Technical Appendix to Accompany

"Pricing to Preclude Sabotage in Regulated Industries"

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This Appendix proves the counterpart to Proposition 1 in the paper for the setting where the regulator seeks to maximize expected consumer surplus ($E\{S\}$) and a fraction of the V's expected profit ($E\{\Pi_0\}$) and E's expected profit ($E\{\Pi_1\}$). Assumptions 1 and 2 in the paper are maintained throughout the ensuing analysis.

The regulator's problem in this setting is:

$$\underset{w,\,p_0}{\text{Maximize}} \quad E\{S\} + \beta_0\, E\{\Pi_0\} + \beta_1\, E\{\Pi_1\} \quad \text{subject to } \; E\{\Pi_0\} \;\geq\; 0$$

$$\text{where } \; \beta_0,\; \beta_1 \in (0,1).$$

For a fixed $\widehat{a}$, call this problem $[RP - \beta\,\widehat{a}\,]$. The associated Lagrangian function is:

$$\mathcal{L} \;=\; E\{S\,(w,p_0,c)\,|\,\widehat{a}\,\} + \;\beta_0\, E\{\,\Pi_0\,(w,p_0,c)\,|\,\widehat{a}\,\} + \beta_1\, E\{\,\Pi_1\,(w,p_0,c)\,|\,\widehat{a}\,\}$$
$$+\;\lambda\, E\{\,\Pi_0\,(w,p_0,c)\,|\,\widehat{a}\,\}\;. \tag{1}$$

From Lemma A1 in the paper:

$$E\{\,S\,(w,p_0,c)\,|\,\widehat{a}\,\} \;=\; -\,w + \frac{B_0 + B_1\,\triangle + \triangle^2}{8\,[\,t_0 + t_1\,]} \tag{2}$$

where B_0 is independent of p_0 and w, $\triangle \equiv p_0 - w$, and from (22) in the paper:

$$B_1 \;=\; 2\,[\,v_1 - v_0\,] - 2\,E\{c\,|\,\widehat{a}\,\} - 6\,t_0 - 8\,t_1\,. \tag{3}$$

$$\text{Define} \quad R \;\equiv\; v_1 - v_0 + t_0 - E\{c\,|\,\widehat{a}\,\}\;. \tag{4}$$

From (13) in the paper:

$$x_1 \;=\; \frac{v_1 - v_0 + t_0 - E\{c\,|\,\widehat{a}\,\} + p_0 - w}{2\,[\,t_0 + t_1\,]} \;=\; \frac{R + \triangle}{2\,[\,t_0 + t_1\,]}\,, \;\text{ and}$$

$$x_0 \;=\; 1 - x_1 \;=\; 1 - \frac{R + \triangle}{2\,[\,t_0 + t_1\,]} \;=\; \frac{2\,[\,t_0 + t_1\,] - R - \triangle}{2\,[\,t_0 + t_1\,]}\,. \tag{5}$$

V's expected profit is:

$$E\{\,\Pi_0\,(w,p_0,c)\,|\,\widehat{a}\,\} \;=\; [\,w - c_u\,]\,x_1 + [\,p_0 - c_d - c_u\,]\,x_0 - F_0$$

$$=\; [\,w - c_u\,]\,[\,1 - x_0\,] + [\,p_0 - c_d - c_u\,]\,x_0 - F_0 \;=\; [\,p_0 - c_d - w\,]\,x_0 + w - c_u - F_0\,. \tag{6}$$

Because $\triangle = p_0 - w$, (5) and (6) provide:

$$E\{\Pi_0(w, p_0, c) \,|\, \widehat{a}\} = \frac{[\triangle - c_d][2(t_0+t_1) - R - \triangle]}{2[t_0+t_1]} + w - c_u - F_0 \tag{7}$$

$$= \frac{-\triangle^2 + c_d \triangle + \triangle[2(t_0+t_1) - R]}{2[t_0+t_1]} - \frac{c_d[2(t_0+t_1) - R]}{2[t_0+t_1]} + w - c_u - F_0\,. \tag{8}$$

From (5) in the paper and from (5) above, E's expected profit is:

$$E\{\Pi_1(w, p_0, c) \,|\, \widehat{a}\} = [E\{p_1 \,|\, \widehat{a}\} - w - E\{c \,|\, \widehat{a}\}]\, x_1 - F_1$$

$$= \left[p_0 + \frac{v_1 - v_0 - p_0 + t_0 + E\{c \,|\, \widehat{a}\} + w}{2} - w - E\{c \,|\, \widehat{a}\}\right] x_1 - F_1$$

$$= \frac{1}{2}[p_0 + v_1 - v_0 + t_0 - E\{c \,|\, \widehat{a}\} - w]\, x_1 - F_1$$

$$= \frac{1}{2}[2(t_0+t_1)\, x_1]\, x_1 - F_1 = [t_0+t_1](x_1)^2 - F_1$$

$$= [t_0+t_1]\left[\frac{R+\triangle}{2(t_0+t_1)}\right]^2 - F_1 = \frac{[R+\triangle]^2}{4[t_0+t_1]} - F_1\,. \tag{9}$$

Consider the following values for $\triangle^*$, w^*, and λ^*:

$$\triangle^* = \frac{2\,c_d + 2\,R\,[\beta_1 - \frac{1}{2}]}{3 - 2\,\beta_1}; \qquad \lambda^* = 1 - \beta_0 > 0\,;$$

$$w^* = c_u + F_0 - \frac{[\triangle^* - c_d][2(t_0+t_1) - R - \triangle^*]}{2[t_0+t_1]}\,. \tag{10}$$

We will show that these values constitute a solution to $[RP - \beta\,\widehat{a}]$ by demonstrating that they solve the associated Kuhn-Tucker conditions.

Lemma T1. $\frac{\partial \mathcal{L}}{\partial w}\Big|_{w=w^*} = 0.$

Proof. Differentiating (2), (8), and (9) provides:

$$\frac{\partial E\{S(w, p_0, c) \,|\, \widehat{a}\}}{\partial w} = -1\,, \quad \frac{\partial E\{\Pi_0(w, p_0, c) \,|\, \widehat{a}\}}{\partial w} = 1\,, \quad \text{and}$$

$$\frac{\partial E\{\Pi_1(w, p_0, c) \,|\, \widehat{a}\}}{\partial w} = 0\,. \tag{11}$$

(1), (10), and (11) imply $\frac{\partial \mathcal{L}}{\partial w}\Big|_{w=w^*} = -1 + \beta_0 + \lambda^* = 0.$ ■

Lemma T2. $\left.\frac{\partial \mathcal{L}}{\partial \triangle}\right|_{\triangle = \triangle^*} = 0\,.$

Proof. (2), (8), and (9) imply:

$$\frac{\partial E\{\, S(w, p_0, c)\,|\,\widehat{a}\,\}}{\partial \triangle} = \frac{B_1 + 2\,\triangle}{8\,[\,t_0 + t_1\,]}\,, \quad \frac{\partial E\{\Pi_1(w, p_0, c)\;|\,\widehat{a}\}}{\partial \triangle} = \frac{R + \triangle}{2\,[\,t_0 + t_1\,]}\,, \text{ and}$$

$$\frac{\partial E\{\Pi_0(w, p_0, c)\;|\,\widehat{a}\}}{\partial \triangle} = \frac{-\,2\,\triangle + 2\,[\,t_0 + t_1\,] + c_d - R}{2\,[\,t_0 + t_1\,]}\,. \tag{12}$$

(1), (10), and (12) imply:

$$\begin{aligned}\left.\frac{\partial \mathcal{L}}{\partial \triangle}\right|_{\triangle\,=\,\triangle^*} &= \frac{B_1 + 2\triangle^*}{8\,[\,t_0 + t_1\,]} + \frac{\beta_0 + \lambda^*}{2\,[\,t_0 + t_1\,]}\,[\,-\,2\,\triangle^* + 2\,(t_0 + t_1) + c_d - R\,] + \frac{\beta_1\,[\,R + \triangle^*\,]}{2\,[\,t_0 + t_1\,]} \\ &= \frac{B_1 + 2\triangle^*}{8\,[\,t_0 + t_1\,]} + \frac{-\,2\,\triangle^* + 2\,[\,t_0 + t_1\,] + c_d - R}{2\,[\,t_0 + t_1\,]} + \frac{\beta_1\,[\,R + \triangle^*\,]}{2\,[\,t_0 + t_1\,]} \;=\; 0 \\ &\Leftrightarrow\;\; B_1 + 2\,\triangle^* + 4\,[\,-\,2\,\triangle^* + 2\,(t_0 + t_1) + c_d - R\,] + 4\,\beta_1\,[\,R + \triangle^*\,] \;=\; 0 \\ &\Leftrightarrow\;\; B_1 - 6\,\triangle^* + 8\,[\,t_0 + t_1\,] + 4\,c_d - 4\,R + 4\,\beta_1\,[\,R + \triangle^*\,] \;=\; 0\,. \end{aligned} \tag{13}$$

(3) and (13) imply:

$$\begin{aligned}\left.\frac{\partial \mathcal{L}}{\partial \triangle}\right|_{\triangle\,=\,\triangle^*} = 0 \;\;&\Leftrightarrow\;\; 2\,[\,v_1 - v_0\,] - 2\,E\{c\,|\,\widehat{a}\,\} - 6\,t_0 - 8\,t_1 - 6\,\triangle^* \\ &\qquad + \; 8\,[\,t_0 + t_1\,] + 4\,c_d - 4\,R + 4\,\beta_1\,[\,R + \triangle^*\,] \;=\; 0 \\ &\Leftrightarrow\;\; 2\,[\,v_1 - v_0\,] - 2\,E\{c\,|\,\widehat{a}\,\} + 2\,t_0 - 6\,\triangle^* + 4\,c_d - 4\,R + 4\,\beta_1\,R + 4\,\beta_1\,\triangle^* \;=\; 0 \\ &\Leftrightarrow\;\; [\,6 - 4\,\beta_1\,]\,\triangle^* \;=\; 2\,[\,v_1 - v_0\,] - 2\,E\{c\,|\,\widehat{a}\,\} + 2\,t_0 + 4\,c_d - 4\,R + 4\,\beta_1\,R \\ &\Leftrightarrow\;\; [\,6 - 4\,\beta_1\,]\,\triangle^* \;=\; 2\,[\,v_1 - v_0 - E\{c\,|\,\widehat{a}\,\} + t_0\,] + 4\,c_d - 4\,R + 4\,\beta_1\,R \\ &\Leftrightarrow\;\; [\,6 - 4\,\beta_1\,]\,\triangle^* \;=\; 2\,R + 4\,c_d - 4\,R + 4\,\beta_1\,R \;=\; 4\,c_d - 2\,R + 4\,\beta_1\,R \\ &\Leftrightarrow\;\; \triangle^* \;=\; \frac{2\,c_d + 2\,R\left[\,\beta_1 - \frac{1}{2}\,\right]}{3 - 2\,\beta_1}\,. \quad \blacksquare \end{aligned} \tag{14}$$

Lemma T3. $\left.\frac{\partial \mathcal{L}}{\partial \lambda}\right|_{\lambda = \lambda^*} = 0\,.$

Proof. From (7):

$$\left.\frac{\partial \mathcal{L}}{\partial \lambda}\right|_{\lambda\,=\,\lambda^*} = \frac{[\,\triangle^* - c_d\,]\,[\,2\,(t_0 + t_1) - R - \triangle^*\,]}{2\,[\,t_0 + t_1\,]} + w^* - c_u - F_0 \;=\; 0\,. \tag{15}$$

The last equality in (15) reflects (10). ■

Lemmas T1 – T3 demonstrate that $(\triangle^*, w^*, \lambda^*)$ solve the Kuhn-Tucker conditions associated with $[RP - \beta\,\widehat{a}\,]$, provided $w^* \geq 0$. It remains to establish that $w^* \geq 0$ and to determine when $\triangle^* \in (0, c_d)$.

Lemma T4. $\triangle^* \in (0, c_d)$ if:

$$c_d \;>\; -\,R\,[\,\beta_1 - \frac{1}{2}\,] \;\text{ and }\; [\,c_d + R\,]\;[\,\beta_1 - \frac{1}{2}\,] \;<\; 0\,. \tag{16}$$

<u>Proof</u>. $3 - 2\,\beta_1 \;>\; 0$ because $\beta_1 < 1$ by assumption. Therefore, (14) implies:

$$\triangle^* \;>\; 0 \quad\Leftrightarrow\quad c_d + R\,[\,\beta_1 - \frac{1}{2}\,] \;>\; 0 \quad\Leftrightarrow\quad c_d \;>\; -\,R\,[\,\beta_1 - \frac{1}{2}\,]\,. \tag{17}$$

From (14):

$$\triangle^* - c_d \;=\; \frac{2\,c_d + 2\,R\left[\,\beta_1 - \frac{1}{2}\,\right] - c_d\,[\,3 - 2\,\beta_1\,]}{3 - 2\beta_1} \;=\; \frac{-\,c_d + 2\,R\left[\,\beta_1 - \frac{1}{2}\,\right] + 2\,\beta_1\,c_d}{3 - 2\,\beta_1}$$

$$=\; \frac{2\,c_d\left[\,\beta_1 - \frac{1}{2}\,\right] + 2\,R\left[\,\beta_1 - \frac{1}{2}\,\right]}{3 - 2\,\beta_1} \;=\; \frac{2\,[\,c_d + R\,]\left[\,\beta_1 - \frac{1}{2}\,\right]}{3 - 2\,\beta_1}$$

$$\Rightarrow\quad \triangle^* - c_d \;<\; 0 \quad\Leftrightarrow\quad [\,c_d + R\,]\;[\,\beta_1 - \frac{1}{2}\,] \;<\; 0\,. \quad ■ \tag{18}$$

Lemma T5. $\triangle^* \in (0, c_d)$ if $\beta_1 \;<\; \frac{1}{2}$ and

$$c_d \;>\; \max\left\{ [\,v_0 - v_1 - t_0 + E\,\{c\,|\,\widehat{a}\,\}\,]\;[\,\beta_1 - \frac{1}{2}\,],\; v_0 - v_1 - t_0 + E\,\{c\,|\,\widehat{a}\,\} \right\}. \tag{19}$$

<u>Proof</u>. If $\beta_1 < \frac{1}{2}$, then (16) implies that $\triangle^* \in (0, c_d)$ if $c_d \;>\; -\,R\left[\,\beta_1 - \frac{1}{2}\,\right]$ and $c_d \;>\; -\,R$. Substituting for R from (4) implies that these inequalities hold if (19) holds. ■

Conclusion T1. $\triangle^* \in (0, c_d)$ if $\beta_1 \;<\; \frac{1}{2}\,$.

<u>Proof</u>. From Lemma T5, $\triangle^* \in (0, c_d)$ if $\beta_1 \;<\; \frac{1}{2}\,$,

$$c_d \;>\; v_0 - v_1 - t_0 + E\,\{\,c\,|\,\widehat{a}\,\}\;,\;\text{ and} \tag{20}$$

$$c_d \;>\; [\,\frac{1}{2} - \beta_1\,]\;[\,v_1 - v_0 + t_0 - E\,\{\,c\,|\,\widehat{a}\,\}\,]\,. \tag{21}$$

(20) and (21) hold when $\beta_1 \;<\; \frac{1}{2}$ if:

$$c_d \;>\; v_0 - v_1 - t_0 + E\,\{\,c\,|\,\widehat{a}\,\} \quad\text{and} \tag{22}$$

$$c_d \; > \; \frac{1}{2} \, [\, v_1 - v_0 + t_0 - E\,\{\, c \,|\, \widehat{a} \,\}\,] \, . \tag{23}$$

(22) and (23) hold when Assumption 2 holds, as demonstrated in the proof of Proposition 1 in the paper. ■

Conclusion T2. $w^* \; > \; 0$ if $\beta_1 \; < \; \frac{1}{2}$.

<u>Proof</u>. Because $E\,\{\, \Pi_0\,(w, p_0, c) \,|\, \widehat{a}\,\} \; \geq \; 0$ at the solution to $[RP - \beta\,\widehat{a}]$, (7) implies:

$$\frac{[\, \triangle^* - c_d \,]\,[\, 2\,(t_0 + t_1) - R - \triangle^* \,]}{2\,[\, t_0 + t_1 \,]} + w^* - c_u - F_0 \; \geq \; 0 \, . \tag{24}$$

If $w^* \leq 0$, then (24) implies:

$$\frac{[\, \triangle^* - c_d \,]\,[\, 2\,(t_0 + t_1) - R - \triangle^* \,]}{2\,[\, t_0 + t_1 \,]} \; \geq \; 0 \, . \tag{25}$$

From (4):

$$\begin{aligned} 2\,[\, t_0 + t_1 \,] - R - \triangle^* \; &= \; 2\,[\, t_0 + t_1 \,] - [\, v_1 - v_0 + t_0 - E\,\{c \,|\, \widehat{a}\,\}\,] - \triangle^* \\ &= \; v_0 - v_1 + 2\,t_1 + t_0 + E\,\{c \,|\, \widehat{a}\,\} - \triangle^* \; > \; 0 \, . \end{aligned} \tag{26}$$

The inequality in (26) reflects Assumption 1.

Because $\triangle^* - c_d \; < \; 0$ from Conclusion T1, (26) implies:

$$\frac{[\, \triangle^* - c_d \,]\,[\, 2\,(t_0 + t_1) - R - \triangle^* \,]}{2\,[\, t_0 + t_1 \,]} \; < \; 0 \, . \tag{27}$$

(27) contradicts (25). Therefore, it cannot be the case that $w^* \leq 0$. ■