p})^2}\,\frac{\partial \overline{p}^*}{\partial r} - \frac{\partial \widetilde{V}(\overline{p}^*)}{\partial \overline{p}} - r\,\frac{\partial^2 \widetilde{V}(\overline{p}^*)}{\partial (\overline{p})^2}\,\frac{\partial \overline{p}^*}{\partial r} \;=\; 0$$

$$\Rightarrow\;\; \frac{\partial \overline{p}^*}{\partial r} \;=\; \frac{\frac{\partial \widetilde{V}(\overline{p}^*)}{\partial \overline{p}}}{\frac{\partial^2 \widetilde{S}(\overline{p}^*)}{\partial (\overline{p})^2} - r\,\frac{\partial^2 \widetilde{V}(\overline{p}^*)}{\partial (\overline{p})^2}} \;\stackrel{s}{=}\; -\,\frac{\partial \widetilde{V}(\overline{p}^*)}{\partial \overline{p}}\,. \tag{257}$$

The last conclusion in (257) holds because Lemmas 2 and 3 imply that $\frac{\partial^2 \widetilde{S}(\overline{p}^*)}{\partial (\overline{p})^2} < 0$ and $\frac{\partial^2 \widetilde{V}(\overline{p}^*)}{\partial (\overline{p})^2} > 0$ when $\overline{p}^* \in (\overline{p}_0, \overline{p}_d)$.

It remains to prove that $\frac{\partial \widetilde{V}_2(\overline{p}^*)}{\partial \overline{p}} \geq 0$. To do so, suppose that $\frac{\partial \widetilde{V}_2(\overline{p}^*)}{\partial \overline{p}} < 0$. Then:

$$\overline{p}^* \;<\; \overline{p}_{Vm}\,. \tag{258}$$

(258) holds because: (i) $\widetilde{V}_2(\overline{p})$ is a strictly convex function of $\overline{p}$, from Lemma 2; and (ii) $\frac{\partial \widetilde{V}_2(\overline{p}_{Vm})}{\partial \overline{p}} = 0$, from (198). Furthermore, because $\widetilde{S}_2(\overline{p})$ is a strictly concave function of $\overline{p}$, from Lemma 3:

$$\frac{\partial \widetilde{S}_2(\overline{p})}{\partial \overline{p}} \;>\; 0 \;\;\text{for all}\;\; \overline{p} \;<\; \overline{p}_{SM}\,. \tag{259}$$

Observe that:

$$\overline{p}^* \;<\; \overline{p}_{Vm} \;<\; \overline{p}_{SM}\,. \tag{260}$$

The first inequality in (260) reflects (258). The second inequality in (260) reflects Lemma 3. (236), (259), and (260) imply:

$$\frac{\partial \widetilde{S}_2\,(\overline{p}^*)}{\partial \overline{p}} \;>\; 0\,. \tag{261}$$

$\frac{\partial \widetilde{S}_2(\overline{p}^*)}{\partial \overline{p}} > 0$ (from (261)), $\frac{\partial \widetilde{V}_2(\overline{p}^*)}{\partial \overline{p}} < 0$ (by assumption), and $\overline{p}^* \in (\overline{p}_0, \overline{p}_d)$ (by assumption) imply that consumer surplus increases and R's revenue declines as $\overline{p}$ increases above $\overline{p}^*$. Therefore, $\overline{p}^*$ cannot be the welfare-maximizing value of $\overline{p}$. Hence, by contradiction, $\frac{\partial V_2(\overline{p}^*)}{\partial \overline{p}} \geq 0$. Consequently, (257) implies that $\frac{\partial \overline{p}^*}{\partial r} \leq 0$. ■

Proposition 6. *When $\overline{p}^* \in (\overline{p}_0, \overline{p}_d)$: (i) $\frac{d\overline{p}^*}{dc_A} > 0$; (ii) $\frac{d\overline{p}^*}{dk_A} > 0$; (iii) $\frac{d\overline{p}^*}{dc} > 0$; and (iv) $\frac{d\overline{p}^*}{dc_N} < 0$.*

Proof. (174) implies that consumer surplus is:

$$\begin{aligned} S \;&=\; a\,Q - \frac{1}{2}\,b\,Q^2 - p\,[\,q + q_N\,] - \overline{p}\;q_A \\ &=\; a\,Q - \frac{1}{2}\,b\,Q^2 - p\,[\,q + q_N + q_A\,] + [\,p - \overline{p}\,]\,q_A \\ &=\; a\,Q - \frac{1}{2}\,b\,Q^2 - [\,a - b\,Q\,]\,Q + [\,p - \overline{p}\,]\,q_A \\ &=\; \frac{1}{2}\,b\,Q^2 + [\,p - \overline{p}\,]\,q_A \;=\; \frac{1}{2}\,b\,Q^2 + [\,a - b\,Q - \overline{p}\,]\,q_A \end{aligned}$$

$$= \ \frac{b}{2}\,Q^2 + [\,a - \overline{p}\,]\,q_A - b\,Q\,q_A\,. \tag{262}$$

(262) implies that $\overline{p}^*$ is the solution to:

$$\underset{\overline{p}}{\text{Maximize}} \ \ W \ = \ \frac{b}{2}\,Q^2 + [\,a - \overline{p}\,]\,q_A - b\,Q\,q_A - \ r\,\overline{p}\ q_A - r\,a\,q_N + r\,b\,Q\,q_N\,. \tag{263}$$

(108) and (263) imply that for $\overline{p} \in [\,\overline{p}_0, \overline{p}_d\,]$:

$$\begin{aligned}
\frac{dW}{d\overline{p}} \ = \ & b\,Q\left[\frac{[\,b+k\,]\,[\,b+k_N\,]}{D}\right] + [\,a-\overline{p}\,]\left[\frac{3\,b^2 + 2\,b\left[\,k + k_N + k^R\,\right] + k\left[\,k_N + k^R\,\right]}{D}\right] \\
& - \ q_A - b\,Q\left[\frac{3\,b^2 + 2\,b\left[\,k + k_N + k^R\,\right] + k\left[\,k_N + k^R\,\right]}{D}\right] - b\,q_A\left[\frac{[\,b+k\,]\,[\,b+k_N\,]}{D}\right] \\
& - r\,q_A - r\,\overline{p}\left[\frac{3\,b^2 + 2\,b\left[\,k + k_N + k^R\,\right] + k\left[\,k_N + k^R\,\right]}{D}\right] \\
& - r\,a\left[-\frac{b\left[\,b + 2\,k^R\,\right] + k\left[\,b + k^R\,\right]}{D}\right] \\
& + \ \ r\,b\,Q\left[-\frac{b\left[\,b + 2\,k^R\,\right] + k\left[\,b + k^R\,\right]}{D}\right] + r\,b\,q_N\left[\frac{[\,b+k\,]\,[\,b+k_N\,]}{D}\right] \ = \ 0
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \ \ & b\,[\,b+k\,]\,[\,b+k_N\,]\,Q + [\,a-\overline{p}\,]\left\{\,3\,b^2 + 2\,b\left[\,k + k_N + k^R\,\right] + k\left[\,k_N + k^R\,\right]\,\right\} \\
& - \ D\,q_A - b\left\{\,3\,b^2 + 2\,b\left[\,k + k_N + k^R\,\right] + k\left[\,k_N + k^R\,\right]\,\right\}Q - b\,[\,b+k\,]\,[\,b+k_N\,]\,q_A \\
& - r\,D\,q_A - r\,\overline{p}\left\{\,3\,b^2 + 2\,b\left[\,k + k_N + k^R\,\right] + k\left[\,k_N + k^R\,\right]\,\right\} \\
& + r\,a\,\left\{\,b\left[\,b + 2\,k^R\,\right] + k\left[\,b + k^R\,\right]\,\right\} \\
& - r\,b\left\{\,b\left[\,b + 2\,k^R\,\right] + k\left[\,b + k^R\,\right]\,\right\}Q + r\,b\,[\,b+k\,]\,[\,b+k_N\,]\,q_N \ = \ 0
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \ \ & \{\,b\,[\,b+k\,]\,[\,b+k_N\,] - b\left[\,3\,b^2 + 2\,b\left(k + k_N + k^R\right) + k\left(k_N + k^R\right)\,\right] \\
& \quad - r\,b\left[\,b\left(b + 2\,k^R\right) + k\left(b + k^R\right)\,\right]\,\}\,Q \\
& \quad - \{\,D + b\,[\,b+k\,]\,[\,b+k_N\,] + r\,D\,\}\ q_A + r\,b\,[\,b+k\,]\,[\,b+k_N\,]\ q_N \\
& \quad - \{\,3\,b^2 + 2\,b\left[\,k + k_N + k^R\,\right] + k\left[\,k_N + k^R\,\right] \\
& \qquad\qquad + \ r\left[\,3\,b^2 + 2\,b\left(k + k_N + k^R\right) + k\ \left(k_N + k^R\right)\,\right]\,\}\,\overline{p}
\end{aligned}$$

$$+ \{ 3b^2 + 2b\left[k + k_N + k^R\right] + k\left[k_N + k^R\right]$$
$$+ \; r\left[b\left(b + 2k^R\right) + k\left(b + k^R\right)\right]\} a \; = \; 0\,. \tag{264}$$

The coefficient on Q in (264) is:

$$b[b+k][b+k_N] - b\left[3b^2 + 2b\left(k + k_N + k^R\right) + k\left(k_N + k^R\right)\right]$$
$$- \, r\,b\left[b\left(b + 2k^R\right) + k\left(b + k^R\right)\right]$$

$$= \; b[b^2 + b\,k_N + b\,k + k\,k_N - 3b^2 - 2b\,k - 2b\,k_N - 2b\,k^R - k\,k_N - k\,k^R$$
$$- \, r\,b\left(b + 2k^R\right) - rk\left(b + k^R\right)]$$

$$= \; b\left[-2b^2 - b\,k_N - b\,k - 2b\,k^R - k\,k^R - b^2 r - 2b\,r\,k^R - b\,r\,k - r\,k\,k^R\right]$$

$$= \; -b[2b^2 + b\,k + b\,k_N + 2b\,k^R + k\,k^R + b^2 r$$
$$+ \; 2b\,r\,k^R + b\,r\,k + r\,k\,k^R] \; < \; 0\,. \tag{265}$$

(2) implies that the coefficient on $-q_A$ in (264) is:

$$[1+r]D + b[b+k][b+k_N]$$
$$= \; [1+r]\left\{[2b+k]\left[k_N\left(k_A + k^R\right) + k_A\,k^R\right] + b\,k_A[3b+2k] - b^2[b+k]\right\}$$
$$+ \; b[b+k][b+k_N] \; > \; 0 \;\; \text{because} \;\; D \; > \; 0\,. \tag{266}$$

(264) – (266) imply that if $\overline{p}^* \in (\overline{p}_0, \overline{p}_d)$, $\overline{p}^*$ is determined by:

$$G - g\,\overline{p}^* \; = \; 0, \;\; \text{where} \tag{267}$$

$$G \; \equiv \; r\,b[b+k][b+k_N]\,q_N$$
$$+ \{3b^2 + 2b\left[k + k_N + k^R\right] + k\left[k_N + k^R\right]$$
$$+ \; r\left[b\left(b + 2k^R\right) + k\left(b + k^R\right)\right]\} a$$
$$- \; b[2b^2 + b\,k + b\,k_N + 2b\,k^R + k\,k^R + b^2 r$$
$$+ \; 2b\,r\,k^R + b\,r\,k + r\,k\,k^R]\,Q$$
$$- \; \{[1+r]D + b[b+k][b+k_N]\}\,q_A\,, \;\; \text{and}$$

$$g \; \equiv \; \{3b^2 + 2b\left[k + k_N + k^R\right] + k\left[k_N + k^R\right]$$
$$+ \; r\left[3b^2 + 2b\left(k + k_N + k^R\right) + k\left(k_N + k^R\right)\right]\} \; > \; 0\,. \tag{268}$$

To prove that $\frac{d\overline{p}^*}{dc_A} > 0$, observe from (268) that $\frac{dg}{d\overline{p}} = 0$. Therefore, (267) implies that for parameter x:

$$[\, G_x - \overline{p}^* \; g_x \,] \, dx + [\, G_{\overline{p}} - \; g \,] \, d\,\overline{p}^* \;=\; 0 \quad \Rightarrow \quad \frac{d\,\overline{p}^*}{dx} \;=\; \frac{G_x - \; \overline{p} \; g_x}{g - G_{\overline{p}}} \,. \tag{269}$$

(2) and (268) imply that because $D > 0$:

$$\begin{aligned} G_{c_A} \;=\;& r\, b\, [\, b + k \,]\, [\, b + k_N \,] \, \frac{dq_N}{dc_A} \\ & - b\, [\, 2\, b^2 + b\, k + b\, k_N + 2\, b\, k^R + k\, k^R + b^2 r \\ & \qquad + \; 2\, b\, r\, k^R + b\, r\, k + r\, k\, k^R \,] \, \frac{dQ}{dc_A} \\ & - \; \{\, [\, 1 + r \,]\, D + b\, [\, b + k \,]\, [\, b + k_N \,] \,\} \, \frac{dq_A}{dc_A} \\ & > \; 0 \;\; \text{if} \;\; \frac{dq_A}{dc_A} \; < \; 0 \,, \;\; \frac{dq_N}{dc_A} \; > \; 0 \,, \;\; \frac{dQ}{dc_A} \; < \; 0 \,. \end{aligned} \tag{270}$$

(270) implies that because $D > 0$:

$$G_{c_A} \; > \; 0 \,. \tag{271}$$

(2) and (268) imply that because $D > 0$:

$$\begin{aligned} G_{\overline{p}} \;=\;& r\, b\, [\, b + k \,]\, [\, b + k_N \,] \, \frac{dq_N}{d\overline{p}} \\ & - \; b\, [\, 2\, b^2 + b\, k + b\, k_N + 2\, b\, k^R + k\, k^R + b^2 r \\ & \qquad + \; 2\, b\, r\, k^R + b\, r\, k + r\, k\, k^R \,] \, \frac{dQ}{d\overline{p}} \\ & - \; \{\, [\, 1 + r \,]\, D + b\, [\, b + k \,]\, [\, b + k_N \,] \,\} \, \frac{dq_A}{d\overline{p}} \\ & < \; 0 \;\; \text{if} \;\; \frac{dq_A}{d\overline{p}} \; > \; 0 \,, \;\; \frac{dq_N}{d\overline{p}} \; < \; 0 \,, \text{and} \;\; \frac{dQ}{d\overline{p}} \; > \; 0 \,. \end{aligned} \tag{272}$$

(108) implies that because $D > 0$:

$$\frac{dq_A}{d\overline{p}} \; > \; 0 \,, \;\; \frac{dq_N}{d\overline{p}} \; < \; 0 \,, \;\; \text{and} \;\; \frac{dQ}{d\overline{p}} \; > \; 0 \,. \tag{273}$$

(272) and (273) imply that because $D > 0$:

$$G_{\overline{p}} \; < \; 0 \,. \tag{274}$$

(268) implies:

$$g_{c_A} \;=\; 0 \,. \tag{275}$$

(268), (269), (271), (274), and (275) imply that because $D > 0$:

$$\frac{d\,\overline{p}^*}{dc_A} \;=\; \frac{G_{c_A}}{g - G_{\overline{p}}} \; > \; 0 \,.$$

To prove that $\frac{d\overline{p}^*}{dc} > 0$, observe that (2) and (268) imply that because $D > 0$:

$$\begin{aligned} G_c \; &= \; r\, b\, [\, b + k\,]\, [\, b + k_N\,]\, \frac{dq_N}{dc} \\ &\quad - b\, [\, 2\, b^2 + b\, k + b\, k_N + 2\, b\, k^R + k\, k^R + b^2 r \\ &\qquad\qquad + \; 2\, b\, r\, k^R + b\, r\, k + r\, k\, k^R\,]\, \frac{dQ}{dc} \\ &\quad - \; \{\, [\, 1 + r\,]\, D + b\, [\, b + k\,]\, [\, b + k_N\,]\,\}\, \frac{dq_A}{dc} \\ &> \; 0 \;\; \text{if} \;\; \frac{dq_A}{dc} \; < \; 0\,, \;\; \frac{dq_N}{dc} \; > \; 0\,, \text{ and } \; \frac{dQ}{dc} \; < \; 0\,. \end{aligned} \tag{276}$$

(276) implies that because $D > 0$:

$$G_c \; > \; 0\,. \tag{277}$$

(268) implies:

$$g_c \; = \; 0\,. \tag{278}$$

(268), (269), (274), (277), and (278) imply that because $D > 0$:

$$\frac{d\,\overline{p}^*}{dc} \; = \; \frac{G_c}{g - G_{\overline{p}}} \; > \; 0\,.$$

To prove that $\frac{\partial \overline{p}^*}{\partial c_N} < 0$, observe that (195) implies that for $\overline{p} \in (\overline{p}_0, \overline{p}_d)$:

$$\frac{\partial V_2(\,\overline{p}\,)}{\partial \overline{p}} \; = \; q_A + \overline{p}\, \frac{\partial q_A}{\partial \overline{p}} + P(Q)\, \frac{\partial q_N}{\partial \overline{p}} - b\, q_N\, \frac{\partial Q}{\partial \overline{p}} \tag{279}$$

$$\begin{aligned} \Rightarrow \;\; \frac{\partial^2 V_2(\,\overline{p}\,)}{\partial \overline{p}\, \partial c_N} \; &= \; \frac{\partial q_A}{\partial c_N} + \overline{p}\, \frac{\partial^2 q_A}{\partial \overline{p}\, \partial c_N} + P\,(Q)\, \frac{\partial^2 q_N}{\partial \overline{p}\, \partial c_N} \\ &\quad - \; b\, \frac{\partial Q}{\partial c_N}\, \frac{\partial q_N}{\partial \overline{p}} - b\, q_N\, \frac{\partial^2 Q}{\partial \overline{p}\, \partial c_N} - b\, \frac{\partial q_N}{\partial c_N}\, \frac{\partial Q}{\partial \overline{p}} \\ &= \; \frac{\partial q_A}{\partial c_N} - b\, \frac{\partial Q}{\partial c_N}\, \frac{\partial q_N}{\partial \overline{p}} - b\, \frac{\partial q_N}{\partial c_N}\, \frac{\partial Q}{\partial \overline{p}}\,. \end{aligned} \tag{280}$$

The last equality in (280) holds because $\frac{\partial^2 q_A}{\partial \overline{p}\, \partial c_N} = \frac{\partial^2 q_N}{\partial \overline{p}\, \partial c_N} = \frac{\partial^2 Q}{\partial \overline{p}\, \partial c_N} = 0$ when $\overline{p} \in (\overline{p}_0, \overline{p}_d)$, from Lemma A2.

(2) and Lemma A2 imply that when $\overline{p} \in (\overline{p}_0, \overline{p}_d)$:

$$\begin{aligned} &\frac{\partial Q}{\partial c_N}\, \frac{\partial q_N}{\partial \overline{p}} + \frac{\partial q_N}{\partial c_N}\, \frac{\partial Q}{\partial \overline{p}} \\ &\overset{s}{=} \; [\, b + k\,]\, [\, k_A - b\,]\, \left[\, b\left(b + 2\, k^R\right) + k\left(b + k^R\right)\,\right] - [\, 2\, b + k\,]\, \left[\, k_A + k^R\,\right]\, [\, b + k\,]\, [\, b + k_N\,] \\ &\overset{s}{=} \; [\, k_A - b\,]\, \left[\, b\left(b + 2\, k^R\right) + k\left(b + k^R\right)\,\right] - [\, 2\, b + k\,]\, \left[\, k_A + k^R\,\right]\, [\, b + k_N\,] \end{aligned}$$

$$= k_A \left[b \left(b + 2\, k^R \right) + k \left(b + k^R \right) \right] - b \left[b \left(b + 2\, k^R \right) + k \left(b + k^R \right) \right]$$
$$- \, k_A \left[2\, b + k \right] \left[b + k_N \right] - k^R \left[b + k_N \right] \left[2\, b + k \right]$$
$$= k_A \left[b \left(b + 2\, k^R \right) + k \left(b + k^R \right) - (2\, b + k) \; (b + k_N) \right]$$
$$- \, b \left[b \left(b + 2\, k^R \right) + k \left(b + k^R \right) \right] - k^R \left[b + k_N \right] \left[2\, b + k \right]. \tag{281}$$

The coefficient on k_A in (281) is:

$$b \left[b + 2\, k^R \right] + k \left[b + k^R \right] - \left[2\, b + k \right] \left[b + k_N \right]$$
$$= b^2 + 2\, b\, k^R + b\, k + k\, k^R - 2\, b^2 - 2\, b\, k_N - k\, b - k\, k_N$$
$$= 2\, b\, k^R + k\, k^R - b^2 - 2\, b\, k_N - k\, k_N\,. \tag{282}$$

(281) and (282) imply that because $k_A \leq k_N$:

$$\frac{\partial Q}{\partial c_N} \; \frac{\partial q_N}{\partial \overline{p}} + \frac{\partial q_N}{\partial c_N} \; \frac{\partial Q}{\partial \overline{p}}$$
$$\stackrel{s}{=} k_A \left[2\, b\, k^R + k\, k^R - b^2 - 2\, b\, k_N - k\, k_N \right] - b \left[b \left(b + 2\, k^R \right) + k \left(b + k^R \right) \right]$$
$$- \, k^R \left[b + k_N \right] \left[2\, b + k \right]$$
$$\leq k_A \left[2\, b\, k^R + k\, k^R - b^2 - 2\, b\, k_N - k\, k_N \right] - b \left[b \left(b + 2\, k^R \right) + k \left(b + k^R \right) \right]$$
$$- \, k^R \left[b + k_A \right] \left[2\, b + k \right]$$
$$= k_A \left[2\, b\, k^R + k\, k^R - b^2 - 2\, b\, k_N - k\, k_N - k^R \left(2\, b + k \right) \right]$$
$$- \, b \left[b \left(b + 2\, k^R \right) + k \left(b + k^R \right) \right] - k^R\, b \, \left[2\, b + k \right]$$
$$= k_A \left[- b^2 - 2\, b\, k_N - k k_N \right] - b \left[b \left(b + 2\, k^R \right) + k \left(b + k^R \right) \right] - k^R b \left[2\, b + k \right] \; < \; 0\,. \tag{283}$$

Because $\frac{\partial q_A}{\partial c_N} \; > \; 0$ when $\overline{p} \in (\overline{p}_0, \overline{p}_d)$, from Lemma A2, (280) and (283) imply:

$$\frac{\partial^2 V_2(\overline{p})}{\partial \overline{p}\, \partial c_N} \;=\; \frac{\partial q_A}{\partial c_N} - b \left[\frac{\partial Q}{\partial c_N} \; \frac{\partial q_N}{\partial \overline{p}} + \frac{\partial q_N}{\partial c_N} \; \frac{\partial Q}{\partial \overline{p}} \right] \; > \; 0\,. \tag{284}$$

(234) and (279) imply:

$$\frac{\partial S_2(\overline{p})}{\partial \overline{p}} \;=\; a \, \frac{\partial Q}{\partial \overline{p}} - b\, Q \, \frac{\partial Q}{\partial \overline{p}} - q_A - \overline{p} \, \frac{\partial q_A}{\partial \overline{p}} - P\,(Q) \left[\frac{\partial q_N}{\partial \overline{p}} + \frac{\partial q}{\partial \overline{p}} \right] + b \, \frac{\partial Q}{\partial \overline{p}} \, \left[\, q_N + q \, \right]$$
$$= a \, \frac{\partial Q}{\partial \overline{p}} - b \; Q \, \frac{\partial Q}{\partial \overline{p}} - P\,(Q) \; \frac{\partial q}{\partial \overline{p}} + b \, \frac{\partial Q}{\partial \overline{p}} \; q$$

$$- \; q_A - \overline{p}\, \frac{\partial q_A}{\partial \overline{p}} - P\,(Q)\; \frac{\partial q_N}{\partial \overline{p}} + b\, \frac{\partial Q}{\partial \overline{p}}\; q_N$$

$$= \; a\; \frac{\partial Q}{\partial \overline{p}} - b\; Q\; \frac{\partial Q}{\partial \overline{p}} - P\,(Q)\; \frac{\partial q}{\partial \overline{p}} + b\; \frac{\partial Q}{\partial \overline{p}}\; q - \frac{\partial V_2(\overline{p}\,)}{\partial \overline{p}}$$

$$\Rightarrow \quad \frac{\partial^2 S_2(\overline{p}\,)}{\partial \overline{p}\,\partial c_N} \; = \; a\; \frac{\partial^2 Q}{\partial \overline{p}\,\partial c_N} - b\,Q\; \frac{\partial^2 Q}{\partial \overline{p}\,\partial c_N} - b\,\frac{\partial Q}{\partial \overline{p}}\; \frac{\partial Q}{\partial c_N} - \frac{\partial^2 V_2(\overline{p}\,)}{\partial \overline{p}\,\partial c_N}$$

$$- \;\; P\,(Q)\; \frac{\partial^2 q}{\partial \overline{p}\,\partial c_N} + b\,\frac{\partial Q}{\partial c_N}\; \frac{\partial q}{\partial \overline{p}} + b\,\frac{\partial^2 Q}{\partial \overline{p}\partial c_N}\, q + b\,\frac{\partial Q}{\partial \overline{p}}\; \frac{\partial q}{\partial c_N}\,. \tag{285}$$

Lemma A2 implies that $\frac{\partial^2 Q}{\partial \overline{p}\,\partial c_N} = \frac{\partial^2 q}{\partial \overline{p}\partial c_N} = 0$ when $\overline{p} \in (\overline{p}_0, \overline{p}_d)$. Therefore, because $Q = Q^R + q$, (285) implies:

$$\frac{\partial^2 S_2(\overline{p}\,)}{\partial \overline{p}\,\partial c_N} \; = \; -\,b\; \frac{\partial Q}{\partial \overline{p}}\; \frac{\partial Q}{\partial c_N} - \frac{\partial^2 V_2(\overline{p}\,)}{\partial \overline{p}\,\partial c_N} + b\; \frac{\partial Q}{\partial c_N}\; \frac{\partial q}{\partial \overline{p}} + b\; \frac{\partial Q}{\partial \overline{p}}\; \frac{\partial q}{\partial c_N}$$

$$= \; -\,b\; \frac{\partial Q}{\partial \overline{p}}\; \frac{\partial Q^R}{\partial c_N} - \frac{\partial^2 V_2(\overline{p}\,)}{\partial \overline{p}\,\partial c_N} + b\; \frac{\partial Q}{\partial c_N}\; \frac{\partial q}{\partial \overline{p}}\,. \tag{286}$$

(280) and (286) imply that because $Q^R = q_A + q_N$:

$$\frac{\partial^2 S_2(\overline{p}\,)}{\partial \overline{p}\,\partial c_N} \; = \; -\,b\; \frac{\partial Q}{\partial \overline{p}}\; \frac{\partial Q^R}{\partial c_N} - \left[\, \frac{\partial q_A}{\partial c_N} - b\; \frac{\partial Q}{\partial c_N}\; \frac{\partial q_N}{\partial \overline{p}} - b\; \frac{\partial q_N}{\partial c_N}\; \frac{\partial Q}{\partial \overline{p}} \,\right] + b\; \frac{\partial Q}{\partial c_N}\; \frac{\partial q}{\partial \overline{p}}$$

$$= \; -\,b\; \frac{\partial Q}{\partial \overline{p}} \left[\, \frac{\partial Q^R}{\partial c_N} - \frac{\partial q_N}{\partial c_N} \,\right] - \frac{\partial q_A}{\partial c_N} + b\; \frac{\partial Q}{\partial c_N} \left[\, \frac{\partial q}{\partial \overline{p}} + \frac{\partial q_N}{\partial \overline{p}} \,\right]$$

$$= \; -\,b\; \frac{\partial Q}{\partial \overline{p}}\; \frac{\partial q_A}{\partial c_N} - \frac{\partial q_A}{\partial c_N} + b\; \frac{\partial Q}{\partial c_N} \left[\, \frac{\partial q}{\partial \overline{p}} + \frac{\partial q_N}{\partial \overline{p}} \,\right]. \tag{287}$$

(2), (287), and Lemma A2 imply:

$$\frac{\partial^2 S_2(\overline{p})}{\partial \overline{p}\,\partial c_N} \; \overset{s}{=} \; -\,b\,[\,b+k\,]\,[\,b+k_N\,]\,[\,2\,b+k\,]\,\left[\,b+k^R\,\right] - \,[\,2\,b+k\,]\,\left[\,b+k^R\,\right] D$$

$$+ \;\; b\,[\,-\,(b+k)\,(k_A-b)\,]\,\left[\,-b\,(b+k_N) - b\left(b+2\,k^R\right) - k\left(b+k^R\right)\,\right]$$

$$= \; -\,b\,[\,b+k\,]\,[\,b+k_N\,]\,[\,2\,b+k\,]\,\left[\,b+k^R\,\right] - \,[\,2\,b+k\,]\,\left[\,b+k^R\,\right] D$$

$$+ \;\; b\,[\,b+k\,]\,[\,k_A-b\,]\,\left[\,b\,(b+k_N) + b\left(b+2\,k^R\right) + k\left(b+k^R\right)\,\right]. \tag{288}$$

(2) and (288) imply:

$$\frac{\partial^2 S_2(\overline{p})}{\partial \overline{p}\,\partial c_N} \; = \; -\,b\,[\,b+k\,]\,[\,b+k_N\,]\,[\,2\,b+k\,]\,\left[\,b+k^R\,\right]$$

$$+\ b\,[\,b+k\,]\,[\,k_A-b\,]\left[\,b\,(b+k_N)+b\left(b+2\,k^R\right)+k\left(b+k^R\right)\,\right]$$

$$-\ [\,2\,b+k\,]\left[\,b+k^R\,\right]$$

$$\cdot\ \left[\,(2\,b+k)\left(k_N\left[\,k_A+k^R\,\right]+k_A k^R\right)+b\,k_A\,(3\,b+2\,k)-b^2\,(b+k)\,\right]$$

$$=\ b\,[\,b+k\,]\,[\,k_A-b\,]\left[\,b\,(b+k_N)+b\left(b+2\,k^R\right)+k\left(b+k^R\right)\,\right]$$

$$-\ [\,2\,b+k\,]\left[\,b+k^R\,\right]$$

$$\cdot\left[\,b\,(b+k)\,(b+k_N)+(2\,b+k)\left(k_N\left[\,k_A+k^R\,\right]+k_A\,k^R\right)+b\,k_A\,(3\,b+2\,k)-b^2\,(b+k)\,\right]$$

$$=\ b\,[\,b+k\,]\,[\,k_A-b\,]\left[\,b\,(b+k_N)+b\left(b+2\,k^R\right)+k\left(b+k^R\right)\,\right]$$

$$-\,[\,2\,b+k\,]\left[\,b+k^R\,\right]\left[\,b\,(b+k)\ k_N+(2\,b+k)\left(k_N\left[\,k_A+k^R\,\right]+k_A k^R\right)+b\,k_A\,(3\,b+2\,k)\,\right]$$

$$=\ -\,b\,[\,b+k\,]\,b\left[\,b\,(b+k_N)\ +b\left(b+2\,k^R\right)+k\left(b+k^R\right)\,\right]$$

$$+\ b\,[\,b+k\,]\,k_A\left[\,b\,(b+k_N)+b\left(b+2\,k^R\right)+k\left(b+k^R\right)\,\right]$$

$$-\ [\,2\,b+k\,]\left[\,b+k^R\,\right]\left[\,b\,(b+k)\,k_N+(2\,b+k)\left(k_N\left[\,k_A+k^R\,\right]+k_A k^R\right)\,\right]$$

$$-\ k_A\,[\,2\,b+k\,]\left[\,b+k^R\,\right]\ b\ [\,3\,b+2\,k\,]$$

$$=\ -\,b\,[\,b+k\,]\,b\left[\,b\,(\,b+k_N)+b\left(b+2\,k^R\right)+k\left(b+k^R\right)\,\right]$$

$$-\ [\,2\,b+k\,]\left[\,b+k^R\,\right]\left[\,b\,(b+k)\ k_N+(2\,b+k)\ k_N\,k^R\,\right]$$

$$+\ k_A\,\{\,b\,[\,b+k\,]\left[\,b\,(b+k_N)+b\left(b+2\,k^R\right)+k\left(b+k^R\right)\,\right]$$

$$-\ [\,2\,b+k\,]\left[\,b+k^R\,\right]\left[(\,2\,b+k)\ \left(k_N+k^R\right)+b\,(3\,b+2\,k)\,\right]\,\}\,. \tag{289}$$

The coefficient on k_A in (289) is:

$$b\,[\,b+k\,]\left[\,b\,(b+k_N)\ +b\left(b+2\,k^R\right)+k\left(b+k^R\right)\,\right]$$

$$-\ [\,2\,b+k\,]\left[\,b+k^R\,\right]\left[\,(2\,b+k)\ \left(k_N+k^R\right)\ +b\,(3\,b+2\,k)\,\right]$$

$$=\ k_N\left[\,b\,(b+k)\ b-(2\,b+k)\ \left(b+k^R\right)\ (2\,b+k)\,\right]$$

$$+\ b\,[\,b+k\,]\left[\,b^2\ +b\left(b+2\,k^R\right)+k\left(b+k^R\right)\,\right]$$

$$-\ [\,2\,b+k\,]\left[\,b+k^R\,\right]\left[\,(2\,b+k)\ k^R\ +b\,(3\,b+2\,k)\,\right]$$

$$=\ k_N\left[\,b\,(b+k)\ b-(2\,b+k)\ \left(b+k^R\right)\ (2\,b+k)\,\right]$$

$$+\ k^R\left[\,b\,(b+k)\ (2\,b+k)-(2\,b+k)\ \left(b+k^R\right)\ (2\,b+k)\,\right]$$

$$+ \, b \, [\, b + k \,] \left[\, 2 \, b^2 + k \, b \, \right] - [\, 2 \, b + k \,] \left[\, b + k^R \, \right] b \, [\, 3 \, b + 2 \, k \,] \, . \tag{290}$$

The coefficient on k_N in (290) is:

$$b \, [\, b + k \,] \, b - \left[\, 2 \, b + k \, \right] \left[\, b + k^R \, \right] [\, 2 \, b + k \,] \ < \ 0 \, . \tag{291}$$

The inequality in (291) holds because $b < b + k^R$, $b + k < 2\,b + k$, and $b < 2\,b + k$.

The coefficient on k^R in (290) is:

$$b \, [\, b + k \,] \, [\, 2 \, b + k \,] - \left[\, 2 \, b + k \, \right] \left[\, b + k^R \, \right] [\, 2 \, b + k \,] \ < \ 0 \, . \tag{292}$$

The inequality in (292) holds because $b < b + k^R$ and $b + k < 2\,b + k$.

The last line in (290) is:

$$\begin{aligned} & b \, [\, b + k \,] \left[\, 2 \, b^2 + k \, b \, \right] - [\, 2 \, b + k \,] \left[\, b + k^R \, \right] b \, [\, 3 \, b + 2 \, k \,] \\ & = \ b^2 \, [\, b + k \,] \, [\, 2 \, b + k \,] - [\, 2 \, b + k \,] \left[\, b + k^R \, \right] b \, [\, 3 \, b + 2 \, k \,] \\ & \stackrel{s}{=} \ b \, [\, b + k \,] - \left[\, b + k^R \, \right] [\, 3 \, b + 2 \, k \,] \ < \ 0 \, . \end{aligned} \tag{293}$$

The inequality in (293) holds because $b < b + k^R$ and $b + k < 3\,b + 2\,k$.

(290) – (293) imply that the coefficient on k_A in (289) is negative. Therefore, (289) implies:

$$\frac{\partial^2 S_2(\,\overline{p}\,)}{\partial \, \overline{p} \, \partial c_N} \ < \ 0 \, . \tag{294}$$

$\overline{p}^*$ satisfies:

$$\frac{\partial S_2(\overline{p}^*)}{\partial \, \overline{p}} - r \, \frac{\partial V_2(\overline{p}^*)}{\partial \overline{p}} \ = \ 0 \, . \tag{295}$$

Totally differentiating (295) with respect to c_N provides:

$$\frac{\partial^2 S_2(\,\overline{p}^*)}{(\partial \overline{p}\,)^2} \, \frac{\partial \, \overline{p}^*}{\partial c_N} + \frac{\partial^2 S_2(\,\overline{p}^*)}{\partial \overline{p} \, \partial c_N} - r \left[\frac{\partial^2 V_2(\,\overline{p}^*)}{(\partial \overline{p})^2} \, \frac{\partial \, \overline{p}^*}{\partial c_N} + \frac{\partial^2 V_2(\,\overline{p}^*)}{\partial \overline{p} \, \partial c_N} \right] \ = \ 0$$

$$\Rightarrow \ \frac{\partial \, \overline{p}^*}{\partial c_N} \ = \ - \, \frac{\frac{\partial^2 S_2(\overline{p}^*)}{\partial \overline{p} \, \partial c_N} - d \, \frac{\partial^2 V_2(\overline{p}^*)}{\partial \overline{p} \, \partial c_N}}{\frac{\partial^2 S_2(\overline{p}^*)}{(\partial \overline{p})^2} - r \, \frac{\partial^2 V_2(\overline{p}^*)}{(\partial \overline{p})^2}} \ = \ - \, \frac{\frac{\partial^2 S_2(\overline{p}^*)}{\partial \overline{p} \, \partial c_N} - r \, \frac{\partial^2 V_2(\overline{p}^*)}{\partial \overline{p} \, \partial c_N}}{\frac{\partial^2 W_2(\overline{p}^*)}{(\partial \overline{p})^2}} \ < \ 0 \, .$$

The inequality follows from (284) and (294), because $\frac{\partial^2 W_2(\overline{p}^*)}{(\partial \overline{p})^2} \ < \ 0$ (from (173) and Lemmas 2 and 3).

To prove that $\frac{d\overline{p}^*}{dk_A} > 0$, observe that (2) implies:

$$\frac{\partial D}{\partial k_A} \ = \ [\, 2 \, b + k \,] \left[\, k_N + k^R \, \right] + b \, [\, 3 \, b + 2 \, k \,] \ > \ 0 \, . \tag{296}$$

(20) and (296) imply that for $\overline{p} \in (\overline{p}_0, \overline{p}_d)$:

$$\frac{dq_A}{dk_A} \;=\; -\,\frac{q_A}{D}\,\frac{\partial D}{\partial k_A} \;<\; 0\,. \tag{297}$$

(29) and (297) imply that for $\overline{p} \in (\overline{p}_0, \overline{p}_d)$:

$$\begin{aligned}\frac{\partial q_N}{\partial k_A} \;&=\; -\left[\frac{k_A + k^R}{b + k^R}\right]\frac{\partial q_A}{\partial k_A} - \left[\frac{1}{b + k^R}\right] q_A \\ &=\; \left[\frac{k_A + k^R}{b + k^R}\right]\frac{q_A}{D}\,\frac{\partial D}{\partial k_A} - \left[\frac{1}{b + k^R}\right] q_A \;\stackrel{s}{=}\; \left[\,k_A + k^R\,\right]\frac{1}{D}\,\frac{\partial D}{\partial k_A} - 1\,. \end{aligned}\tag{298}$$

(2), (296), and (298) imply that $\frac{\partial q_N}{\partial k_A} > 0$ because:

$$\begin{aligned}&\frac{\partial q_N}{\partial k_A} \;>\; 0 \;\;\Leftrightarrow\;\; \left[\,k_A + k^R\,\right]\frac{\partial D}{\partial k_A} \;>\; D \\ \Leftrightarrow\;\; &[\,2\,b + k\,]\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R\,\right] + b\,k_A[\,3\,b + 2\,k\,] - b^2\,[\,b + k\,] \\ &<\; \left[\,k_A + k^R\,\right]\left[\,(2\,b + k)\left(k_N + k^R\right) + b\,(3\,b + 2\,k)\,\right] \\ \Leftrightarrow\;\; &[\,2\,b + k\,]\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R\,\right] + b\,k_A[\,3\,b + 2\,k\,] - b^2\,[\,b + k\,] \\ &<\; [\,2\,b + k\,]\left[\,k_N + k^R\,\right]\,\left[\,k_A + k^R\,\right] + b\left[\,k_A + k^R\,\right][\,3\,b + 2\,k\,] \\ \Leftrightarrow\;\; &[\,2\,b + k\,]\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R\,\right] + b\,k_A[\,3\,b + 2\,k\,] - b^2\,[\,b + k\,] \\ &<\; [\,2\,b + k\,]\left[\,k_N\left(k_A + k^R\right) + k^R\,k_A + \left(k^R\right)^2\,\right] + b\left[\,k_A + k^R\,\right][\,3\,b + 2\,k\,]. \end{aligned}\tag{299}$$

It is apparent that the inequality in (299) holds.

Because $Q(\overline{p}\,)$ is linear in $\overline{p}$:

$$Q(\overline{p}\,) \;=\; Q(\overline{p}_0) + \frac{\partial Q}{\partial \overline{p}}\,[\,\overline{p} - \overline{p}_0\,] \quad \text{for } \overline{p} \in (\overline{p}_0, \overline{p}_d)\,. \tag{300}$$

(24) implies that for $\overline{p} \in (\overline{p}_0, \overline{p}_d)$:

$$\frac{\partial Q}{\partial \overline{p}} \;=\; \frac{[\,b + k\,]\,[\,b + k_N\,]}{D}\,. \tag{301}$$

(296) and (301) imply:

$$\frac{\partial Q}{\partial \overline{p}\,\partial k_A} \;=\; -\,\frac{[\,b + k\,][\,b + k_N\,]}{D^2}\,\frac{\partial D}{\partial k_A} \;<\; 0\,. \tag{302}$$

(6) implies that $\overline{p}_0$ does not vary with k_A. Lemma A1 implies that $Q(\overline{p}_0)$ does not vary with k_A. Therefore, (300) and (302) imply that for $\overline{p} \in (\overline{p}_0, \overline{p}_d)$:

$$\frac{\partial Q(\overline{p}\,)}{\partial k_A} \;=\; \frac{\partial Q}{\partial \overline{p}\,\partial k_A}\,[\,\overline{p} - \overline{p}_0\,] \;<\; 0\,. \tag{303}$$

In summary, (297), (299), and (303) imply:

$$\frac{dq_A}{dk_A} \; < \; 0\,, \quad \frac{dq_N}{dk_A} \; > \; 0\,, \quad \text{and} \quad \frac{dQ}{dk_A} \; < \; 0 \;\; \text{for all} \;\; \overline{p} \in (\overline{p}_0, \overline{p}_d)\,. \tag{304}$$

(20) implies that for $\overline{p} \in (\overline{p}_0, \overline{p}_d)$:

$$\begin{aligned} D\, q_A \;\; = \;\; & \left[\, 3\, b^2 + 2\, b \left(k + k_N + k^R \right) + k \, \left(k_N + k^R \right) \,\right] [\, \overline{p} - c_A \,] \\ & + b \left[\, b + k^R \,\right] [\, a - c \,] - [\, 2\, b + k \,] \, \left[\, b + k^R \,\right] [\, a - c_N \,] \end{aligned}$$

which is not a function of k_A. Therefore, (268) implies:

$$\begin{aligned} G_{k_A} \;\; = \;\; & r\, b\, [\, b + k \,] \, [\, b + k_N \,] \, \frac{\partial q_N}{\partial k_A} \\ & - \;\; b \left[\, 2\, b^2 + b\, k + b\, k_N + 2\, b\, k^R + k\, k^R + b^2\, r + 2\, b\, r\, k^R + b\, r\, k + r\, k\, k^R \,\right] \, \frac{\partial Q}{\partial k_A} \\ & - \, b\, [\, b + k \,] \, [\, b + k_N \,] \, \frac{\partial q_A}{\partial k_A}\,. \end{aligned} \tag{305}$$

(304) and (305) imply:

$$G_{k_A} \; > \; 0\,. \tag{306}$$

(268) implies:

$$g_{k_A} \; = \; 0\,. \tag{307}$$

(268), (269), (274), (306), and (307) imply:

$$\frac{d\overline{p}^*}{dk_A} \;\; = \;\; \frac{G_{k_A}}{g - G_{\overline{p}}} \;\; > \;\; 0\,. \;\; \blacksquare$$

B. Benchmark Setting where R is a Monopoly Supplier.

R's problem, [M], when it is the sole supplier of the product is:

$$\begin{aligned} & \underset{q_A \geq 0,\; q_N \geq 0}{\text{Maximize}} \quad P_A(\, q_A + q_N \,)\, q_A + [\, a - b\, (\, q_A + q_N \,)\,]\, q_N - C^R(q_A, q_N) \\ & \qquad \text{where} \quad P_A(Q) \;\; = \;\; \begin{cases} \overline{p} & \text{if} \;\; P(Q) \; \geq \; \overline{p} \\ P(Q) & \text{if} \;\; P(Q) \; < \; \overline{p}\,. \end{cases} \end{aligned} \tag{308}$$

<u>Definitions</u>

$$\overline{p}_{0M} \;\; \equiv \;\; c_A + \frac{a - c_N}{2\, b + k_N + k^R} \left[\, b + k^R \,\right]. \tag{309}$$

$$\overline{p}_{dM} \;\; \equiv \;\; \frac{a \left[\, \left(b + k^R \right) (\, k_A + k_N \,) - b\, k_N + k_A\, k_N \,\right] + b\, c_N\, [\, k_A - b \,] + b\, c_A\, [\, b + k_N \,]}{[\, 2\, b + k^R \,]\, [\, k_A + k_N \,] - b\, k_N + k_N\, k_A}\,. \tag{310}$$

$$\overline{p}_{bM} \equiv \frac{a\left[\left(b+k^R\right)(k_A+k_N)+k_N\,k_A\right]+b\,[\,c_N\,k_A+c_A\,k_N\,]}{[\,2\,b+k^R\,]\,[\,k_A+k_N\,]+k_N\,k_A}\,. \tag{311}$$

Assumptions

1. $\left[\,k_A+k^R\,\right]\left[\,2\,b+k_N+k^R\,\right] > \left[\,b+k^R\,\right]^2$.
2. $0 \leq c_A \leq c_N < a$.
3. $0 \leq k_A \leq k_N$.

We characterize equilibrium outcomes when R is the monopoly supplier as follows.

Lemma 4 establishes that the price cap does not bind, and so has no impact on equilibrium outputs or prices, when $\overline{p} > \overline{p}_{bM}$.

Lemma 5 establishes that the market price of oil that is shipped without using the Alliance input is $\overline{p}$ when $\overline{p} \in [\,\overline{p}_{dM}\,,\,\overline{p}_{bM}\,]$, where $\overline{p}_{dM} < \overline{p}_{bM}$.

Lemma 6 establishes that the market price of oil exceeds $\overline{p}$ and R ships a positive amount of oil using the Alliance input (so $q_A > 0$) when $\overline{p} \in (\overline{p}_{0M}, \overline{p}_{dM})$.

Lemma 7 establishes that R does not ship any oil using the Alliance input and the market price exceeds $\overline{p}$ when $\overline{p} \leq \overline{p}_{0M}$.

Lemma 4. *If $\overline{p} > \overline{p}_{bM}$, then in equilibrium:*

$$q_A = \frac{[\,a-c_A\,]\left[\,2\,b+k_N+k^R\,\right]-[\,a-c_N\,]\left[\,2\,b+k^R\,\right]}{[\,2\,b+k^R\,]\,[\,k_A+k_N\,]+k_A\,k_N}\,,$$

$$q_N = \frac{[\,a-c_N\,]\left[\,2\,b+k_A+k^R\,\right]-[\,a-c_A\,]\left[\,2\,b+k^R\,\right]}{[\,2\,b+k^R\,]\,[\,k_A+k_N\,]+k_A\,k_N}\,, \quad \textit{and}$$

$$Q^R = \frac{[\,a-c_N\,]\,k_A+[\,a-c_A\,]\,k_N}{[\,2\,b+k^R\,]\,[\,k_A+k_N\,]+k_A\,k_N}\,.$$

Proof. (308) implies that when the price cap does not bind, [M] is:

$$\underset{q_A,\,q_N}{\text{Maximize}} \quad [\,a-b\,(\,q_A+q_N\,)\,]\,[\,q_A+q_N\,]-c_A\,q_A-\frac{k_A}{2}\,[\,q_A\,]^2$$
$$-\,c_N\,q_N-\frac{k_N}{2}\,[\,q_N\,]^2-\frac{k^R}{2}\,[\,q_A+q_N\,]^2\,. \tag{312}$$

Differentiating (312) with respect to q_A provides:[3]

$$a - b\,[\,q_A + q_N\,] - b\,[\,q_A + q_N\,] - c_A - k_A q_A - k^R\,[\,q_A + q_N\,] \;=\; 0$$

$$\Rightarrow \quad a - \big[\,2\,b + k^R\,\big]\,[\,q_A + q_N\,] - c_A - k_A\,q_A \;=\; 0\,. \tag{313}$$

Differentiating (312) with respect to q_N provides:

$$a - b\,[\,q_A + q_N\,] - b\,[\,q_A + q_N\,] - c_N - k_N\,q_N - k^R\,[\,q_A + q_N\,] \;=\; 0$$

$$\Rightarrow \quad a - \big[\,2\,b + k^R\,\big]\,q_A - \big[\,2\,b + k_N + k^R\,\big]\,q_N - c_N \;=\; 0$$

$$\Rightarrow \quad q_N \;=\; \frac{a - \big[\,2\,b + k^R\,\big]\,q_A - c_N}{2\,b + k_N + k^R}\,. \tag{314}$$

(313) and (314) imply:

$$a - \big[\,2\,b + k^R\,\big]\left[\,q_A + \frac{a - \big(2\,b + k^R\big)\,q_A - c_N}{2\,b + k_N + k^R}\,\right] - c_A - k_A\,q_A \;=\; 0$$

$$\Leftrightarrow \quad a\,\big[\,2\,b + k_N + k^R\,\big] - \big[\,2\,b + k^R\,\big]\,\Big[\,q_A\,\big(2\,b + k_N + k^R\big) + a - \big(2\,b + k^R\big)\,q_A - c_N\,\Big]$$
$$-\;\; c_A\,\big[\,2\,b + k_N + k^R\,\big] - k_A\,q_A\,\big[\,2\,b + k_N + k^R\,\big] \;=\; 0$$

$$\Leftrightarrow \quad a\,\big[\,2\,b + k_N + k^R\,\big] - \big[\,2\,b + k^R\,\big]\,[\,q_A\,k_N + a - c_N\,]$$
$$-\;\; c_A\,\big[\,2\,b + k_N + k^R\,\big] - k_A\,q_A\,\big[\,2\,b + k_N + k^R\,\big] \;=\; 0$$

$$\Leftrightarrow \quad [\,a - c_A\,]\,\big[\,2\,b + k_N + k^R\,\big] - \big[\,2\,b + k^R\,\big]\,[\,q_A\,k_N + a - c_N\,]$$
$$-\;\; k_A\,q_A\,\big[\,2\,b + k_N + k^R\,\big] \;=\; 0$$

$$\Leftrightarrow \quad [\,a - c_A\,]\,\big[\,2\,b + k_N + k^R\,\big] - \Big[\,\big(2\,b + k^R\big)\,(k_A + k_N) + k_A\,k_N\,\Big]\,q_A$$
$$-\;[\,a - c_N\,]\,\big[\,2\,b + k^R\,\big] \;=\; 0$$

$$\Leftrightarrow \quad q_A \;=\; \frac{[\,a - c_A\,]\,\big[\,2\,b + k_N + k^R\,\big] - [\,a - c_N\,]\,\big[\,2\,b + k^R\,\big]}{\big[\,2\,b + k^R\,\big]\,[\,k_A + k_N\,] + k_A\,k_N}\,. \tag{315}$$

By symmetry:

$$q_N \;=\; \frac{[\,a - c_N\,]\,\big[\,2\,b + k_A + k^R\,\big] - [\,a - c_A\,]\,\big[\,2\,b + k^R\,\big]}{\big[\,2\,b + k^R\,\big]\,[\,k_A + k_N\,] + k_A\,k_N}\,. \tag{316}$$

(315) and (316) imply:

$$Q^R \;=\; q_A + q_N$$

[3]It is readily verified that the determinant of the Hessian associated with [M] is $\big[\,2\,b + k^R\,\big]\,k_N + \big[\,2\,b + k^R\,\big]\,k_A > 0$.

$$= \frac{1}{[\,2\,b + k^R\,]\,[\,k_A + k_N\,] + k_A\,k_N} \{\,[\,a - c_A\,]\left[\,2\,b + k_N + k^R\,\right] - [\,a - c_N\,]\left[\,2\,b + k^R\,\right]$$

$$+ \,[\,a - c_N\,]\left[\,2\,b + k_A + k^R\,\right] - [\,a - c_A\,]\left[\,2\,b + k^R\,\right]\,\}$$

$$= \frac{[\,a - c_N\,]\,k_A + [\,a - c_A\,]\,k_N}{[\,2\,b + k^R\,]\,[\,k_A + k_N\,] + k_A\,k_N}\,. \tag{317}$$

(317) implies:

$$P(Q) \;=\; a - b\,Q^R \;=\; a - b\,\frac{[\,a - c_N\,]\,k_A + [\,a - c_A\,]\,k_N}{[\,2\,b + k^R\,]\,[\,k_A + k_N\,] + k_A\,k_N}$$

$$= \frac{a\left[\,\left(2\,b + k^R\right)(k_A + k_N) + k_A\,k_N\,\right] - b\,[\,a - c_N\,]\,k_A - b\,[\,a - c_A\,]\,k_N}{[\,2\,b + k^R\,]\,[\,k_A + k_N\,] + k_A\,k_N}$$

$$= \frac{a\,\left[\,\left(b + k^R\right)(k_A + k_N) + k_N\,k_A\,\right] + a\,b\,[\,k_A + k_N\,] - a\,b\,[\,k_A + k_N\,] + b\,[\,c_N\,k_A + c_A\,k_N\,]}{[\,2\,b + k^R\,]\,[\,k_A + k_N\,] + k_N\,k_A}$$

$$= \frac{a\left[\,\left(b + k^R\right)(k_A + k_N) + k_N\,k_A\,\right] + b\,[\,c_N\,k_A + c_A\,k_N\,]}{[\,2\,b + k^R\,]\,[\,k_A + k_N\,] + k_N\,k_A} \;=\; \overline{p}_{bM}\,. \;\; \blacksquare$$

Lemma 5. *Suppose* $\overline{p} \in (\,\overline{p}_{dM}, \overline{p}_{bM}\,]$*, where* $\overline{p}_{dM} < \overline{p}_{bM}$*. Then in equilibrium:*

$$q_A \;=\; \frac{b\,[\,c_N - c_A\,] + k_N\,[\,a - \overline{p}\,]}{b\,[\,k_A + k_N\,]}\,;\;\; q_N \;=\; \frac{[\,a - \overline{p}\,]\,k_A - b\,[\,c_N - c_A\,]}{b\,[\,k_A + k_N\,]}\,;$$

$$P(Q) \;=\; \overline{p}\,;\;\; \textit{and}\;\; Q^R \;=\; \frac{a - \overline{p}}{b}\,. \tag{318}$$

<u>Proof</u>. To prove that $\overline{p}_{dM} < \overline{p}_{bM}$, observe that (310) and (311) imply:

$$\overline{p}_{bM} \;>\; \overline{p}_{dM} \;\;\Leftrightarrow\;\; \frac{a\,\left[\,\left(b + k^R\right)(k_A + k_N) + k_N\,k_A\,\right] + b\,[\,c_N\,k_A + c_A\,k_N\,]}{[\,2\,b + k^R\,]\,[\,k_A + k_N\,] + k_N\,k_A}$$

$$> \frac{a\left[\,\left(b + k^R\right)(\,k_A + k_N\,) - b\,k_N + k_A\,k_N\,\right] + b\,c_N\,[\,k_A - b\,] + b\,c_A\,[\,b + k_N\,]}{[\,2\,b + k^R\,]\,[\,k_A + k_N\,] - b\,k_N + k_N\,k_A}$$

$$\Leftrightarrow \frac{a\,\left[\,\left(b + k^R\right)(k_A + k_N) + k_N\,k_A\,\right] + b\,[\,c_N\,k_A + c_A\,k_N\,]}{[\,2\,b + k^R\,]\,[\,k_A + k_N\,] + k_N\,k_A}$$

$$> \frac{a\left[\,\left(b + k^R\right)(\,k_A + k_N\,) + k_A\,k_N\,\right] - a\,b\,k_N + b\,[\,c_N\,k_A + c_A\,k_N\,] - b^2\,[\,c_N - c_A\,]}{[\,2\,b + k^R\,]\,[\,k_A + k_N\,] + k_N\,k_A - b\,k_N}\,. \tag{319}$$

<u>Definitions</u>. $N_1 \;\equiv\; a\left[\,\left(b + k^R\right)(k_A + k_N) + k_N\,k_A\,\right] + b\,[\,c_N\,k_A + c_A\,k_N\,] \;>\; 0\,.$

$$D_1 \;\equiv\; \left[\,2\,b + k^R\,\right][\,k_A + k_N\,] + k_N\,k_A \;>\; 0\,. \tag{320}$$

(319) and (320) imply:

$$\overline{p}_{bM} \;>\; \overline{p}_{dM} \quad \Leftrightarrow \quad \frac{N_1}{D_1} \;>\; \frac{N_1 - a\,b\,k_N - b^2\,[\,c_N - c_A\,]}{D_1 - b\,k_N}\,. \tag{321}$$

(320) implies:

$$D_1 - b\,k_N \;=\; \left[\,2\,b + k^R\,\right] k_A + \left[\,b + k^R\,\right] k_N + k_N\,k_A \;>\; 0\,. \tag{322}$$

(320) and (322) imply that (321) holds if $N_1 - a\,b\,k_N - b^2\,[\,c_N - c_A\,] \leq 0$.

Suppose $N_1 - a\,b\,k_N - b^2\,[\,c_N - c_A\,] > 0$. Then (321) implies:

$$\begin{aligned}
\overline{p}_{bM} \;>\; \overline{p}_{dM} \quad &\Leftrightarrow \quad N_1\,D_1 - b\,k_N\,N_1 \;>\; D_1\,N_1 - a\,b\,k_N\,D_1 - b^2\,[\,c_N - c_A\,]\,D_1 \\
&\Leftrightarrow \quad a\,b\,k_N\,D_1 + b^2\,[\,c_N - c_A\,]\,D_1 \;>\; b\,k_N\,N_1 \\
&\Leftrightarrow \quad a\,k_N\,D_1 + b\,[\,c_N - c_A\,]\,D_1 \;>\; k_N\,N_1 \\
&\Leftrightarrow \quad [\,a\,k_N + b\,(c_N - c_A)\,]\,D_1 \;>\; k_N\,N_1.
\end{aligned} \tag{323}$$

(320) and (323) imply:

$$\begin{aligned}
\overline{p}_{bM} \;>\; \overline{p}_{dM} \quad \Leftrightarrow \quad & [\,a\,k_N + b\,(c_N - c_A)\,]\left\{\,\left[\,2\,b + k^R\,\right][\,k_A + k_N\,] + k_N\,k_A\,\right\} \\
& \quad > \; a\,k_N\left[\,\left(b + k^R\right)(k_A + k_N) + k_N\,k_A\,\right] + b\,k_N\,[\,c_N\,k_A + c_A\,k_N\,]
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \quad & a\,k_N\left\{\,\left[\,2\,b + k^R\,\right][\,k_A + k_N\,] + k_N\,k_A\,\right\} + b\,[\,c_N - c_A\,]\left\{\,\left[\,2\,b + k^R\,\right][\,k_A + k_N\,] + k_N\,k_A\,\right\} \\
& \quad > \; a\,k_N\left[\,b + k^R\,\right][\,k_A + k_N\,] + a\,(k_N)^2\,k_A + b\,k_N\,[\,c_N\,k_A + c_A\,k_N\,]
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \quad & a\,k_N\left[\,2\,b + k^R\,\right][\,k_A + k_N\,] + a\,(k_N)^2\,k_A + b\,[\,c_N - c_A\,]\left[\,2\,b + k^R\,\right][\,k_A + k_N\,] \\
& \quad + \; b\,k_N\,k_A\,[\,c_N - c_A\,] \\
& \quad > \; a\,k_N\left[\,b + k^R\,\right][\,k_A + k_N\,] + a\,(k_N)^2\,k_A + b\,c_N\,k_N\,k_A + b\,c_A\,(k_N)^2
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \quad & a\,k_N\,b\,[\,k_A + k_N\,] + b\,[\,c_N - c_A\,]\left[\,2\,b + k^R\,\right][\,k_A + k_N\,] + b\,k_N\,k_A\,[\,c_N - c_A\,] \\
& \quad > \; b\,c_N\,k_N\,k_A + b\,c_A\,(k_N)^2
\end{aligned}$$

$$\Leftrightarrow \quad a\,k_N\,b\,[\,k_A + k_N\,] + b\,[\,c_N - c_A\,]\left[\,2\,b + k^R\,\right][\,k_A + k_N\,] - b\,k_N\,k_A\,c_A \;>\; b\,c_A\,(k_N)^2$$

$$\Leftrightarrow \quad a\,k_N\,b\,[\,k_A + k_N\,] + b\,[\,c_N - c_A\,]\left[\,2\,b + k^R\,\right][\,k_A + k_N\,] \;>\; b\,c_A\,k_N\,[\,k_A + k_N\,]$$

$$\Leftrightarrow \quad [\,a - c_A\,]\,k_N\,b\,[\,k_A + k_N\,] + b\,[\,c_N - c_A\,]\left[\,2\,b + k^R\,\right][\,k_A + k_N\,] \;>\; 0\,. \tag{324}$$

The inequality in (324) holds because $a >\, c_N \geq\, c_A$, by assumption.

Next observe that [M] can be stated as:

$$\underset{q_A,\, Q^R}{\text{Maximize}} \;\; \Pi^R \;=\; \big[\, P_A(Q^R) - c_A \,\big]\, q_A + \big[\, P(Q^R) - c_N \,\big]\, \big[\, Q^R - q_A \,\big]$$

$$- \, \frac{k_A}{2}\, [\, q_A \,]^2 - \frac{k_N}{2}\, \big[\, Q^R - q_A \,\big]^2 - \frac{k^R}{2}\, \big[\, Q^R \,\big]^2$$

$$\text{where } \;\; P_A(Q^R) \;=\; \begin{cases} \overline{p} & \text{if } \; P(Q^R) \, \geq \, \overline{p} \\ P(Q^R) & \text{if } \; \overline{p} \, > \, P(Q^R)\,. \end{cases} \tag{325}$$

Then when $q_A > 0$ and there exists a range of $\overline{p}$ for which $P(Q) = \overline{p}$, the necessary conditions for a solution to [M] are:

$$\frac{\partial \Pi^R}{\partial q_A} \;=\; P_A(Q^R) - c_A - k_A\, q_A - \big[\, P(Q^R) - c_N \,\big] + k_N\, \big[\, Q^R - q_A \,\big] \;=\; 0 \;\; \text{and} \tag{326}$$

$$\frac{\partial^+ \Pi^R}{\partial Q^R} \;\leq\; 0 \;\; \text{and} \;\; \frac{\partial^- \Pi^R}{\partial Q^R} \;\geq\; 0 \;\; \text{for all } \; \overline{p} \in [\, \overline{p}_{dM},\, \overline{p}_{bM} \,]\,. \tag{327}$$

Recall that $\frac{\partial^- \Pi^R}{\partial Q^R}$ denotes the left-sided derivative of Π^R with respect to Q^R, which is relevant when $P_A(\cdot) = \overline{p}$, and $\frac{\partial^+ \Pi^R}{\partial Q^R}$ denotes the right-sided derivative of Π^R with respect to Q^R, which is relevant when $P_A(\cdot) = P(Q^R)$.

If $P_A(Q^R) = \overline{p}$, then (326) implies:

$$\overline{p} - c_A - k_A\, q_A - [\, \overline{p} - c_N \,] + k_N\, \big[\, Q^R - q_A \,\big] \;=\; 0$$

$$\Leftrightarrow \;\; c_N - c_A - k_A\, q_A + k_N\, Q^R - k_N\, q_A \;=\; 0\,. \tag{328}$$

If $P_A(Q^R) = \overline{p}$, then $Q^R = \frac{a - \overline{p}}{b}$, so (328) implies:

$$c_N - c_A - k_A\, q_A + k_N \left[\frac{a - \overline{p}}{b} \right] - k_N\, q_A \;=\; 0$$

$$\Leftrightarrow \;\; c_N - c_A + k_N \left[\frac{a - \overline{p}}{b} \right] \;=\; q_A\, [\, k_A + k_N \,]$$

$$\Leftrightarrow \;\; q_A \;=\; \frac{c_N - c_A + k_N \left[\frac{a - \overline{p}}{b} \right]}{k_A + k_N} \;=\; \frac{b\, [\, c_N - c_A \,] + k_N\, [\, a - \overline{p} \,]}{b\, [\, k_A + k_N \,]}\,. \tag{329}$$

(329) implies that when $Q^R = \frac{a - \overline{p}}{b}$:

$$q_N \;=\; Q^R - q_A \;=\; \frac{a - \overline{p}}{b} - \frac{b\, [\, c_N - c_A \,] + k_N\, [\, a - \overline{p} \,]}{b\, [\, k_A + k_N \,]}$$

$$=\; \frac{a\, [\, k_A + k_N \,] - \overline{p}\, [\, k_A + k_N \,] - b\, [\, c_N - c_A \,] - k_N\, [\, a - \overline{p} \,]}{b\, [\, k_A + k_N \,]}$$

$$= \frac{a\,k_A - \overline{p}\,k_A - b\,[\,c_N - c_A\,]}{b\,[\,k_A + k_N\,]} \;=\; \frac{[\,a - \overline{p}\,]\,k_A - b\,[\,c_N - c_A\,]}{b\,[\,k_A + k_N\,]}\,. \tag{330}$$

It remains to prove that the inequalities in (327) hold. (325) implies:

$$\frac{\partial^+\Pi^R}{\partial Q^R} \;\leq\; 0 \quad\Leftrightarrow\quad -\,b\,q_A - b\left[\,Q^R - q_A\,\right] + a - b\,Q^R - c_N - k_N\left[\,Q^R - q_A\,\right] - k^R\,Q^R \;\leq\; 0$$

$$\Leftrightarrow\quad a - 2\,b\,Q^R - c_N - k_N\,Q^R + k_N\,q_A - k^R\,Q^R \;\leq\; 0$$

$$\Leftrightarrow\quad a - b\,Q^R - b\,Q^R - c_N - k_N\left[\,Q^R - q_A\,\right] - k^R\,Q^R \;\leq\; 0$$

$$\Leftrightarrow\quad \overline{p} - \left[\,b + k^R\,\right]Q^R - c_N - k_N\,q_N \;\leq\; 0$$

$$\Leftrightarrow\quad \overline{p} \;\leq\; \left[\,b + k^R\,\right]Q^R + c_N + k_N\,q_N\,. \tag{331}$$

(330) and (331) imply:

$$\frac{\partial^+\Pi^R}{\partial Q^R} \;\leq\; 0 \quad\Leftrightarrow\quad \overline{p} \;\leq\; \left[\,b + k^R\,\right]\,\frac{a - \overline{p}}{b} + c_N + k_N\,\frac{[\,a - \overline{p}\,]\,k_A - b\,[\,c_N - c_A\,]}{b\,[\,k_A + k_N\,]}$$

$$\Leftrightarrow\quad \overline{p}\,b\,[\,k_A + k_N\,] \;\leq\; \left[\,b + k^R\,\right][\,a - \overline{p}\,]\,[\,k_A + k_N\,] + c_N\,b\,[\,k_A + k_N\,] + \;k_N\left[\,(a - \overline{p})\,k_A - b\,(c_N - c_A)\,\right]$$

$$\Leftrightarrow\quad \overline{p}\,b\,[\,k_A + k_N\,] \;\leq\; [\,a - \overline{p}\,]\left[\,\left(b + k^R\right)(k_A + k_N) + k_N\,k_A\,\right] + \;c_N\,b\,[\,k_A + k_N\,] - k_N\,b\,[\,c_N - c_A\,]$$

$$\Leftrightarrow\quad \overline{p}\left[\,b\,(k_A + k_N) + \left(b + k^R\right)(k_A + k_N) + k_N\,k_A\,\right] \;\leq\; a\left[\,\left(b + k^R\right)(k_A + k_N) + k_N\,k_A\,\right] + c_N\,b\,[\,k_A + k_N\,] - k_N\,b\,[\,c_N - c_A\,]$$

$$\Leftrightarrow\quad \overline{p}\left[\,\left(2\,b + k^R\right)(k_A + k_N) + k_N\,k_A\,\right] \;\leq\; a\left[\,\left(b + k^R\right)(k_A + k_N) + k_N\,k_A\,\right] + b\,c_N\,[\,k_A + k_N - k_N\,] + b\,k_N\,c_A$$

$$\Leftrightarrow\quad \overline{p} \;\leq\; \frac{a\,\left[\,\left(b + k^R\right)(k_A + k_N) + k_N\,k_A\,\right] + b\,c_N\,k_A + b\,c_A\,k_N}{[\,2\,b + k^R\,]\,[\,k_A + k_N\,] + k_N\,k_A} \;=\; \overline{p}_{bM}\,. \tag{332}$$

(325) implies:

$$\frac{\partial^-\Pi^R}{\partial Q^R} \;\geq\; 0 \quad\Leftrightarrow\quad a - 2\,b\,Q^R - c_N + b\,q_A - k_N\left[\,Q^R - q_A\,\right] - k^R\,Q^R \;\geq\; 0$$

$$\Leftrightarrow\quad a - 2\,b\,Q^R - c_N + b\,q_A - k_N\,q_N - k^R\,Q^R \;\geq\; 0$$

$$\Leftrightarrow\quad \overline{p} - b\,Q^R - c_N + b\,q_A - k_N\,q_N - k^R\,Q^R \;\geq\; 0$$

$$\Leftrightarrow\quad \overline{p} \;\leq\; \left[\,b + k^R\,\right]Q^R + c_N - b\,q_A + k_N\,q_N \;\geq\; 0$$

$$\Leftrightarrow \quad \overline{p} - \left[b + k^R \right] Q^R - c_N + b\, q_A - k_N\, q_N \;\geq\; 0\,. \tag{333}$$

(329), (330), and (333) imply:

$$\frac{\partial^- \Pi^R}{\partial Q^R} \;\geq\; 0 \quad \Leftrightarrow \quad \overline{p} \;\geq\; \left[b + k^R \right] \frac{a - \overline{p}}{b} + c_N - b\, \frac{b\,[\, c_N - c_A \,] + k_N\,[\, a - \overline{p} \,]}{b\,[\, k_A + k_N \,]}$$

$$+\; k_N\, \frac{[\, a - \overline{p} \,]\, k_A - b\,[\, c_N - c_A \,]}{b\,[\, k_A + k_N \,]}$$

$$\Leftrightarrow \quad \overline{p}\, b\,[\, k_A + k_N \,] \;\geq\; \left[b + k^R \right] [\, a - \overline{p} \,]\,[\, k_A + k_N \,] + c_N\, b\,[\, k_A + k_N \,]$$

$$-\;\; b\,[\, b\,(c_N - c_A) + k_N\,(a - \overline{p}\,) \,] + k_N\,[\, (a - \overline{p}\,)\, k_A - b\,(c_N - c_A) \,]$$

$$\Leftrightarrow \quad \overline{p}\, b\,[\, k_A + k_N \,] \;\geq\; [\, a - \overline{p} \,] \left[\left(b + k^R \right) (k_A + k_N) - b\, k_N + k_N\, k_A \right] + c_N\, b\,[\, k_A + k_N \,]$$

$$-\; b^2\,[\, c_N - c_A \,] - k_N\, b\,[\, c_N - c_A \,]$$

$$\Leftrightarrow \quad \overline{p} \left[b\,(k_A + k_N) + \left(b + k^R \right) (k_A + k_N) - b\, k_N + k_N\, k_A \right]$$

$$\geq\; a \left[\left(b + k^R \right) (k_A + k_N) - b\, k_N + k_N\, k_A \right] + c_N\, b\,[\, k_A + k_N \,]$$

$$-\; b^2\,[\, c_N - c_A \,] - k_N\, b\,[\, c_N - c_A \,]$$

$$\Leftrightarrow \quad \overline{p} \left[\left(2\, b + k^R \right) (k_A + k_N) - b\, k_N + k_N\, k_A \right]$$

$$\geq\; a \left[\left(b + k^R \right) (k_A + k_N) - b\, k_N + k_N\, k_A \right] + c_N \left[b\,(k_A + k_N) - b^2 - k_N\, b \right]$$

$$+\;\; \left[b^2 + b\, k_N \right] c_A$$

$$\Leftrightarrow \quad \overline{p} \;\geq\; \frac{a \left[\left(b + k^R \right) (k_A + k_N) - b\, k_N + k_N\, k_A \right] + b\, c_N\,[\, k_A + k_N - b - k_N \,] + [\, b + k_N \,]\, b c_A}{[\, 2\, b + k^R \,]\,[\, k_A + k_N \,] - b\, k_N + k_N\, k_A}$$

$$\Leftrightarrow \quad \overline{p} \;\geq\; \frac{a \left[\left(b + k^R \right) (k_A + k_N) - b\, k_N + k_N\, k_A \right] + b\, c_N\,[\, k_A - b \,] + b\, c_A\,[\, b + k_N \,]}{[\, 2\, b + k^R \,]\,[\, k_A + k_N \,] - b\, k_N + k_N\, k_A}$$

$$\Leftrightarrow \quad \overline{p} \;\geq\; \overline{p}_{dM}\,. \quad \blacksquare$$

Lemma 6. *If* $\overline{p} \in (\,\overline{p}_{0M}, \overline{p}_{dM}\,]$, *then in equilibrium:*

$$q_A \;=\; \frac{\left[2\, b + k_N + k^R \right] [\, \overline{p} - c_A \,] - \left[b + k^R \right] [\, a - c_N \,]}{[\, k_A + k^R \,]\,[\, 2\, b + k_N + k^R \,] - [\, b + k^R \,]^2}\,,$$

$$q_N \;=\; \frac{\left[k_A + k^R \right] [\, a - c_N \,] - \left[b + k^R \right] [\, \overline{p} - c_A \,]}{[\, k_A + k^R \,]\,[\, 2\, b + k_N + k^R \,] - [\, b + k^R \,]^2}\,, \quad \textit{and}$$

$$Q^R \;\equiv\; q_A + q_N \;=\; \frac{[\,k_A - b\,][\,a - c_N\,] + [\,b + k_N\,][\,\overline{p} - c_A\,]}{[\,k_A + k^R\,][\,2\,b + k_N + k^R\,] - [\,b + k^R\,]^2}\,.$$

Proof. (308) implies that if $q_A > 0$ and $\overline{p} < P(Q)$, then [M] is:

$$\underset{q_A,\,q_N}{\text{Maximize}} \;\; \overline{p}\, q_A + [\,a - b\,(\,q_A + q_N\,)\,]\, q_N - c_A\, q_A - \frac{k_A}{2}\,[\,q_A\,]^2$$
$$-\; c_N\, q_N - \frac{k_N}{2}\,[\,q_N\,]^2 - \frac{k^R}{2}\,[\,q_A + q_N\,]^2\,.$$

The necessary conditions for an interior solution to [M] when $\overline{p} < P(Q)$ are:[4]

$$q_A: \;\; \overline{p} - b\,q_N - c_A - k_A\,q_A - k^R\,[\,q_A + q_N\,] \;=\; 0\,; \tag{334}$$

$$q_N: \;\; a - b\,[\,q_A + q_N\,] - b\,q_N - c_N - k_N\,q_N - k^R\,[\,q_A + q_N\,] \;=\; 0\,. \tag{335}$$

(334) implies:

$$\overline{p} - b\,q_N - c_A - k^R\,q_N \;=\; \left[\,k_A + k^R\,\right] q_A \;\;\Rightarrow\;\; q_A \;=\; \frac{\overline{p} - c_A}{k_A + k^R} - \left[\,\frac{b + k^R}{k_A + k^R}\,\right] q_N\,. \tag{336}$$

(335) implies:

$$a - b\,q_A - c_N - k^R\,q_A \;=\; \left[\,2\,b + k_N + k^R\,\right] q_N$$

$$\Rightarrow\;\; q_N \;=\; \frac{a - c_N}{2\,b + k_N + k^R} - \left[\,\frac{b + k^R}{2\,b + k_N + k^R}\,\right] q_A\,. \tag{337}$$

(334) also implies:

$$\overline{p} - c_A - k_A\,q_A - k^R\,q_A \;=\; \left[\,b + k^R\,\right] q_N \;\;\Rightarrow\;\; q_N \;=\; \frac{\overline{p} - c_A}{b + k^R} - \left[\,\frac{k_A + k^R}{b + k^R}\,\right] q_A\,. \tag{338}$$

(337) and (338) imply:

$$\frac{a - c_N}{2\,b + k_N + k^R} - \left[\,\frac{b + k^R}{2\,b + k_N + k^R}\,\right] q_A \;=\; \frac{\overline{p} - c_A}{b + k^R} - \left[\,\frac{k_A + k^R}{b + k^R}\,\right] q_A$$

$$\Rightarrow\;\; \left[\,\frac{b + k^R}{2\,b + k_N + k^R} - \frac{k_A + k^R}{b + k^R}\,\right] q_A \;=\; \frac{a - c_N}{2\,b + k_N + k^R} - \frac{\overline{p} - c_A}{b + k^R}$$

$$\Rightarrow\;\; \left\{\,\left[\,b + k^R\,\right]^2 - \left[\,k_A + k^R\,\right]\left[\,2\,b + k_N + k^R\,\right]\,\right\} q_A$$

$$=\;\; \left[\,b + k^R\,\right][\,a - c_N\,] - \left[\,2\,b + k_N + k^R\,\right][\,\overline{p} - c_A\,]$$

$$\Rightarrow\;\; q_A \;=\; \frac{\left[\,b + k^R\,\right][\,a - c_N\,] - \left[\,2\,b + k_N + k^R\,\right][\,\overline{p} - c_A\,]}{[\,b + k^R\,]^2 - [\,k_A + k^R\,][\,2\,b + k_N + k^R\,]}$$

[4]It is readily verified that the Hessian associated with this problem is $\left[\,k_A + k^R\,\right]\left[\,2\,b + k_N + k^R\,\right] - \left[\,b + k^R\,\right]^2 > 0$. This inequality holds, by assumption.

$$\Rightarrow \quad q_A \;=\; \frac{\left[\,2\,b + k_N + k^R\,\right]\left[\,\overline{p} - c_A\,\right] - \left[\,b + k^R\,\right]\left[\,a - c_N\,\right]}{\left[\,k_A + k^R\,\right]\left[\,2\,b + k_N + k^R\,\right] - \left[\,b + k^R\,\right]^2}\,. \tag{339}$$

(338) and (339) imply:

$$q_N \;=\; \frac{\overline{p} - c_A}{b + k^R} - \left[\frac{k_A + k^R}{b + k^R}\right]\left[\frac{\left(2\,b + k_N + k^R\right)\left(\overline{p} - c_A\right) - \left(b + k^R\right)\left(a - c_N\right)}{\left(k_A + k^R\right)\left(2\,b + k_N + k^R\right) - \left(b + k^R\right)^2}\right]$$

$$= \frac{\left[\,\overline{p} - c_A\,\right]\left[\left(k_A + k^R\right)\left(2\,b + k_N + k^R\right) - \left(b + k^R\right)^2\right]}{\left[\,b + k^R\,\right]\left[\left(k_A + k^R\right)\left(2\,b + k_N + k^R\right) - \left(b + k^R\right)^2\right]}$$

$$- \left[\frac{k_A + k^R}{b + k^R}\right]\left[\frac{\left(2\,b + k_N + k^R\right)\left(\overline{p} - c_A\right) - \left(b + k^R\right)\left(a - c_N\right)}{\left(k_A + k^R\right)\left(2\,b + k_N + k^R\right) - \left(b + k^R\right)^2}\right]$$

$$= \frac{\left[\,k_A + k^R\,\right]\left[\,b + k^R\,\right]\left[\,a - c_N\,\right] - \left[\,b + k^R\,\right]^2\left[\,\overline{p} - c_A\,\right]}{\left[\,b + k^R\,\right]\left[\left(k_A + k^R\right)\left(2\,b + k_N + k^R\right) - \left(b + k^R\right)^2\right]}$$

$$= \frac{\left[\,k_A + k^R\,\right]\left[\,a - c_N\,\right] - \left[\,b + k^R\,\right]\left[\,\overline{p} - c_A\,\right]}{\left[\,k_A + k^R\,\right]\left[\,2\,b + k_N + k^R\,\right] - \left[\,b + k^R\,\right]^2}\,. \tag{340}$$

(339) and (340) imply:

$$Q \;=\; Q^R \;=\; q_A + q_N \;=\; \frac{\left[\,k_A - b\,\right]\left[\,a - c_N\,\right] + \left[\,b + k_N\,\right]\left[\,\overline{p} - c_A\,\right]}{\left[\,k_A + k^R\,\right]\left[\,2\,b + k_N + k^R\,\right] - \left[\,b + k^R\,\right]^2}\,. \tag{341}$$

It remains to show that $q_A > 0$ and $\overline{p} \le P(Q)$ when $\overline{p} \in (\overline{p}_{0M}, \overline{p}_{dM}\,]$. (339) implies that $q_A > 0$ if:

$$\left[\,2\,b + k_N + k^R\,\right]\left[\,\overline{p} - c_A\,\right] - \left[\,b + k^R\,\right]\left[\,a - c_N\,\right] \;>\; 0$$

$$\Leftrightarrow \quad \overline{p} \;>\; c_A + \frac{\left[\,b + k^R\,\right]\left[\,a - c_N\,\right]}{2\,b + k_N + k^R} \;=\; \overline{p}_{0M}\,.$$

(341) implies that $P(Q^R) \ge \overline{p}$ if:

$$a - b\,\frac{\left[\,k_A - b\,\right]\left[\,a - c_N\,\right] \,+\, \left[\,b + k_N\,\right]\left[\,\overline{p} - c_A\,\right]}{\left[\,k_A + k^R\,\right]\left[\,2\,b + k_N + k^R\,\right] - \left[\,b + k^R\,\right]^2} \;\ge\; \overline{p}$$

$$\Leftrightarrow \quad a - b\,\frac{\left[\,k_A - b\,\right]\left[\,a - c_N\,\right] - \left[\,b + k_N\,\right]c_A}{\left[\,k_A + k^R\,\right]\left[\,2\,b + k_N + k^R\,\right] - \left[\,b + k^R\,\right]^2}$$

$$\ge\; \overline{p} + \overline{p}\,\frac{b\left[\,b + k_N\,\right]}{\left[\,k_A + k^R\,\right]\left[\,2\,b + k_N + k^R\,\right] - \left[\,b + k^R\,\right]^2}$$

$$\Leftrightarrow \quad a + \frac{-b\,[\,k_A - b\,]\,[\,a - c_N\,] \, + b\,[\,b + k_N\,]\,c_A}{[\,k_A + k^R\,]\,[\,2\,b + k_N + k^R\,] - [\,b + k^R\,]^2}$$

$$\geq \; \overline{p} \; \frac{\left[\,k_A + k^R\,\right] \left[\,2\,b + k_N + k^R\,\right] - \left[\,b + k^R\,\right]^2 + b\,\left[\,b + k_N\,\right]}{[\,k_A + k^R\,]\,[\,2\,b + k_N + k^R\,] - [\,b + k^R\,]^2}$$

$$\Leftrightarrow \quad a + \frac{-b\,[\,k_A - b\,]\,a + b\,[\,k_A - b\,]\,c_N + b\,[\,b + k_N\,]\,c_A}{[\,k_A + k^R\,]\,[\,2\,b + k_N + k^R\,] - [\,b + k^R\,]^2}$$

$$\geq \; \overline{p} \; \frac{\left[\,k_A + k^R\,\right] \left[\,2\,b + k_N + k^R\,\right] - \left[\,b + k^R\,\right]^2 + b\,[\,b + k_N\,]}{[\,k_A + k^R\,]\,[\,2\,b + k_N + k^R\,] - [\,b + k^R\,]^2}$$

$$\Leftrightarrow \quad \frac{a\left[\,\left(k_A + k^R\right)\left(2\,b + k_N + k^R\right) - \left(b + k^R\right)^2\,\right] - b\,[\,k_A - b\,]\,a + b\,[\,k_A - b\,]\,c_N + b\,[\,b + k_N\,]\,c_A}{[\,k_A + k^R\,]\,[\,2\,b + k_N + k^R\,] - [\,b + k^R\,]^2}$$

$$\geq \; \overline{p} \; \frac{\left[\,k_A + k^R\,\right] \left[\,2\,b + k_N + k^R\,\right] - \left[\,b + k^R\,\right]^2 + b\,[\,b + k_N\,]}{[\,k_A + k^R\,]\,[\,2\,b + k_N + k^R\,] - [\,b + k^R\,]^2}$$

$$\Leftrightarrow \quad a\left[\,\left(k_A + k^R\right)\left(2\,b + k_N + k^R\right) - \left(b + k^R\right)^2\,\right] - b\,[\,k_A - b\,]\,a + b\,[\,k_A - b\,]\,c_N + b\,[\,b + k_N\,]\,c_A$$

$$\geq \; \overline{p} \left\{ \left[\,k_A + k^R\,\right] \left[\,2\,b + k_N + k^R\,\right] - \left[\,b + k^R\,\right]^2 + b\,[\,b + k_N\,] \right\}$$

$$\Leftrightarrow \quad a\left[\,\left(k_A + k^R\right)\left(2\,b + k_N + k^R\right) - \left(b + k^R\right)^2 - b\,(k_A - b)\,\right] + b\,[\,k_A - b\,]\,c_N + b\,[\,b + k_N\,]\,c_A$$

$$\geq \; \overline{p} \left\{ \left[\,k_A + k^R\,\right] \left[\,2\,b + k_N + k^R\,\right] - \left[\,b + k^R\,\right]^2 + b\,[\,b + k_N\,] \right\}. \tag{342}$$

Observe that:

$$\left[\,k_A + k^R\,\right] \left[\,2\,b + k_N + k^R\,\right] - \left[\,b + k^R\,\right]^2 - b\,[\,k_A - b\,]$$

$$= \; 2\,b\,k_A + k_A\,k_N + k_A\,k^R + 2\,b\,k^R + k^R\,k_N + \left[\,k^R\,\right]^2 - b^2 - 2\,b\,k^R - \left[\,k^R\,\right]^2 - b\,k_A + b^2$$

$$= \; b\,k_A + k_A\,k_N + k_A\,k^R + k^R\,k_N + b\,k_N - b\,k_N \; = \; \left[\,b + k^R\,\right]\,[\,k_A + k_N\,] - b\,k_N + k_A\,k_N\,. \tag{343}$$

Further observe that:

$$\left[\,k_A + k^R\,\right] \left[\,2\,b + k_N + k^R\,\right] - \left[\,b + k^R\,\right]^2 + b\,[\,b + k_N\,]$$

$$= \; 2\,b\,k_A + k_A\,k_N + k_A\,k^R + 2\,b\,k^R + k^R\,k_N + \left[\,k^R\,\right]^2 - b^2 - 2\,b\,k^R - \left[\,k^R\,\right]^2 + b^2 + b\,k_N$$

$$= \; 2\,b\,k_A + k_A\,k_N + k_A\,k^R + k^R\,k_N + b\,k_N$$

$$= \; 2\,b\,k_A + k_A\,k_N + k_A\,k^R + k^R\,k_N - b\,k_N + 2\,b\,k_N$$

$$= \; 2\,b\,k_A + 2\,b\,k_N + k_A\,k^R + k^R\,k_N - b\,k_N + k_A\,k_N$$

$$= \left[2b + k^R \right] [k_A + k_N] - b k_N + k_N k_A \,. \tag{344}$$

(342) – (344) imply that $P(Q^R) \geq \overline{p}$ if:

$$a \left\{ \left[b + k^R \right] [k_A + k_N] - b k_N + k_A k_N \right\} + b [k_A - b] c_N + b [b + k_N] c_A$$

$$\geq \overline{p} \left\{ \left[2b + k^R \right] [k_A + k_N] - b k_N + k_N k_A \right\}$$

$$\Leftrightarrow \overline{p} \leq \frac{a \left[\left(b + k^R \right) (k_A + k_N) - b k_N + k_A k_N \right] + b [k_A - b] c_N + b [b + k_N] c_A}{[2b + k^R] [k_A + k_N] - b k_N + k_N k_A}$$

$$\Leftrightarrow \overline{p} \leq \overline{p}_{dM} \,.$$

Thus, $q_A > 0$ and $\overline{p} \leq P(Q)$ when $\overline{p} \in (\overline{p}_{0M}, \overline{p}_{dM}]$ ■

Lemma 7. *If $\overline{p} \leq \overline{p}_{0M}$, then in equilibrium:*

$$q_A = 0 \quad and \quad q_N = \frac{a - c_N}{2b + k_N + k^R} \,.$$

Proof. (308) implies that when $q_A = 0$, R's problem is:

$$\underset{q_N \geq 0}{\text{Maximize}} \; [a - b q_N - c_N] q_N - \frac{k_N}{2} [q_N]^2 - \frac{k^R}{2} [q_N]^2 \,. \tag{345}$$

(345) implies that R's profit-maximizing choice of $q_N > 0$ is determined by:

$$a - 2b q_N - c_N - k_N q_N - k^R q_N = 0 \;\Rightarrow\; q_N = \frac{a - c_N}{2b + k_N + k^R} \,. \tag{346}$$

(346) implies:

$$P(Q) = a - b \left[\frac{a - c_N}{2b + k_N + k^R} \right] = \frac{a \left[2b + k_N + k^R \right] - b [a - c_N]}{2b + k_N + k^R}$$

$$= \frac{a \left[b + k_N + k^R \right] + b c_N}{2b + k_N + k^R} \,. \tag{347}$$

(309) and (347) imply:

$$\overline{p}_{0M} < P(Q) \Leftrightarrow c_A + \frac{\left[b + k^R \right] [a - c_N]}{2b + k_N + k^R} < \frac{a \left[2b + k_N + k^R \right] - b [a - c_N]}{2b + k_N + k^R}$$

$$\Leftrightarrow \frac{\left[b + k^R \right] [a - c_N]}{2b + k_N + k^R} < \frac{[a - c_A] \left[2b + k_N + k^R \right] - b [a - c_N]}{2b + k_N + k^R}$$

$$\Leftrightarrow \frac{\left[2b + k^R \right] [a - c_N]}{2b + k_N + k^R} < \frac{[a - c_A] \left[2b + k_N + k^R \right]}{2b + k_N + k^R}$$

$$\Leftrightarrow \left[2b + k^R \right] [a - c_N] < [a - c_A] \left[2b + k_N + k^R \right] .$$

The last inequality here holds because $c_A \leq c_N < a$ and $k_N > 0$. Therefore, $\overline{p} < P(Q)$ when $\overline{p} \leq \overline{p}_{0M}$.

It remains to show that $q_A = 0$ when $\overline{p} \leq \overline{p}_{0M}$. Because $\overline{p} < P(Q)$ when $\overline{p} \leq \overline{p}_{0M}$, $q_A = 0$ if R's profit declines as q_A increases above 0, i.e., if:

$$\frac{\partial}{\partial q_A} \left\{ [\,\overline{p} - c_A\,]\, q_A + [\,a - b\,(q_A + q_N) - c_N\,]\, q_N - \frac{k_A}{2}\,[\,q_A\,]^2 - \frac{k_N}{2}\,[\,q_N\,]^2 - \frac{k^R}{2}\,[\,q_N + q_A\,]^2 \right\} \Bigg|_{q_A = 0} \leq 0$$

$$\Leftrightarrow \;\; \overline{p} - c_A - b\,q_N - k^R q_N \;\leq\; 0 \;\;\Leftrightarrow\;\; \overline{p} \;\leq\; c_A + \left[\, b + k^R \,\right] q_N$$

$$\Leftrightarrow \;\; \overline{p} \;\leq\; c_A + \frac{a - c_N}{2\,b + k_N + k^R} \left[\, b + k^R \,\right] \;=\; \overline{p}_{0M}\,. \tag{348}$$

The last inequality in (348) reflects (314). ■

To compare the effects of a price cap on R's revenue under monopoly and duopoly, suppose the maintained assumptions in both the monopoly and duopoly settings hold. Also let $Q^R_{dbM}(\overline{p})$ denote R's total output when $\overline{p} \in (\overline{p}_{dM}, \overline{p}_{bM}\,]$. Then (318) implies:

$$Q^R_{dbM}(\overline{p}) \;=\; \frac{a - \overline{p}}{b} \;\;\Rightarrow\;\; \frac{\partial Q^R_{dbM}(\overline{p})}{\partial \overline{p}} \;=\; -\,\frac{1}{b}\,. \tag{349}$$

Let $Q^R_{db}(\overline{p})$ denote R's total output when $\overline{p} \in (\overline{p}_d, \overline{p}_b\,]$ and R faces a rival. Recall from Lemma A3 that:

$$Q^R_{db}(\overline{p}) \;=\; \frac{a\,[\,b + k\,] + b\,c - \overline{p}\,[\,2\,b + k\,]}{b\,[\,b + k\,]} \;\;\Rightarrow\;\; \frac{\partial Q^R_{db}(\overline{p})}{\partial \overline{p}} \;=\; -\,\frac{2\,b + k}{b\,[\,b + k\,]}\,. \tag{350}$$

Conclusion 1. $\frac{\partial Q^R_{db}(\overline{p})}{\partial \overline{p}} < \frac{\partial Q^R_{dbM}(\overline{p}_M)}{\partial \overline{p}} < 0$ *when* $\overline{p}_M \in (\overline{p}_{dM}, \overline{p}_{bM}\,]$ *and* $\overline{p} \in (\overline{p}_d, \overline{p}_b\,]$.

<u>Proof</u>. (349) and (350) imply:

$$\frac{\partial Q^R_{dbM}(\overline{p})}{\partial \overline{p}} > \frac{\partial Q^R_{db}(\overline{p})}{\partial \overline{p}} \;\;\Leftrightarrow\;\; -\,\frac{1}{b} \;>\; -\,\frac{2\,b + k}{b\,[\,b + k\,]} \;\;\Leftrightarrow\;\; b + k \;<\; 2\,b + k \;\;\Leftrightarrow\;\; b \;>\; 0\,. \;\; ■$$

Define $V_{dmM}(\overline{p})$ to be R's revenue when $\overline{p} \in (\overline{p}_{dM}, \overline{p}_{bM}]$. Then (349) implies:

$$V_{dmM}(\overline{p}) \;=\; \overline{p}\; Q^R_{dbM}(\overline{p}) \;=\; \overline{p} \left[\frac{a - \overline{p}}{b} \right]. \tag{351}$$

(350) implies that when R faces a rival, R's revenue when $\overline{p} \in (\overline{p}_d, \overline{p}_b]$ is:

$$V_{dm}(\overline{p}) \;=\; \overline{p} \left[\frac{a\,(b + k) + b\,c - \overline{p}\,(2\,b + k)}{b\,(b + k)} \right]. \tag{352}$$

Conclusion 2. $\frac{\partial V_{dm}(\overline{p})}{\partial \overline{p}} < \frac{\partial V_{dmM}(\overline{p})}{\partial \overline{p}}$ *when* $\overline{p} \in (\overline{p}_{dM}, \overline{p}_{bM}]$ *and* $\overline{p} \in (\overline{p}_d, \overline{p}_b]$.

Proof. (351) implies:

$$\frac{\partial V_{dbM}(\overline{p})}{\partial \overline{p}} = \frac{a - 2\,\overline{p}}{b}. \tag{353}$$

(352) implies:

$$\frac{\partial V_{db}(\overline{p})}{\partial \overline{p}} = \frac{a\,[\,b+k\,] + b\,c - 2\,\overline{p}\,[\,2\,b+k\,]}{b\,[\,b+k\,]}. \tag{354}$$

(353) and (354) imply:

$$\frac{\partial V_{db}(\overline{p})}{\partial \overline{p}} < \frac{\partial V_{dbM}(\overline{p})}{\partial \overline{p}} \Leftrightarrow \frac{a\,[\,b+k\,] + b\,c - 2\,\overline{p}\,[\,2\,b+k\,]}{b\,[\,b+k\,]} < \frac{a - 2\,\overline{p}}{b}$$

$$\Leftrightarrow a\,[\,b+k\,] + b\,c - 2\,\overline{p}\,[\,2\,b+k\,] < [\,b+k\,]\,[\,a - 2\,\overline{p}\,]$$

$$\Leftrightarrow b\,c + \overline{p}\,[\,2\,(b+k) - 2\,(2\,b+k)\,] < 0 \Leftrightarrow b\,c < 2\,b\,\overline{p} \Leftrightarrow \overline{p} > \frac{c}{2}.$$

The last inequality here holds because, by assumption, $\overline{p} \geq \overline{p}_d > c$. ■

C. Welfare Analysis Involving Profit Rather than Revenue.

Now consider the duopoly setting where R's profit replaces R's revenue in the welfare function. Call this the $W - \Pi$ *setting*. Welfare in the $W - \Pi$ setting is:

$$W_{\Pi}(\overline{p}) \equiv S(\overline{p}) - r\,\Pi^R(\overline{p}) \tag{355}$$

where $r > 0$ is a parameter and $S(\cdot)$, which denotes consumer surplus, is:

$$S(\overline{p}) = a\,Q - \frac{b}{2}\,Q^2 - \overline{p}\,q_A - P(Q)\,[\,q_N + q\,]. \tag{356}$$

Definitions. $S_{0d}(\overline{p})$ is consumer surplus when $\overline{p} \in [\overline{p}_0, \overline{p}_d]$.

$S_{db}(\overline{p})$ is consumer surplus when $\overline{p} \in (\overline{p}_d, \overline{p}_b)$.

$\Pi^R_{0d}(\overline{p})$ is R's profit when $\overline{p} \in [\overline{p}_0, \overline{p}_d]$.

$\Pi^R_{db}(\overline{p})$ is R's profit when $\overline{p} \in (\overline{p}_d, \overline{p}_b)$. $\Pi^R_b(\overline{p}_b)$ is R's profit when $\overline{p} = \overline{p}_b$.

$W_{\Pi 0d}(\overline{p})$ is welfare when $\overline{p} \in [\overline{p}_0, \overline{p}_d]$.

$W_{\Pi db}(\overline{p})$ is welfare when $\overline{p} \in (\overline{p}_d, \overline{p}_b)$.

Lemma 8. *In the* $W - \Pi$ *setting:*

$$\frac{\partial^2 \Pi^R_{db}(\overline{p})}{\partial\,(\overline{p})^2} < 0\,, \quad \frac{\partial^2 S_{db}(\overline{p})}{\partial\,(\overline{p})^2} > 0\,, \quad \textit{and} \quad \frac{\partial^2 W_{\Pi db}(\overline{p})}{\partial\,(\overline{p})^2} > 0 \quad \textit{for } \overline{p} \in (\overline{p}_d, \overline{p}_b)\,. \tag{357}$$

Proof. Lemma A3 implies that when $\overline{p} \in (\overline{p}_d, \overline{p}_b)$, so $P(Q) = \overline{p}$:

$$Q = \frac{a - \overline{p}}{b} \quad \Rightarrow \quad \frac{\partial Q}{\partial \overline{p}} = -\frac{1}{b}. \tag{358}$$

(356) and (358) imply that for $\overline{p} \in (\overline{p}_d, \overline{p}_b)$, where $P(Q) = \overline{p}$:

$$\frac{\partial S_{db}(\overline{p})}{\partial \overline{p}} = a \frac{\partial Q}{\partial \overline{p}} - b\, Q \frac{\partial Q}{\partial \overline{p}} - Q - \overline{p} \frac{\partial Q}{\partial \overline{p}} = -\frac{a}{b} + Q - Q + \frac{\overline{p}}{b}$$

$$= -\frac{a - \overline{p}}{b} \quad \Rightarrow \quad \frac{\partial^2 S_{db}(\overline{p})}{\partial (\overline{p})^2} = \frac{1}{b} > 0. \tag{359}$$

(110) implies that R's revenue in the $W - \Pi$ setting when $\overline{p} \in (\overline{p}_d, \overline{p}_b)$ is:

$$\frac{\partial V_{dp}(\overline{p})}{\partial \overline{p}} = \frac{a\,[\,b + k\,] + b\,c - 2\,\overline{p}\,[\,2\,b + k\,]}{b\,[\,b + k\,]} \quad \Rightarrow \quad \frac{\partial^2 V_{db}(\overline{p})}{\partial (\overline{p})^2} = -\frac{2\,[\,2\,b + k\,]}{b\,[\,b + k\,]} < 0. \tag{360}$$

Let $C^R_{db}(\overline{p})$ denote R's total cost as a function of the price cap $\overline{p}$ when $\overline{p} \in (\overline{p}_d, \overline{p}_b)$. Then:

$$\frac{\partial C^R_{db}(\overline{p})}{\partial \overline{p}} = c_A \frac{\partial q_A}{\partial \overline{p}} + c_N \frac{\partial q_N}{\partial \overline{p}} + k_A\, q_A \frac{\partial q_A}{\partial \overline{p}} + k_N\, q_N \frac{\partial q_N}{\partial \overline{p}} + k^R\,[\,q_A + q_N\,] \left[\frac{\partial q_A}{\partial \overline{p}} + \frac{\partial q_N}{\partial \overline{p}} \right].$$

Lemma A3 implies that $\frac{\partial q_N}{\partial \overline{p}}$ and $\frac{\partial q_A}{\partial \overline{p}}$ do not vary with $\overline{p}$ when $\overline{p} \in (\overline{p}_d, \overline{p}_b)$. Therefore:

$$\frac{\partial^2 C^R_{db}(\overline{p})}{\partial (\overline{p})^2} = k_A \left[\frac{\partial q_A}{\partial \overline{p}} \right]^2 + k_N \left[\frac{\partial q_N}{\partial \overline{p}} \right]^2 + k^R \left[\frac{\partial q_A}{\partial \overline{p}} + \frac{\partial q_N}{\partial \overline{p}} \right]^2 \geq 0. \tag{361}$$

(360) and (361) imply:

$$\frac{\partial^2 \Pi^R_{db}(\overline{p})}{\partial (\overline{p})^2} = \frac{\partial^2 V_{db}(\overline{p})}{\partial (\overline{p})^2} - \frac{\partial^2 C^R_{db}(\overline{p})}{\partial (\overline{p})^2} < 0. \tag{362}$$

(355), (359), and (362) imply:

$$\frac{\partial^2 W_{\Pi db}(\overline{p})}{\partial (\overline{p})^2} = \frac{\partial^2 S_{db}(\overline{p})}{\partial (\overline{p})^2} - r\, \frac{\partial^2 \Pi^R_{db}(\overline{p})}{\partial (\overline{p})^2} > 0. \;\; ■$$

Lemma 9. $\Pi^R_{0d}(\overline{p}_0) < \Pi^R_b(\overline{p}_b)$ *in the* $W - \Pi$ *setting.*

Proof. Lemma A1 implies that $Q^R(\overline{p}_0) = q_N(\overline{p}_0)$ because $q_A(\overline{p}_0) = 0$.

Definition. $D_N \equiv \left[\,2\,b + k_N + k^R\,\right][\,2\,b + k\,] - b^2$. (363)

(363) and Lemmas A1 and A3 imply:

$$Q^R(\overline{p}_0) = \frac{[\,a - c_N\,]\,[\,2\,b + k\,] - b\,[\,a - c\,]}{D_N} < \frac{[\,b + k\,]\,[\,a - \overline{p}_b\,] - b\,[\,\overline{p}_b - c\,]}{b\,[\,b + k\,]} = Q^R(\overline{p}_b)$$

$$\Leftrightarrow \quad \frac{a\,[\,b+k\,]+a\,b-c_N\,[\,2\,b+k\,]-b\,a+b\,c}{D_N} \;<\; \frac{[\,b+k\,]\,a-[\,b+k\,]\,\overline{p}_b-b\,\overline{p}_b+b\,c}{b\;[\,b+k\,]}$$

$$\Leftrightarrow \quad \frac{a\,[\,b+k\,]+b\,c-c_N\,[\,2\,b+k\,]}{D_N} \;<\; \frac{[\,b+k\,]\,a+b\,c-[\,2\,b+k\,]\,\overline{p}_b}{b\;[\,b+k\,]}$$

$$\Leftrightarrow \quad \frac{a\,[\,b+k\,]+b\,c-c_N\,[\,2\,b+k\,]}{D_N}\;b\,[\,b+k\,]-[\,b+k\,]\,a-b\,c \;<\; -\,[\,2\,b+k\,]\,\overline{p}_b$$

$$\Leftrightarrow \quad \frac{a\,[\,b+k\,]+b\,c}{2\,b+k}-\frac{a\,[\,b+k\,]+b\,c-c_N\,[\,2\,b+k\,]}{[\,2\,b+k\,]\,D_N}\;b\,[\,b+k\,] \;>\; \overline{p}_b$$

$$\Leftrightarrow \quad \frac{a\,[\,b+k\,]+b\,c}{2\,b+k}-\frac{[\,a\,(b+k)+b\,c\,]\,b\;[\,b+k\,]-c_N\,[\,2\,b+k\,]\,b\;[\,b+k\,]}{[\,2\,b+k\,]\,D_N} \;>\; \overline{p}_b$$

$$\Leftrightarrow \quad \frac{1}{[\,2\,b+k\,]\,D_N}\,\Big\{\,[\,a\,(b+k)+b\,c\,]\,\Big[\,\big(2\,b+k_N+k^R\big)\,(2\,b+k)-b^2-b\;(b+k)\,\Big]$$
$$+\,c_N\,[\,2\,b+k\,]\,b\;[\,b+k\,]\,\Big\} \;>\; \overline{p}_b$$

$$\Leftrightarrow \quad \frac{1}{[\,2\,b+k\,]\,D_N}\,\Big\{\,[\,a\,(b+k)+b\,c\,]\,\Big[\,\big(2\,b+k_N+k^R\big)\,(2\,b+k)-b\,(2\,b+k)\,\Big]$$
$$+\,c_N\,[\,2\,b+k\,]\,b\;[\,b+k\,]\,\Big\} \;>\; \overline{p}_b$$

$$\Leftrightarrow \quad \frac{[\,a\,(b+k)+b\,c\,]\,\big[\,2\,b+k_N+k^R-b\,\big]+c_N\,b\,[\,b+k\,]}{D_N} \;>\; \overline{p}_b$$

$$\Leftrightarrow \quad \frac{[\,a\,(b+k)+b\,c\,]\,\big[\,b+k_N+k^R\,\big]+c_N\,b\,[\,b+k\,]}{D_N} \;>\; \overline{p}_b\,. \tag{364}$$

As established in the proof of Proposition 4, just below (149), $\overline{p}_b$ is increasing in k_A. Therefore, (8) implies that when $k_A \leq k_N$:

$$\overline{p}_b \;\leq\; \frac{[\,a\,(b+k)+b\,c\,]\,\Big[\,2\,k_N\,\big(b+k^R\big)+(k_N)^2\,\Big]+2\,b\,c_N\,[\,b+k\,]\,k_N}{2\,b\,[\,b+k\,]\,k_N\;+(k_N)^2\;[\,2\,b+k\,]+2\,k_N\,[\,2\,b+k\,]\,[\,b+k^R\,]}$$

$$=\; \frac{[\,a\,(b+k)+b\,c\,]\,\Big[\,2\,\big(b+k^R\big)+k_N\,\Big]+2\,b\,c_N\,[\,b+k\,]}{2\,b\;[\,b+k\,]+k_N\;[\,2\,b+k\,]+2\;[\,2\,b+k\,]\,[\,b+k^R\,]}$$

$$=\; \frac{[\,a\,(b+k)+b\,c\,]\,\Big[\,b+k^R+\frac{k_N}{2}\,\Big]+b\,c_N\,[\,b+k\,]}{b\,[\,b+k\,]+\frac{k_N}{2}\;[\,2\,b+k\,]+[\,2\,b+k\,]\,[\,b+k^R\,]}$$

$$=\; \frac{[\,a\,(b+k)+b\,c\,]\,\Big[\,b+k^R+\frac{k_N}{2}\,\Big]+b\,c_N\,[\,b+k\,]}{[\,2\,b+k\,]\,\Big[\,b+k^R+\frac{k_N}{2}\,\Big]+b\,[\,b+k\,]}$$

$$=\; \frac{[\,a\,(b+k)+b\,c\,]\,\Big[\,b+k^R+\frac{k_N}{2}\,\Big]+b\,c_N\,[\,b+k\,]}{[\,2\,b+k\,]\,\Big[\,2\,b+k^R+\frac{k_N}{2}\,\Big]-b^2}\,. \tag{365}$$

The last equality in (365) holds because:

$$[\,2\,b + k\,]\left[b + k^R + \frac{k_N}{2}\right] + b\,[\,b + k\,] \;=\; [\,2\,b + k\,]\left[2\,b + k^R + \frac{k_N}{2}\right] - b\,[\,2\,b + k\,] + b\,[\,b + k\,]$$

$$=\; [\,2\,b + k\,]\left[2\,b + k^R + \frac{k_N}{2}\right] - 2\,b^2 - b\,k + b^2 + b\,k \;=\; [\,2\,b + k\,]\left[2\,b + k^R + \frac{k_N}{2}\right] - b^2.$$

(363), (364), and (365) imply that $Q^R(\overline{p}_0) \; < Q^R(\overline{p}_b)$ if:

$$\frac{[\,a\,(b + k) + b\,c\,]\left[\,b + k^R + \frac{k_N}{2}\,\right] + b\,c_N\,[\,b + k\,]}{[\,2\,b + k\,]\left[\,2\,b + k^R + \frac{k_N}{2}\,\right] - b^2}$$

$$< \; \frac{[\,a\,(b + k) + b\,c\,]\left[\,b + k^R + k_N\,\right] + b\,c_N\,[\,b + k\,]}{[\,2\,b + k\,]\,[\,2\,b + k^R + k_N\,] - b^2}\,. \tag{366}$$

The proof of Lemma A7 establishes that this inequality holds, so $Q^R(\overline{p}_0) \; < Q^R(\overline{p}_b)$.

(12) implies that the rival's output q is determined by:

$$a - b\left[\,Q^R(\,\overline{p}\,) + q(\,\overline{p}\,)\,\right] - c - b\,q(\,\overline{p}\,) - k\,q(\,\overline{p}\,) \;=\; 0 \;\;\Rightarrow\;\; q(\,\overline{p}\,) \;=\; \frac{a - b\,Q^R(\,\overline{p}\,) - c}{2\,b + k}\,. \tag{367}$$

Because $Q^R(\overline{p}_0) < Q^R(\overline{p}_b)$, (367) implies that $q(\overline{p}_0) > q(\overline{p}_b)$. R's profit when $\overline{p} = \overline{p}_0$ is:

$$\begin{aligned}
\Pi^R_{0d}(\overline{p}_0) \;&=\; q_N(\overline{p}_0)\,[\,P(q_N(\overline{p}_0) + q(\overline{p}_0)) - c_N\,] - \frac{k_N}{2}\,[\,q_N(\overline{p}_0)\,]^2 - \frac{k^R}{2}\,[\,q_N(\overline{p}_0)\,]^2 \\
&<\; q_N(\overline{p}_0)\,[\,P(q_N(\overline{p}_0) + q(\overline{p}_b)\,) - c_N\,] - \frac{k_N}{2}\,[\,q_N(\overline{p}_0)\,]^2 - \frac{k^R}{2}\,[\,q_N(\overline{p}_0)\,]^2 \\
&\leq\; \max_{q_N\,\geq\,0,\,q_A\,\geq\,0} \;\{q_N\,[\,P(q_N + q + q_A) - c_N\,] + q_A\,[\,P(q_N + q + q_A) - c_A\,] \\
&\qquad - \frac{k_A}{2}\,[\,q_A\,]^2 - \frac{k_N}{2}\,[\,q_N\,]^2 - \frac{k^R}{2}\,[\,q_A + q_N\,]^2\,\} \;=\; \Pi^R_b(\overline{p}_b)\,. \;\blacksquare
\end{aligned}$$

Definition. $\overline{p}^*_\Pi \;\equiv\; \arg\max\, W_\Pi(\,\overline{p}\,)\,.$

Conclusion 3. $\overline{p}^*_\Pi \;\in\; [\,\overline{p}_0,\,\overline{p}_d\,]\,.$

Proof. Lemma 8 implies that $\overline{p}^*_\Pi \notin (\overline{p}_d, \overline{p}_b)$. Equilibrium outcomes do not vary with $\overline{p}$ when $\overline{p} < \overline{p}_0$ (because $q_A = 0$ for all such $\overline{p}$). Similarly, equilibrium outcomes do not vary with $\overline{p}$ when $\overline{p} > \overline{p}_b$ (because the price cap does not bind for all such $\overline{p}$). Therefore: (i) welfare does not vary with $\overline{p}$ when $\overline{p} \leq \overline{p}_0$; and (ii) welfare does not vary with $\overline{p}$ when $\overline{p} \geq \overline{p}_b$. Consequently, $\overline{p}^*_\Pi \in [\,\overline{p}_0, \overline{p}_d\,] \bigcup \overline{p}_b\,.$

It remains to show that $\overline{p}^*_\Pi \neq \overline{p}_b$. The proof of Lemma 9 establishes that:

$$Q^R(\overline{p}_0) \;<\; Q^R(\overline{p}_b) \tag{368}$$

where $Q^R(\,\overline{p}\,)$ is R's total output when the price cap is $\overline{p}$. Proposition 2 implies:

$$Q^R(\overline{p}_b) \;<\; Q^R(\overline{p}_d)\,. \tag{369}$$

(368) and (369) imply that $Q^R(\overline{p}_0) < Q^R(\overline{p}_b) < Q^R(\overline{p}_d)$. Lemma A2 implies that $Q^R(\overline{p})$ is continuous and monotonically increasing in $\overline{p}$ for $\overline{p} \in (\overline{p}_0, \overline{p}_d)$. Therefore, the Intermediate Value Theorem implies that there exists a $\overline{p}_E \in (\overline{p}_0, \overline{p}_d)$ such that:

$$Q^R(\overline{p}_E) \;=\; Q^R(\overline{p}_b)\,. \tag{370}$$

(12) implies that the rival's output q is determined by:

$$a - b\left[\, Q^R(\overline{p}) + q(\overline{p}\,)\,\right] - c - b\, q(\overline{p}) - k\, q(\overline{p}\,) \;=\; 0\,. \tag{371}$$

(370) and (371) imply:

$$q(\overline{p}_E) \;=\; q(\overline{p}_b)\,. \tag{372}$$

(370) and (372) imply:

$$Q(\overline{p}_E) \;=\; Q(\overline{p}_b) \;\text{ and }\; P(Q(\overline{p}_E)\,) \;=\; P(Q(\overline{p}_b)\,)\,. \tag{373}$$

Observe that:

$$\begin{aligned}
\Pi^R_{0d}(\overline{p}_E) \;&=\; \overline{p}_E\, q_A(\overline{p}_E) + P(Q(\overline{p}_E)\,)\, q_N(\overline{p}_E) - C^R(q_A(\overline{p}_E), q_N(\overline{p}_E)\,) \\
&<\; P(Q(\overline{p}_E)\,)\, q_A(\overline{p}_E) + P(Q(\overline{p}_E)\,)\, q_N(\overline{p}_E) - C^R(q_A(\overline{p}_E), q_N(\overline{p}_E)\,) \\
&=\; P(Q(\overline{p}_b)\,)\, q_A(\overline{p}_b) + P(Q(\overline{p}_b)\,)\, q_N(\overline{p}_b) - C^R(q_A(\overline{p}_b), q_N(\overline{p}_b)\,) \;=\; \Pi^R_b(\overline{p}_b).
\end{aligned} \tag{374}$$

The inequality in (374) holds because $\overline{p}_E < P(Q(\overline{p}_E)\,)$, since $\overline{p}_E \in (\overline{p}_0, \overline{p}_d)$. The penultimate equality in (374) reflects (373). The last equality in (374) holds because $P(Q(\overline{p}_b)\,) = \overline{p}_b$.

(356) and (373) imply:

$$\begin{aligned}
S_{0d}(\overline{p}_E) \;&=\; a\, Q(\overline{p}_E) - \frac{b}{2}\, Q(\overline{p}_E)^2 - P(Q(\overline{p}_E)\,)\, \left[\, q(\overline{p}_E) + q_N(\overline{p}_E)\,\right] - \overline{p}_E\, q_A(\overline{p}_E) \\
&>\; a\, Q(\overline{p}_E) - \frac{b}{2}\, Q(\overline{p}_E)^2 - P(Q(\overline{p}_E)\,)\, \left[\, q(\overline{p}_E) + q_N(\overline{p}_E) + q_A(\overline{p}_E)\,\right] \\
&=\; a\, Q(\overline{p}_E) - \frac{b}{2}\, Q(\overline{p}_b)^2 - P(Q(\overline{p}_E)\,)\, Q(\overline{p}_E) \\
&=\; a\, Q(\overline{p}_E) - \frac{b}{2}\, Q(\overline{p}_b)^2 - P\left(Q(\overline{p}_b)\right)\, Q(\overline{p}_b) \;=\; S(\overline{p}_b)\,.
\end{aligned} \tag{375}$$

The inequality in (375) holds because $\overline{p}_E < P\left(Q(\overline{p}_E)\,\right)$, since $\overline{p}_E \in (\overline{p}_0, \overline{p}_d)$. (374) and (375) imply that consumer surplus is higher and R's profit is lower in the $W - \Pi$ setting when $\overline{p} = \overline{p}_E$ than when $\overline{p} = \overline{p}_b$. Therefore, welfare is strictly greater when $\overline{p} = \overline{p}_E$ than when $\overline{p} = \overline{p}_b$, so $\overline{p}^{\,*}_{\Pi} \neq \overline{p}_b$. ■

D. Benchmark Setting with Exogenous Prices.

Finally, consider the benchmark *exogenous price setting* in which the price of output supplied using A's input is set (exogenously) at $\overline{p}_A$ and the price of output supplied without using A's input is set (exogenously) at $\overline{p}_N$.

Conclusion 4. *R reduces both q_A and $Q^R = q_A + q_N$ as $\overline{p}_A$ declines in the exogenous price setting.*

Proof. In the exogenous price setting, R chooses q_A and q_N to:

$$\text{Maximize } \; \overline{p}_A \, q_A + \overline{p}_N \, q_N - c_A \, q_A - \frac{k_A}{2} [\, q_A \,]^2 - c_N \, q_N - \frac{k_N}{2} [\, q_N \,]^2 - \frac{k^R}{2} [\, q_A + q_N \,]^2 .$$

The necessary conditions for a solution to this problem, [P-E], are:

$$\overline{p}_A - c_A - k_A \, q_A - k^R [\, q_A + q_N \,] \; \leq \; 0 \qquad q_A [\, \cdot \,] \; = \; 0 \, ;$$

$$\overline{p}_N - c_N - k_N \, q_N - k^R [\, q_A + q_N \,] \; \leq \; 0 \qquad q_N [\, \cdot \,] \; = \; 0 \, . \tag{376}$$

(376) implies that if $q_A = 0$ and $q_N > 0$ at the solution to [P-E]:

$$\overline{p}_N - c_N - k_N \, q_N - k^R \, q_N \;\; = \;\; 0 \;\; \Rightarrow \;\; q_N \;\; = \;\; \frac{\overline{p}_N - c_N}{k_N + k^R}$$

$$\Rightarrow \;\; \frac{\partial q_N}{\partial \overline{p}_N} \;\; = \;\; \frac{1}{k_N + k^R} \;\; > \;\; 0 \;\;\; \text{and} \;\;\; \frac{\partial q_N}{\partial \overline{p}_A} \;\; = \;\; 0 \, .$$

(376) also implies that if $q_N = 0$ and $q_A > 0$ at the solution to [P-E]:

$$\overline{p}_A - c_A - k_A \, q_A - k^R \, q_A \;\; = \;\; 0 \;\; \Rightarrow \;\; q_A \;\; = \;\; \frac{\overline{p}_A - c_A}{k_A + k^R}$$

$$\Rightarrow \;\; \frac{\partial q_A}{\partial \overline{p}_A} \;\; = \;\; \frac{1}{k_A + k^R} \;\; > \;\; 0 \;\;\; \text{and} \;\;\; \frac{\partial q_A}{\partial \overline{p}_N} \;\; = \;\; 0 \, .$$

(376) further implies that if $q_A > 0$ and $q_N > 0$ at the solution to [P-E]:

$$\overline{p}_A - c_A - k_A \, q_A \;\; = \;\; k^R [\, q_A + q_N \,] \;\;\; \Rightarrow \;\;\; q_A \left[\, k_A + k^R \,\right] \;\; = \;\; \overline{p}_A - c_A - k^R \, q_N$$

$$\Rightarrow \;\; q_A \;\; = \;\; \frac{\overline{p}_A - c_A - k^R \, q_N}{k_A + k^R} \, ; \;\; \text{and} \tag{377}$$

$$\overline{p}_N - c_N - k_N \, q_N \;\; = \;\; k^R [\, q_A + q_N \,] \;\;\; \Rightarrow \;\;\; q_N \left[\, k_N + k^R \,\right] \;\; = \;\; \overline{p}_N - c_N - k^R \, q_A$$

$$\Rightarrow \;\; q_N \;\; = \;\; \frac{\overline{p}_N - c_N - k^R \, q_A}{k_N + k^R} \, . \tag{378}$$

(377) and (378) imply:

$$q_N \;\; = \;\; \frac{\overline{p}_N - c_N}{k_N + k^R} - \frac{k^R}{k_N + k^R} \left[\frac{\overline{p}_A - c_A - k^R \, q_N}{k_A + k^R} \right]$$

$$\Rightarrow \quad q_N \left[1 - \frac{\left(k^R\right)^2}{(k_A + k^R)\,(k_N + k^R)} \right] \;=\; \frac{\overline{p}_N - c_N}{k_N + k^R} - \frac{k^R\,[\,\overline{p}_A - c_A\,]}{[\,k_A + k^R\,]\,[\,k_N + k^R\,]}$$

$$\Rightarrow \quad q_N \left[\left(k_A + k^R\right) \left(k_N + k^R\right) - \left(k^R\right)^2 \right] \;=\; [\,\overline{p}_N - c_N\,] \left[\,k_A + k^R\,\right] - k^R\,[\,\overline{p}_A - c_A\,]$$

$$\Rightarrow \quad q_N \;=\; \frac{[\,\overline{p}_N - c_N\,] \left[\,k_A + k^R\,\right] - k^R\,[\,\overline{p}_A - c_A\,]}{[\,k_A + k^R\,]\,[\,k_N + k^R\,] - (k^R)^2}\,. \tag{379}$$

(377) and (379) imply:

$$q_A \;=\; \frac{\overline{p}_A - c_A}{k_A + k^R} - \frac{k^R}{k_A + k^R} \left\{ \frac{[\,\overline{p}_N - c_N\,] \left[\,k_A + k^R\,\right] - k^R\,[\,\overline{p}_A - c_A\,]}{[\,k_A + k^R\,]\,[\,k_N + k^R\,] - (k^R)^2} \right\}$$

$$= \frac{1}{[\,k_A + k^R\,] \left\{ [\,k_A + k^R\,]\,[\,k_N + k^R\,] - (k^R)^2 \right\}}$$

$$\cdot \left\{ [\,\overline{p}_A - c_A\,] \left\{ \left[\,k_A + k^R\,\right] \left[\,k_N + k^R\,\right] - \left(k^R\right)^2 \right\} - k^R\,[\,\overline{p}_N - c_N\,] \left[\,k_A + k^R\,\right] + \left(k^R\right)^2 [\,\overline{p}_A - c_A\,] \right\}$$

$$= \frac{[\,\overline{p}_A - c_A\,] \left[\,k_A + k^R\,\right] \left[\,k_N + k^R\,\right] - k^R\,[\,\overline{p}_N - c_N\,] \left[\,k_A + k^R\,\right]}{[\,k_A + k^R\,] \left\{ [\,k_A + k^R\,]\,[\,k_N + k^R\,] - (k^R)^2 \right\}}$$

$$= \frac{[\,\overline{p}_A - c_A\,] \left[\,k_N + k^R\,\right] - k^R\,[\,\overline{p}_N - c_N\,]}{[\,k_A + k^R\,]\,[\,k_N + k^R\,] - (k^R)^2}\,. \tag{380}$$

(379) and (380) imply:

$$\frac{\partial q_A}{\partial \overline{p}_A} \;=\; \frac{k_N + k^R}{[\,k_A + k^R\,]\,[\,k_N + k^R\,] - (k^R)^2} \;>\; 0\,, \;\; \text{and}$$

$$\frac{\partial\,(q_A + q_N)}{\partial \overline{p}_A} \;=\; \frac{k_N + k^R - k^R}{[\,k_A + k^R\,]\,[\,k_N + k^R\,] - (k^R)^2}$$

$$= \frac{k_N}{[\,k_A + k^R\,]\,[\,k_N + k^R\,] - (k^R)^2} \;>\; 0\,. \;\; \blacksquare \tag{381}$$

E. The *Simplified Cost Setting.*

<u>Definitions</u>.

$$\overline{p}_2 \;\equiv\; \frac{b\,c_A\,[\,b + k\,] + a\,[\,b + k\,]\,k^R + b\,c\,k^R}{[\,2\,b + k\,]\,[\,b + k^R\,] - b^2}\,.$$

$$\overline{p}_3 \;\equiv\; \frac{a\left[\,b+k^R\,\right][\,b+k\,]+b\,c_A\,[\,b+k\,]+b\,c\left[\,b+k^R\,\right]}{[\,2\,b+k^R\,]\,[\,2\,b+k\,]-b^2}. \tag{382}$$

Assumptions

1. $C^R(q_A, q_N) \;=\; c_A\, q_A + c_N\, q_N + \frac{k^R}{2}\,[\,q_A + q_N\,]^2$, where $k^R > 0$.
2. $c_N \;>\; c_A \;>\; 0\,.$
3. $a \;>\; \max\{\,c, c_N\,\}\,.$
4. $[\,a - c_A\,]\,[\,2\,b + k\,] \;>\; b\,[\,a - c\,]$.
5. $\overline{p}_2 \;>\; c\,.$
6. All other model features are as specified in the main analysis.

Lemma 10. $\overline{p}_3 \,>\, \overline{p}_2$.

Proof. (382) implies that the lemma holds if:

$$\left\{a\,[\,b+k\,]\left[\,b+k^R\,\right]+b\,c_A\,[\,b+k\,]+b\,c\left[\,b+k^R\,\right]\right\}\left\{[\,2\,b+k\,]\left[\,b+k^R\,\right]-b^2\right\}$$
$$>\;\left\{a\,[\,b+k\,]\,k^R+b\,c_A\,[\,b+k\,]+b\,c\,k^R\right\}\left\{\left[\,2\,b+k^R\,\right][\,2\,b+k\,]-b^2\right\}$$

$$\Leftrightarrow\;\; a\,[\,b+k\,]\,[\,2\,b+k\,]\left[\,b+k^R\,\right]^2 - a\,b^2\,[\,b+k\,]\left[\,b+k^R\,\right]$$
$$+\,b\,c_A\,[\,b+k\,]\,[\,2\,b+k\,]\left[\,b+k^R\,\right]-b^3\,c_A\,[\,b+k\,]$$
$$+\;b\,c\,[\,2\,b+k\,]\left[\,b+k^R\,\right]^2-b^3\,c\left[\,b+k^R\,\right]$$
$$>\;a\,[\,b+k\,]\,[\,2\,b+k\,]\,k^R\left[\,2\,b+k^R\,\right]-a\,b^2\,[\,b+k\,]\,k^R$$
$$+\,b\,c_A\,[\,b+k\,]\,[\,2\,b+k\,]\left[\,2\,b+k^R\,\right]-b^3\,c_A\,[\,b+k\,]$$
$$+\;b\,c\,[\,2\,b+k\,]\,k^R\left[\,2\,b+k^R\,\right]-b^3\,c\,k^R$$

$$\Leftrightarrow\;\; a\,[\,b+k\,]\,[\,2\,b+k\,]\left[b^2+2\,b\,k^R+\left(k^R\right)^2-2\,b\,k^R-\left(k^R\right)^2\right]-a\,b^3\,[\,b+k\,]$$
$$-\,b^2\,c_A\,[\,b+k\,]\,[\,2\,b+k\,]$$
$$+\;b\,c\,[\,2\,b+k\,]\left[b^2+2\,b\,k^R+\left(k^R\right)^2-2\,b\,k^R-\left(k^R\right)^2\right]-b^4\,c\;>\;0$$

$$\Leftrightarrow\;\; a\,b^2\,[\,b+k\,]\,[\,2\,b+k\,]-a\,b^3\,[\,b+k\,]-\,b^2\,c_A\,[\,b+k\,]\,[\,2\,b+k\,]$$
$$+\,b^3\,c\,[\,2\,b+k\,]-b^4\,c\;>\;0$$

$$\Leftrightarrow \quad a\,b^2\,[\,b+k\,]\,[\,2\,b+k-b\,] + b^2\,c_A\,[\,2\,b+k\,]\,[\,b-(b+k)\,] - b^4\,c \;>\; 0$$

$$\Leftrightarrow \quad a\,b^2\,[\,b+k\,]^2 - b^2\,k\,c_A\,[\,2\,b+k\,] - b^4\,c \;>\; 0$$

$$\Leftrightarrow \quad a\,b^2\left[\,b^2+2\,b\,k+k^2\,\right] - 2\,b^3\,k\,c_A - b^2\,k^2\,c_A - b^4\,c \;>\; 0$$

$$\Leftrightarrow \quad a\,b^4 + 2\,a\,b^3\,k + a\,b^2\,k^2 - 2\,b^3\,k\,c_A - b^2\,k^2\,c_A - b^4\,c \;>\; 0$$

$$\Leftrightarrow \quad [\,a-c\,]\,b^4 + 2\,b^3\,k\,[\,a-c_A\,] + b^2\,k^2\,[\,a-c_A\,] \;>\; 0\,.$$

The last inequality here holds because $a > c$ and $a > c_A$, by assumption. ■

Conclusion 5. $q_A = 0$ *or* $q_A = 0$ *in equilibrium in the simplified cost setting.*

Proof. Let $\Pi^R(q_a, q_n)$ denote R's profit when it supplies q_a units using A's input and q_n units without using this input. We will show in three distinct cases that if R is initially supplying strictly positive amounts of output both using A's input and not using A's input (so $q_A > 0$ and $q_N > 0$), then R can strictly increase its profit by setting $q_A = 0$ or $q_N = 0$.

Case 1. $P_A(Q) - c_A \;>\; P(Q) - c_N$. If $q_A > 0$ and $q_N > 0$ in this case, then:

$$\Pi^R(q_A, q_N) \;=\; [\,P_A(Q) - c_A\,]\,q_A + [\,P(Q) - c_N\,]\,q_N - \frac{k^R}{2}\,[\,q_A + q_N\,]^2 \tag{383}$$

$$<\; [\,P_A(Q) - c_A\,]\,[\,q_A + q_N\,] - \frac{k^R}{2}\,[\,q_A + q_N\,]^2 \;=\; \Pi^R(q_A + q_N, 0)\,. \tag{384}$$

(384) implies that R could increase its profit by selling its entire output, $Q^R = q_A + q_N$, using A's input, and supplying no output without using the input.

Case 2. $P_A(Q) - c_A \;<\; P(Q) - c_N$. If $q_A > 0$ and $q_N > 0$ in this case, then:

$$\Pi^R(q_A, q_N) \;=\; [\,P_A(Q) - c_A\,]\,q_A + [\,P(Q) - c_N\,]\,q_N - \frac{k^R}{2}\,[\,q_A + q_N\,]^2$$

$$<\; [\,P(Q) - c_N\,]\,[\,q_A + q_N\,] - \frac{k^R}{2}\,[\,q_A + q_N\,]^2 \;=\; \Pi^R(0, q_A + q_N)\,. \tag{385}$$

(385) implies that R could increase its profit by selling its entire output, $Q^R = q_A + q_N$, without using A's input, and supplying no output using the input.

Case 3. $P_A(Q) - c_A \;=\; P(Q) - c_N$. In this case:

$$P(Q) - P_A(Q) \;=\; c_N - c_A \;>\; 0 \;\;\Rightarrow\;\; P_A(Q) \;<\; P(Q) \;\;\Rightarrow\;\; P_A(Q) \;=\; \overline{p}\,. \tag{386}$$

(386) implies that if $q_A > 0$ and $q_N > 0$ in this case, then:

$$\frac{\partial \Pi^R(q_A, q_N)}{\partial q_A} \;=\; -\,b\,q_N + \overline{p} - c_A - k^R\,[\,q_A + q_N\,] \;=\; 0$$

$$\Rightarrow\;\; \overline{p} - c_A - k^R\,[\,q_A + q_N\,] \;=\; b\,q_N \;>\; 0\,. \tag{387}$$

(386) implies that in this case:

$$\begin{aligned}\Pi^R(q_A, q_N) &= [\, P_A(Q) - c_A\,]\, q_A + [\, P(Q) - c_N\,]\, q_N - \frac{k^R}{2}\,[\, q_A + q_N\,]^2 \\ &= [\, P_A(Q) - c_A\,]\,[\, q_A + q_N\,] - \frac{k^R}{2}\,[\, q_A + q_N\,]^2 \\ &= [\, \overline{p} - c_A\,]\,[\, q_A + q_N\,] - \frac{k^R}{2}\,[\, q_A + q_N\,]^2 \;=\; \Pi^R(q_A + q_N, 0)\,. \end{aligned} \tag{388}$$

(388) implies that R would secure the same profit if it sold its entire output, $Q^R = q_A + q_N$, using A's input. (388) also implies:

$$\frac{\partial \Pi^R(q_A + q_N, 0)}{\partial q_A} \;=\; \overline{p} - c_A - k^R\,[\, q_A + q_N\,] \;>\; 0\,. \tag{389}$$

The inequality in (389) reflects (387). (389) implies that R could increase its profit by marginally increasing its total output and selling the entire output using A's input. ■

Lemma 11. *Suppose $\overline{p} \geq \overline{p}_3$. Then in equilibrium in the simplified cost setting:*

$$\overline{p} \;\geq\; P(Q), \;\; q^* \;=\; \frac{[\, a - c\,]\,\left[\, 2\,b + k^R\,\right] - b\,[\, a - c_A\,]}{[\, 2\,b + k^R\,]\,[\, 2\,b + k\,] - b^2}\,, \;\; q_N^* \;=\; 0\,, \; \textit{and}$$

$$q_A^* \;=\; \frac{[\, a - c_A\,]\,[\, 2\,b + k\,] - b\,[\, a - c\,]}{[\, 2\,b \,+ k^R\,]\,[\, 2\,b + k\,] - b^2}\,. \tag{390}$$

Proof. Consider a putative equilibrium in which outputs are as specified in (390). Then in this equilibrium, for $Q^* = q^* + q_A^* + q_N^*$:

$$\begin{aligned} P(Q^*) \;&=\; a - b\,[\, q_A^* + q_N^* + q^*\,] \;=\; a - b\, q_A^* - b\, q^* \\ &= a - b\left[\frac{(\, a - c_A\,)\,(2\,b + k) - b\,(a - c)}{(2\,b + k^R)\,(2\,b + k) - b^2}\right] - b\left[\frac{(a - c)\,\left(2\,b + k^R\right) - b\,(a - c_A)}{(2\,b + k^R)\,(2\,b + k) - b^2}\right] \\ &= a - b\left[\frac{(a - c_A)\,(2\,b + k) - b\,(a - c) + (a - c)\,\left(2\,b + k^R\right) - b\,(a - c_A)}{(2\,b + k^R)\,(2\,b + k) - b^2}\right] \\ &= a - b\left[\frac{(a - c_A)\,(b + k) + (a - c)\,\left(b + k^R\right)}{(2\,b + k^R)\,(2\,b + k) - b^2}\right] \\ &= \frac{a\left[\,\left(2\,b + k^R\right)(2\,b + k) - b^2\,\right] - b\,[\, a - c_A\,]\,[\, b + k\,] - b\,[\, a - c\,]\left[\, b + k^R\,\right]}{[\, 2\,b + k^R\,]\,[\, 2\,b + k\,] - b^2} \\ &= \frac{a\left[\,\left(2\,b + k^R\right)(2\,b + k) - b^2 - b\,(b + k) - b\left(b + k^R\right)\,\right] + b\, c_A\,[\, b + k\,] + b\, c\left[\, b + k^R\,\right]}{[\, 2\,b + k^R\,]\,[\, 2\,b + k\,] - b^2} \end{aligned}$$

$$= \frac{a\left[\left(2b+k^R\right)(2b+k) - b(2b+k) - b\left(b+k^R\right)\right] + b\,c_A\,[\,b+k\,] + b\,c\left[\,b+k^R\,\right]}{[\,2b+k^R\,]\,[\,2b+k\,] - b^2}$$

$$= \frac{a\left[\left(b+k^R\right)(2b+k) - b\left(b+k^R\right)\right] + b\,c_A\,[\,b+k\,] + b\,c\left[\,b+k^R\,\right]}{[\,2b+k^R\,]\,[\,2b+k\,] - b^2}$$

$$= \frac{a\left[\,b+k^R\,\right][\,b+k\,] + b\,c_A\,[\,b+k\,] + b\,c\left[\,b+k^R\,\right]}{[\,2b+k^R\,]\,[\,2b+k\,] - b^2} = \overline{p}_3\,. \tag{391}$$

The last inequality in (391) reflects (382). (391) implies that $\overline{p} \geq P(Q^*)$ in the putative equilibrium when $\overline{p} \geq \overline{p}_3$.

We now establish that R cannot increase its profit by deviating unilaterally to supply q_A and q_N that differ from their values in (390). (383) and (391) imply that R's profit in the simplified cost setting when $\overline{p} \geq \overline{p}_3$ is:

$$\Pi^R(q_A, q_N) = [\,P_A(Q) - c_A\,]\,q_A + [\,P(Q) - c_N\,]\,q_N - \frac{k^R}{2}\,[\,q_A + q_N\,]^2$$

$$= [\,a - b\,Q - c_A\,]\,q_A + [\,a - b\,Q - c_N\,]\,q_N - \frac{k^R}{2}\,[\,q_A + q_N\,]^2$$

$$\Rightarrow \frac{\partial \Pi^R(q_A, q_N)}{\partial q_A} = a - b\,[\,q + q_A + q_N\,] - c_A - b\,q_A - b\,q_N - k^R\,[\,q_A + q_N\,]$$

$$= a - b\,q - 2\,b\,q_A - 2\,b\,q_N - c_A - k^R\,[\,q_A + q_N\,] \tag{392}$$

$$\Rightarrow \frac{\partial^2 \Pi^R(q_A, q_N)}{\partial\,(q_A)^2} = -\,2\,b - k^R \;<\; 0\,. \tag{393}$$

(393) implies that $\Pi^R(q_A, q_N)$ is strictly concave in q_A.

(390) and (392) imply that in the putative equilibrium:

$$\left.\frac{\partial \Pi^R(q_A, q_N)}{\partial q_A}\right|_{q_A = q_A^*,\, q_N = q_N^*} = a - b\,q^* - 2\,b\,q_A^* - c_A - k^R\,q_A^* = 0$$

$$\Leftrightarrow \; a - c_A - b\,q^* = \left[\,2\,b + k^R\,\right] q_A^*$$

$$\Leftrightarrow \; [\,a - c_A\,]\left[\left(2\,b + k^R\right)(2\,b + k) - b^2\,\right] - b\left[\,(a - c)\left(2\,b + k^R\right) - b\,(a - c_A)\,\right]$$

$$= \left[\,2\,b + k^R\,\right]\left[\,(a - c_A)\,(2\,b + k) - b\,(a - c)\,\right]$$

$$\Leftrightarrow \; [\,a - c_A\,]\left[\left(2\,b + k^R\right)(2\,b + k) - b^2 - \left(2\,b + k^R\right)(2\,b + k) + b^2\,\right]$$

$$- \,b\,[\,a - c\,]\left[\,2\,b + k^R - \left(2\,b + k^R\right)\right] = 0$$

$$\Leftrightarrow \; [\,a - c_A\,]\,0 - b\,[\,a - c\,]\,0 = 0\,. \tag{394}$$

The last equality in (394) holds. (393) and (394) imply:

$$[\,P(Q) - c_A\,]\; q_A - \frac{k^R}{2}\,[\,q_A\,]^2 \;\leq\; \Pi^R(q_A^*, 0) \;\text{ for all } q_A\,. \tag{395}$$

(395) implies that R cannot increase its profit by deviating from the putative equilibrium by producing distinct q_A and q_N because (383) implies:

$$\begin{aligned}
\Pi^R(q_A, q_N) \;&=\; [\,P(Q) - c_N\,]\, q_N + [\,P_A(Q) - c_A\,]\; q_A - \frac{k^R}{2}\,[\,q_A + q_N\,]^2 \\
&\leq\; [\,P(Q) - c_N\,]\; q_N + [\,P(Q) - c_A\,]\; q_A - \frac{k^R}{2}\,[\,q_A + q_N\,]^2 \\
&<\; [\,P(Q) - c_A\,]\,[\,q_A + q_N\,] - \frac{k^R}{2}\,[\,q_A + q_N\,]^2 \;\leq\; \Pi^R(q_A^*, 0)\,.
\end{aligned} \tag{396}$$

The first inequality in (396) holds because $P_A(Q) \leq P(Q)$ for all Q. The second inequality in (396) holds because $c_A < c_N$, by assumption. The last inequality in (396) reflects (395). (396) implies that R cannot increase its profit by undertaking such a deviation.

Finally, we establish that the rival cannot increase its profit by unilaterally deviating to supply some output other than the q^* specified in (390). The rival's profit is:

$$\Pi(q) \;=\; [\,P(Q) - c\,]\,q - \frac{k}{2}\,q^2 \;=\; [\,a - b(q + q_A + q_N) - c\,]\,q - \frac{k}{2}\,q^2$$

$$\Rightarrow\; \frac{\partial \Pi(q)}{\partial q} \;=\; a - b\,Q - c - b\,q - k\,q \tag{397}$$

$$\Rightarrow\; \frac{\partial^2 \Pi(q)}{\partial q^2} \;=\; -\,2\,b - k \;<\; 0\,. \tag{398}$$

(390) and (397) imply that at the putative equilibrium:

$$\begin{aligned}
\left.\frac{\partial \Pi(q)}{\partial q}\right|_{q\,=\,q^*} \;&=\; a - b\,q_A^* - 2\,b\,q^* - c - k\,q^* \;=\; a - [\,2\,b + k\,]\,q^* - b\,q_A^* \\
&=\; a - [\,2\,b + k\,]\left[\frac{(a-c)\left(2\,b + k^R\right) - b\,(a - c_A)}{(2\,b + k^R)\,(2\,b + k) - b^2}\right] \\
&\qquad - \,b\left[\frac{(a - c_A)\,(2\,b + k) - b\,(a - c)}{(2\,b + k^R)\,(2\,b + k) - b^2}\right] - c \\
&=\; [\,a - c\,]\left[\,\left(2\,b + k^R\right)(2\,b + k) - b^2\,\right] - [\,2\,b + k\,]\left[\,(a-c)\left(2\,b + k^R\right) - b\,(a - c_A)\,\right] \\
&\qquad - \,b\,[\,(a - c_A)\,(2\,b + k) - b\,(a - c)\,] \;=\; 0 \\
&=\; [\,a - c\,]\left[\,\left(2\,b + k^R\right)(2\,b + k) - b^2 - (2\,b + k)\left(2\,b + k^R\right) + b^2\,\right] \\
&\qquad - \,b\,[\,a - c_A\,]\,[\,2\,b + k \,-\, (2\,b + k)\,] \;=\; 0
\end{aligned} \tag{399}$$

(398) and (399) imply that the rival cannot increase its profit by unilaterally deviating from the putative equilibrium. Therefore, the outputs in (390) constitute an equilibrium because neither firm can increase its profit by undertaking a unilateral deviation. ■

Lemma 12. *In the simplified cost setting, there exists a* $\widetilde{\widetilde{\overline{p}}} \in [\,\overline{p}_2, \overline{p}_3)$ *such that in equilibrium, for all* $\overline{p} \in (\widetilde{\widetilde{\overline{p}}}, \overline{p}_3\,]$*:*

$$P(Q^*) \;=\; \overline{p},\;\; q_A^* \;=\; \frac{a\,[\,b+k\,]+b\,c-\,\overline{p}\,[\,2\,b+k\,]}{b\,[\,b+k\,]},\;\; q_N^* \;=\; 0,\;\text{and}\;\; q^* \;=\; \frac{\overline{p}-c}{b+k}. \tag{400}$$

Proof. Suppose an equilibrium exits in which prices and outputs are as specified in (400). We will show that neither R nor the rival can increase its profit by deviating unilaterally from the putative equilibrium. We first consider three possible deviations by R.

Deviation 1. R deviates unilaterally to set $q_A \neq q_A^*$ and $q_N \;=\; 0$.

(383) implies:

$$\Pi^R(q_A, 0) \;=\; [\,P_A(Q) - c_A\,]\,q_A - \frac{k^R}{2}\,[\,q_A\,]^2 \tag{401}$$

$$\Rightarrow\;\; \frac{\partial \Pi^R(q_A,0)}{\partial q_A} \;=\; P_A(Q) - c_A + q_A\,\frac{\partial P_A(Q)}{\partial q_A} - k^R\,q_A \tag{402}$$

$$\Rightarrow\;\; \frac{\partial^2 \Pi^R(q_A,0)}{\partial\,(q_A)^2} \;=\; 2\,\frac{\partial P_A(Q)}{\partial q_A} - k^R \;<\; 0\,. \tag{403}$$

The inequality in (403) holds because $\frac{\partial P_A(Q)}{\partial q_A} \in \{\,0, -\,b\,\}$. (403) implies that the identified deviation will not increase R's profit if:

$$\frac{\partial^+ \Pi^R(q_A^*,0)}{\partial q_A} \;\leq\; 0 \;\leq\; \frac{\partial^- \Pi^R(q_A^*,0)}{\partial q_A}\,. \tag{404}$$

If (404) holds, then R's profit declines as q_A increases above q_A^* or as q_A declines below q_A^*, given q^* and $q_N^* = 0$.

(400) and (402) imply that at the putative equilibrium:

$$\begin{aligned}
&\left.\frac{\partial^+ \Pi^R(q_A,0)}{\partial q_A}\right|_{q_A = q_A^*} \;=\; P_A(Q^*) \,-\, c_A + q_A^*\,\frac{\partial P_A(Q^*)}{\partial q_A} \,-\, k^R\,q_A^* \\
&=\; \overline{p} \,-\, c_A - \left[\,b+k^R\,\right] q_A^* \;=\; \overline{p} - c_A - \left[\,b+k^R\,\right]\left[\frac{a\,(b+k)+b\,c-\overline{p}\,(2\,b+k)}{b\,(b+k)}\right] \\
&=\; \frac{\overline{p}\,b\,[\,b+k\,] - b\,c_A\,[\,b+k\,] - a\,[\,b+k\,]\left[\,b+k^R\,\right] - b\,c\left[\,b+k^R\,\right] + \overline{p}\,[\,2\,b+k\,]\left[\,b+k^R\,\right]}{b\,[\,b+k\,]} \\
&=\; \frac{\overline{p}\left[\,b\,(b+k)+(2\,b+k)\left(b+k^R\right)\right] - b\,c_A\,[\,b+k\,] - a\,[\,b+k\,]\left[\,b+k^R\,\right] - b\,c\left[\,b+k^R\,\right]}{b\,[\,b+k\,]} \;\leq\; 0
\end{aligned} \tag{405}$$

$$\Leftrightarrow \;\; \overline{p} \;\le\; \frac{a\,[\,b+k\,]\left[\,b+k^R\,\right]+b\,c_A\,[\,b+k\,]+b\,c\left[\,b+k^R\,\right]}{b\,[\,b+k\,]+[\,2\,b+k\,]\,[\,b+k^R\,]}$$

$$\Leftrightarrow \;\; \overline{p} \;\le\; \frac{a\,[\,b+k\,]\left[\,b+k^R\,\right]+b\,c_A\,[\,b+k\,]+b\,c\left[\,b+k^R\,\right]}{b\,[\,b+k\,]+[\,2\,b+k\,]\,[\,2\,b+k^R\,]-b\,[\,2\,b+k\,]}$$

$$\Leftrightarrow \;\; \overline{p} \le \frac{a\,[\,b+k\,]\left[\,b+k^R\,\right]+b\,c_A\,[\,b+k\,]+b\,c\left[\,b+k^R\,\right]}{[\,2\,b+k\,]\,[\,2\,b+k^R\,]-b^2} \;=\; \overline{p}_3. \tag{406}$$

The first equality in (405) holds because $P_A(Q) = P(Q) = a - b\,Q$ as q_A increases above q_A^* (thereby reducing $P(Q)$ below $\overline{p}$). The last equality in (406) reflects (382).

(400) and (402) also imply that at the putative equilibrium:

$$\left.\frac{\partial^- \Pi^R(q_A,0)}{\partial q_A}\right|_{q_A=q_A^*} \;=\; P_A(Q^*)-c_A+q_A^*\,\frac{\partial P_A(Q^*)}{\partial q_A}-k^R\,q_A^* \;=\; \overline{p}-c_A-k^R\,q_A^* \;\ge\; 0 \tag{407}$$

$$\Leftrightarrow \;\; \overline{p}-c_A-k^R\left[\frac{a\,(b+k)+b\,c-\overline{p}\,(2\,b+k)}{b\,(\,b+k\,)}\right] \;\ge\; 0$$

$$\Leftrightarrow \;\; \overline{p}-c_A-\left[\frac{a\,(b+k)\,k^R+b\,c\,k^R-\overline{p}\,(2\,b+k)\;k^R}{b\,(b+k)}\right] \;\ge\; 0$$

$$\Leftrightarrow \;\; \frac{[\,\overline{p}-c_A\,]\,b\,[\,b+k\,]-a\,[\,b+k\,]\,k^R-b\,c\,k^R+\overline{p}\,[\,2\,b+k\,]\,k^R}{b\,[\,b+k\,]} \;\ge\; 0$$

$$\Leftrightarrow \;\; \frac{\overline{p}\left[\,b\,(b+k)+(2\,b+k)\,k^R\,\right]-b\,c_A\,[\,b+k\,]-a\,[\,b+k\,]\,k^R-b\,c\,k^R}{b\,[\,b+k\,]} \;\ge\; 0$$

$$\Leftrightarrow \;\; \overline{p} \;\ge\; \frac{b\,c_A\,[\,b+k\,]+a\,[\,b+k\,]\,k^R+b\,c\,k^R}{b\,[\,b+k\,]+[\,2\,b+k\,]\,k^R}$$

$$\Leftrightarrow \;\; \overline{p} \;\ge\; \frac{b\,c_A\,[\,b+k\,]+a\,[\,b+k\,]\,k^R+b\,c\,k^R}{b\,[\,2\,b+k\,]+[\,2\,b+k\,]\,k^R-b^2}$$

$$\Leftrightarrow \;\; \overline{p} \;\ge\; \frac{b\,c_A\,[\,b+k\,]+a\,[\,b+k\,]\,k^R+b\,c\,k^R}{[\,2\,b+k\,]\,[\,b+k^R\,]-b^2} \;=\; \overline{p}_2\,. \tag{408}$$

The second equality in (407) holds because $P_A(Q) = \overline{p}$ as q_A declines below q_A^* (thereby increasing $P(Q)$ above $\overline{p}$). The last equality in (408) reflects (382). (404), (406), and (408) imply that R cannot increase its profit by undertaking a deviation of this kind when $\overline{p} \in [\,\overline{p}_2, \overline{p}_3\,]$.

Deviation 2. R deviates unilaterally to set $q_A = 0$ and $q_N > 0$.

Let $q_A^*(\overline{p})$ denote q_A^* as defined in (400) as a function of $\overline{p}$. (382) and (400) imply:

$$q_A^*(\overline{p}_3) \;=\; \frac{a\,[\,b+k\,]+b\,c-\overline{p}_3\,[\,2\,b+k\,]}{b\,[\,b+k\,]}$$

$$= \frac{a\,[\,b+k\,] + b\,c - \frac{a\,[\,b+k\,]\left[\,b+k^R\right] + b\,c_A\,[\,b+k\,] + b\,c\left[\,b+k^R\right]}{[\,2\,b+k\,][\,2\,b+k^R\,] - b^2}\,[\,2\,b+k\,]}{b\,[\,b+k\,]}$$

$$= \frac{\left[\,(\,2\,b+k\,)\left(2\,b+k^R\right) - b^2\,\right][\,a\,(b+k) + b\,c\,]}{b\,[\,b+k\,]\,[\,(2\,b+k)\,(2\,b+k^R) - b^2\,]}$$

$$- \frac{\left[\,a\,(b+k)\left(b+k^R\right) + b\,c_A\,(b+k) + b\,c\left(b+k^R\right)\,\right][\,2\,b+k\,]}{b\,[\,b+k\,]\,[\,(2\,b+k)\,(2\,b+k^R) - b^2\,]}$$

$$= \frac{\left[\,(2\,b+k)\left(2\,b+k^R\right) - b^2\,\right][\,a\,(b+k) + b\,c\,]}{b\,[\,b+k\,]\,[\,(2\,b+k)\,(2\,b+k^R) - b^2\,]}$$

$$- \frac{\left\{\,[\,a\,(b+k) + b\,c\,]\left[\,b+k^R\,\right] + b\,c_A\,[\,b+k\,]\,\right\}[\,2\,b+k\,]}{b\,[\,b+k\,]\,[\,(2\,b+k)\,(2\,b+k^R) - b^2\,]}$$

$$= \frac{[\,(2\,b+k)\,b - b^2\,]\,[\,a\,(b+k) + b\,c\,] - b\,c_A\,[\,b+k\,]\,[\,2\,b+k\,]}{b\,[\,b+k\,]\,[\,(2\,b+k)\,(2\,b+k^R) - b^2\,]}$$

$$= \frac{[\,2\,b+k-b\,]\,[\,a\,(b+k) + b\,c\,] - c_A\,[\,b+k\,]\,[\,2\,b+k\,]}{[\,b+k\,]\,[\,(2\,b+k)\,(2\,b+k^R) - b^2\,]}$$

$$= \frac{[\,b+k\,]\,[\,a\,(b+k) + b\,c\,] - c_A\,[\,b+k\,]\,[\,2\,b+k\,]}{[\,b+k\,]\,[\,(2\,b+k)\,(2\,b+k^R) - b^2\,]}$$

$$= \frac{a\,[\,b+k\,] + b\,c - c_A\,[\,2\,b+k\,]}{[\,2\,b+k\,]\,[\,2\,b+k^R\,] - b^2} = \frac{[\,a-c_A\,]\,[\,2\,b+k\,] - a\,b + b\,c}{[\,2\,b+k\,]\,[2\,b+k^R] - b^2}$$

$$= \frac{[\,a-c_A\,]\,[\,2\,b+k\,] - b\,[\,a-c\,]}{[\,2\,b+k^R\,]\,[\,2\,b+k\,] - b^2} = q^*_{A4} \tag{409}$$

where q^*_{A4} reflects the value of q^*_A identified in (390). Let q_N^{max} denote the value of q_N that maximizes R's profit when $q_A = 0$, given q. Then, given q, for all q_N:

$$\Pi^R(0, q_N) = [\,P(q+q_N) - c_N\,]\,q_N - \frac{k^R}{2}\,[\,q_N\,]^2 \le \Pi^R(0, q_N^{max})\,. \tag{410}$$

q_N^{max} exists because it is apparent from (410) that $\Pi^R(0, q_N)$ is a strictly concave function of q_N. Observe that, given q^* as specified in (390):

$$\Pi^R(0, q_N^{max}) < [\,P(q^* + q_N^{max}) - c_A\,]\,q_N^{max} - \frac{k^R}{2}\,[\,q_N^{max}\,]^2$$
$$\le \Pi^R(q^*_{A4}, 0) = \Pi^R(q^*_A(\overline{p}_3), 0). \tag{411}$$

The first inequality in (411) holds because $c_A < c_N$, by assumption. The second inequality in (411) holds because, from Lemma 11, $q^*_{A4} = q^*_A(\overline{p}_3)$ is the value of q_A that maximizes R's profit when $q_N = 0$ and $q = q^*$. The equality in (411) reflects (409).

R's profit when $q = q^*$, $q_N = 0$, and $q_A = q^*_A(\overline{p})$ is:

$$\Pi^R(q_A^*(\overline{p}), 0) \;=\; [\, P_A(\, q_A^*(\overline{p}\,) + q^*(\overline{p}\,)\,) - c_A\,]\; q_A^*(\overline{p}\,) - \frac{k^R}{2}\; [\, q_A^*(\overline{p}\,)\,]^2\,. \tag{412}$$

(400) and (412) imply that $\Pi^R(q_A^*(\overline{p}), 0)$ is a continuous function of $\overline{p}$. Therefore, (411) implies that there exists a $\overline{p}' \leq \overline{p}_3$ such that, for all $\overline{p} \in (\overline{p}', \overline{p}_3\,]$:

$$\Pi^R(q_A^*(\overline{p}), 0) \;>\; \Pi^R(0, q_N^{max}) \;\geq\; \Pi^R(0, q_N) \;\text{ for all } q_N\,.$$

Consequently, when $\overline{p} \in (\overline{p}', \overline{p}_3\,]$, R cannot increase its profit by undertaking a deviation of the specified type.

Deviation 3. R deviates unilaterally to set $q_A > 0$ and $q_N > 0$.

Conclusion 5 establishes that such a deviation generates less profit for R than R secures by setting $q_A = 0$ or $q_N = 0$. Therefore, the foregoing findings regarding Deviations 1 and 2 imply that R cannot cannot increase its profit by undertaking a unilateral deviation in which $q_A > 0$ and $q_N > 0$.

In summary, R cannot increase its profit by undertaking a unilateral deviation when $\overline{p} \in [\,\widetilde{\overline{p}}, \overline{p}_3\,]$ where $\widetilde{\overline{p}} = \max\{\overline{p}', \overline{p}_2\}$. It remains to establish that the rival cannot increase its profit by changing q unilaterally. The rival's profit is:

$$\Pi(q) \;=\; [\, a - b\,Q - c\,]\, q - \frac{k}{2}\, q^2 \;\;\Rightarrow\;\; \frac{\partial \Pi(q)}{\partial q} \;=\; a - b\,Q - c - b\,q - k\,q \tag{413}$$

$$\Rightarrow\;\; \frac{\partial^2 \Pi(q)}{\partial q^2} \;=\; -\,2\,b - k \;<\; 0\,. \tag{414}$$

(413) implies that when $P(Q) = \overline{p}$:

$$\left.\frac{\partial \Pi(q)}{\partial q}\right|_{q\,=\,q^*} \;=\; a - b\,Q - c - b\,q^* - k\,q^*$$

$$=\; \overline{p} - c - b\,q^* - k\,q^* \;=\; 0 \;\;\Leftrightarrow\;\; q^* \;=\; \frac{\overline{p} - c}{b + k}\,. \tag{415}$$

(400) implies that the last equality in (415) holds at the putative equilibrium. (414) and (415) imply that the rival cannot increase its profit by deviating from the proposed equilibrium.

We have proved that neither firm can increase its profit by deviating from the outputs specified in (400). Therefore, these outputs constitute an equilibrium. ■

Conclusion 6. *In the simplified cost setting, there exists a* $\widetilde{\overline{p}} \in [\,\overline{p}_2, \overline{p}_3\,)$ *such that R's revenue,* $V(\overline{p}\,)$*, is strictly decreasing in* $\overline{p}$ *for* $\overline{p} \in (\widetilde{\overline{p}}, \overline{p}_3\,]$.

Proof. Let $\widetilde{\overline{p}}$ denote the smallest $\overline{p}$ for which the outcomes identified in (400) prevail in equilibrium. (400) implies that when $\overline{p} \in (\widetilde{\overline{p}}, \overline{p}_3\,]$:

$$V(\overline{p}\,) \;=\; \overline{p}\; q_A \;=\; \overline{p}\; \left[\, \frac{a\,(b+k) + b\,c - \overline{p}\;(2\,b + k)}{b\,(b+k)} \,\right]$$

$$\Rightarrow \quad \frac{\partial V(\overline{p})}{\partial \overline{p}} \;=\; \frac{a\,[\,b+k\,]+b\,c-2\,\overline{p}\;[\,2\,b+k\,]}{b\,[\,b+k\,]}$$

$$\Rightarrow \quad \frac{\partial V(\overline{p})}{\partial \overline{p}} \;<\; 0 \quad \Leftrightarrow \quad a\,[\,b+k\,]+b\,c-2\,\overline{p}\;[\,2\,b+k\,] \;<\; 0$$

$$\Leftrightarrow \quad \overline{p} \;>\; \frac{a\,[\,b+k\,]+b\,c}{2\,[\,2\,b+k\,]} \;\equiv\; \overline{p}_{V_{3M}}\,. \tag{416}$$

(382) and (416) imply:

$$\overline{p}_{V_{3M}} \;<\; \overline{p}_3 \quad \Leftrightarrow \quad \frac{a\,[\,b+k\,]+bc}{2\,[\,2\,b+k\,]} \;<\; \frac{a\left[\,b+k^R\,\right][\,b+k\,]+bc_A\,[\,b+k\,]+b\,c\left[\,b+k^R\,\right]}{[\,2\,b+k^R\,]\,[\,2\,b+k\,]-b^2}$$

$$\Leftrightarrow \quad \frac{a\,[\,b+k\,]+b\,c}{2\,[\,2\,b+k\,]} \;<\; \frac{[\,a\,(b+k\,)+b\,c]\left[\,b+k^R\,\right]+b\,c_A\,[\,b+k\,]}{[\,2\,b+k^R\,]\,[\,2\,b+k\,]-b^2}\,. \tag{417}$$

The inequality in (417) holds if:

$$\frac{a\,[\,b+k\,]+b\,c}{2\,[\,2\,b+k\,]} \;<\; \frac{[\,a\,(b+k\,)+b\,c]\left[\,b+k^R\,\right]}{[\,2\,b+k^R\,]\,[\,2\,b+k\,]-b^2}$$

$$\Leftrightarrow \quad \frac{1}{2\,[\,2\,b+k\,]} \;<\; \frac{b+k^R}{[\,2\,b+k^R\,]\,[\,2\,b+k\,]-b^2}$$

$$\Leftrightarrow \quad \left[\,2\,b+k^R\,\right][\,2\,b+k\,]-b^2 \;<\; 2\,[\,2\,b+k\,]\left[\,b+k^R\,\right]$$

$$\Leftrightarrow \quad \left[\,b+k^R\,\right][\,2\,b+k\,]+b\,[\,2\,b+k\,]-b^2 \;<\; 2\,[\,2\,b+k\,]\left[\,b+k^R\,\right]$$

$$\Leftrightarrow \quad b\,[\,2\,b+k\,]-b^2 \;<\; [\,2\,b+k\,]\left[\,b+k^R\,\right] \quad \Leftrightarrow \quad -\,b^2 \;<\; [\,2\,b+k\,]\,k^R.$$

The last inequality here holds. Therefore, (416) and (417) imply that $\frac{\partial V(\overline{p})}{\partial \overline{p}} < 0$ when $\overline{p} \in (\max\{\,\widetilde{\overline{p}}, \overline{p}_{VM}\,\}, \overline{p}_3\,]$. ■

F. The Modified Baseline Setting.

Table TA1 reports how equilibrium outcomes change as parameter values change in the modified baseline setting. Recall that $P(Q) = m\,Q^{-\frac{1}{\varepsilon}}$ where $m = 10^6$ and $\varepsilon = 2$ in this setting.

Parameter Variation	$\frac{\overline{p}_b - \overline{p}_d}{\overline{p}_b}$	$\frac{V(\overline{p}_d) - V(\overline{p}_b)}{V(\overline{p}_b)}$	$\overline{p}^*$	$\frac{\overline{p}^*}{\overline{p}_b}$	$\frac{W(\overline{p}^*) - W(\overline{p}_b)}{W(\overline{p}_b)}$
$1.50\,m$	0.05	0.14	88.06	0.95	0.01
$0.50\,m$	0.06	0.15	43.01	0.95	0.01
$1.10\,\varepsilon$	0.04	0.13	121.56	0.96	0.01
$0.90\,\varepsilon$	0.07	0.16	34.82	0.93	0.02
$1.50\,c_A$	0.05	0.14	67.59	0.95	0.01
$0.50\,c_A$	0.05	0.14	67.12	0.95	0.01
$1.50\,k_A$	0.04	0.12	70.64	0.96	0.01
$0.5\,k_A$	0.07	0.19	61.51	0.93	0.03
$1.50\,k^R$	0.05	0.14	69.00	0.95	0.01
$0.50\,k^R$	0.05	0.15	65.72	0.95	0.02
$1.50\,c_N$	0.05	0.15	67.59	0.95	0.02
$0.50\,c_N$	0.05	0.14	67.36	0.95	0.01
$1.50\,k_N$	0.06	0.16	68.29	0.94	0.02
$0.50\,k_N$	0.04	0.12	65.48	0.96	0.01
$1.50\,c$	0.05	0.14	67.59	0.95	0.02
$0.50\,c$	0.05	0.14	67.36	0.95	0.01
$1.50\,k$	0.07	0.16	70.64	0.94	0.02
$0.50\,k$	0.03	0.12	61.27	0.97	0.01

Table TA1. The Effects of Changing Baseline Parameters.

The first column in Table A4 identifies the single parameter that is changed in the modified baseline setting and the amount by which it is changed. All other parameters remain at their levels in the baseline setting.[5] The remaining columns in Table TA1 identify outcomes that arise in equilibrium, corresponding to the entries in Table A1 in the Appendix of the paper. The welfare calculations in the last column assume that $r = \frac{1}{2}$.

[5] For example, the first row of data in Table TA1 records the outcomes that arise in equilibrium when m is increased by 50% above its level in the modified baseline setting with $P(Q) = m\,Q^{-\frac{1}{\varepsilon}}$, holding all other parameters at their values in this setting. Table TA1 considers relatively limited variation in ε because the second order condition for R's problem is violated if ε becomes too small.

F. Supplemental Figures.

Figure A1 below complements Figure 2 in the paper by illustrating how consumer surplus $(S(\cdot))$ and R's revenue $(V(\cdot))$ vary with the price cap $(\overline{p})$ in settings where $\overline{p}_{Vm} \in (\overline{p}_0, \overline{p}_d)$ and $\overline{p}_{SM} \in (\overline{p}_0, \overline{p}_d)$ Recall that $\overline{p}_{Vm}$ is the value of the price cap at which R's equilibrium revenue is minimized and $\overline{p}_{SM}$ is the value of the price cap at which equilibrium consumer surplus is maximized.

Figure A2 below complements Figures 2 and 4 in the paper by depicting how consumer surplus $(S(\cdot))$, R's revenue $(V(\cdot))$, and welfare $(W(\cdot) = S(\cdot) - \frac{1}{2} V(\cdot))$ vary with $\overline{p}$ in the baseline setting.

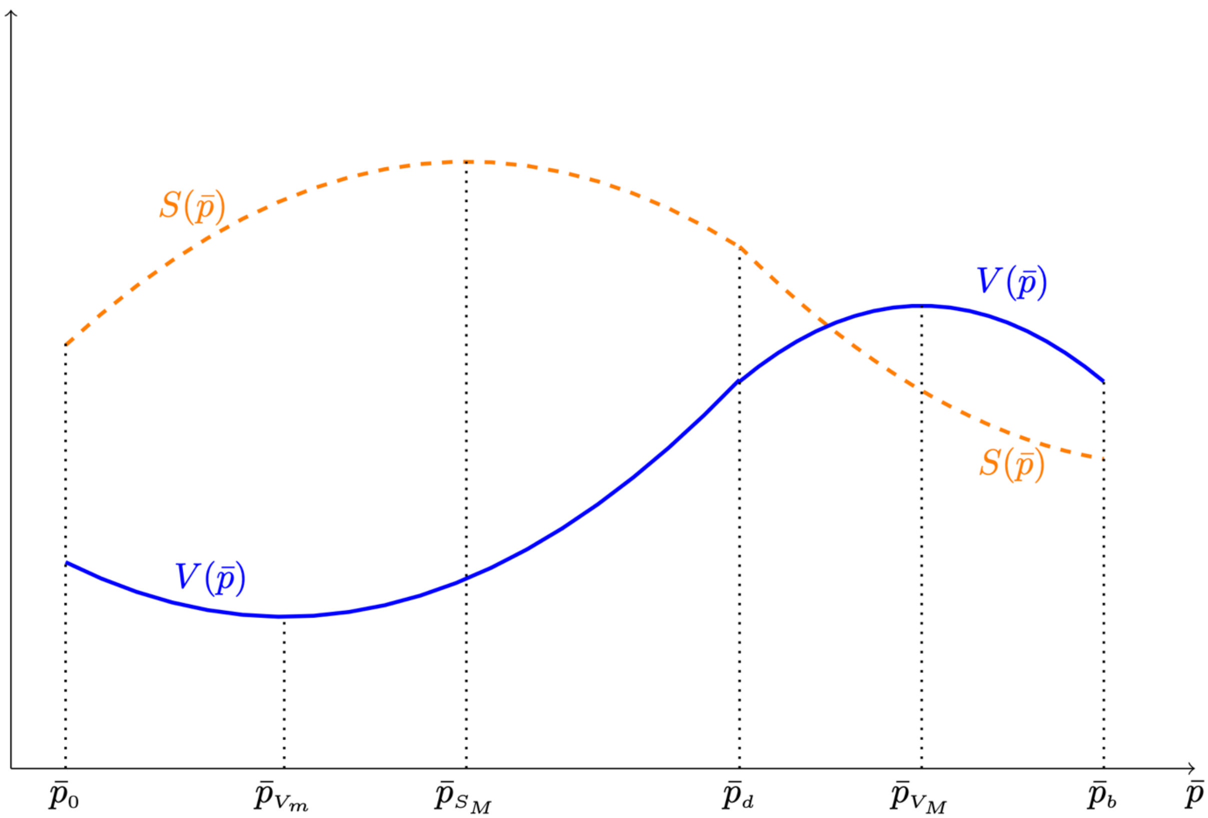

Figure A1. Consumer Surplus, $S(\overline{p})$, and R's Revenue, $V(\overline{p})$, when $\overline{p}_0 < \overline{p}_{Vm} < \overline{p}_{SM} < \overline{p}_d < \overline{p}_b$.

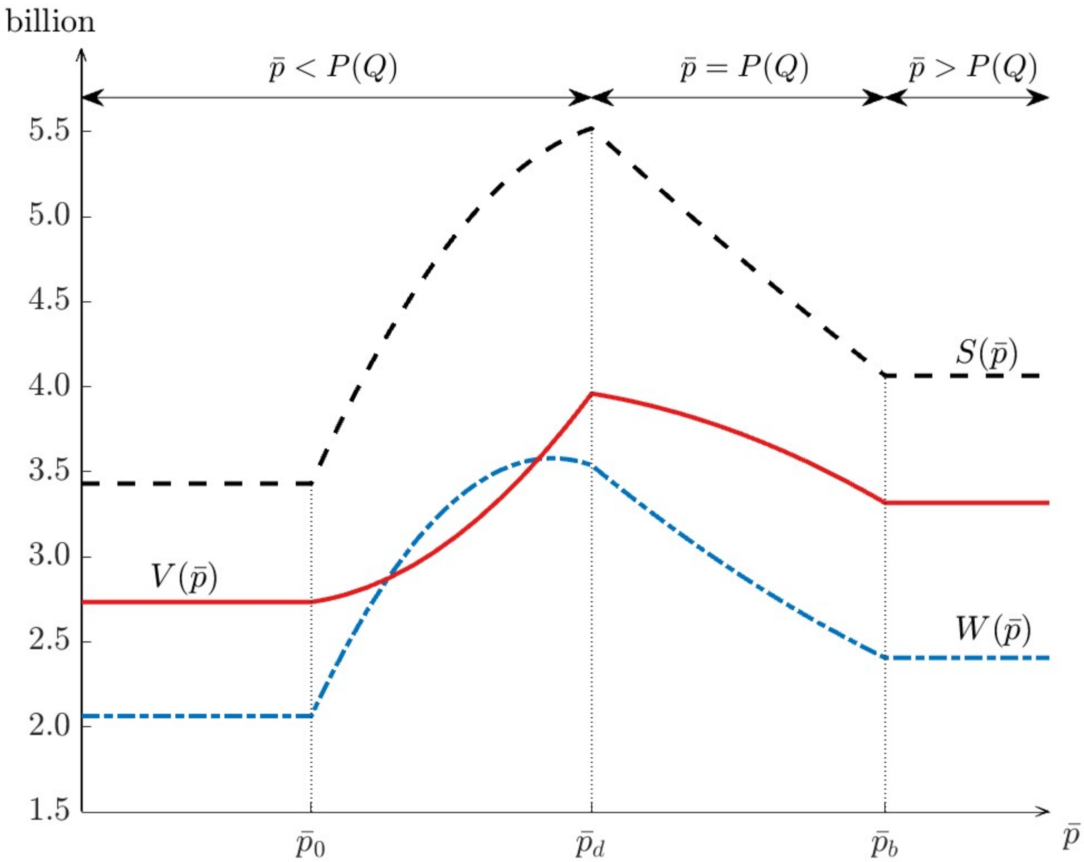

Figure A2. Consumer Surplus, $S(\overline{p})$, Welfare, $W(\overline{p})$, and R's Revenue, $V(\overline{p})$, in the Baseline Setting.

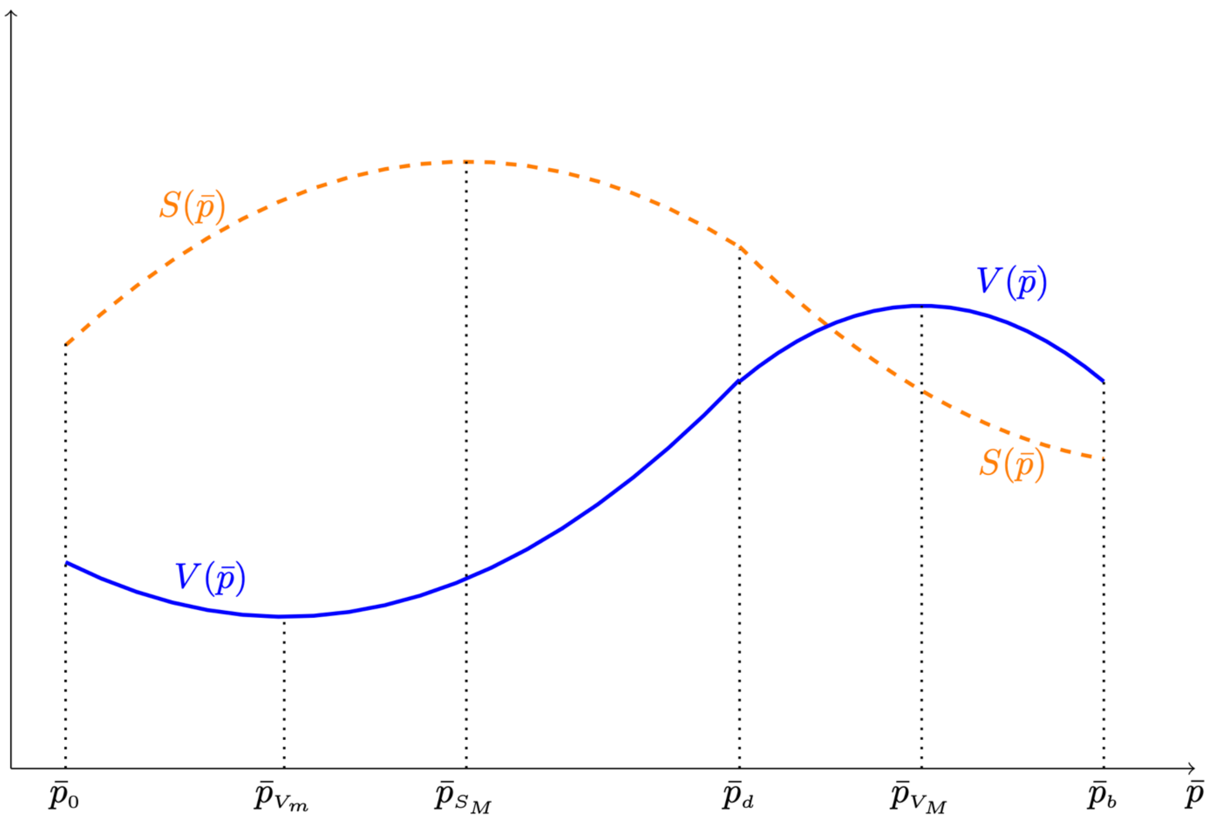

Figure A1. Consumer Surplus, $S(\overline{p})$, and R's Revenue, $V(\overline{p})$, when $\overline{p}_0 < \overline{p}_{Vm} < \overline{p}_{SM} < \overline{p}_d < \overline{p}_b$.

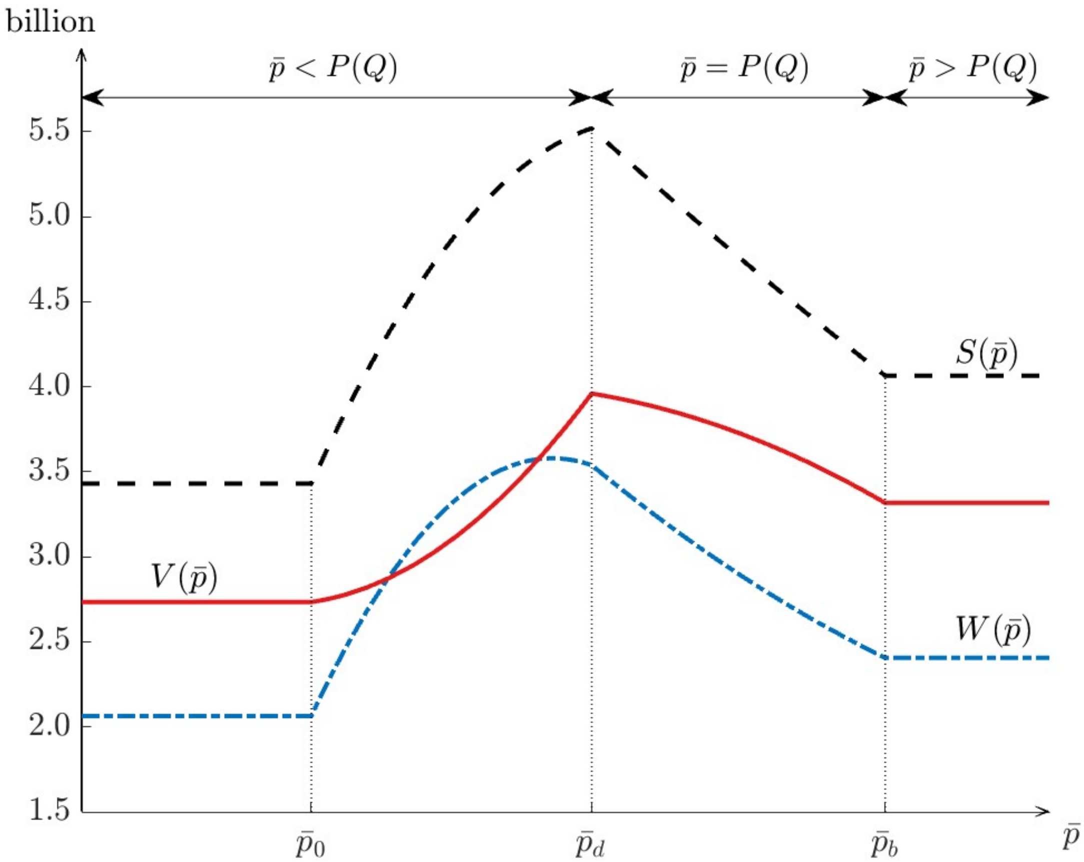

Figure A2. Consumer Surplus, $S(\overline{p})$, Welfare, $W(\overline{p})$, and R's Revenue, $V(\overline{p})$, in the Baseline Setting.