Technical Appendix A to Accompany

"The Impact of Vertical Integration on Losses from Collusion"

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Part I of this Appendix provides detailed proofs of the formal conclusions in the paper. Part II characterizes outcomes under ubiquitous collusion, where U1, U2, D1, and D2 all collude to maximize their joint profit. Part III illustrates the qualitative changes that can arise when U1 and U2 are able to set two-part tariffs. Part IV provides additional conclusions from numerical solutions. We begin by reproducing the key equations in the paper.

Equations from the Text

$$U(\cdot) \;=\; \alpha_1\, q_1 + \alpha_2\, q_2 - \frac{1}{2}\left[\, (q_1)^2 + 2\,\gamma\, q_1\, q_2 + (q_2)^2 \,\right] - p_1\, q_1 - p_2\, q_2\,. \tag{1}$$

$$P_i(q_i, q_j) \;=\; \alpha_i - q_i - \gamma\, q_j \quad \text{for } i,\, j \in \{1, 2\} \;\; (j \neq i). \tag{2}$$

$$\pi_i(q_i, q_j) \;=\; \left[\, P_i(q_i, q_j) - w_i - c_i^d \,\right] q_i\,. \tag{3}$$

$$\Pi_i(\widetilde{q}_1, \widetilde{q}_2) \;=\; [\, \widetilde{w}_1 - c^u \,]\, \widetilde{q}_1 + [\, \widetilde{w}_2 - c^u \,]\, \widetilde{q}_2\,. \tag{4}$$

(i) $q_i^{nQ} \;=\; \dfrac{2\,\Delta_i - \gamma\,\Delta_j}{4-\gamma^2}$ and $\pi_i^{nQ} \;=\; \left[\dfrac{2\,\Delta_i - \gamma\,\Delta_j}{4-\gamma^2}\right]^2$ for $\dfrac{\Delta_2}{\Delta_1} \in (\dfrac{\gamma}{2}, \dfrac{2}{\gamma})$

under downstream quantity competition; and

(ii) $q_i^{nP} \;=\; \dfrac{[\,2-\gamma^2\,]\,\Delta_i - \gamma\,\Delta_j}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}$ and $\pi_i^{nP} \;=\; \dfrac{1}{1-\gamma^2}\left[\dfrac{(\,2-\gamma^2\,)\,\Delta_i - \gamma\,\Delta_j}{4-\gamma^2}\right]^2$

for $\dfrac{\Delta_2}{\Delta_1} \in (\dfrac{\gamma}{2-\gamma^2}, \dfrac{2-\gamma^2}{\gamma})$ under downstream price competition. (5)

$$Maximize \quad \left[\, \alpha_i - q_i - \gamma\, q_j - w_i - c_i^d \,\right] q_i \quad \Rightarrow \quad q_i \;=\; \frac{1}{2}\left[\, \alpha_i - w_i - c_i^d \,\right] - \frac{\gamma}{2}\, q_j\,. \tag{6}$$

$$q_i(w_i, w_j) \;=\; \frac{2\left[\, \alpha_i - w_i - c_i^d \,\right] - \gamma\left[\, \alpha_j - w_j - c_j^d \,\right]}{4-\gamma^2} \quad \text{for } i,\, j \in \{1, 2\} \;\; (j \neq i). \tag{7}$$

$$Maximize \quad \left[\, \alpha_1 - q_1 - \gamma\, q_2 - w_1 - c_1^d \,\right] q_1 + [\, w_1 - c^u \,]\, q_1 + [\, w_2 - c^u \,]\, q_2\,. \tag{8}$$

$$q_1(w_2) \;=\; \frac{2\,\Delta_1 - \gamma\left[\, \alpha_2 - w_2 - c_2^d \,\right]}{4\,-\,\gamma^2} \quad \text{and} \quad q_2(w_2) \;=\; \frac{2\left[\, \alpha_2 - w_2 - c_2^d \,\right] - \gamma\,\Delta_1}{4-\gamma^2}\,. \tag{9}$$

$$\left[\, \alpha_1 - q_1(w_2) - \gamma\, q_2(w_2) - c^u - c_1^d \,\right] q_1(w_2) + [\, w_2 - c^u \,]\, q_2(w_2)\,. \tag{10}$$

I. Proofs of Formal Conclusions

Lemma 1. *Suppose U1 and U2 do not collude. Then both in the presence of downstream price competition and in the presence of downstream quantity competition, U1 and U2 both offer input price c^u to: (i) D1 and D2 under VS; and (ii) to D2 under VI.*

Proof**.** The proof of conclusion (ii) follows from Findings 1A – 1C below. The proof of conclusion (i) is similar but more straightforward, and so is omitted. For convenience, the proof of conclusion (ii): (a) assumes Di purchases the input from Ui ($i = 1, 2$) when indifferent between purchasing the input from U1 and U2; and (b) focuses on settings where D1 and D2 both produce strictly positive output in equilibrium.

Finding 1A. *Suppose D2 buys the input from U2 at unit price w_2 whereas D1 secures the input from U1. Then in the absence of collusion under VI, the combined equilibrium profit of U1 and D1 under downstream quantity competition is*:

$$\Pi_1^I(w_2) \;=\; \left[\frac{2\,\Delta_1 - \gamma\left(\alpha_2 - w_2 - c_2^d\right)}{4-\gamma^2} \right]^2 . \tag{11}$$

Proof. Standard techniques (see the proof of Lemma 4) reveal that the equilibrium outputs of D1 and D2 in this setting are, respectively:

$$q_1 \;=\; \frac{2\,\Delta_1 - \gamma\left[\alpha_2 - w_2 - c_2^d\right]}{4-\gamma^2} \quad \text{and} \quad q_2 \;=\; \frac{2\left[\alpha_2 - w_2 - c_2^d\right] - \gamma\,\Delta_1}{4-\gamma^2} \tag{12}$$

$$\Rightarrow \; P_1 \;=\; \alpha_1 - q_1 - \gamma\, q_2$$

$$=\; \frac{1}{4-\gamma^2}\left\{ \left[4-\gamma^2\right]\alpha_1 - \left[2-\gamma^2\right]\Delta_1 - \gamma\left[\alpha_2 - w_2 - c_2^d\right] \right\} \tag{13}$$

$$\Rightarrow \; P_1 - c^u - c_1^d \;=\; \frac{2\,\Delta_1 - \gamma\left[\alpha_2 - w_2 - c_2^d\right]}{4-\gamma^2} . \tag{14}$$

(12) and (14) imply that the combined profit of U1 and D1 is as specified in (11). □

Finding 1B. *Suppose D1 and D2 secure the input from U1 at unit price w_1. Then in the absence of collusion under VI, the combined equilibrium profit of U1 and D1 under downstream quantity competition is*:

$$\widetilde{\Pi}_1^I(w_2) \;=\; \left[w_1 - c^u\right]\left[\frac{2\left(\alpha_2 - w_1 - c_2^d\right) - \gamma\,\Delta_1}{4-\gamma^2} \right] + \left[\frac{2\,\Delta_1 - \gamma\left(\alpha_2 - w_1 - c_2^d\right)}{4-\gamma^2} \right]^2 . \tag{15}$$

Proof. (12) implies that the equilibrium outputs of D1 and D2 in this setting are:

$$q_1 \;=\; \frac{2\,\Delta_1 - \gamma\left[\,\alpha_2 - w_1 - c_2^d\,\right]}{4-\gamma^2} \quad \text{and} \quad q_2 \;=\; \frac{2\left[\,\alpha_2 - w_1 - c_2^d\,\right] - \gamma\,\Delta_1}{4-\gamma^2}. \tag{16}$$

Therefore, the equilibrium downstream price is:

$$P_1 \;=\; \frac{1}{4-\gamma^2}\left\{\left[\,4-\gamma^2\,\right]\alpha_1 - \left[\,2-\gamma^2\,\right]\Delta_1 - \gamma\left[\,\alpha_2 - w_1 - c_2^d\,\right]\right\}$$

$$\Rightarrow \;\; P_1 - c^u - c_1^d \;=\; \frac{2\,\Delta_1 - \gamma\left[\,\alpha_2 - w_1 - c_2^d\,\right]}{4-\gamma^2}\,. \tag{17}$$

(16) and (17) imply that the combined profit of U1 and D1 is as specified in (15). □

Finding 1C. *In the absence of collusion and in the presence of downstream quantity competition under VI, D2 purchases the input from U2 at unit price $w_2 = c^u$ and D1 secures the input from U1.*

Proof. Suppose U2 sets $w_2 > c^u$ at a level that generates positive equilibrium output for D2 if D2 purchases the input from U2 and D1 secures the input from U1. Then if U1 sets $w_1 \geq w_2$, D2 will purchase the input from U2 and D1 will secure the input from U1. Consequently, the combined profit of U1 and D1 will be $\Pi_1^I(w_2)$, as specified in (11). If U1 sets w_1 just below w_2, D2 will purchase the input from U1 and D1 will secure the input from U1. Consequently, the combined profit of U1 and D1 will be nearly $\widetilde{\Pi}_1^I(w_2)$, as specified in (15). (11) and (15) imply:

$$\widetilde{\Pi}_1^I(w_2) \;>\; \Pi_1^I(w_2) \quad \Leftrightarrow \quad \left[\,w_1 - c^u\,\right]\left[\frac{2\left(\alpha_2 - w_2 - c_2^d\right) - \gamma\,\Delta_1}{4-\gamma^2}\right] \;>\; 0\,. \tag{18}$$

The last inequality in (18) reflects (12). Consequently, U1 will find it most profitable to slightly undercut any such $w_2 > c^u$. In response, U2 will find it most profitable to match the lower price set by U1. Therefore, the equilibrium value of the input price offered to D2 cannot exceed c^u.

U2 will secure negative profit if it sets $w_2 < c^u$ and either D1 or D2 purchases the input from U2. In contrast, U2 can secure zero profit in equilibrium by setting $w_2 = c^u$. This is the case because U1 will not reduce w_1 below c^u in response. Doing so would both reduce D1's downstream profit and cause U1 to incur a loss on its sale of the input to D2. Therefore, the equilibrium outcome is as specified in the Finding. □

The corresponding findings under downstream price competition are proved in analogous fashion. Together, the findings ensure the conclusions in the Lemma hold. ■

Lemma 2. *Suppose U1 and U2 do not collude. Then under VS and under VI, for* $i,\, j \in \{1, 2\}$ $(j \neq i)$*, D*i*'s equilibrium output and profit are, respectively:*[1]

$$(i)\quad q_i^{nQ} \;=\; \frac{2\,\Delta_i - \gamma\,\Delta_j}{4-\gamma^2} \quad and \quad \pi_i^{nQ} \;=\; \left[\frac{2\,\Delta_i - \gamma\,\Delta_j}{4-\gamma^2}\right]^2 \quad for \quad \frac{\Delta_2}{\Delta_1} \in \left(\frac{\gamma}{2}, \frac{2}{\gamma}\right)$$

under downstream quantity competition; and

$$(ii)\quad q_i^{nP} \;=\; \frac{[\,2-\gamma^2\,]\,\Delta_i - \gamma\,\Delta_j}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \quad and \quad \pi_i^{nP} \;=\; \frac{1}{1-\gamma^2}\left[\frac{(\,2-\gamma^2\,)\,\Delta_i - \gamma\,\Delta_j}{4-\gamma^2}\right]^2$$

$$for \quad \frac{\Delta_2}{\Delta_1} \in \left(\frac{\gamma}{2-\gamma^2}, \frac{2-\gamma^2}{\gamma}\right) \quad under\ downstream\ price\ competition.$$

Proof. To prove conclusion (i) in the Lemma, first consider VS. (2) implies that $\frac{\partial P_i(\cdot)}{\partial q_i} = -\,1$. Therefore, (3) implies that when Di incurs unit cost c_i, its profit-maximizing choice of q_i is determined by:

$$P_i(\cdot) - c_i - q_i \;=\; 0 \quad\Leftrightarrow\quad \alpha_i - q_i - \gamma\,q_j - c_i - q_i \;=\; 0$$

$$\Rightarrow\quad q_i \;=\; \frac{1}{2}\,[\,\alpha_i - c_i - \gamma\,q_j\,]\,. \tag{19}$$

By symmetry:

$$q_j \;=\; \frac{1}{2}\,[\,\alpha_j - c_j - \gamma\,q_i\,]\,. \tag{20}$$

(19) and (20) imply:

$$q_i \;=\; \frac{1}{2}\,[\,\alpha_i - c_i\,] - \frac{\gamma}{4}\,[\,\alpha_j - c_j - \gamma\,q_i\,]$$

$$\Rightarrow\quad q_i\left[1 - \frac{\gamma^2}{4}\right] \;=\; \frac{1}{4}\,[\,2\,\alpha_i - 2\,c_i - \gamma\,(\,\alpha_j - c_j\,)\,]$$

$$\Rightarrow\quad q_i\left[\frac{4-\gamma^2}{4}\right] \;=\; \frac{1}{4}\,[\,2\,(\,\alpha_i - c_i\,) - \gamma\,(\,\alpha_j - c_j\,)\,]\,. \tag{21}$$

(21) implies that Di's profit-maximizing output is as specified in (7).

$w_1 = w_2 = c^u$ when U1 and U2 do not collude. Therefore, (7) implies:

$$q_i^{nQ} \;=\; \frac{2\,\Delta_i - \gamma\,\Delta_j}{4-\gamma^2}\,. \tag{22}$$

The first line in (19) implies that $P_i(\cdot) - w_i - c_i^d = q_i$. Therefore, (3) and (22) imply:

[1] The superscript "n" in equation (5) denotes the setting with no collusion between U1 and U2. The supercript "Q" denotes downstream quantity competition. The supercript "P" denotes downstream price competition.

$$\pi_i^{nQ} \;=\; \left[\, P_i(\cdot) - w_i - c_i^d \,\right] q_i^n \;=\; [\, q_i^n \,]^2 \;=\; \left[\frac{2\,\Delta_i - \gamma\,\Delta_j}{4-\gamma^2} \right]^2 . \tag{23}$$

As explained in the text, outputs and profit are the same under VS and VI when $w_1 = w_2 = c^u$ (so U1 and U2 secure no upstream profit).

To prove conclusion (ii) in the Lemma, first consider VS. (Again, outcomes are the same under VI because $w_1 = w_2 = c^u$ in the absence of collusion). Under VS with downstream price competition, D_i chooses P_i to:

$$\text{Maximize} \quad \left[\, P_i - w_i - c_i^d \,\right] q_i \, . \tag{24}$$

(24) implies that Di's profit-maximizing choice of P_i is determined by:

$$q_i + \left[\, P_i - w_i - c_i^d \,\right] \frac{\partial q_i}{\partial P_i} \;=\; 0$$

$$\Rightarrow \; \frac{[\,\alpha_i - P_i\,] - \gamma\,[\,\alpha_j - P_j\,]}{1-\gamma^2} - \left[\, P_i - w_i - c_i^d \,\right] \frac{1}{1-\gamma^2} \;=\; 0$$

$$\Rightarrow \; P_i \;=\; \frac{\alpha_i - \gamma\,\alpha_j + w_i + c_i^d}{2} + \frac{\gamma}{2}\, P_j \, . \tag{25}$$

(25) implies that in equilibrium:

$$P_i \;=\; \frac{\alpha_i - \gamma\,\alpha_j + w_i + c_i^d}{2} + \frac{\gamma}{2} \left[\frac{\alpha_j - \gamma\,\alpha_i + w_j + c_j^d}{2} + \frac{\gamma}{2} P_i \right]$$

$$\Rightarrow \; P_i \left[1 - \frac{\gamma^2}{4} \right] \;=\; \frac{\alpha_i - \gamma\,\alpha_j + w_i + c_i^d}{2} + \frac{\gamma}{2} \left[\frac{\alpha_j - \gamma\,\alpha_i + w_j + c_j^d}{2} \right]$$

$$\Rightarrow \; P_i \left[\frac{4-\gamma^2}{4} \right] \;=\; \frac{[\,2-\gamma^2\,]\,\alpha_i - \gamma\,\alpha_j + 2\,w_i + 2\,c_i^d + \gamma\,w_j + \gamma\,c_j^d}{4}$$

$$\Rightarrow \; P_i \;=\; \frac{[\,2-\gamma^2\,]\,\alpha_i - \gamma\,\alpha_j + 2\,w_i + 2\,c_i^d + \gamma\,w_j + \gamma\,c_j^d}{4-\gamma^2} \, . \tag{26}$$

(2) implies:

$$q_i \;=\; \alpha_i - P_i - \gamma\,q_j \;=\; \alpha_i - P_i - \gamma\,[\,\alpha_j - P_j - \gamma\,q_i\,]$$

$$\Rightarrow \; q_i \left[\, 1 - \gamma^2 \,\right] \;=\; \alpha_i - \gamma\,\alpha_j - P_i + \gamma\,P_j$$

$$\Rightarrow \; q_i \;=\; \frac{\alpha_i - \gamma\,\alpha_j}{1-\gamma^2} - \frac{1}{1-\gamma^2}\, P_i + \frac{\gamma}{1-\gamma^2}\, P_j \, . \tag{27}$$

(26) and (27) imply:

$$q_i = \frac{\alpha_i - \gamma\,\alpha_j}{1-\gamma^2} - \frac{1}{1-\gamma^2}\; P_i + \frac{\gamma}{1-\gamma^2}\; P_j$$

$$= \frac{\alpha_i - \gamma\,\alpha_j}{1-\gamma^2} - \left[\frac{1}{1-\gamma^2}\right] \frac{[\,2-\gamma^2\,]\,\alpha_i - \gamma\,\alpha_j + 2\,w_i + 2\,c_i^d + \gamma\,w_j + \gamma\,c_j^d}{4-\gamma^2}$$

$$+ \frac{\gamma}{1-\gamma^2} \frac{[\,2-\gamma^2\,]\,\alpha_j - \gamma\,\alpha_i + 2\,w_j + 2\,c_j^d + \gamma\,w_i + \gamma\,c_i^d}{4-\gamma^2}$$

$$= \frac{[\,4-\gamma^2\,]\,\alpha_i - \gamma\,[\,4-\gamma^2\,]\;\alpha_j}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} - \frac{[\,2-\gamma^2\,]\,\alpha_i - \gamma\,\alpha_j + 2\,w_i + 2\,c_i^d + \gamma\,w_j + \gamma\,c_j^d}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}$$

$$+ \frac{\gamma\,[\,2-\gamma^2\,]\,\alpha_j - \gamma^2\,\alpha_i + 2\,\gamma\,w_j + 2\,\gamma\,c_j^d + \gamma^2 w_i + \gamma^2 c_i^d}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}$$

$$= \frac{1}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \left\{ \left[\,4-\gamma^2\,\right]\alpha_i - \gamma\left[\,4-\gamma^2\,\right]\;\alpha_j - \left[\,2-\gamma^2\,\right]\alpha_i + \gamma\,\alpha_j - 2\,w_i - 2\,c_i^d - \gamma\,w_j \right.$$

$$\left. -\;\gamma\,c_j^d + \gamma\left[\,2-\gamma^2\,\right]\alpha_j - \gamma^2\,\alpha_i + 2\,\gamma\,w_j + 2\,\gamma\,c_j^d + \gamma^2 w_i + \gamma^2 c_i^d \right\}$$

$$= \frac{1}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \left\{ \left[\,2-\gamma^2\,\right]\;\alpha_i - \gamma\,\alpha_j - \left[\,2-\gamma^2\,\right]w_i - \left[\,2-\gamma^2\,\right]c_i^d + \gamma\,w_j + \gamma\,c_j^d \right\}. \tag{28}$$

Under VI, D1 chooses P_1 to:

$$\text{Maximize} \quad \left[\,P_1 - c^u - c_1^d\,\right] q_1 + [\,w_2 - c^u\,]\,q_2\,. \tag{29}$$

(27) and (29) imply D1's profit-maximizing choice of P_1 is determined by:

$$q_1 + \left[\,P_1 - c^u - c_1^d\,\right] \frac{\partial q_1}{\partial P_1} + [\,w_2 - c^u\,]\,\frac{\partial q_2}{\partial P_1} = 0$$

$$\Rightarrow \quad \frac{[\,\alpha_1 - P_1\,] - \gamma\,[\,\alpha_2 - P_2\,]}{1-\gamma^2} - \left[\,P_1 - c^u - c_1^d\,\right]\frac{1}{1-\gamma^2} + [\,w_2 - c^u\,]\,\frac{\gamma}{1-\gamma^2} = 0$$

$$\Rightarrow \quad \left[\frac{2}{1-\gamma^2}\right] P_1 = \frac{\alpha_1 + c^u + c_1^d}{1-\gamma^2} + \frac{\gamma\,[\,w_2 - c^u - \alpha_2\,]}{1-\gamma^2} + \left[\frac{\gamma}{1-\gamma^2}\right] P_2$$

$$\Rightarrow \quad P_1 = \frac{\alpha_1 + c^u + c_1^d}{2} + \frac{\gamma\,[\,w_2 - c^u - \alpha_2\,]}{2} + \frac{\gamma}{2}\; P_2\,. \tag{30}$$

D2 chooses P_2 to:

$$\text{Maximize} \quad \left[\,P_2 - w_2 - c_2^d\,\right] q_2. \tag{31}$$

(27) and (31) imply D2's profit-maximizing choice of P_2 is determined by:

$$q_2 + \left[\, P_2 - w_2 - c_2^d \,\right] \frac{\partial q_2}{\partial P_2} \;=\; 0$$

$$\Rightarrow\quad \frac{\alpha_2 - P_2 - \gamma\,[\,\alpha_1 - P_1\,]}{1-\gamma^2} - \left[\, P_2 - w_2 - c_2^d \,\right] \frac{1}{1-\gamma^2} \;=\; 0$$

$$\Rightarrow\quad P_2 \;=\; \frac{\alpha_2 - \gamma\,\alpha_1 + w_2 + c_2^d}{2} + \frac{\gamma}{2}\, P_1\,. \tag{32}$$

(30) and (32) imply:

$$P_1 \;=\; \frac{\alpha_1 + c^u + c_1^d}{2} + \frac{\gamma\,[\,w_2 - c^u - \alpha_2\,]}{2} + \frac{\gamma}{2}\left[\frac{\alpha_2 - \gamma\,\alpha_1 + w_2 + c_2^d}{2} + \frac{\gamma}{2} P_1\right]$$

$$\Rightarrow\quad P_1 \left[1 - \frac{\gamma^2}{4}\right] \;=\; \frac{\alpha_1 + c^u + c_1^d}{2} + \frac{\gamma\,[\,w_2 - c^u - \alpha_2\,]}{2} + \frac{\gamma\left[\,\alpha_2 - \gamma\,\alpha_1 + w_2 + c_2^d\,\right]}{4}$$

$$\Rightarrow\quad P_1 \left[\frac{4-\gamma^2}{4}\right] \;=\; \frac{[\,2-\gamma^2\,]\;\alpha_1 + 2\,[\,1-\gamma\,]\,c^u + 2\,c_1^d + 3\,\gamma\,w_2 - \gamma\,\alpha_2 + \gamma\,c_2^d}{4}$$

$$\Rightarrow\quad P_1 \;=\; \frac{[\,2-\gamma^2\,]\;\alpha_1 + 2\,[\,1-\gamma\,]\,c^u + 2\,c_1^d + 3\,\gamma\,w_2 - \gamma\,\alpha_2 + \gamma\,c_2^d}{4-\gamma^2}\,. \tag{33}$$

(32) and (33) imply:

$$P_2 \;=\; \frac{\alpha_2 - \gamma\,\alpha_1 + w_2 + c_2^d}{2} + \frac{\gamma}{2}\left[\frac{(\,2-\gamma^2\,)\;\alpha_1 + 2\,(\,1-\gamma\,)\,c^u + 2\,c_1^d + 3\,\gamma\,w_2 - \gamma\,\alpha_2 + \gamma\,c_2^d}{4-\gamma^2}\right]$$

$$\Rightarrow\quad P_2 \;=\; \frac{2\,[\,2-\gamma^2\,]\,\alpha_2 + [\,4+2\,\gamma^2\,]\,w_2 + 4\,c_2^d - 2\,\gamma\,\alpha_1 + 2\,\gamma\,c_1^d + 2\,\gamma\,c^u - 2\,\gamma^2\,c^u}{2\,[\,4-\gamma^2\,]}$$

$$\Rightarrow\quad P_2 \;=\; \frac{2\,[\,2-\gamma^2\,]\,\alpha_2 + [\,4+2\,\gamma^2\,]\,w_2 + 4\,c_2^d - 2\,\gamma\,\Delta_1 - 2\,\gamma^2\,c^u}{2\,[\,4-\gamma^2\,]}\,. \tag{34}$$

(27), (33), and (34) imply:

$$q_1 \;=\; \frac{[\,\alpha_1 - P_1\,] - \gamma\,[\,\alpha_2 - P_2\,]}{1-\gamma^2}$$

$$= \frac{\alpha_1 - \gamma\,\alpha_2}{1-\gamma^2} - \frac{1}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\left\{\left[\,2-\gamma^2\,\right]\;\alpha_1 + 2\,[\,1-\gamma\,]\,c^u + 2\,c_1^d + 3\,\gamma\,w_2 - \gamma\,\alpha_2 + \gamma\,c_2^d\right\}$$

$$+ \frac{\gamma}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\left\{2\left[\,2-\gamma^2\,\right]\alpha_2 + \left[\,4+2\,\gamma^2\,\right]w_2 + 4\,c_2^d - 2\,\gamma\,\Delta_1 - 2\,\gamma^2\,c^u\right\}$$

$$= \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\,\{\,2\left[\,4-\gamma^2\,\right]\alpha_1 - 2\left[\,4-\gamma^2\,\right]\gamma\,\alpha_2 - 2\left[\,2-\gamma^2\,\right]\,\alpha_1 - 4\,[\,1-\gamma\,]\,c^u - 4\,c_1^d$$

$$-\;\; 6\,\gamma\,w_2 + 2\,\gamma\,\alpha_2 - 2\,\gamma\,c_2^d + 2\,\gamma\left[\,2-\gamma^2\,\right]\alpha_2$$

$$+\;\; \gamma\left[\,4+2\,\gamma^2\,\right]w_2 + 4\,\gamma\,c_2^d - 2\,\gamma^2\,\Delta_1 - 2\,\gamma^3\,c^u\,\}$$

$$= \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\,\{\,4\,\alpha_1 - 4\,c_1^d - 2\,\gamma\,\alpha_2 + 2\,\gamma\,c_2^d - 2\left[\,2-2\,\gamma+\gamma^3\,\right]c^u$$

$$-\;\; 2\,\gamma\left[\,1-\gamma^2\,\right]w_2 - 2\,\gamma^2\,\Delta_1\,\}$$

$$= \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\,\{\,4\,\alpha_1 - 4\,c_1^d - 4\,c^u - 2\,\gamma\,\alpha_2 + 2\,\gamma\,c_2^d + 2\,\gamma\,c^u - 2\left[\,-\gamma+\gamma^3\,\right]c^u$$

$$-\;\; 2\,\gamma\left[\,1-\gamma^2\,\right]w_2 - 2\,\gamma^2\,\Delta_1\,\}$$

$$= \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\,\{\,4\,\Delta_1 - 2\,\gamma\,\Delta_2 + 2\,\gamma\left[\,1-\gamma^2\,\right]c^u - 2\,\gamma\left[\,1-\gamma^2\,\right]w_2 - 2\,\gamma^2\,\Delta_1\,\}$$

$$= \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\,\{\,2\left[\,2-\gamma^2\,\right]\Delta_1 - 2\,\gamma\,\Delta_2 + 2\,\gamma\left[\,1-\gamma^2\,\right]c^u - 2\,\gamma\left[\,1-\gamma^2\,\right]w_2\,\}; \qquad (35)$$

$$q_2 \;=\; \frac{\alpha_2 - P_2 - \gamma\,[\,\alpha_1 - P_1\,]}{1-\gamma^2}$$

$$= \frac{\alpha_2 - \gamma\,\alpha_1}{1-\gamma^2} + \frac{\gamma}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\,\{\left[\,2-\gamma^2\,\right]\,\alpha_1 + 2\,[\,1-\gamma\,]\,c^u + 2\,c_1^d + 3\,\gamma\,w_2 - \gamma\,\alpha_2 + \gamma\,c_2^d\,\}$$

$$-\; \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\,\{\,2\left[\,2-\gamma^2\,\right]\alpha_2 + \left[\,4+2\,\gamma^2\,\right]w_2 + 4\,c_2^d - 2\,\gamma\,\Delta_1 - 2\,\gamma^2\,c^u\,\}$$

$$= \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\,\{\,2\left[\,4-\gamma^2\,\right]\alpha_2 - 2\left[\,4-\gamma^2\,\right]\gamma\,\alpha_1 + 2\,\gamma\left[\,2-\gamma^2\,\right]\,\alpha_1 + 4\,\gamma\,[\,1-\gamma\,]\,c^u$$

$$+\;\; 4\,\gamma\,c_1^d + 6\,\gamma^2\,w_2 - 2\,\gamma^2\,\alpha_2 + 2\,\gamma^2\,c_2^d$$

$$-\;\; 2\left[\,2-\gamma^2\,\right]\alpha_2 - \left[\,4+2\,\gamma^2\,\right]w_2 - 4\,c_2^d + 2\,\gamma\,\Delta_1 + 2\,\gamma^2\,c^u\,\}$$

$$= \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\,\{\,2\left[\,2-\gamma^2\,\right]\alpha_2 - 2\left[\,2-\gamma^2\,\right]c_2^d - 4\,\gamma\,\alpha_1 + 4\,\gamma\,c_1^d + 2\,\gamma\,[\,2-\gamma\,]\,c^u$$

$$-\;\; 4\left[\,1-\gamma^2\,\right]w_2 + 2\,\gamma\,\Delta_1\,\}$$

$$= \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\,\{\,2\left[\,2-\gamma^2\,\right]\alpha_2 - 2\left[\,2-\gamma^2\,\right]c_2^d - 2\left[\,2-\gamma^2\,\right]c^u - 4\,\gamma\,\alpha_1 + 4\,\gamma\,c_1^d + 4\,\gamma\,c^u$$

$$+\;\; 2\left[\,2-\gamma^2\,\right]c^u - 2\,\gamma^2\,c^u - 4\left[\,1-\gamma^2\,\right]w_2 + 2\,\gamma\,\Delta_1\,\}$$

$$= \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\,\{\,2\left[\,2-\gamma^2\,\right]\Delta_2 - 2\,\gamma\,\Delta_1 + 4\left[\,1-\gamma^2\,\right]\,c^u - 4\left[\,1-\gamma^2\,\right]w_2\,\}. \qquad (36)$$

$w_1 = w_2 = c^u$ when U1 and U2 do not collude. Therefore, (28), (35), (36) imply:

$$q_i^{nP} = \frac{[\,2-\gamma^2\,]\,\Delta_i - \gamma\,\Delta_j}{[\,1-\gamma^2\,][\,4-\gamma^2\,]}\,. \tag{37}$$

Di's downstream profit under price competition is:

$$\pi_i\,(P_i, P_j) = \left[\,P_i - w_i - c_i^d\,\right] q_i\,(P_i, P_j)\,. \tag{38}$$

(27) implies that $\frac{\partial q_i(\cdot)}{\partial P_i} = -\frac{1}{1-\gamma^2}$. Therefore, (38) implies that Di's profit-maximizing choice of P_i is determined by:

$$q_i(\cdot) - \frac{1}{1-\gamma^2}\left[\,P_i - w_i - c_i^d\,\right] = 0 \;\;\Rightarrow\;\; P_i - w_i - c_i^d = \left[\,1-\gamma^2\,\right] q_i(\cdot)\,. \tag{39}$$

(37), (38), and (39) imply:

$$\pi_i^{nP} = \left[\,1-\gamma^2\,\right]\left[\,q_i^{nP}\,\right]^2 = \frac{1}{1-\gamma^2}\left[\frac{(\,2-\gamma^2\,)\,\Delta_i - \gamma\,\Delta_j}{4-\gamma^2}\right]^2. \;\blacksquare \tag{40}$$

Lemma 3. *$w_i^{SQ} = c^u + \frac{1}{2}\,\Delta_i$ and $q_i^{SQ} = \frac{2\,\Delta_i - \gamma\,\Delta_j}{2\,[\,4-\gamma^2\,]}$ for all $\frac{\Delta_2}{\Delta_1} \in (\,\frac{\gamma}{2}, \frac{2}{\gamma}\,)$, for $i,\, j \in \{1,2\}$ ($j \neq i$).*

Proof. (4) and (7) imply that U1 and U2 choose w_1 and w_2 to maximize:

$$[\,w_1 - c^u\,]\,\frac{2\left[\,\alpha_1 - w_1 - c_1^d\,\right] - \gamma\left[\,\alpha_2 - w_2 - c_2^d\,\right]}{4-\gamma^2}$$
$$+\;[\,w_2 - c^u\,]\,\frac{2\left[\,\alpha_2 - w_2 - c_2^d\,\right] - \gamma\left[\,\alpha_1 - w_1 - c_1^d\,\right]}{4-\gamma^2}\,. \tag{41}$$

(41) implies that the profit-maximizing value of w_i ($i \in \{1,2\}$) is determined by:

$$[\,w_i - c^u\,][\,-2\,] + 2\left[\,\alpha_i - w_i - c_i^d\,\right] - \gamma\left[\,\alpha_j - w_j - c_j^d\,\right] + \gamma\,[\,w_j - c^u\,] = 0$$
$$\Rightarrow\;\; 4\,w_i = [\,2-\gamma\,]\,c^u + 2\left[\,\alpha_i - c_i^d\,\right] - \gamma\left[\,\alpha_j - c_j^d\,\right] + 2\,\gamma\,w_j$$
$$\Rightarrow\;\; 4\,w_i = [\,4-2\,\gamma\,]\,c^u + 2\left[\,\alpha_i - c^u - c_i^d\,\right] - \gamma\left[\,\alpha_j - c^u - c_j^d\,\right] + 2\,\gamma\,w_j$$
$$\Rightarrow\;\; w_i = \left[\frac{2-\gamma}{2}\right] c^u + \frac{\Delta_i}{2} - \frac{\gamma\,\Delta_j}{4} + \frac{\gamma}{2}\,w_j\,. \tag{42}$$

By symmetry, (42) implies:

$$w_i = \left[\frac{2-\gamma}{2}\right] c^u + \frac{\Delta_i}{2} - \frac{\gamma\,\Delta_j}{4} + \frac{\gamma}{2}\left[\left(\frac{2-\gamma}{2}\right) c^u + \frac{\Delta_j}{2} - \frac{\gamma\,\Delta_i}{4} + \frac{\gamma}{2}\,w_i\right]$$

$$\Rightarrow \quad w_i \left[1 - \frac{\gamma^2}{4} \right] \;=\; \left[\frac{4 - 2\,\gamma}{4} \right] c^u + \frac{2\,\Delta_i}{4} - \frac{\gamma\,\Delta_j}{4} + \left[\frac{\gamma\,(\,2 - \gamma\,)}{4} \right] c^u + \frac{\gamma\,\Delta_j}{4} - \frac{\gamma^2\,\Delta_i}{8}$$

$$\Rightarrow \quad w_i \left[\frac{4 - \gamma^2}{4} \right] \;=\; \left[\frac{4 - \gamma^2}{4} \right] c^u + \frac{[\,4 - \gamma^2\,]\,\Delta_i}{8} \quad \Rightarrow \quad w_i \;=\; c^u + \frac{1}{2}\,\Delta_i\,. \tag{43}$$

(7) and (43) imply:

$$q_i^{SQ} \;=\; \frac{1}{4 - \gamma^2} \left\{ 2\left[\,\alpha_i - c_i^d\,\right] - \gamma\left[\,\alpha_j - c_j^d\,\right] - 2\,w_i + \gamma\,w_j \right\}$$

$$= \frac{1}{4 - \gamma^2} \left\{ 2\left[\,\alpha_i - c_i^d\,\right] - \gamma\left[\,\alpha_j - c_j^d\,\right] - 2\left[c^u + \frac{1}{2}\,\Delta_i \right] + \gamma\left[c^u + \frac{1}{2}\,\Delta_j \right] \right\}$$

$$= \frac{1}{4 - \gamma^2} \left\{ 2\left[\,\alpha_i - c^u - c_i^d\,\right] - \gamma\left[\,\alpha_j - c^u - c_j^d\,\right] - 2\left[\frac{1}{2}\,\Delta_i \right] + \gamma\left[\frac{1}{2}\,\Delta_j \right] \right\}$$

$$= \frac{1}{4 - \gamma^2} \left[2\,\Delta_i \, - \gamma\,\Delta_j - \Delta_i \, + \, \frac{\gamma}{2}\,\Delta_j \right] \;=\; \frac{\Delta_i \, - \frac{\gamma}{2}\,\Delta_j}{4 - \gamma^2} \;=\; \frac{2\,\Delta_i \, - \, \gamma\,\Delta_j}{2\,[\,4 - \gamma^2\,]}\,. \quad \blacksquare \tag{44}$$

Proposition 1. $L_2^{SQ} \;=\; \frac{3}{4}\left[\frac{2\,\Delta_2 - \gamma\,\Delta_1}{4 - \gamma^2}\right]^2$ *for all* $\frac{\Delta_2}{\Delta_1} \in (\,\frac{\gamma}{2}, \frac{2}{\gamma}\,)$.

Proof. The first line in (19) implies that $P_2(\cdot) - w_2 - c_2^d \;=\; q_2$. Therefore, (3) and (44) imply:

$$\pi_2^{SQ} \;=\; \left[\,q_2^{SQ}\,\right]^2 \;=\; \left[\frac{2\,\Delta_2 \, - \, \gamma\,\Delta_1}{2\,(\,4 - \gamma^2\,)} \right]^2 \;=\; \frac{1}{4}\left[\frac{2\,\Delta_2 \, - \, \gamma\,\Delta_1}{4 \, - \, \gamma^2} \right]^2. \tag{45}$$

(23) and (45) imply:

$$L_2^{SQ} \;=\; \left[\frac{2\,\Delta_2 \, - \, \gamma\,\Delta_1}{4 \, - \, \gamma^2} \right]^2 - \frac{1}{4}\left[\frac{2\,\Delta_2 \, - \, \gamma\,\Delta_1}{4 \, - \, \gamma^2} \right]^2 \;=\; \frac{3}{4}\left[\frac{2\,\Delta_2 \, - \, \gamma\,\Delta_1}{4 \, - \, \gamma^2} \right]^2. \quad \blacksquare$$

Lemma 4. $w_2^{IQ} \;=\; c^u + \frac{4\left[\,2 - \gamma^2\,\right]\Delta_2 + \gamma^3\,\Delta_1}{2\,[\,8 - 3\,\gamma^2\,]}$, $q_1^{IQ} \;=\; \frac{\left[\,8 - \gamma^2\,\right]\Delta_1 - 2\,\gamma\,\Delta_2}{2\,[\,8 - 3\,\gamma^2\,]}$, *and* $q_2^{IQ} \;=\; \frac{2\,[\,\Delta_2 - \gamma\,\Delta_1\,]}{8 - 3\,\gamma^2}$ *when* $\frac{\Delta_2}{\Delta_1} \in (\,\gamma, \frac{2}{\gamma}\,)$.

Proof. Under VI, D1 chooses q_1 to:

$$\text{Maximize} \quad \left[\,P_1 - c^u - c_1^d\,\right] q_1 + \left[\,w_2 - c^u\,\right] q_2. \tag{46}$$

(2) and (46) imply that q_1^{IQ} is determined by:

$$P_1 - c^u - c_1^d + q_1 \frac{\partial P_1(\cdot)}{\partial q_1} = 0 \quad \Rightarrow \quad q_1 = P_1 - c^u - c_1^d$$

$$\Rightarrow \quad q_1 = \alpha_1 - q_1 - \gamma \, q_2 - c^u - c_1^d$$

$$\Rightarrow \quad q_1 = \frac{1}{2} \left[\alpha_1 - c^u - c_1^d \right] - \frac{\gamma}{2} q_2 = \frac{\Delta_1}{2} - \frac{\gamma}{2} q_2 \, . \tag{47}$$

(20) implies that D2's choice of q_2 under VI is:

$$q_2 = \frac{1}{2} \left[\alpha_2 - w_2 - c_2^d \right] - \frac{\gamma}{2} q_1 \, . \tag{48}$$

(47) and (48) imply:

$$q_1 = \frac{1}{2} \Delta_1 - \frac{\gamma}{4} \left[\alpha_2 - w_2 - c_2^d - \gamma \, q_1 \right]$$

$$\Rightarrow \quad q_1 \left[1 - \frac{\gamma^2}{4} \right] = \frac{1}{4} \left[2 \, \Delta_1 - \gamma \left(\alpha_2 - w_2 - c_2^d \right) \right]$$

$$\Rightarrow \quad q_1 = \frac{2 \, \Delta_1 - \gamma \left[\alpha_2 - w_2 - c_2^d \right]}{4 - \gamma^2} \, . \tag{49}$$

(48) and (49) imply:

$$q_2 = \frac{1}{2} \left[\alpha_2 - w_2 - c_2^d \right] - \frac{\gamma}{2} \left[\frac{2 \, \Delta_1 - \gamma \left(\alpha_2 - w_2 - c_2^d \right)}{4 - \gamma^2} \right]$$

$$= \frac{1}{2 \left[4 - \gamma^2 \right]} \left[\left(4 - \gamma^2 \right) \left(\alpha_2 - w_2 - c_2^d \right) - 2 \, \gamma \, \Delta_1 + \gamma^2 \left(\alpha_2 - w_2 - c_2^d \right) \right]$$

$$= \frac{4 \left[\alpha_2 - w_2 - c_2^d \right] - 2 \, \gamma \, \Delta_1}{2 \left[4 - \gamma^2 \right]} = \frac{2 \left[\alpha_2 - w_2 - c_2^d \right] - \gamma \, \Delta_1}{4 - \gamma^2} \, . \tag{50}$$

(2), (49), and (50) imply:

$$P_1 = \alpha_1 - q_1 - \gamma \, q_2 = \frac{1}{4 - \gamma^2} \left\{ \left[4 - \gamma^2 \right] \alpha_1 - 2 \, \Delta_1 + \gamma \left[\alpha_2 - w_2 - c_2^d \right] \right.$$

$$\left. - \; 2 \, \gamma \left[\alpha_2 - w_2 - c_2^d \right] + \gamma^2 \, \Delta_1 \right\}$$

$$= \frac{1}{4 - \gamma^2} \left\{ \left[4 - \gamma^2 \right] \alpha_1 - \left[2 - \gamma^2 \right] \Delta_1 - \gamma \left[\alpha_2 - w_2 - c_2^d \right] \right\} . \tag{51}$$

(49) – (51) imply:

$$\frac{\partial P_1}{\partial w_2} \;=\; \frac{\gamma}{4-\gamma^2}\,; \qquad \frac{\partial q_1}{\partial w_2} \;=\; \frac{\gamma}{4-\gamma^2}\,; \qquad \frac{\partial q_2}{\partial w_2} \;=\; -\,\frac{2}{4-\gamma^2}\,. \tag{52}$$

Under VI, U1 and U2 choose w_2 to:

$$\text{Maximize} \quad \left[\, P_1(\cdot) - c^u - c_1^d \,\right] q_1(\cdot) + [\, w_2 - c^u \,]\, q_2(\cdot)\,. \tag{53}$$

(53) implies that U1 and U2's profit-maximizing choice of w_2 is determined by:

$$q_1 \frac{\partial P_1(\cdot)}{\partial w_2} + \left[\, P_1 - c^u - c_1^d \,\right] \frac{\partial q_1(\cdot)}{\partial w_2} + q_2 + [\, w_2 - c^u \,]\, \frac{\partial q_2(\cdot)}{\partial w_2} \;=\; 0\,. \tag{54}$$

(52) and (54):

$$q_1 \left[\frac{\gamma}{4-\gamma^2} \right] + \left[\, P_1 - c^u - c_1^d \,\right] \left[\frac{\gamma}{4-\gamma^2} \right] + q_2 - [\, w_2 - c^u \,] \left[\frac{2}{4-\gamma^2} \right] \;=\; 0$$

$$\Rightarrow \quad \gamma\, q_1 + \gamma \left[\, P_1 - c^u - c_1^d \,\right] + \left[\, 4 - \gamma^2 \,\right] q_2 - 2\,[\, w_2 - c^u \,] \;=\; 0\,. \tag{55}$$

(51) implies:

$$P_1 \;=\; \frac{1}{4-\gamma^2} \left\{ \left[\, 4-\gamma^2 \,\right] \alpha_1 - \left[\, 2-\gamma^2 \,\right] \left[\, \alpha_1 - c^u - c_1^d \,\right] - \gamma \left[\, \alpha_2 - w_2 - c_2^d \,\right] \right\}$$

$$=\; \frac{1}{4-\gamma^2} \left\{ 2\,\alpha_1 + \left[\, 2-\gamma^2 \,\right] c_1^d - \gamma\,\alpha_2 + \gamma\, c_2^d + \left[\, 2-\gamma^2 \,\right] c^u + \gamma\, w_2 \right\}. \tag{56}$$

(56) implies:

$$\begin{aligned} P_1 - c^u - c_1^d \;=\; & \frac{1}{4-\gamma^2} \left\{ 2\,\alpha_1 + \left[\, 2-\gamma^2 - \left(4-\gamma^2 \right) \right] c_1^d - \gamma\,\alpha_2 + \gamma\, c_2^d \right. \\ & \qquad\qquad \left. + \left[\, 2-\gamma^2 - \left(4-\gamma^2 \right) \right] c^u + \gamma\, w_2 \right\} \\ =\; & \frac{1}{4-\gamma^2} \left[\, 2\,\alpha_1 - \, 2\, c_1^d - \gamma\,\alpha_2 + \gamma\, c_2^d - 2\, c^u + \gamma\, w_2 \,\right]. \end{aligned} \tag{57}$$

(49), (50), (55), and (57) imply:

$$\frac{1}{4-\gamma^2} \left\{ 2\,\gamma\,\Delta_1 - \gamma^2 \left[\, \alpha_2 - w_2 - c_2^d \,\right] + 2\,\gamma\,\alpha_1 - 2\,\gamma\, c_1^d - 2\,\gamma\, c^u - \gamma^2 \left[\, \alpha_2 - c_2^d \,\right] + \gamma^2\, w_2 \right\}$$

$$+\, 2 \left[\, \alpha_2 - w_2 - c_2^d \,\right] - \gamma\,\Delta_1 - 2\,[\, w_2 - c^u \,] \;=\; 0$$

$$\Rightarrow \quad \frac{1}{4-\gamma^2} \left[\, 4\,\gamma\,\Delta_1 - 2\,\gamma^2 \left(\alpha_2 - c_2^d \right) + 2\,\gamma^2\, w_2 \,\right]$$

$$+ \, 2 \left[\alpha_2 - c_2^d \right] - \gamma \, \Delta_1 - 4 \, w_2 + 2 \, c^u \;\; = \;\; 0$$

$$\Rightarrow \quad \frac{1}{4 - \gamma^2} \, \left[4 \, \gamma \, \Delta_1 - 2 \, \gamma^2 \left(\alpha_2 - c^u - c_2^d \right) + 2 \, \gamma^2 \, (\, w_2 - c^u \,) \right]$$

$$+ \, 2 \left[\alpha_2 - c^u - c_2^d \right] - \gamma \, \Delta_1 - 4 \, w_2 + 4 \, c^u \;\; = \;\; 0$$

$$\Rightarrow \quad \frac{1}{4 - \gamma^2} \, \left[4 \, \gamma \, \Delta_1 - 2 \, \gamma^2 \, \Delta_2 \, + 2 \, \gamma^2 \, (\, w_2 - c^u \,) \right] + 2 \, \Delta_2 - \gamma \, \Delta_1 - 4 \, [\, w_2 - c^u \,] \;\; = \;\; 0$$

$$\Rightarrow \quad 4 \, \gamma \, \Delta_1 - 2 \, \gamma^2 \, \Delta_2 \, + 2 \, \gamma^2 \, [\, w_2 - c^u \,]$$

$$+ \, 2 \left[4 - \gamma^2 \right] \Delta_2 - \gamma \left[4 - \gamma^2 \right] \Delta_1 - 4 \left[4 - \gamma^2 \right] [\, w_2 - c^u \,] \;\; = \;\; 0$$

$$\Rightarrow \quad \left[16 - 6 \, \gamma^2 \right] [\, w_2 - c^u \,] \;\; = \;\; \left[8 - 4 \, \gamma^2 \right] \Delta_2 + \gamma^3 \, \Delta_1$$

$$\Rightarrow \quad 2 \left[8 - 3 \, \gamma^2 \right] [\, w_2 - c^u \,] \;\; = \;\; 4 \left[2 - \gamma^2 \right] \Delta_2 + \gamma^3 \, \Delta_1$$

$$\Rightarrow \quad w_2 \;\; = \;\; c^u + \frac{4 \, [\, 2 - \gamma^2 \,] \, \Delta_2 + \gamma^3 \, \Delta_1}{2 \, [\, 8 - 3 \, \gamma^2 \,]} \, . \tag{58}$$

(49) and (58) imply:

$$q_1^{IQ} \;\; = \;\; \frac{2 \, \Delta_1 - \gamma \left[\alpha_2 - c^u - c_2^d \right] + \gamma \, [\, w_2 - c^u \,]}{4 - \gamma^2} \;\; = \;\; \frac{2 \, \Delta_1 - \gamma \, \Delta_2 \, + \gamma \, \frac{4 \left[2 - \gamma^2 \right] \Delta_2 + \gamma^3 \, \Delta_1}{2 \, [\, 8 - 3 \, \gamma^2 \,]}}{4 - \gamma^2}$$

$$= \;\; \frac{2 \, [\, 8 \, - 3 \, \gamma^2 \,] \, [\, 2 \, \Delta_1 - \gamma \, \Delta_2 \,] + 4 \, \gamma \, [\, 2 - \gamma^2 \,] \, \Delta_2 + \gamma^4 \, \Delta_1}{2 \, [\, 8 - 3 \, \gamma^2 \,] \, [\, 4 - \gamma^2 \,]}$$

$$= \;\; \frac{[\, 4 \, (\, 8 \, - \, 3 \, \gamma^2 \,) + \gamma^4 \,] \, \Delta_1 + [\, 4 \, \gamma \, (\, 2 - \gamma^2 \,) - 2 \, \gamma \, (\, 8 - 3 \, \gamma^2 \,) \,] \, \Delta_2}{2 \, [\, 8 - 3 \, \gamma^2 \,] \, [\, 4 - \gamma^2 \,]}$$

$$= \;\; \frac{[\, 32 \, - 12 \, \gamma^2 \, + \gamma^4 \,] \, \Delta_1 + [\, 8 \, \gamma - 4 \, \gamma^3 - 16 \, \gamma + 6 \, \gamma^3 \,] \, \Delta_2}{2 \, [\, 8 - 3 \, \gamma^2 \,] \, [\, 4 - \gamma^2 \,]}$$

$$= \;\; \frac{[\, 8 \, - \gamma^2 \,] \, [\, 4 - \gamma^2 \,] \, \Delta_1 - [\, 8 \, \gamma - 2 \, \gamma^3 \,] \, \Delta_2}{2 \, [\, 8 - 3 \, \gamma^2 \,] \, [\, 4 - \gamma^2 \,]} \;\; = \;\; \frac{[\, 8 \, - \gamma^2 \,] \, \Delta_1 - 2 \, \gamma \, \Delta_2}{2 \, [\, 8 - 3 \, \gamma^2 \,]} \, . \tag{59}$$

(50) and (58) imply:

$$q_2^{IQ} \;\; = \;\; \frac{2 \left[\alpha_2 - c^u - c_2^d \right] - \gamma \, \Delta_1 - 2 \, [\, w_2 - c^u \,]}{4 - \gamma^2} \;\; = \;\; \frac{2 \, \Delta_2 - \gamma \, \Delta_1 - 2 \;\; \frac{4 \left[2 - \gamma^2 \right] \Delta_2 + \gamma^3 \, \Delta_1}{2 \, [\, 8 - 3 \, \gamma^2 \,]}}{4 - \gamma^2}$$

$$= \;\; \frac{[\, 8 - 3 \, \gamma^2 \,] \, [\, 2 \, \Delta_2 - \gamma \, \Delta_1 \,] - 4 \, [\, 2 - \gamma^2 \,] \, \Delta_2 - \gamma^3 \, \Delta_1}{[\, 8 - 3 \, \gamma^2 \,] \, [\, 4 - \gamma^2 \,]}$$

$$= \frac{[\,2\,(8-3\,\gamma^2\,) - 4\,(\,2-\gamma^2\,)\,]\,\Delta_2 - [\,\gamma\,(8-3\,\gamma^2\,) + \gamma^3\,]\,\Delta_1}{[\,8-3\,\gamma^2\,]\,[\,4-\gamma^2\,]}$$

$$= \frac{[\,8-2\,\gamma^2\,]\,\Delta_2 - \gamma\,[\,8-2\,\gamma^2\,]\,\Delta_1}{[\,8-3\,\gamma^2\,]\,[\,4-\gamma^2\,]} = \frac{2\,[\,\Delta_2 - \gamma\,\Delta_1\,]}{8-3\,\gamma^2}\,. \quad \blacksquare \tag{60}$$

Proposition 2. $L_2^{IQ} = \begin{cases} \left[\frac{2\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 & \textit{when } \frac{\Delta_2}{\Delta_1} \in (\frac{\gamma}{2}, \gamma) \\ \left[\frac{2\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 - \left[\frac{2\,(\,\Delta_2 - \gamma\,\Delta_1\,)}{8-3\,\gamma^2}\right]^2 & \textit{when } \frac{\Delta_2}{\Delta_1} \in (\gamma, \frac{2}{\gamma})\,. \end{cases}$

<u>Proof</u>. The first line in (19) implies that $P_2(\cdot) - w_2 - c_2^d = q_2$. Therefore, (3) and (60) imply:

$$\pi_2^{IQ} = \left[q_2^{IQ}\right]^2 = \left[\frac{2\,(\,\Delta_2 - \gamma\,\Delta_1\,)}{8-3\,\gamma^2}\right]^2. \tag{61}$$

(23) and (61) imply that when $\frac{\Delta_2}{\Delta_1} \in (\gamma, \frac{2}{\gamma})$:

$$L_2^{IQ} = \pi_2^n - \pi_2^{IQ} = \left[\frac{2\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right] - \left[\frac{2\,(\,\Delta_2 - \gamma\,\Delta_1\,)}{8-3\,\gamma^2}\right]^2.$$

(60) implies that $q_2^{IQ} = 0$ when $\Delta_2 - \gamma\,\Delta_1 \leq 0 \;\Leftrightarrow\; \frac{\Delta_2}{\Delta_1} \leq \gamma$. D2's profit is 0 in this case. Therefore, (23) implies:

$$L_2^{IQ} = \pi_2^n - 0 = \left[\frac{2\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 \text{ when } \frac{\Delta_2}{\Delta_1} \in (\frac{\gamma}{2}, \gamma). \quad \blacksquare$$

Theorem 1. $L_2^{IQ} > L_2^{SQ}$ *for all* $\frac{\Delta_2}{\Delta_1} \in (\frac{\gamma}{2}, \frac{2}{\gamma})$.

<u>Proof</u>. Propositions 1 and 2 imply that when $\frac{\Delta_2}{\Delta_1} \in (\frac{\gamma}{2}, \gamma)$:

$$L_2^{IQ} - L_2^{SQ} = \left[\frac{2\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 - \frac{3}{4}\left[\frac{2\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 = \frac{1}{4}\left[\frac{2\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 > 0\,.$$

Propositions 1 and 2 imply that when $\frac{\Delta_2}{\Delta_1} \in (\gamma, \frac{2}{\gamma})$:

$$L_2^{IQ} - L_2^{SQ} = \left[\frac{2\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 - \left[\frac{2\,(\,\Delta_2 - \gamma\,\Delta_1\,)}{8-3\,\gamma^2}\right]^2 - \frac{3}{4}\left[\frac{2\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2$$

$$= \frac{1}{4}\left[\frac{2\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 - \left[\frac{2\,(\,\Delta_2 - \gamma\,\Delta_1\,)}{8-3\,\gamma^2}\right]^2 > 0$$

$$\Leftrightarrow \quad \frac{2\,\Delta_2 \,-\, \gamma\,\Delta_1}{2\,[\,4-\gamma^2\,]} \;>\; \frac{2\,[\,\Delta_2 - \gamma\,\Delta_1\,]}{8-3\,\gamma^2} \quad \Leftrightarrow \quad \frac{\Delta_2}{\Delta_1} \;<\; \frac{8-\gamma^2}{2\,\gamma}\,. \tag{62}$$

The inequality in (62) holds when $\frac{\Delta_2}{\Delta_1} \in (\gamma, \frac{2}{\gamma})$ because:

$$\frac{2}{\gamma} \;<\; \frac{8-\gamma^2}{2\,\gamma} \;\Leftrightarrow\; 4 \;<\; 8-\gamma^2 \;\Leftrightarrow\; \gamma^2 \;<\; 4\,. \;\blacksquare$$

Lemma 5. *$w_i^{SP} = c^u + \frac{1}{2}\,\Delta_i$ and $q_i^{SP} = \frac{[\,2-\gamma^2\,]\,\Delta_i - \gamma\,\Delta_j}{2\,[\,1-\gamma^2\,][\,4-\gamma^2\,]}$ for $i,\, j \in \{1,2\}$ ($j \neq i$) when $\frac{\Delta_2}{\Delta_1} \in (\frac{\gamma}{2-\gamma^2}, \frac{2-\gamma^2}{\gamma})$.*

<u>Proof</u>. (4) and (28) imply that U1 and U2 choose w_1 and w_2 to maximize:

$$[\,w_1 - c^u\,]\;\frac{[\,2-\gamma^2\,]\,[\,\alpha_1 - w_1 - c_1^d\,] - \gamma\,[\,\alpha_2 - w_2 - c_2^d\,]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}$$
$$+\;[\,w_2 - c^u\,]\;\frac{[\,2-\gamma^2\,]\,[\,\alpha_2 - w_2 - c_2^d\,] - \gamma\,[\,\alpha_1 - w_1 - c_1^d\,]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\,. \tag{63}$$

(63) implies that the profit-maximizing value of w_i ($i \in \{1,2\}$) is determined by:

$$[\,w_i - c^u\,]\,[-\left(2-\gamma^2\right)] + [\,2-\gamma^2\,]\,[\,\alpha_i - w_i - c_i^d\,] - \gamma\,[\,\alpha_j - w_j - c_j^d\,] + \gamma\,[\,w_j - c^u\,] \;=\; 0$$
$$\Rightarrow\; 2\,[\,2-\gamma^2\,]\;w_i \;=\; [\,2-\gamma^2-\gamma\,]\,c^u + [\,2-\gamma^2\,]\,[\,\alpha_i - c_i^d\,] - \gamma\,[\,\alpha_j - c_j^d\,] + 2\,\gamma\,w_j$$
$$\Rightarrow\; 2\,[\,2-\gamma^2\,]\;w_i \;=\; 2\,[\,2-\gamma^2-\gamma\,]\,c^u + [\,2-\gamma^2\,]\,[\,\alpha_i - c^u - c_i^d\,]$$
$$-\,\gamma\,[\,\alpha_j - c^u - c_j^d\,] + 2\,\gamma\,w_j$$
$$\Rightarrow\; w_i \;=\; \left[\frac{2-\gamma^2-\gamma}{2-\gamma^2}\right] c^u + \frac{\Delta_i}{2} - \frac{\gamma\,\Delta_j}{2\,[\,2-\gamma^2\,]} + \frac{\gamma}{2-\gamma^2}\,w_j\,. \tag{64}$$

(64) implies that in equilibrium:

$$w_i \;=\; \left[\frac{2-\gamma^2-\gamma}{2-\gamma^2}\right] c^u + \frac{\Delta_i}{2} - \frac{\gamma\,\Delta_j}{2\,[\,2-\gamma^2\,]}$$
$$+\;\frac{\gamma}{2-\gamma^2}\left[\left(\frac{2-\gamma^2-\gamma}{2-\gamma^2}\right) c^u + \frac{\Delta_j}{2} - \frac{\gamma\,\Delta_i}{2\,(\,2-\gamma^2\,)} + \frac{\gamma}{2-\gamma^2}\,w_i\right]$$
$$\Rightarrow\; w_i\left[1 - \frac{\gamma^2}{(\,2-\gamma^2\,)^2}\right] \;=\; \left[\frac{2-\gamma^2-\gamma}{2-\gamma^2}\right] c^u + \frac{\Delta_i}{2} - \frac{\gamma\,\Delta_j}{2\,[\,2-\gamma^2\,]}$$

$$+ \; \frac{\gamma}{2-\gamma^2} \left[\left(\frac{2-\gamma^2-\gamma}{2-\gamma^2} \right) c^u + \frac{\Delta_j}{2} - \frac{\gamma \, \Delta_i}{2\,(\,2-\gamma^2\,)} \right]$$

$$\Rightarrow \;\; w_i \left[\frac{(\,2-\gamma^2\,)^2 - \gamma^2}{(\,2-\gamma^2\,)^2} \right] \; = \; \frac{2-\gamma^2+\gamma}{2-\gamma^2} \left[\frac{2-\gamma^2-\gamma}{2-\gamma^2} \right] c^u + \left[\frac{1}{2} - \frac{\gamma^2}{2\,(\,2-\gamma^2\,)^2} \right] \Delta_i$$

$$\Rightarrow \;\; w_i \left[\frac{(\,2-\gamma^2\,)^2 - \gamma^2}{(\,2-\gamma^2\,)^2} \right] \; = \; \left[\frac{(\,2-\gamma^2\,)^2 - \gamma^2}{(\,2-\gamma^2\,)^2} \right] c^u + \left[\frac{(\,2-\gamma^2\,)^2 - \gamma^2}{2\,(\,2-\gamma^2\,)^2} \right] \Delta_i$$

$$\Rightarrow \;\; w_i \; = \; c^u + \frac{1}{2} \, \Delta_i \, . \tag{65}$$

(28) and (65) imply:

$$q_i^{SP} \; = \; \frac{1}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \left\{ \,[\,2-\gamma^2\,]\,[\,\alpha_i - c_i^d\,] - \gamma\,[\,\alpha_j - c_j^d\,] - [\,2-\gamma^2\,]\; w_i + \gamma\, w_j \,\right\}$$

$$= \; \frac{1}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \Bigg\{ \,[\,2-\gamma^2\,]\,[\,\alpha_i - c_i^d\,] - \gamma\,[\,\alpha_j - c_j^d\,]$$

$$- \;\; [\,2-\gamma^2\,] \left[c^u + \frac{1}{2}\,\Delta_i \right] + \gamma \left[c^u + \frac{1}{2}\,\Delta_j \right] \Bigg\}$$

$$= \; \frac{1}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \Bigg\{ \,[\,2-\gamma^2\,]\,[\,\alpha_i - c^u - c_i^d\,] - \gamma\,[\,\alpha_j - c^u - c_j^d\,]$$

$$- \;\; [\,2-\gamma^2\,] \left[\frac{1}{2}\,\Delta_i \right] + \gamma \left[\frac{1}{2}\,\Delta_j \right] \Bigg\}$$

$$= \; \frac{1}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \left[\left(2-\gamma^2 \right) \, \Delta_i \, - \gamma\,\Delta_j - \frac{2-\gamma^2}{2}\Delta_i \, + \, \frac{\gamma}{2}\,\Delta_j \right]$$

$$= \; \frac{\frac{2-\gamma^2}{2}\Delta_i - \frac{\gamma}{2}\,\Delta_j}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \; = \; \frac{[\,2-\gamma^2\,]\,\Delta_i - \gamma\,\Delta_j}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \, . \;\; \blacksquare \tag{66}$$

Proposition 3.

$$L_2^{SP} \;\; = \;\; \begin{cases} \frac{1}{1-\gamma^2} \left[\frac{(\,2-\gamma^2\,)\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2} \right]^2 & \textit{when} \;\; \frac{\Delta_2}{\Delta_1} \in (\,\frac{\gamma}{2}, \frac{\gamma}{2-\gamma^2}\,) \\ \frac{3}{4\,[\,1-\gamma^2\,]} \left[\frac{(\,2-\gamma^2\,)\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2} \right]^2 & \textit{when} \;\; \frac{\Delta_2}{\Delta_1} \in (\frac{\gamma}{2-\gamma^2}, \frac{2-\gamma^2}{\gamma}) \\ \frac{1}{1-\gamma^2} \left[\frac{(\,2-\gamma^2\,)\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2} \right]^2 - \left[\frac{\Delta_2}{4} \right]^2 & \textit{when} \;\; \frac{\Delta_2}{\Delta_1} \in (\,\frac{2-\gamma^2}{\gamma}, \frac{2}{\gamma}\,). \end{cases}$$

Proof. (66) implies:

$$q_1^{SP} \; > \; 0 \;\; \text{and} \; q_2^{SP} \; > \; 0 \;\; \text{when} \; \frac{\Delta_2}{\Delta_1} \; \in \; (\frac{\gamma}{2 \, - \, \gamma^2}, \frac{2 \, - \, \gamma^2}{\gamma}). \tag{67}$$

(38), (39), (66), and (67) imply:

$$\pi_2^{SP} \; = \; \left[1 - \gamma^2 \right] \left[q_2^{SP} \right]^2 \; = \; \frac{1}{1 \, - \, \gamma^2} \left[\frac{(2 \, - \, \gamma^2) \, \Delta_2 - \gamma \, \Delta_1}{2 \, (4 \, - \, \gamma^2)} \right]^2 . \tag{68}$$

(40) and (68) imply:

$$\begin{aligned} L_2^{SP} \; &= \; \pi_2^{nP} - \pi_2^{SP} \\ &= \; \frac{1}{1 - \gamma^2} \left[\frac{(2 - \gamma^2) \, \Delta_2 - \gamma \, \Delta_1}{4 - \gamma^2} \right]^2 - \frac{1}{1 \, - \, \gamma^2} \left[\frac{(2 - \gamma^2) \, \Delta_2 - \gamma \, \Delta_1}{2 \, (4 - \gamma^2)} \right]^2 \\ &= \; \frac{3}{4 \, [1 - \gamma^2]} \left[\frac{(2 - \gamma^2) \, \Delta_2 - \gamma \, \Delta_1}{4 - \gamma^2} \right]^2 . \end{aligned}$$

(66) implies:

$$q_2^{SP} \; < \; 0 \;\; \text{if} \;\; \frac{\Delta_2}{\Delta_1} \; < \; \frac{\gamma}{2 - \gamma^2} ; \;\; q_1^{SP} \; < \; 0 \;\; \text{if} \;\; \frac{\Delta_2}{\Delta_1} \; > \; \frac{2 - \gamma^2}{\gamma} . \tag{69}$$

(69) implies that when $\frac{\Delta_2}{\Delta_1} < \frac{\gamma}{2-\gamma^2}$ under VS, D2 is driven from the market in the presence of downstream price competition. D_2's profit is zero in this event. Therefore, (40) implies:

$$L_2^{SP} \; = \; \pi_2^{nP} \; = \; \frac{1}{1 - \gamma^2} \left[\frac{(2 - \gamma^2) \, \Delta_2 - \gamma \, \Delta_1}{4 - \gamma^2} \right]^2 \;\; \text{when} \;\; \frac{\Delta_2}{\Delta_1} \; \in \; (\frac{\gamma}{2}, \frac{\gamma}{2 - \gamma^2}).$$

(69) implies that when $\frac{\Delta_2}{\Delta_1} > \frac{2-\gamma^2}{\gamma}$ under VS, D1 is driven from the market in the presence of downstream price competition and upstream collusion. (2) implies:

$$q_2 \; = \; \alpha_2 - P_2 \;\; \text{when} \;\; q_1^{SP} \; = \; 0 \, . \tag{70}$$

(70) implies $\frac{\partial q_2}{\partial P_2} = -1$. Therefore, (38) implies that D2's profit-maximizing choice of P_2 is determined by:

$$q_2(\cdot) - \left[P_2 - w_2 - c_2^d \right] \; = \; 0 \;\; \Rightarrow \;\; P_2 - w_2 - c_2^d \; = \; q_2(\cdot) \, . \tag{71}$$

(70) and (71) imply:

$$q_2 \; = \; \frac{\alpha_2 - w_2 - c_2^d}{2} \, . \tag{72}$$

(4) and (72) imply that when $q_1^{SP} = 0$, U1 and U2 choose w_2 to maximize:

$$[w_2 - c^u] \left[\frac{\alpha_2 - w_2 - c_2^d}{2} \right] . \tag{73}$$

(73) implies that the profit-maximizing value of w_2 in this case is determined by:

$$\frac{\alpha_2 - w_2 - c_2^d}{2} - \frac{w_2 - c^u}{2} = 0 \quad \Rightarrow \quad w_2 = \frac{\alpha_2 - c_2^d + c^u}{2}. \tag{74}$$

(72) and (74) imply:

$$q_2 = \frac{\alpha_2 - c_2^d}{2} - \frac{\alpha_2 - c_2^d + c^u}{4} = \frac{\alpha_2 - c_2^d - c^u}{4} = \frac{\Delta_2}{4}. \tag{75}$$

(38), (71), and (75) imply that when $q_1^{SP} = 0$:

$$\pi_2^{SP} = \left[q_2^{SP} \right]^2 = \left[\frac{\Delta_2}{4} \right]^2. \tag{76}$$

(40), (69), and (76) imply:

$$L_2^{SP} = \frac{1}{1-\gamma^2} \left[\frac{(2-\gamma^2)\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2} \right]^2 - \left[\frac{\Delta_2}{4} \right]^2 \quad \text{when } \frac{\Delta_2}{\Delta_1} \in \left(\frac{2-\gamma^2}{\gamma}, \frac{2}{\gamma} \right). \quad \blacksquare$$

Lemma 6. $w_2^{IP} = c^u + \frac{\gamma^3\,\Delta_1 + 8\,\Delta_2}{2\,[\,8+\gamma^2\,]}$, $q_1^{IP} = \frac{[\,8-\gamma^2-\gamma^4\,]\Delta_1 - 6\,\gamma\Delta_2}{2\,[\,1-\gamma^2\,]\,[\,8+\gamma^2\,]}$, *and*

$$q_2^{IP} = \frac{[\,2+\gamma^2\,]\,\Delta_2 - \gamma\,[\,2+\gamma^2\,]\,\Delta_1}{[\,8+\gamma^2\,]\,[\,1-\gamma^2\,]} \ \ \textit{when} \ \frac{\Delta_2}{\Delta_1} \in \left(\gamma, \frac{8-\gamma^2-\gamma^4}{6\,\gamma}\right).$$

<u>Proof</u>. (33), (35), and (36) imply:

$$\frac{\partial P_1}{\partial w_2} = \frac{3\,\gamma}{4-\gamma^2}; \qquad \frac{\partial q_1}{\partial w_2} = -\frac{\gamma}{4-\gamma^2}; \qquad \frac{\partial q_2}{\partial w_2} = -\frac{2}{4-\gamma^2}. \tag{77}$$

Under VI, U1 and U2 choose w_2 to:

$$\text{Maximize} \quad \left[P_1(\cdot) - c^u - c_1^d \right] q_1(\cdot) + [\,w_2 - c^u\,]\, q_2(\cdot). \tag{78}$$

(78) implies that U1 and U2's profit-maximizing choice of w_2 is determined by:

$$q_1 \frac{\partial P_1(\cdot)}{\partial w_2} + \left[P_1 - c^u - c_1^d \right] \frac{\partial q_1(\cdot)}{\partial w_2} + q_2 + [\,w_2 - c^u\,] \frac{\partial q_2(\cdot)}{\partial w_2} = 0. \tag{79}$$

(77) and (79) imply:

$$q_1 \left[\frac{3\,\gamma}{4-\gamma^2} \right] - \left[P_1 - c^u - c_1^d \right] \left[\frac{\gamma}{4-\gamma^2} \right] + q_2 - [\,w_2 - c^u\,] \left[\frac{2}{4-\gamma^2} \right] = 0$$

$$\Rightarrow \quad 3\,\gamma\, q_1 - \gamma \left[P_1 - c^u - c_1^d \right] + \left[4 - \gamma^2 \right] q_2 - 2\,[\,w_2 - c^u\,] = 0. \tag{80}$$

(33) implies:

$$P_1 - c^u - c_1^d \;=\; \frac{1}{4-\gamma^2}\,\{\,[\,2-\gamma^2\,]\;\alpha_1 + 2\,[\,1-\gamma\,]\,c^u + 2\,c_1^d + 3\,\gamma\,w_2 - \gamma\,\alpha_2 + \gamma\,c_2^d$$

$$-\;\;[\,4-\gamma^2\,]\,c^u - [\,4-\gamma^2\,]\,c_1^d\,\}$$

$$=\;\frac{1}{4-\gamma^2}\{\,[\,2-\gamma^2\,]\;\alpha_1 - [\,2-\gamma^2\,]\,c_1^d - \gamma\,\alpha_2 + \gamma\,c_2^d + [\,-2\,\gamma - 2 + \gamma^2\,]\,c^u + 3\,\gamma\,w_2\,\}$$

$$=\;\frac{1}{4-\gamma^2}\{\,[\,2-\gamma^2\,]\;\alpha_1 - [\,2-\gamma^2\,]\,c_1^d - [\,2-\gamma^2\,]\,c^u - \gamma\,\alpha_2 + \gamma\,c_2^d + \gamma\,c^u - 3\,\gamma\,c^u + 3\,\gamma\,w_2\,\}$$

$$=\;\frac{1}{4-\gamma^2}\{\,[\,2-\gamma^2\,]\;\Delta_1 - \gamma\,\Delta_2 - 3\,\gamma\,c^u + 3\,\gamma\,w_2\,\}. \tag{81}$$

(35), (36), (77), (80), and (81) imply:

$$\frac{3\,\gamma}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\{\,2\,[\,2-\gamma^2\,]\,\Delta_1 - 2\,\gamma\,\Delta_2 + 2\,\gamma\,[\,1-\gamma^2\,]\,c^u - 2\,\gamma\,[\,1-\gamma^2\,]\,w_2\,\}$$

$$-\;\frac{\gamma}{4-\gamma^2}\{\,[\,2-\gamma^2\,]\;\Delta_1 - \gamma\,\Delta_2 - 3\,\gamma\,c^u + 3\,\gamma\,w_2\,\} - 2\,[\,w_2 - c^u\,]$$

$$+\;\frac{1}{2\,[\,1-\gamma^2\,]}\{\,2\,[\,2-\gamma^2\,]\,\Delta_2 - 2\,\gamma\,\Delta_1 + 4\,[\,1-\gamma^2\,]\;c^u - 4\,[\,1-\gamma^2\,]\,w_2\,\}\;=\;0$$

$$\Rightarrow\;\; 3\,\gamma\,[\,2\,(\,2-\gamma^2\,)\,\Delta_1 - 2\,\gamma\,\Delta_2 + 2\,\gamma\,(\,1-\gamma^2\,)\,c^u - 2\,\gamma\,(\,1-\gamma^2\,)\,w_2\,]$$

$$-\;2\,\gamma\,[\,1-\gamma^2\,]\,[\,(\,2-\gamma^2\,)\;\Delta_1 - \gamma\,\Delta_2 - 3\,\gamma\,c^u + 3\,\gamma\,w_2\,] - 4\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]\,[\,w_2 - c^u\,]$$

$$+\;\;[\,4-\gamma^2\,]\,[\,2\,(\,2-\gamma^2\,)\,\Delta_2 - 2\,\gamma\,\Delta_1 + 4\,(\,1-\gamma^2\,)\;c^u - 4\,(\,1-\gamma^2\,)\,w_2\,]\;\;=\;0$$

$$\Rightarrow\;\; 6\,\gamma\,[\,2-\gamma^2\,]\,\Delta_1 - 6\,\gamma^2\,\Delta_2 + 6\,\gamma^2\,[\,1-\gamma^2\,]\,c^u - 6\,\gamma^2\,[\,1-\gamma^2\,]\,w_2$$

$$-\;2\,\gamma\,[\,1-\gamma^2\,]\,[\,2-\gamma^2\,]\;\Delta_1 + 2\,\gamma^2\,[\,1-\gamma^2\,]\;\Delta_2 + 6\,\gamma^2\,[\,1-\gamma^2\,]\;c^u - 6\,\gamma^2\,[\,1-\gamma^2\,]\;w_2$$

$$-\;4\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]\,[\,w_2 - c^u\,] + 2\,[\,2-\gamma^2\,]\,[\,4-\gamma^2\,]\,\Delta_2 - 2\,\gamma\,[\,4-\gamma^2\,]\;\Delta_1$$

$$+\;4\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]\;c^u - 4\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]\,w_2\;=\;0$$

$$\Rightarrow\;\; 2\,\gamma\,[\,3\,(\,2-\gamma^2\,) - (\,1-\gamma^2\,)\,(\,2-\gamma^2\,) - (\,4-\gamma^2\,)\,]\;\Delta_1$$

$$+\;2\,[\,\gamma^2\,(\,1-\gamma^2\,) - 3\,\gamma^2 + (\,2-\gamma^2\,)\,(\,4-\gamma^2\,)\,]\,\Delta_2$$

$$+\;4\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]\,[\,c^u - w_2\,] + 6\,\gamma^2\,[\,1-\gamma^2\,]\,[\,c^u - w_2\,]$$

$$+\;4\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]\,[\,c^u - w_2\,] + 6\,\gamma^2\,[\,1-\gamma^2\,]\,[\,c^u - w_2\,]\;=\;0$$

$$\Rightarrow \;\; 2\gamma \left[\left(2+\gamma^2 \right) \left(2-\gamma^2 \right) - \left(4-\gamma^2 \right) \right] \; \Delta_1$$

$$+ \, 2 \left[\gamma^2 \left(-2-\gamma^2 \right) + \left(2-\gamma^2 \right) \left(4-\gamma^2 \right) \right] \Delta_2$$

$$+ \, 8 \left[1-\gamma^2 \right] \left[4-\gamma^2 \right] \left[c^u - w_2 \right] + 12\,\gamma^2 \left[1-\gamma^2 \right] \left[c^u - w_2 \right] \; = \; 0$$

$$\Rightarrow \;\; 2\gamma \left[\left(4-\gamma^4 \right) - \left(4-\gamma^2 \right) \right] \; \Delta_1 + 2 \left[-2\,\gamma^2 - \gamma^4 + 8 + \gamma^4 - 6\,\gamma^2 \right] \Delta_2$$

$$+ \, 4 \left[1-\gamma^2 \right] \left[8 - 2\,\gamma^2 + 3\,\gamma^2 \right] \left[c^u - w_2 \right] \; = \; 0$$

$$\Rightarrow \;\; 2\,\gamma^3 \left[1-\gamma^2 \right] \; \Delta_1 + 16 \left[1-\gamma^2 \right] \Delta_2 + 4 \left[1-\gamma^2 \right] \left[8+\gamma^2 \right] \left[c^u - w_2 \right] \; = \; 0$$

$$\Rightarrow \;\; \gamma^3 \, \Delta_1 + 8\,\Delta_2 + 2 \left[8+\gamma^2 \right] \left[c^u - w_2 \right] \; = \; 0 \;\; \Rightarrow \;\; w_2^{IP} \; = \; c^u + \frac{\gamma^3\,\Delta_1 + 8\,\Delta_2}{2\,[\,8\,+\,\gamma^2\,]}. \tag{82}$$

(35) and (82) imply:

$$q_1^{IP} \; = \; \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \left\{ 2 \left[2-\gamma^2 \right] \Delta_1 - 2\,\gamma\,\Delta_2 + 2\,\gamma \left[1-\gamma^2 \right] c^u \right.$$

$$\left. - \;\; 2\,\gamma \left[1-\gamma^2 \right] \left[c^u + \frac{\gamma^3\,\Delta_1 \,+\, 8\,\Delta_2}{2\,(\,8\,+\,\gamma^2\,)} \right] \right\}$$

$$= \; \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \left\{ 2 \left[2-\gamma^2 \right] \Delta_1 - 2\,\gamma\,\Delta_2 - \gamma \left[1-\gamma^2 \right] \left[\frac{\gamma^3\,\Delta_1 \,+\, 8\,\Delta_2}{8\,+\,\gamma^2} \right] \right\}$$

$$= \; \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]\,[\,8+\gamma^2\,]} \left\{ 2 \left[2-\gamma^2 \right] \left[8\,+\,\gamma^2 \right] \Delta_1 - 2\,\gamma \left[8\,+\,\gamma^2 \right] \; \Delta_2 \right.$$

$$\left. - \;\; \gamma^4 \left[1-\gamma^2 \right] \; \Delta_1 \; - 8\,\gamma \left[1-\gamma^2 \right] \; \Delta_2 \right\}$$

$$= \; \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]\,[\,8+\gamma^2\,]} \left\{ \; \left[2 \left(16 - 6\,\gamma^2 - \gamma^4 \right) - \gamma^4 + \gamma^6 \right] \Delta_1 \right.$$

$$\left. - \, 2\,\gamma \left[8\,+\,\gamma^2 + 4 \left(1-\gamma^2 \right) \right] \Delta_2 \right\}$$

$$= \; \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]\,[\,8+\gamma^2\,]} \; \left\{ \left[32 - 12\,\gamma^2 - 3\,\gamma^4 + \gamma^6 \right] \Delta_1 - 6\,\gamma \left[4-\gamma^2 \right] \Delta_2 \right\}$$

$$= \; \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]\,[\,8+\gamma^2\,]} \; \left\{ \left[4-\gamma^2 \right] \left[8-\gamma^2-\gamma^4 \right] \Delta_1 - 6\,\gamma \left[4-\gamma^2 \right] \Delta_2 \right\}$$

$$= \; \frac{[\,8-\gamma^2-\gamma^4\,]\,\Delta_1 - 6\,\gamma\Delta_2}{2\,[\,1-\gamma^2\,]\,[\,8+\gamma^2\,]}\,. \tag{83}$$

(36) and (82) imply:

$$q_2^{IP} = \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\Big\{\,2\,\big[\,2-\gamma^2\,\big]\,\Delta_2 - 2\,\gamma\,\Delta_1 + 4\,\big[\,1-\gamma^2\,\big]\;c^u$$

$$-\;4\,\big[\,1-\gamma^2\,\big]\left[\,c^u + \frac{\gamma^3\,\Delta_1\,+\,8\,\Delta_2}{2\,[\,8\,+\,\gamma^2\,]}\,\right]\Big\}$$

$$= \frac{1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\left\{\,2\,\big[\,2-\gamma^2\,\big]\,\Delta_2 - 2\,\gamma\,\Delta_1 - 4\,\big[\,1-\gamma^2\,\big]\left[\,\frac{\gamma^3\,\Delta_1\,+\,8\,\Delta_2}{2\,[\,8\,+\,\gamma^2\,]}\,\right]\right\}$$

$$= \frac{1}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}\left\{\,\big[\,2-\gamma^2\,\big]\,\Delta_2 - \gamma\,\Delta_1 - \frac{\gamma^3\,[\,1-\gamma^2\,]\;\Delta_1\;+8\,[\,1-\gamma^2\,]\;\Delta_2}{[\,8\,+\,\gamma^2\,]}\,\right\}$$

$$= \frac{1}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]\,[\,8\,+\,\gamma^2\,]}\,\{\,\big[\,2-\gamma^2\,\big]\,\big[\,8\,+\,\gamma^2\,\big]\,\Delta_2 - \gamma\,\big[\,8\,+\,\gamma^2\,\big]\;\Delta_1$$

$$-\;\gamma^3\,\big[\,1-\gamma^2\,\big]\;\Delta_1\;-8\,\big[\,1-\gamma^2\,\big]\;\Delta_2\,\}$$

$$= \frac{1}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]\,[\,8\,+\,\gamma^2\,]}\,\{\,\big[\,16-\gamma^4-6\,\gamma^2-8+8\,\gamma^2\,\big]\,\Delta_2$$

$$-\;\gamma\,\big[\,\gamma^2-\gamma^4+8\,+\,\gamma^2\,\big]\;\Delta_1\,\}$$

$$= \frac{1}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]\,[\,8\,+\,\gamma^2\,]}\,\{\,\big[\,8-\gamma^4+2\,\gamma^2\,\big]\,\Delta_2 - \gamma\,\big[\,8-\gamma^4+2\,\gamma^2\,\big]\;\Delta_1\,\}$$

$$= \frac{[\,8-\gamma^4+2\,\gamma^2\,]\,[\,\Delta_2-\gamma\,\Delta_1\,]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]\,[\,8\,+\,\gamma^2\,]} \;=\; \frac{[\,2+\gamma^2\,]\,[\,4-\gamma^2\,]\,[\,\Delta_2-\gamma\,\Delta_1\,]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]\,[\,8\,+\,\gamma^2\,]}$$

$$= \frac{[\,2+\gamma^2\,]\,[\,\Delta_2-\gamma\,\Delta_1\,]}{[\,1-\gamma^2\,]\,[\,8\,+\,\gamma^2\,]}.\quad\blacksquare \tag{84}$$

Proposition 4.

$$L_2^{IP} = \begin{cases} \frac{1}{1-\gamma^2}\left[\,\frac{(\,2-\gamma^2\,)\,\Delta_2-\gamma\,\Delta_1}{4-\gamma^2}\,\right]^2 & \textit{when}\ \frac{\Delta_2}{\Delta_1}\in(\,\frac{\gamma}{2},\gamma\,) \\ \frac{1}{1-\gamma^2}\left[\,\frac{(\,2-\gamma^2\,)\,\Delta_2-\gamma\,\Delta_1}{4-\gamma^2}\,\right]^2 - \frac{1}{1-\gamma^2}\left[\,\frac{(\,2+\gamma^2\,)\,\Delta_2-\gamma\,(\,2+\gamma^2\,)\,\Delta_1}{8+\gamma^2}\,\right]^2 & \\ & \textit{when}\ \frac{\Delta_2}{\Delta_1}\in(\,\gamma,\frac{8-\gamma^2-\gamma^4}{6\,\gamma}\,) \\ \frac{1}{1-\gamma^2}\left[\,\frac{(\,2-\gamma^2\,)\,\Delta_2-\gamma\,\Delta_1}{4-\gamma^2}\,\right]^2 - \left[\,\frac{\Delta_2}{4}\,\right]^2 & \textit{when}\ \frac{\Delta_2}{\Delta_1}\in(\,\frac{8-\gamma^2-\gamma^4}{6\,\gamma},\frac{2}{\gamma}\,). \end{cases}$$

Proof. (83) and (84) imply:

$$q_1^{IP} \;>\; 0 \;\text{ and } q_2^{IP} \;>\; 0 \;\text{ when } \frac{\Delta_2}{\Delta_1} \;\in\; (\gamma, \frac{8-\gamma^2-\gamma^4}{6\,\gamma}). \tag{85}$$

(38), (39), and (84) imply:

$$\pi_2^{IP} \;=\; \left[1-\gamma^2\right]\left[q_2^{IP}\right]^2 \;=\; \frac{1}{1-\gamma^2}\left[\frac{(\,2\,+\,\gamma^2\,)\,(\Delta_2-\gamma\,\Delta_1)}{8+\gamma^2}\right]^2. \tag{86}$$

(40) and (86) imply:

$$L_2^{IP} \;=\; \pi_2^{nP} - \pi_2^{IP}$$

$$= \; \frac{1}{1-\gamma^2}\left[\frac{(\,2-\gamma^2\,)\,\Delta_2-\gamma\,\Delta_1}{4-\gamma^2}\right]^2 - \frac{1}{1-\gamma^2}\left[\frac{(\,2+\gamma^2\,)\,(\,\Delta_2-\gamma\,\Delta_1\,)}{8+\gamma^2}\right]^2.$$

(83) and (84) imply:

$$q_1^{IP} \;<\; 0 \;\text{ if }\; \frac{\Delta_2}{\Delta_1} \;>\; \frac{8-\gamma^2-\gamma^4}{6\,\gamma}; \quad q_2^{IP} \;<\; 0 \;\text{ if }\; \frac{\Delta_2}{\Delta_1} \;<\; \gamma\,. \tag{87}$$

(87) implies that when $\frac{\Delta_2}{\Delta_1} < \gamma$ under VS, D2 is driven from the market in the presence of downstream price competition. D_2's profit is zero in this event. Therefore, (40) implies:

$$L_2^{IP} \;=\; \pi_2^{nP} \;=\; \frac{1}{1-\gamma^2}\left[\frac{(\,2-\gamma^2\,)\,\Delta_2-\gamma\,\Delta_1}{4-\gamma^2}\right]^2 \quad\text{when } \frac{\Delta_2}{\Delta_1} \in (\frac{\gamma}{2}, \gamma).$$

(87) implies that when $\frac{\Delta_2}{\Delta_1} > \frac{8-\gamma^2-\gamma^4}{6\,\gamma}$ under VS, D1 is driven from the market in the presence of downstream price competition. (2) implies:

$$q_2 \;=\; \alpha_2 - P_2 \;\text{ when }\; q_1^{IP} \;=\; 0\,. \tag{88}$$

(88) implies $\frac{\partial q_2}{\partial P_2} = -1$. Therefore, (38) implies that D2's profit-maximizing choice of P_2 is determined by:

$$q_2(\cdot) - \left[\,P_2 - w_2 - c_2^d\,\right] \;=\; 0 \quad\Rightarrow\quad P_2 - w_2 - c_2^d \;=\; q_2(\cdot)\,. \tag{89}$$

(88) and (89) imply:

$$q_2 \;=\; \frac{\alpha_2 - w_2 - c_2^d}{2}\,. \tag{90}$$

(4) and (90) imply that when $q_1^{IP} = 0$, U1 and U2 choose w_2 to maximize:

$$[\,w_2 - c^u\,]\left[\frac{\alpha_2 - w_2 - c_2^d}{2}\right]. \tag{91}$$

(91) implies that the profit-maximizing value of w_2 is determined by:

$$\frac{\alpha_2 - w_2 - c_2^d}{2} - \frac{w_2 - c^u}{2} \;=\; 0 \quad\Rightarrow\quad w_2 \;=\; \frac{\alpha_2 - c_2^d + c^u}{2}\,. \tag{92}$$

(90) and (92) imply:

$$q_2 \;=\; \frac{\alpha_2 - c_2^d}{2} - \frac{\alpha_2 - c_2^d + c^u}{4} \;=\; \frac{\alpha_2 - c_2^d - c^u}{4} \;=\; \frac{\Delta_2}{4}\,. \tag{93}$$

(38), (89), (93) imply that when $q_1^{IP} = 0$:

$$\pi_2^{IP} \;=\; \left[\, q_2^{IP} \,\right]^2 \;=\; \left[\, \frac{\Delta_2}{4} \,\right]^2. \tag{94}$$

(40) and (94) imply:

$$L_2^{IP} \;=\; \frac{1}{1-\gamma^2}\left[\, \frac{(\,2-\gamma^2\,)\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2} \,\right]^2 - \left[\, \frac{\Delta_2}{4} \,\right]^2 \quad \text{when } \frac{\Delta_2}{\Delta_1} \in (\,\frac{8-\gamma^2-\gamma^4}{6\,\gamma}, \frac{2}{\gamma}\,). \;\blacksquare$$

Theorem 2. *$L_2^{IP} \geq L_2^{SP}$ for $\frac{\Delta_2}{\Delta_1} \in (\,\frac{\gamma}{2}, m(\gamma)\,)$, with strict inequality for $\frac{\Delta_2}{\Delta_1} \in (\,\frac{\gamma}{2-\gamma^2}, m(\gamma)\,)$. $L_2^{SP} \geq L_2^{IP}$ for $\frac{\Delta_2}{\Delta_1} \in (\,m(\gamma), \frac{2}{\gamma}\,)$, with strict inequality for $\frac{\Delta_2}{\Delta_1} \in (\,m(\gamma), \frac{2-\gamma^2}{\gamma}\,)$.*[2]

Proof. Observe that:

$$\frac{\gamma}{2-\gamma^2} \;>\; \frac{\gamma}{2} \;\Leftrightarrow\; 2 \;>\; 2-\gamma^2 \;\Leftrightarrow\; \gamma^2 \;>\; 0\,;$$

$$\gamma \;\geq\; \frac{\gamma}{2-\gamma^2} \;\Leftrightarrow\; 2-\gamma^2 \;\geq\; 1 \;\Leftrightarrow\; \gamma^2 \;\leq\; 1\,;$$

$$\frac{8-\gamma^2-\gamma^4}{6\,\gamma} \;>\; \gamma \;\Leftrightarrow\; 8-\gamma^2-\gamma^4 \;>\; 6\,\gamma^2 \;\Leftrightarrow\; 8-7\,\gamma^2-\gamma^4 \;>\; 0$$
$$\Leftrightarrow\; 7\left[\,1-\gamma^2\,\right] + 1 - \gamma^4 \;>\; 0\,;$$

$$\frac{2-\gamma^2}{\gamma} \;\geq\; \frac{8-\gamma^2-\gamma^4}{6\,\gamma} \;\Leftrightarrow\; 12-6\,\gamma^2 \;\geq\; 8-\gamma^2-\gamma^4 \;\Leftrightarrow\; 4-5\,\gamma^2+\gamma^4 \;\geq\; 0$$
$$\Leftrightarrow\; 4-4\,\gamma^2-\gamma^2+\gamma^4 \;\geq\; 0 \;\Leftrightarrow\; 4\left[\,1-\gamma^2\,\right] - \gamma^2\left[\,1-\gamma^2\,\right] \;\geq\; 0$$
$$\Leftrightarrow\; \left[\,1-\gamma^2\,\right]\left[\,4-\gamma^2\,\right] \;\geq\; 0\,;$$

$$\frac{2-\gamma^2}{\gamma} \;<\; \frac{2}{\gamma} \;\Leftrightarrow\; 2-\gamma^2 \;<\; 2 \;\Leftrightarrow\; \gamma^2 \;>\; 0\,. \tag{95}$$

[2] $L_2^{IP} = L_2^{SP}$ when $\frac{\Delta_2}{\Delta_1} \in (\,\frac{\gamma}{2}, \frac{\gamma}{2-\gamma^2}\,) \cup (\,\frac{2-\gamma^2}{\gamma}, \frac{2}{\gamma}\,)$. This is the case because in the presence of downstream price competition: (i) D2 is driven from the market under both VS and VI when $\frac{\Delta_2}{\Delta_1} \in (\,\frac{\gamma}{2}, \frac{\gamma}{2-\gamma^2}\,)$; and (ii) D2 is the monopoly producer under both VS and VI when $\frac{\Delta_2}{\Delta_1} \in (\,\frac{2-\gamma^2}{\gamma}, \frac{2}{\gamma}\,)$.

(95) implies:

$$\frac{\gamma}{2} \;<\; \frac{\gamma}{2-\gamma^2} \;\leq\; \gamma \;<\; \frac{8-\gamma^2-\gamma^4}{6\,\gamma} \;\leq\; \frac{2-\gamma^2}{\gamma} \;<\; \frac{2}{\gamma}\,. \tag{96}$$

(96) and Proposition 3 imply that when $\frac{\Delta_2}{\Delta_1} \in \left(\frac{\gamma}{2}, \frac{\gamma}{2-\gamma^2}\right)$:

$$\begin{aligned} L_2^{IP} - L_2^{SP} \;=\; & \frac{1}{1-\gamma^2}\left[\frac{(2-\gamma^2)\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 \\ & - \frac{1}{1-\gamma^2}\left[\frac{(2-\gamma^2)\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 \;=\; 0\,. \end{aligned} \tag{97}$$

(96) and Propositions 3 and 4 imply that when $\frac{\Delta_2}{\Delta_1} \in \left(\frac{\gamma}{2-\gamma^2}, \gamma\right)$:

$$\begin{aligned} L_2^{IP} - L_2^{SP} \;&=\; \frac{1}{1-\gamma^2}\left[\frac{(2-\gamma^2)\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 - \frac{3}{4\,[1-\gamma^2]}\left[\frac{(2-\gamma^2)\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 \\ &=\; \frac{1}{4\,[1-\gamma^2]}\left[\frac{(2-\gamma^2)\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 \;>\; 0\,. \end{aligned} \tag{98}$$

(96) and Propositions 3 and 4 imply that when $\frac{\Delta_2}{\Delta_1} \in \left(\frac{2-\gamma^2}{\gamma}, \frac{2}{\gamma}\right)$:

$$\begin{aligned} L_2^{IP} - L_2^{SP} \;=\; & \frac{1}{1-\gamma^2}\left[\frac{(2-\gamma^2)\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 - \left[\frac{\Delta_2}{4}\right]^2 \\ & - \frac{1}{1-\gamma^2}\left[\frac{(2-\gamma^2)\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 + \left[\frac{\Delta_2}{4}\right]^2 \;=\; 0\,. \end{aligned} \tag{99}$$

(96) and Propositions 3 and 4 imply that when $\frac{\Delta_2}{\Delta_1} \in \left(\frac{8-\gamma^2-\gamma^4}{6\,\gamma}, \frac{2-\gamma^2}{\gamma}\right)$:

$$\begin{aligned} & L_2^{IP} - L_2^{SP} \\ &= \frac{1}{1-\gamma^2}\left[\frac{(2-\gamma^2)\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 - \left[\frac{\Delta_2}{4}\right]^2 - \frac{3}{4\,[1-\gamma^2]}\left[\frac{(2-\gamma^2)\,\Delta_2 - \gamma\,\Delta_1}{4-\gamma^2}\right]^2 \\ &= \frac{1}{1-\gamma^2}\left[\frac{(2-\gamma^2)\,\Delta_2 - \gamma\,\Delta_1}{2\,(4-\gamma^2)}\right]^2 - \left[\frac{\Delta_2}{4}\right]^2 . \end{aligned} \tag{100}$$

(96) implies that when $\frac{\Delta_2}{\Delta_1} \in \left(\frac{8-\gamma^2-\gamma^4}{6\,\gamma}, \frac{2-\gamma^2}{\gamma}\right)$:

$$\frac{\Delta_2}{\Delta_1} \;>\; \frac{\gamma}{2-\gamma^2} \quad\Rightarrow\quad \left[\,2-\gamma^2\,\right]\Delta_2 \;>\; \gamma\,\Delta_1\,. \tag{101}$$

(100) and (101) imply:

$$L_2^{IP} \ < \ L_2^{SP} \quad \Leftrightarrow \quad \left[1 - \gamma^2 \right] \left[\frac{(2-\gamma^2)\,\Delta_2 - \gamma\,\Delta_1}{2\,(1-\gamma^2)\,(4-\gamma^2)} \right]^2 \ < \ \left[\frac{\Delta_2}{4} \right]^2$$

$$\Leftrightarrow \quad \sqrt{1-\gamma^2} \left[\frac{(2-\gamma^2)\,\Delta_2 - \gamma\,\Delta_1}{2\,(1-\gamma^2)\,(4-\gamma^2)} \right] \ < \ \frac{\Delta_2}{4}$$

$$\Leftrightarrow \quad \frac{[\,2-\gamma^2\,]\,\Delta_2 - \gamma\,\Delta_1}{\sqrt{1-\gamma^2}\,[\,4-\gamma^2\,]} \ < \ \frac{\Delta_2}{2} \quad \Leftrightarrow \quad 2\left[\,2-\gamma^2\,\right]\Delta_2 - 2\,\gamma\,\Delta_1 \ < \ \sqrt{1-\gamma^2}\left[\,4-\gamma^2\,\right]\Delta_2$$

$$\Leftrightarrow \quad 2\left[\,2-\gamma^2\,\right]\Delta_2 - \sqrt{1-\gamma^2}\left[\,4-\gamma^2\,\right]\Delta_2 \ < \ 2\,\gamma\,\Delta_1$$

$$\Leftrightarrow \quad \left[2\left(2-\gamma^2\right) - \sqrt{1-\gamma^2}\,\left(4-\gamma^2\right) \right]\Delta_2 \ < \ 2\,\gamma\,\Delta_1\,. \tag{102}$$

Observe that:

$$2\left[\,2-\gamma^2\,\right] \ > \ \sqrt{1-\gamma^2}\,\left[\,4-\gamma^2\,\right] \quad \Leftrightarrow \quad 4\left[\,2-\gamma^2\,\right]^2 \ > \ \left[\,1-\gamma^2\,\right]\left[\,4-\gamma^2\,\right]^2$$

$$\Leftrightarrow \quad 4\left[\,4+\gamma^4-4\,\gamma^2\,\right] \ > \ \left[\,1-\gamma^2\,\right]\left[\,16+\gamma^4-8\,\gamma^2\,\right]$$

$$\Leftrightarrow \quad 16+4\,\gamma^4-16\,\gamma^2 \ > \ 16\left[\,1-\gamma^2\,\right]+\gamma^4\left[\,1-\gamma^2\,\right]-8\,\gamma^2\left[\,1-\gamma^2\,\right]$$

$$\Leftrightarrow \quad 16+4\,\gamma^4-16\,\gamma^2 \ > \ 16-16\,\gamma^2+\gamma^4-\gamma^6-8\,\gamma^2+8\,\gamma^4$$

$$\Leftrightarrow \quad 0 \ > \ -\gamma^6-8\,\gamma^2+5\,\gamma^4 \quad \Leftrightarrow \quad 0 \ > \ -\gamma^4-8+5\,\gamma^2$$

$$\Leftrightarrow \quad 8-5\,\gamma^2+\gamma^4 \ > \ 0\,. \tag{103}$$

The inequality in (103) holds because $8-5\,\gamma^2+\gamma^4 \ > \ 8-5+0 \ > \ 0$.

(102) and (103) imply:

$$L_2^{IP} \ < \ L_2^{SP} \quad \Leftrightarrow \quad \frac{\Delta_2}{\Delta_1} \ < \ \frac{2\,\gamma}{2\,[\,2-\gamma^2\,]-\sqrt{1-\gamma^2}\,[\,4-\gamma^2\,]}\,. \tag{104}$$

Observe that:

$$\frac{2\,\gamma}{2\,[\,2-\gamma^2\,]-\sqrt{1-\gamma^2}\,[\,4-\gamma^2\,]} \ > \ \frac{2-\gamma^2}{\gamma}$$

$$\Leftrightarrow \quad 2\,\gamma^2 \ > \ \left[\,2-\gamma^2\,\right]\left[2\left(2-\gamma^2\right) - \sqrt{1-\gamma^2}\left(4-\gamma^2\right) \right]$$

$$\Leftrightarrow \quad 2\,\gamma^2 \ > \ 2\left[\,2-\gamma^2\,\right]^2 - \sqrt{1-\gamma^2}\left[\,4-\gamma^2\,\right]\left[\,2-\gamma^2\,\right]$$

$$\Leftrightarrow \quad \sqrt{1-\gamma^2}\left[\,4-\gamma^2\,\right]\left[\,2-\gamma^2\,\right] \ > \ 2\left[\,2-\gamma^2\,\right]^2 - 2\,\gamma^2$$

$$\Leftrightarrow \quad \left[1-\gamma^2\right]\left[4-\gamma^2\right]^2\left[2-\gamma^2\right]^2 \;>\; \left[2\left(2-\gamma^2\right)^2-2\gamma^2\right]^2$$

$$\Leftrightarrow \quad \left[1-\gamma^2\right]\left[2+2-\gamma^2\right]^2\left[2-\gamma^2\right]^2 \;>\; \left[2\left(2-\gamma^2\right)^2-2\gamma^2\right]^2$$

$$\Leftrightarrow \quad \left[1-\gamma^2\right]\left[4+\left(2-\gamma^2\right)^2+4\left(2-\gamma^2\right)\right]\left[2-\gamma^2\right]^2 \;>\; \left[2\left(2-\gamma^2\right)^2-2\gamma^2\right]^2$$

$$\Leftrightarrow \quad \left[1-\gamma^2\right]4\left[2-\gamma^2\right]^2+\left[1-\gamma^2\right]\left[2-\gamma^2\right]^4+\left[1-\gamma^2\right]4\left[2-\gamma^2\right]^3$$
$$> \; 4\left[2-\gamma^2\right]^4+4\gamma^4-8\gamma^2\left[2-\gamma^2\right]^2$$

$$\Leftrightarrow \quad 4\left[1-\gamma^2+2\gamma^2\right]\left[2-\gamma^2\right]^2+4\left[1-\gamma^2\right]\left[2-\gamma^2\right]^3 \;>\; \left[3+\gamma^2\right]\left[2-\gamma^2\right]^4+4\gamma^4$$

$$\Leftrightarrow \quad 4\left[1+\gamma^2\right]\left[2-\gamma^2\right]^2+4\left[1-\gamma^2\right]\left[2-\gamma^2\right]^3 \;>\; \left[3+\gamma^2\right]\left[2-\gamma^2\right]^4+4\gamma^4$$

$$\Leftrightarrow \quad \left[2-\gamma^2\right]^2\left\{4\left[1+\gamma^2\right]+4\left[1-\gamma^2\right]\left[2-\gamma^2\right]-\left[3+\gamma^2\right]\left[2-\gamma^2\right]^2\right\} \;>\; 4\gamma^4$$

$$\Leftrightarrow \quad \left[2-\gamma^2\right]^2\left\{4+4\gamma^2+4\left[2+\gamma^4-3\gamma^2\right]-\left[3+\gamma^2\right]\left[4+\gamma^4-4\gamma^2\right]\right\} \;>\; 4\gamma^4$$

$$\Leftrightarrow \quad \left[2-\gamma^2\right]^2\left\{4+4\gamma^2+8+4\gamma^4-12\gamma^2-\left[3+\gamma^2\right]\left[4+\gamma^4-4\gamma^2\right]\right\} \;>\; 4\gamma^4$$

$$\Leftrightarrow \quad \left[2-\gamma^2\right]^2\left\{12+4\gamma^4-8\gamma^2-\left[3+\gamma^2\right]\left[4+\gamma^4-4\gamma^2\right]\right\} \;>\; 4\gamma^4$$

$$\Leftrightarrow \quad \left[2-\gamma^2\right]^2\left\{4+2\gamma^4+2\left[4+\gamma^4-4\gamma^2\right]-\left[3+\gamma^2\right]\left[4+\gamma^4-4\gamma^2\right]\right\} \;>\; 4\gamma^4$$

$$\Leftrightarrow \quad \left[2-\gamma^2\right]^2\left\{4+2\gamma^4-\left[1+\gamma^2\right]\left[4+\gamma^4-4\gamma^2\right]\right\} \;>\; 4\gamma^4$$

$$\Leftrightarrow \quad \left[2-\gamma^2\right]^2\left\{4+2\gamma^4-4\left[1+\gamma^2\right]-\gamma^4\left[1+\gamma^2\right]+4\gamma^2\left[1+\gamma^2\right]\right\} \;>\; 4\gamma^4$$

$$\Leftrightarrow \quad \left[2-\gamma^2\right]^2\left\{4+2\gamma^4-4-4\gamma^2-\gamma^4-\gamma^6+4\gamma^2+4\gamma^4\right\} \;>\; 4\gamma^4$$

$$\Leftrightarrow \quad \left[2-\gamma^2\right]^2\left[5\gamma^4-\gamma^6\right] \;>\; 4\gamma^4 \quad \Leftrightarrow \quad \gamma^4\left[2-\gamma^2\right]^2\left[5-\gamma^2\right] \;>\; 4\gamma^4$$

$$\Leftrightarrow \quad \left[2-\gamma^2\right]^2\left[5-\gamma^2\right] \;>\; 4\,. \tag{105}$$

The last inequality in (105) holds because $\left[2-\gamma^2\right]^2\left[5-\gamma^2\right] > \left[2-1^2\right]^2\left[5-1^2\right] = 4$.

(96), (104), and (105) imply:

$$L_2^{IP} \;<\; L_2^{SP} \;\; \text{when} \;\; \frac{\Delta_2}{\Delta_1} \;\in\; \left(\frac{8-\gamma^2-\gamma^4}{6\gamma}, \frac{2-\gamma^2}{\gamma}\right). \tag{106}$$

Lemma 2 implies:

$$L_2^{SP} \lesseqgtr L_2^{IP} \Leftrightarrow \pi_2^{nP} - \pi_2^{SP} \lesseqgtr \pi_2^{nP} - \pi_2^{IP} \Leftrightarrow \pi_2^{IP} \lesseqgtr \pi_2^{SP} . \tag{107}$$

(66), (83), (84), and (96) imply:

$$q_1^{SP} > 0, \; q_2^{SP} > 0, \; q_1^{IP} > 0, \text{ and } q_2^{IP} > 0 \text{ when } \frac{\Delta_2}{\Delta_1} \in \left(\gamma, \frac{8 - \gamma^2 - \gamma^4}{6\,\gamma} \right). \tag{108}$$

(38) and (39) imply:

$$\pi_2^{SP} = \left[1 - \gamma^2 \right] \left[q_2^{SP} \right]^2 \text{ and } \pi_2^{IP} = \left[1 - \gamma^2 \right] \left[q_2^{IP} \right]^2 . \tag{109}$$

(28), (36), and (108) imply that when $\frac{\Delta_2}{\Delta_1} \in \left(\gamma, \frac{8-\gamma^2-\gamma^4}{6\,\gamma} \right)$,

$$q_2^{SP} = \frac{[\,2-\gamma^2\,]\left[\,\alpha_2 - w_2^{SP} - c_2^d\,\right] - \gamma\left[\,\alpha_1 - w_1^{SP} - c_1^d\,\right]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}; \tag{110}$$

$$\begin{aligned} q_2^{IP} &= \frac{[\,2-\gamma^2\,]\,\Delta_2 - \gamma\,\Delta_1 + 2\,[\,1-\gamma^2\,]\left[\,c^u - w_2^{IP}\,\right]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \\ &= \frac{[\,2-\gamma^2\,]\,\Delta_2 - \gamma\,\Delta_1 + [\,2-\gamma^2\,]\left[\,c^u - w_2^{IP}\,\right] - \gamma^2\left[\,c^u - w_2^{IP}\,\right]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \\ &= \frac{[\,2-\gamma^2\,]\left[\,\alpha_2 - w_2^{IP} - c_2^d\,\right] - \gamma\,\Delta_1 - \gamma^2\left[\,c^u - w_2^{IP}\,\right]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \\ &= \frac{[\,2-\gamma^2\,]\left[\,\alpha_2 - w_2^{IP} - c_2^d\,\right] - \gamma\left[\,\alpha_1 - c^u - c_1^d\,\right] - \gamma^2\left[\,c^u - w_2^{IP}\,\right]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} . \end{aligned} \tag{111}$$

(109) – (111) imply that when $\frac{\Delta_2}{\Delta_1} \in \left(\gamma, \frac{8-\gamma^2-\gamma^4}{6\,\gamma} \right)$:

$$\begin{aligned} & \pi_2^{IP} \lesseqgtr \pi_2^{SP} \Leftrightarrow q_2^{IP} \lesseqgtr q_2^{SP} \\ \Leftrightarrow\; & \left[2-\gamma^2 \right] \left[\alpha_2 - w_2^{IP} - c_2^d \right] - \gamma \left[\alpha_1 - c^u - c_1^d \right] - \gamma^2 \left[c^u - w_2^{IP} \right] \\ & \quad \lesseqgtr \left[2-\gamma^2 \right] \left[\alpha_2 - w_2^{SP} - c_2^d \right] - \gamma \left[\alpha_1 - w_1^{SP} - c_1^d \right] \\ \Leftrightarrow\; & \left[2-\gamma^2 \right] \left[- w_2^{IP} \right] - \gamma \left[-c^u \right] - \gamma^2 \left[c^u - w_2^{IP} \right] \\ & \quad \lesseqgtr \left[2-\gamma^2 \right] \left[- w_2^{SP} \right] - \gamma \left[- w_1^{SP} \right] \\ \Leftrightarrow\; & \left[2-\gamma^2 \right] \left[w_2^{SP} - w_2^{IP} \right] + \gamma^2 \left[w_2^{IP} - c^u \right] \lesseqgtr \gamma \left[w_1^{SP} - c^u \right] \end{aligned}$$

$$\Leftrightarrow \quad \left[\, 2 - \gamma^2 \,\right] \left[\, w_2^{SP} - w_2^{IP} \,\right] + \gamma^2 \left[\, w_2^{IP} - c^u \,\right] \;\lesseqgtr\; \gamma \left[\, w_1^{SP} - c^u \,\right] \,. \tag{112}$$

Lemma 5 and Lemma 6 imply that when $\frac{\Delta_2}{\Delta_1} \in \left(\gamma, \frac{8 - \gamma^2 - \gamma^4}{6\,\gamma} \right)$:

$$w_2^{SP} - w_2^{IP} \;=\; c^u + \frac{1}{2}\,\Delta_2 - c^u - \frac{\gamma^3\,\Delta_1 \,+\, 8\,\Delta_2}{2\,[\,8 \,+\, \gamma^2\,]}$$

$$= \; \frac{\gamma^2\,\Delta_2 - \gamma^3\,\Delta_1}{2\,[\,8 \,+\, \gamma^2\,]} \;=\; \frac{\gamma^2\,[\,\Delta_2 - \gamma\,\Delta_1\,]}{2\,[\,8 \,+\, \gamma^2\,]} \;>\; 0\,;$$

$$w_1^{SP} - c^u \;=\; c^u + \frac{1}{2}\,\Delta_1 - c^u \;=\; \frac{1}{2}\,\Delta_1 \;>\; 0\,;$$

$$w_2^{IP} - c^u \;=\; \frac{\gamma^3\,\Delta_1 \,+\, 8\,\Delta_2}{2\,[\,8 \,+\, \gamma^2\,]} \;>\; 0\,. \tag{113}$$

(112) and (113) imply that when $\frac{\Delta_2}{\Delta_1} \in \left(\gamma, \frac{8 - \gamma^2 - \gamma^4}{6\,\gamma} \right)$:

$$\pi_2^{IP} \;\lesseqgtr\; \pi_2^{SP} \quad \Leftrightarrow \quad \left[\, 2 - \gamma^2 \,\right] \frac{\gamma^2\,[\,\Delta_2 - \gamma\,\Delta_1\,]}{2\,[\,8 \,+\, \gamma^2\,]} + \gamma^2\,\frac{\gamma^3\,\Delta_1 \,+\, 8\,\Delta_2}{2\,[\,8 \,+\, \gamma^2\,]} \;\lesseqgtr\; \gamma\,\frac{1}{2}\,\Delta_1$$

$$\Leftrightarrow \quad \left[\, 2 - \gamma^2 \,\right] \frac{\gamma\,[\,\Delta_2 - \gamma\,\Delta_1\,]}{8 \,+\, \gamma^2} + \gamma\,\frac{\gamma^3\,\Delta_1 \,+\, 8\,\Delta_2}{8 \,+\, \gamma^2} \;\lesseqgtr\; \Delta_1$$

$$\Leftrightarrow \quad \gamma\;\frac{[\,2 - \gamma^2\,] \left[\, \frac{\Delta_2}{\Delta_1} - \gamma \,\right] + \gamma^3 + 8\,\frac{\Delta_2}{\Delta_1}}{8 \,+\, \gamma^2} \;\lesseqgtr\; 1$$

$$\Leftrightarrow \quad \gamma\;\frac{[\,10 - \gamma^2\,]\,\frac{\Delta_2}{\Delta_1} - \gamma\,[\,2 - \gamma^2\,] + \gamma^3}{8 \,+\, \gamma^2} \;\lesseqgtr\; 1$$

$$\Leftrightarrow \quad \gamma\;\frac{[\,10 - \gamma^2\,]\,\frac{\Delta_2}{\Delta_1} - 2\,\gamma\,[\,1 - \gamma^2\,]}{8 \,+\, \gamma^2} \;\lesseqgtr\; 1$$

$$\Leftrightarrow \quad \left[\, 10 - \gamma^2 \,\right] \frac{\Delta_2}{\Delta_1} - 2\,\gamma \left[\, 1 - \gamma^2 \,\right] \;\lesseqgtr\; \frac{8 \,+\, \gamma^2}{\gamma}$$

$$\Leftrightarrow \quad \frac{\Delta_2}{\Delta_1} \;\lesseqgtr\; \frac{8 + \gamma^2}{\gamma\,[\,10 - \gamma^2\,]} + \frac{2\,\gamma\,[\,1 - \gamma^2\,]}{10 - \gamma^2}$$

$$= \; \frac{8 + \gamma^2 + 2\,\gamma^2\,[\,1 - \gamma^2\,]}{\gamma\,[\,10 - \gamma^2\,]} \;=\; m\,(\gamma) \;. \tag{114}$$

Because $m\,(\gamma) \in (\,\gamma, \frac{8 - \gamma^2 - \gamma^4}{6\,\gamma}\,)$ for all $\gamma \in (0, 1)$, as illustrated in Figure 2, (107) and (114) imply:

$$L_2^{IP} \;>\; L_2^{SP} \quad \text{when} \quad \frac{\Delta_2}{\Delta_1} \;\in\; (\,\gamma,\, m(\gamma)\,)\,;$$

$$L_2^{IP} < L_2^{SP} \quad \text{when} \quad \frac{\Delta_2}{\Delta_1} \in \left(m(\gamma), \frac{8 - \gamma^2 - \gamma^4}{6\,\gamma} \right). \quad \blacksquare$$

The complete proof of Corollary 1 is provided in the text.

II. Ubiquitous Collusion

Under ubiquitous collusion, U1, U2, D1, and D2 collude to maximize their combined profit.

Lemma 7. *The following retail prices and outputs arise under ubiquitous collusion:*

$$p_i^U \;=\; \frac{1}{2}\left[\,\alpha_i + c^u + c_i^d\,\right] \quad \textit{and} \quad q_i^U \;=\; \frac{\Delta_i - \gamma\,\Delta_j}{2\,[\,1-\gamma^2\,]} \;\; \textit{for } i \in \{1,2\}\ (j \neq i). \tag{115}$$

Proof. Under ubiquitous collusion, U1, U2, D1, and D2 set prices p_1 and p_2 to:

$$\text{Maximize} \quad \Pi \;=\; \left[\,p_1 - c^u - c_1^d\,\right] q_1 + \left[\,p_2 - c^u - c_2^d\,\right] q_2\,. \tag{116}$$

Recall from (2) that for $i,\, j \in \{1,2\}$ $(j \neq i)$:

$$\begin{aligned}
& p_i \;=\; \alpha_i - q_i - \gamma\, q_j \\
\Rightarrow\;\; & q_i \;=\; \alpha_i - p_i - \gamma\, q_j \;=\; \alpha_i - p_i - \gamma\,[\,\alpha_j - p_j - \gamma\, q_i\,] \\
\Rightarrow\;\; & q_i\left[\,1-\gamma^2\,\right] \;=\; \alpha_i - \gamma\,\alpha_j - p_i + \gamma\, p_j \\
\Rightarrow\;\; & q_i \;=\; \frac{1}{1-\gamma^2}\,[\,\alpha_i - \gamma\,\alpha_j - p_i + \gamma\, p_j\,]\,.
\end{aligned} \tag{117}$$

(116) and (117) imply the colluding producers set p_1 and p_2 to maximize:

$$\begin{aligned}
\Pi \;=\; & \left[\,p_1 - c^u - c_1^d\,\right] \frac{1}{1-\gamma^2}\,[\,\alpha_1 - \gamma\,\alpha_2 - p_1 + \gamma\, p_2\,] \\
& + \left[\,p_2 - c^u - c_2^d\,\right] \frac{1}{1-\gamma^2}\,[\,\alpha_2 - \gamma\,\alpha_1 - p_2 + \gamma\, p_1\,]\,.
\end{aligned} \tag{118}$$

Differentiating (118) implies the optimal collusive prices are determined by:

$$\begin{aligned}
\frac{\partial \Pi}{\partial p_i} \;\overset{s}{=}\; & -\left[\,p_i - c^u - c_i^d\,\right] + \alpha_i - \gamma\,\alpha_j - p_i + \gamma\, p_j + \gamma\left[\,p_j - c^u - c_j^d\,\right] \;=\; 0 \\
\Leftrightarrow\;\; & 2\,p_i \;=\; c^u + c_i^d + \alpha_i - \gamma\,\alpha_j + 2\,\gamma\, p_j - \gamma\, c^u - \gamma\, c_j^d \\
\Leftrightarrow\;\; & p_i \;=\; \gamma\, p_j + \frac{1}{2}\left[\,\alpha_i + c^u + c_i^d - \gamma\left(\,\alpha_j + c^u + c_j^d\,\right)\,\right].
\end{aligned} \tag{119}$$

(119) implies:

$$p_i \;=\; \frac{1}{2}\left[\,\alpha_i + c^u + c_i^d - \gamma\left(\,\alpha_j + c^u + c_j^d\,\right)\,\right]$$

$$+ \gamma \left\{ \gamma p_i + \frac{1}{2} \left[\alpha_j + c^u + c_j^d - \gamma \left(\alpha_i + c^u + c_i^d \right) \right] \right\}$$

$$\Rightarrow \quad p_i \left[1 - \gamma^2 \right] \; = \; \frac{1}{2} \{ \alpha_i + c^u + c_i^d - \gamma \left[\alpha_j + c^u + c_j^d \right]$$

$$+ \; \gamma \left[\alpha_j + c^u + c_j^d \right] - \gamma^2 \left[\alpha_i + c^u + c_i^d \right] \}$$

$$\Rightarrow \quad p_i \left[1 - \gamma^2 \right] \; = \; \frac{1}{2} \left[1 - \gamma^2 \right] \left[\alpha_i + c^u + c_i^d \right] \quad \Rightarrow \quad p_i \; = \; \frac{1}{2} \left[\alpha_i + c^u + c_i^d \right] . \tag{120}$$

(117) and (120) imply the corresponding downstream outputs are:

$$q_i \; = \; \frac{1}{1 - \gamma^2} \left[\alpha_i - \gamma \, \alpha_j - \frac{1}{2} \left(\alpha_i + c^u + c_i^d \right) + \gamma \, \frac{1}{2} \left(\alpha_j + c^u + c_j^d \right) \right]$$

$$= \; \frac{1}{2 \left[1 - \gamma^2 \right]} \left[2 \left(\alpha_i - \gamma \, \alpha_j \right) - \left(\alpha_i + c^u + c_i^d \right) + \gamma \left(\alpha_j + c^u + c_j^d \right) \right]$$

$$= \; \frac{1}{2 \left[1 - \gamma^2 \right]} \left[\alpha_i - 2 \, \gamma \, \alpha_j \; - c^u - c_i^d \; + \gamma \, \alpha_j + \gamma \, c^u + \gamma \, c_j^d \right]$$

$$= \; \frac{1}{2 \left[1 - \gamma^2 \right]} \left[\alpha_i - c^u - c_i^d - \gamma \left(\alpha_j - c^u - c_j^d \right) \right] \; = \; \frac{\Delta_i - \gamma \, \Delta_j}{2 \left[1 - \gamma^2 \right]} . \quad \blacksquare \tag{121}$$

Conclusion 1. *Suppose D1 and D2 each produce positive output both under ubiquitous collusion and in the absence of collusion. Then total industry output is lower under ubiquitous collusion than in the absence of collusion.*

Proof. (115) implies that total industry output under ubiquitous collusion is:

$$q^U \; \equiv \; q_1^U + q_2^U \; = \; \frac{\Delta_1 - \gamma \, \Delta_2 + \Delta_2 - \gamma \, \Delta_1}{2 \left[1 - \gamma^2 \right]} \; = \; \frac{\left[1 - \gamma \right] \left[\Delta_1 + \Delta_2 \right]}{2 \left[1 - \gamma^2 \right]} \; = \; \frac{\Delta_1 + \Delta_2}{2 \left[1 + \gamma \right]} . \tag{122}$$

Lemma 2 implies that in the absence of collusion, under downstream quantity competition:

$$q^{nQ} \; \equiv \; q_1^{nQ} + q_2^{nQ} \; = \; \frac{2 \, \Delta_1 - \gamma \, \Delta_2 + 2 \, \Delta_2 - \gamma \, \Delta_1}{4 - \gamma^2} \; = \; \frac{\left[2 - \gamma \right] \left[\Delta_1 + \Delta_2 \right]}{4 - \gamma^2} . \tag{123}$$

(122) and (123) imply:

$$q^U \; < \; q^{nQ} \quad \Leftrightarrow \quad \frac{1}{2 \left[1 + \gamma \right]} \; < \; \frac{2 - \gamma}{4 - \gamma^2} \quad \Leftrightarrow \quad 4 - \gamma^2 \; < \; 2 \left[1 + \gamma \right] \left[2 - \gamma \right]$$

$$\Leftrightarrow \quad 4 - \gamma^2 \; < \; 2 \left[2 + \gamma - \gamma^2 \right] \quad \Leftrightarrow \quad 4 - \gamma^2 \; < \; 4 + 2 \, \gamma - 2 \, \gamma^2$$

$$\Leftrightarrow \quad \gamma^2 \; < \; 2 \, \gamma \quad \Leftrightarrow \quad \gamma \; < \; 2 \, .$$

Lemma 2 implies that in the absence of collusion, under downstream price competition:

$$q^{nP} \equiv q_1^{nP} + q_2^{nP} = \frac{[\,2-\gamma^2\,]\,\Delta_1 - \gamma\,\Delta_2 + [\,2-\gamma^2\,]\,\Delta_2 - \gamma\,\Delta_1}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}$$

$$= \frac{[\,2-\gamma-\gamma^2\,]\,[\,\Delta_1+\Delta_2\,]}{[\,1-\gamma\,]\,[\,1+\gamma\,]\,[\,4-\gamma^2\,]}\,. \tag{124}$$

(122) and (124) imply:

$$q^U < q^{nP} \;\Leftrightarrow\; \frac{1}{2} < \frac{2-\gamma-\gamma^2}{[\,1-\gamma\,]\,[\,4-\gamma^2\,]} \;\Leftrightarrow\; [\,1-\gamma\,]\left[\,4-\gamma^2\,\right] < 2\left[\,2-\gamma-\gamma^2\,\right]$$

$$\Leftrightarrow\; 4-\gamma^2-4\,\gamma+\gamma^3 < 4-2\,\gamma-2\,\gamma^2 \;\Leftrightarrow\; 2\,\gamma-\gamma^2-\gamma^3 > 0$$

$$\Leftrightarrow\; 2 > \gamma+\gamma^2 \;\Leftrightarrow\; 2 > \gamma\,[\,1+\gamma\,]\,. \;\blacksquare$$

Conclusion 2. *Suppose D1 and D2 each produce positive output both under ubiquitous collusion and under upstream collusion. Then total industry output is higher under ubiquitous collusion than under upstream collusion.*

Proof. Lemma 3 implies that under upstream collusion with vertical separation and downstream quantity competition:

$$q^{SQ} \equiv q_1^{SQ} + q_2^{SQ} = \frac{2\,\Delta_1 - \gamma\,\Delta_2 + 2\,\Delta_2 - \gamma\,\Delta_1}{2\,[\,4-\,\gamma^2\,]} = \frac{[\,2-\gamma\,]\,[\,\Delta_1+\Delta_2\,]}{2\,[\,4-\,\gamma^2\,]}$$

$$= \frac{[\,2-\gamma\,]\,[\,1+\gamma\,]}{4-\gamma^2}\left[\frac{\Delta_1+\Delta_2}{2\,(\,1+\gamma\,)}\right] = \frac{[\,2-\gamma\,]\,[\,1+\gamma\,]}{4-\gamma^2}\,q^U < q^U\,. \tag{125}$$

The last equality in (125) reflects (122). The inequality in (125) holds because:

$$\frac{[\,2-\gamma\,]\,[\,1+\gamma\,]}{4-\gamma^2} < 1 \;\Leftrightarrow\; 2+2\,\gamma-\gamma-\gamma^2 < 4-\gamma^2 \;\Leftrightarrow\; \gamma < 2\,.$$

Lemma 5 implies that under upstream collusion with vertical separation and downstream price competition:

$$q^{SP} \equiv q_1^{SP} + q_2^{SP} = \frac{[\,2-\gamma^2\,]\,\Delta_1 - \gamma\,\Delta_2 + [\,2-\gamma^2\,]\,\Delta_2 - \gamma\,\Delta_1}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}$$

$$= \frac{[\,2-\gamma-\gamma^2\,]\,[\,\Delta_1+\Delta_2\,]}{2\,[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} = \frac{2-\gamma-\gamma^2}{[\,1-\gamma\,]\,[\,4-\gamma^2\,]}\left[\frac{\Delta_1+\Delta_2}{2\,(\,1+\gamma\,)}\right]$$

$$= \frac{2-\gamma-\gamma^2}{[\,1-\gamma\,]\,[\,4-\gamma^2\,]}\,q^U < q^U\,. \tag{126}$$

The last equality in (126) reflects (122). The inequality in (126) holds because:

$$\frac{2-\gamma-\gamma^2}{[1-\gamma][4-\gamma^2]} \;<\; 1 \;\;\Leftrightarrow\;\; 2-\gamma-\gamma^2 \;<\; 4-\gamma^2-4\gamma+\gamma^3$$
$$\Leftrightarrow\;\; z(\gamma) \;\equiv\; 2-3\gamma+\gamma^3 \;>\; 0\,. \tag{127}$$

The inequality in (127) holds for all $\gamma \in (0,1)$ because $z(0) = 2$, $z(1) = 0$, and $z'(\gamma) = -3+3\gamma^2 < 0$.

Lemma 4 implies that under upstream collusion with vertical integration and downstream quantity competition:

$$q^{IQ} \;\equiv\; q_1^{IQ} + q_2^{IQ} \;=\; \frac{[8-\gamma^2]\,\Delta_1 - 2\gamma\,\Delta_2 + 4[\Delta_2 - \gamma\,\Delta_1]}{2\,[8\,-\,3\gamma^2]}$$
$$=\; \frac{[8-4\gamma-\gamma^2]\,\Delta_1 + [4-2\gamma]\,\Delta_2}{2\,[8\,-\,3\gamma^2]}\,. \tag{128}$$

(122) and (128) imply:

$$q^U \;>\; q^{IQ} \;\;\Leftrightarrow\;\; \frac{\Delta_1+\Delta_2}{2[1+\gamma]} \;>\; \frac{[8-4\gamma-\gamma^2]\,\Delta_1 + [4-2\gamma]\,\Delta_2}{2[8-3\gamma^2]}$$

$$\Leftrightarrow\;\; \left[8-3\gamma^2\right]\Delta_1 + \left[8-3\gamma^2\right]\Delta_2$$
$$>\;\; [1+\gamma]\left[8-4\gamma-\gamma^2\right]\Delta_1 + [1+\gamma][4-2\gamma]\,\Delta_2$$

$$\Leftrightarrow\;\; \left[8-3\gamma^2-\left(8-4\gamma-\gamma^2+8\gamma-4\gamma^2-\gamma^3\right)\right]\Delta_1$$
$$+\;\left[8-3\gamma^2-2\left(2+\gamma-\gamma^2\right)\right]\Delta_2 \;>\; 0$$

$$\Leftrightarrow\;\; \left[-3\gamma^2+4\gamma+\gamma^2-8\gamma+4\gamma^2+\gamma^3\right]\Delta_1$$
$$+\;\left[8-3\gamma^2-4-2\gamma+2\gamma^2\right]\Delta_2 \;>\; 0$$

$$\Leftrightarrow\;\; \left[-4\gamma+2\gamma^2+\gamma^3\right]\Delta_1 + \left[4-2\gamma-\gamma^2\right]\Delta_2 \;>\; 0$$

$$\Leftrightarrow\;\; \left[4-2\gamma-\gamma^2\right]\Delta_2 - \gamma\left[4-2\gamma-\gamma^2\right]\Delta_1 \;>\; 0 \;\;\Leftrightarrow\;\; \frac{\Delta_2}{\Delta_1} \;>\; \gamma\,. \tag{129}$$

The last inequality in (129) holds because, from Lemma 4 , $\frac{\Delta_2}{\Delta_1} \in (\gamma, \frac{2}{\gamma})$ when D1 and D2 each produce positive output under upstream collusion, VI, and quantity competition.

Lemma 6 implies that under upstream collusion with vertical integration and downstream price competition:

$$q^{IP} \;\equiv\; q_1^{IP} + q_2^{IP} \;=\; \frac{[8-\gamma^2-\gamma^4]\,\Delta_1 - 6\gamma\,\Delta_2 + 2[2+\gamma^2][\Delta_2 - \gamma\,\Delta_1]}{2[1-\gamma^2][8+\gamma^2]}$$

$$= \frac{[\,8-\gamma^2-\gamma^4-2\,\gamma\,(\,2+\gamma^2\,)\,]\,\Delta_1+[\,4+2\,\gamma^2-6\,\gamma\,]\,\Delta_2}{2\,[\,1-\gamma^2\,]\,[\,8+\gamma^2\,]}$$

$$= \frac{[\,8-4\,\gamma-\gamma^2-2\,\gamma^3-\gamma^4\,]\,\Delta_1+2\,[\,2-3\,\gamma+\gamma^2\,]\,\Delta_2}{2\,[\,1-\gamma^2\,]\,[\,8+\gamma^2\,]}\,. \tag{130}$$

(122) and (130) imply:

$$q^U > q^{IP} \;\Leftrightarrow\; \frac{\Delta_1+\Delta_2}{2\,[\,1+\gamma\,]} > \frac{[\,8-4\,\gamma-\gamma^2-2\,\gamma^3-\gamma^4\,]\,\Delta_1+2\,[\,2-3\,\gamma+\gamma^2\,]\,\Delta_2}{2\,[\,1+\gamma\,]\,[\,1-\gamma\,]\,[\,8+\gamma^2\,]}$$

$$\Leftrightarrow\; [\,1-\gamma\,]\left[\,8+\gamma^2\,\right][\,\Delta_1+\Delta_2\,]$$
$$> \left[\,8-4\,\gamma-\gamma^2-2\,\gamma^3-\gamma^4\,\right]\Delta_1+2\left[\,2-3\,\gamma+\gamma^2\,\right]\Delta_2$$

$$\Leftrightarrow\; \left[\,8-4\,\gamma-\gamma^2-2\,\gamma^3-\gamma^4-(\,1-\gamma\,)\left(\,8+\gamma^2\,\right)\,\right]\Delta_1$$
$$+\left[\,4-6\,\gamma+2\,\gamma^2-(\,1-\gamma\,)\left(\,8+\gamma^2\,\right)\,\right]\Delta_2 < 0$$

$$\Leftrightarrow\; \left[\,8-4\,\gamma-\gamma^2-2\,\gamma^3-\gamma^4-8-\gamma^2+8\,\gamma+\gamma^3\,\right]\Delta_1$$
$$+\left[\,4-6\,\gamma+2\,\gamma^2-8-\gamma^2+8\,\gamma+\gamma^3\,\right]\Delta_2 < 0$$

$$\Leftrightarrow\; \left[\,4\,\gamma-2\,\gamma^2-\gamma^3-\gamma^4\,\right]\Delta_1-\left[\,4-2\,\gamma-\gamma^2-\gamma^3\,\right]\Delta_2 < 0$$

$$\Leftrightarrow\; \gamma\left[\,4-2\,\gamma-\gamma^2-\gamma^3\,\right]\Delta_1-\left[\,4-2\,\gamma-\gamma^2-\gamma^3\,\right]\Delta_2 < 0 \;\Leftrightarrow\; \frac{\Delta_2}{\Delta_1} > \gamma\,. \tag{131}$$

The last inequality in (131) holds because, from Lemma 6, $\frac{\Delta_2}{\Delta_1} \in (\,\gamma,\, \frac{8-\gamma^2-\gamma^4}{6\,\gamma}\,)$ when D1 and D2 each produce positive output under upstream collusion, VI, and price competition. ■

Conclusion 3. *Suppose D1 and D2 each produce positive output both under ubiquitous collusion and under upstream collusion. Then in the presence of VI, D2's output is higher under ubiquitous collusion than under upstream collusion.*

Proof. Lemmas 4 and 7 imply:

$$q_2^U > q_2^{IQ} \;\Leftrightarrow\; \frac{\Delta_2-\gamma\,\Delta_1}{2\,[\,1-\gamma^2\,]} > \frac{2\,[\,\Delta_2-\gamma\,\Delta_1\,]}{8-3\,\gamma^2} \;\Leftrightarrow\; \frac{1}{2\,[\,1-\gamma^2\,]} > \frac{2}{8-3\,\gamma^2}$$

$$\Leftrightarrow\; 8-3\,\gamma^2 > 4\left[\,1-\gamma^2\,\right] \;\Leftrightarrow\; \gamma^2+4 > 0\,.$$

Lemmas 6 and 7 imply:

$$q_2^U \ > \ q_2^{IP} \ \Leftrightarrow \ \frac{\Delta_2 - \gamma\,\Delta_1}{2\,[\,1-\gamma^2\,]} \ > \ \frac{[\,2+\gamma^2\,]\,[\,\Delta_2 - \gamma\,\Delta_1\,]}{[\,1-\gamma^2\,]\,[\,8+\gamma^2\,]} \ \Leftrightarrow \ \frac{1}{2} \ > \ \frac{2+\gamma^2}{8+\gamma^2}$$

$$\Leftrightarrow \ 8+\gamma^2 \ > \ 2\left[\,2+\gamma^2\,\right] \ \Leftrightarrow \ \gamma^2 \ < \ 4\,. \ \blacksquare$$

III. Two-Part Tariffs

We now illustrate the qualitative changes that can arise when U1 and U2 are able to set discriminatory two-part tariffs for the input. We consider the setting in which D1 and D2 engage in price competition.

Recall that the inverse demand function for Di's product is:

$$P_i(q_i, q_j) \;=\; \alpha_i - q_i - \gamma\, q_j \quad \text{for } i,\, j \in \{1, 2\} \;\; (j \neq i). \tag{132}$$

Conclusion 4 characterizes equilibrium outcomes when U1 and U2 collude to maximize their combined profit.

Conclusion 4. *D2's equilibrium profit is* 0 *both under VS and under VI when U1 and U2 can collude and set discriminatory two-part tariffs.*

Proof. Suppose D2's equilibrium profit is $\chi > 0$. Then U1 and U2 can increase their profit by increasing the lump-sum component of D2's tariff (A_2) by χ. Doing so does not change equilibrium outcomes as long as D2 continues to operate. D2 will continue to operate because it secures exactly zero profit under the higher A_2. Because the postulated setting in which D2's equilibrium profit is $\chi > 0$ does not maximize the joint profit of U1 and U2, this setting cannot arise when U1 and U2 collude. ■

Now consider the setting in which U1 and U2 do not collude. We consider a two-stage game in which U1 and U2 first offer two-part tariffs to D1 and D2 (simultaneously and noncooperatively). D1 and D2 then simultaneously decide which offer(s) to accept, and subsequently choose prices (simultaneously and noncooperatively). We assume that when Di is indifferent between accepting Ui's tariff and Uj's tariff, Di accepts Ui's tariff. We consider settings in which D1 and D2 both produce strictly positive output in equilibrium.

Let $(w_j^i,\, A_j^i)$ denote the two-part tariff Ui offers to Dj ($i,\, j \in \{1, 2\}$). We assume that when $A_j^i < 0$, Dj can collect the fixed fee from Ui even if it purchases none of the input from Ui. After selecting two-part tariff $(w_i,\, A_i)$, Di's ($i \in \{1, 2\}$) downstream profit when it sets price P_i and produces output q_i is:

$$\pi_i \;=\; \left[\, P_i - w_i - c_i^d \,\right] q_i - A_i\,. \tag{133}$$

Lemma 8. *Under VS, for* $j,\, l \in \{1, 2\}$ *(*$j \neq l$*), D*j*'s downstream profit: (i) declines as the unit price (*w_j*) or the fixed fee (*A_j*) it faces increases; and (ii) increases as the unit price its rival faces (*w_l*) increases.*

Proof. (132) implies:

$$q_i \;=\; \alpha_i - P_i - \gamma\, q_j \;=\; \alpha_i - P_i - \gamma\, [\, \alpha_j - P_j - \gamma\, q_i \,]$$

$$\Rightarrow \quad q_i(P_i, P_j) \;=\; \frac{\alpha_i - \gamma\, \alpha_j}{1-\gamma^2} - \frac{1}{1-\gamma^2}\, P_i + \frac{\gamma}{1-\gamma^2}\, P_j\,. \tag{134}$$

(133) and (134) imply that under VS, for $i, j \in \{1, 2\}$ $(j \neq i)$, Di chooses P_i to:

$$Maximize \quad \left[\, P_i - w_i - c_i^d \,\right] q_i(P_i, P_j) - A_i\,. \tag{135}$$

(135) implies that Di's profit-maximizing choice of P_i is determined by:

$$q_i + \left[\, P_i - w_i - c_i^d \,\right] \frac{\partial q_i}{\partial P_i} \;=\; 0\,. \tag{136}$$

(134) and (136) imply:

$$\frac{\alpha_i - \gamma\, \alpha_j}{1-\gamma^2} - \frac{1}{1-\gamma^2}\, P_i + \frac{\gamma}{1-\gamma^2}\, P_j - \frac{1}{1-\gamma^2} \left[\, P_i - w_i - c_i^d \,\right] \;=\; 0$$

$$\Rightarrow \quad P_i \;=\; \frac{\alpha_i - \gamma\, \alpha_j + w_i + c_i^d}{2} + \frac{\gamma}{2}\, P_j\,. \tag{137}$$

(137) and Dj's corresponding reaction function imply that in equilibrium:

$$P_i \;=\; \frac{\alpha_i - \gamma\, \alpha_j + w_i + c_i^d}{2} + \frac{\gamma}{2} \left[\frac{\alpha_j - \gamma\, \alpha_i + w_j + c_j^d}{2} + \frac{\gamma}{2}\, P_i \right]$$

$$\Rightarrow \quad \left[1 - \frac{\gamma^2}{4} \right] P_i \;=\; \frac{1}{2} \left[\, \alpha_i - \gamma\, \alpha_j + w_i + c_i^d \,\right] + \frac{\gamma}{4} \left[\, \alpha_j - \gamma\, \alpha_i + w_j + c_j^d \,\right]$$

$$\Rightarrow \quad P_i\,(w_i, w_j) \;=\; \frac{2}{4-\gamma^2} \left[\, \alpha_i - \gamma\, \alpha_j + w_i + c_i^d \,\right] + \frac{\gamma}{4-\gamma^2} \left[\, \alpha_j - \gamma\, \alpha_i + w_j + c_j^d \,\right]. \tag{138}$$

(134) and (138) imply:

$$q_i \;=\; \frac{\alpha_i - \gamma\, \alpha_j}{1-\gamma^2} - \frac{1}{1-\gamma^2} \left[\frac{2}{4-\gamma^2} \left(\, \alpha_i - \gamma\, \alpha_j + w_i + c_i^d \,\right) + \frac{\gamma}{4-\gamma^2} \left(\, \alpha_j - \gamma\, \alpha_i + w_j + c_j^d \,\right) \right]$$

$$+ \frac{\gamma}{1-\gamma^2} \left[\frac{2}{4-\gamma^2} \left(\, \alpha_j - \gamma\, \alpha_i + w_j + c_j^d \,\right) + \frac{\gamma}{4-\gamma^2} \left(\, \alpha_i - \gamma\, \alpha_j + w_i + c_i^d \,\right) \right]$$

$$= \frac{1}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \left\{ \left[\, 4-\gamma^2 \,\right] [\, \alpha_i - \gamma\, \alpha_j \,] - \left[\, 2-\gamma^2 \,\right] \left[\, \alpha_i - \gamma\, \alpha_j + w_i + c_i^d \,\right] \right.$$

$$\left. + \gamma \left[\, \alpha_j - \gamma\, \alpha_i + w_j + c_j^d \,\right] \right\}$$

$$= \frac{1}{[1-\gamma^2][4-\gamma^2]} \left\{ \left[2-\gamma^2\right] \alpha_i - \gamma\,\alpha_j - \left[2-\gamma^2\right] w_i - \left[2-\gamma^2\right] c_i^d + \gamma\, w_j + \gamma\, c_j^d \right\}$$

$$\Rightarrow \quad q_i(w_i, w_j) = \frac{[2-\gamma^2]\left[\alpha_i - w_i - c_i^d\right] - \gamma\left[\alpha_j - w_j - c_j^d\right]}{[1-\gamma^2][4-\gamma^2]}\,. \tag{139}$$

(134) and (136) imply:

$$P_i - w_i - c_i^d = -\frac{q_i}{-\left[\frac{1}{1-\gamma^2}\right]} = \left[1-\gamma^2\right] q_i\,.$$

Therefore, (133) implies:

$$\pi_i(w_i, w_j) = \left[1-\gamma^2\right]\left[q_i(w_i, w_j)\right]^2 - A_i \tag{140}$$

where $q_i(w_i, w_j)$ is given by (139).

(139) and (140) imply that when $q_j > 0$ under VS:

$$\frac{\partial \pi_j}{\partial w_j} = \frac{\partial \pi_j}{\partial q_j}\frac{\partial q_j}{\partial w_j} = -\frac{2[1-\gamma^2][2-\gamma^2]\,q_j}{[1-\gamma^2][4-\gamma^2]} = -\frac{2[2-\gamma^2]\,q_j}{[4-\gamma^2]} < 0\,;$$

$$\frac{\partial \pi_j}{\partial A_j} = -1 < 0\,; \quad \frac{\partial \pi_j}{\partial w_l} = \frac{\partial \pi_j}{\partial q_j}\frac{\partial q_j}{\partial w_l} = \frac{2\,\gamma[1-\gamma^2]\,q_j}{[1-\gamma^2][4-\gamma^2]} = \frac{2\,\gamma\,q_j}{4-\gamma^2} > 0\,. \;\blacksquare \tag{141}$$

Conclusion 5. *Suppose U1 and U2 can set discriminatory two-part tariffs and cannot collude. Then under VS, there exists an equilibrium in which each upstream supplier offers the tariff* $(c^u, 0)$ *to both downstream suppliers.*

Proof. Throughout the ensuing proof, we assume Uk ($k \in \{1, 2\}$) offers the (c^u, 0) tariff to D1 and D2. We will show that Ui ($i \in \{1, 2\}$, $i \neq k$) cannot secure strictly positive profit by offering any tariffs other than (c^u, 0) to D1 and D2. The proof proceeds in two steps. In Step A, we show that Ui cannot secure strictly positive profit by offering tariff (c^u, 0) to Dl and a distinct tariff to Dj ($j, l \in \{1, 2\}$, $j \neq l$). In Step B, we show that Ui cannot secure strictly positive profit by offering tariffs other than (c^u, 0) to both D1 and D2.

A. First suppose Ui offers $\left(\widetilde{w}_j^i, \widetilde{A}_j^i\right) \neq (c^u, 0)$ to Dj and $(c^u, 0)$ to Dl.

Ui can only secure strictly positive profit from the $\left(\widetilde{w}_j^i, \widetilde{A}_j^i\right)$ tariff if $\widetilde{w}_j^i > c^u$ or $\widetilde{A}_j^i > 0$ or both.

If $\widetilde{w}_j^i > c^u$ and $\widetilde{A}_j^i \geq 0$, then Lemma 8 implies that Dj will reject Ui's $\left(\widetilde{w}_j^i, \widetilde{A}_j^i\right)$ offer and accept Uk's $(c^u, 0)$ offer. Consequently, Ui will secure zero profit.

If $\widetilde{w}_j^i > c^u$ and $\widetilde{A}_j^i < 0$, Dj will accept Ui's $\left(\widetilde{w}_j^i, \widetilde{A}_j^i\right)$ offer and also accept Uk's $(c^u, 0)$ offer. By doing so, Dj can increase its profit by collecting the fixed fee from Ui and purchasing all of the input from Uk. Consequently, Ui will secure negative profit.

If $\widetilde{A}_j^i > 0$ and $\widetilde{w}_j^i \geq c^u$, Lemma 8 implies that D$j$ will reject Ui's $\left(\widetilde{w}_j^i, \widetilde{A}_j^i\right)$ offer and accept Uk's $(c^u, 0)$ offer. Consequently, Ui will secure zero profit.

Now suppose $\widetilde{A}_j^i > 0$ and $\widetilde{w}_j^i < c^u$. (139) and (140) imply that Dj will accept Ui's ($\widetilde{w}_j^i$, $\widetilde{A}_j^i$) offer rather than Uk's $(c^u, 0)$ offer if:

$$
\begin{aligned}
& \left[1-\gamma^2\right]\left[q_j\left(\widetilde{w}_j^i, c^u\right)\right]^2 - \widetilde{A}_j^i \;>\; \left[1-\gamma^2\right]\left[q_j\left(c^u, c^u\right)\right]^2 \\
\Leftrightarrow\; & \frac{\widetilde{A}_j^i}{1-\gamma^2} \;<\; \left[q_j\left(\widetilde{w}_j^i, c^u\right)\right]^2 - \left[q_j\left(c^u, c^u\right)\right]^2 \\
\Leftrightarrow\; & \frac{\widetilde{A}_j^i}{1-\gamma^2} \;<\; \left\{\frac{\left[2-\gamma^2\right]\left[\alpha_j - \widetilde{w}_j^i - c_j^d\right] - \gamma\left[\alpha_l - c^u - c_l^d\right]}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]}\right\}^2 \\
& \qquad - \left\{\frac{\left[2-\gamma^2\right]\left[\alpha_j - c^u - c_j^d\right] - \gamma\left[\alpha_l - c^u - c_l^d\right]}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]}\right\}^2 \\
\Leftrightarrow\; & \frac{\widetilde{A}_j^i}{1-\gamma^2} \;<\; \left\{\frac{\left[2-\gamma^2\right]\left[\alpha_j - \widetilde{w}_j^i - c_j^d\right] - \gamma\left[\alpha_l - c^u - c_l^d\right]}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]}\right. \\
& \qquad \left. + \frac{\left[2-\gamma^2\right]\left[\alpha_j - c^u - c_j^d\right] - \gamma\left[\alpha_l - c^u - c_l^d\right]}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]}\right\} \\
& \qquad \cdot \left\{\frac{\left[2-\gamma^2\right]\left[\alpha_j - \widetilde{w}_j^i - c_j^d\right] - \gamma\left[\alpha_l - c^u - c_l^d\right]}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]}\right. \\
& \qquad \left. - \frac{\left[2-\gamma^2\right]\left[\alpha_j - c^u - c_j^d\right] - \gamma\left[\alpha_l - c^u - c_l^d\right]}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]}\right\} \\
\Leftrightarrow\; & \frac{\widetilde{A}_j^i}{1-\gamma^2} \;<\; \left\{\frac{2\left[2-\gamma^2\right]\left[\alpha_j - c_j^d\right] - 2\gamma\left[\alpha_l - c^u - c_l^d\right] - \left[2-\gamma^2\right]\left[c^u + \widetilde{w}_j^i\right]}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]}\right\} \\
& \qquad \cdot \left\{\frac{\left[2-\gamma^2\right]\left[c^u - \widetilde{w}_j^i\right]}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]}\right\} \\
\Leftrightarrow\; & \frac{\widetilde{A}_j^i}{1-\gamma^2} \;<\; \frac{\left[2-\gamma^2\right]\left[c^u - \widetilde{w}_j^i\right]}{\left[1-\gamma^2\right]^2\left[4-\gamma^2\right]^2}\left\{2\left[2-\gamma^2\right]\left[\alpha_j - c^u - c_j^d\right] - 2\gamma\left[\alpha_l - c^u - c_l^d\right]\right.
\end{aligned}
$$

$$+ 2 \left[2 - \gamma^2 \right] c^u - \left[2 - \gamma^2 \right] \left[c^u + \widetilde{w}_j^i \right] \Big\}$$

$$\Leftrightarrow \quad \frac{\widetilde{A}_j^i}{1-\gamma^2} \; < \; \frac{[\,2-\gamma^2\,]\,[\,c^u - \widetilde{w}_j^i\,]}{[\,1-\gamma^2\,]^2\,[\,4-\gamma^2\,]^2} \left[2 \left(2 - \gamma^2 \right) \Delta_j - 2\,\gamma\,\Delta_l + \left(2 - \gamma^2 \right) \left(c^u - \widetilde{w}_j^i \right) \right]$$

$$\Leftrightarrow \quad \widetilde{A}_j^i \; < \; \frac{[\,2-\gamma^2\,]\,[\,2\,(\,[\,2-\gamma^2\,]\;\Delta_j - \gamma\,\Delta_l\,) + (\,2-\gamma^2\,)\left(c^u - \widetilde{w}_j^i\right)\,]\,[\,c^u - \widetilde{w}_j^i\,]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]^2} \,. \tag{142}$$

(139) implies that Ui can only secure strictly positive profit by offering the $\left(\widetilde{w}_j^i, \widetilde{A}_j^i\right)$ tariff to Dj if:

$$\left[\,\widetilde{w}_j^i - c^u\,\right] \left[\frac{(\,2-\gamma^2\,)\left(\alpha_j - \widetilde{w}_j^i - c_j^d\right) - \gamma\left(\alpha_l - c^u - c_l^d\right)}{(\,1-\gamma^2\,)\,(\,4-\gamma^2\,)} \right] + \widetilde{A}_j^i \; > \; 0$$

$$\Leftrightarrow \quad \left[\,\widetilde{w}_j^i - c^u\,\right] \left[\frac{(\,2-\gamma^2\,)\left(\alpha_j - c^u - c_j^d\right) - \gamma\left(\alpha_l - c^u - c_l^d\right) + (\,2-\gamma^2\,)\left(c^u - \widetilde{w}_j^i\right)}{(\,1-\gamma^2\,)\,(\,4-\gamma^2\,)} \right]$$

$$+ \widetilde{A}_j^i \; > \; 0$$

$$\Leftrightarrow \quad \left[\,\widetilde{w}_j^i - c^u\,\right] \left[\frac{(\,2-\gamma^2\,)\;\Delta_j - \gamma\,\Delta_l + (\,2-\gamma^2\,)\left(c^u - \widetilde{w}_j^i\right)}{(\,1-\gamma^2\,)\,(\,4-\gamma^2\,)} \right] + \widetilde{A}_j^i \; > \; 0$$

$$\Leftrightarrow \quad \widetilde{A}_j^i \; > \; \frac{[\,(\,2-\gamma^2\,)\;\Delta_j - \gamma\,\Delta_l + (\,2-\gamma^2\,)\left(c^u - \widetilde{w}_j^i\right)\,]\,[\,c^u - \widetilde{w}_j^i\,]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \,. \tag{143}$$

Observe that:

$$\frac{[\,(\,2-\gamma^2\,)\;\Delta_j - \gamma\,\Delta_l + (\,2-\gamma^2\,)\left(c^u - \widetilde{w}_j^i\right)\,]\,[\,c^u - \widetilde{w}_j^i\,]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]}$$

$$< \; \frac{[\,2-\gamma^2\,]\,[\,2\,(\,[\,2-\gamma^2\,]\;\Delta_j - \gamma\,\Delta_l\,) + (\,2-\gamma^2\,)\left(c^u - \widetilde{w}_j^i\right)\,]\,[\,c^u - \widetilde{w}_j^i\,]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]^2}$$

$$\Leftrightarrow \quad \left[4 - \gamma^2 \right] \left[\left(2 - \gamma^2 \right) \Delta_j - \gamma\,\Delta_l + \left(2 - \gamma^2 \right) \left(c^u - \widetilde{w}_j^i \right) \right]$$

$$< \; \left[2 - \gamma^2 \right] \left[2 \left(\left[2 - \gamma^2 \right] \Delta_j - \gamma\,\Delta_l \right) + \left(2 - \gamma^2 \right) \left(c^u - \widetilde{w}_j^i \right) \right]$$

$$\Leftrightarrow \quad \left[4 - \gamma^2 \right] \left[\left(2 - \gamma^2 \right) \Delta_j - \gamma\,\Delta_l \right] + \left[4 - \gamma^2 \right] \left[2 - \gamma^2 \right] \left[c^u - \widetilde{w}_j^i \right]$$

$$< \; 2 \left[2 - \gamma^2 \right] \left[\left(2 - \gamma^2 \right) \Delta_j - \gamma\,\Delta_l \right] + \left[2 - \gamma^2 \right]^2 \left[c^u - \widetilde{w}_j^i \right]$$

$$\Leftrightarrow \quad \left[4 - \gamma^2 - 2 + \gamma^2 \right] \left[2 - \gamma^2 \right] \left[c^u - \widetilde{w}_j^i \right]$$

$$< \left[4 - 2\gamma^2 - 4 + \gamma^2 \right] \left[\left(2 - \gamma^2 \right) \Delta_j - \gamma \Delta_l \right]$$

$$\Leftrightarrow \quad 2 \left[2 - \gamma^2 \right] \left[c^u - \widetilde{w}_j^i \right] \; < \; -\gamma^2 \left[\left(2 - \gamma^2 \right) \Delta_j - \gamma \Delta_l \right] . \tag{144}$$

(139) implies $[2 - \gamma^2] \Delta_j - \gamma \Delta_l > 0$ for $l, j \in \{1, 2\}$ $(l \neq j)$ to ensure D1 and D2 both produce strictly positive output in equilibrium. Therefore, because $\widetilde{w}_j^i < c^u$, the left-hand side of the inequality in (144) is positive whereas the right-hand side is negative. Therefore, the inequality cannot hold, so there does not exist an $\widetilde{A}_j^i$ that satisfies both (142) and (143). Hence, Ui cannot secure strictly positive profit by offering $\left(\widetilde{w}_j^i, \widetilde{A}_j^i \right) \neq (c^u, 0)$ to Dj and $(c^u, 0)$ to Dl.

B. Now suppose Ui offers $\left(\widetilde{w}_j^i, \widetilde{A}_j^i \right) \neq (c^u, 0)$ to Dj and $\left(\widetilde{w}_l^i, \widetilde{A}_l^i \right) \neq (c^u, 0)$ to Dl.

First suppose $\widetilde{w}_*^i > c^u$ and $\widetilde{A}_*^i \geq 0$ for $* = j$ or $* = l$. Lemma 8 implies that D$*$ will reject Ui's $\left(\widetilde{w}_*^i, \widetilde{A}_*^i \right)$ offer and accept Uk's $(c^u, 0)$ offer. If Ui offers such a tariff to both downstream producers, they will both reject Ui's offer. Ui will secure zero profit in this case. If Ui offers $\left(\widetilde{w}_l^i, \widetilde{A}_l^i \right)$ where $\widetilde{w}_l^i > c^u$ and $\widetilde{A}_l^i \geq 0$ to Dl and offers some tariff $\left(\widetilde{w}_j^i, \widetilde{A}_j^i \right) \neq (c^u, 0)$ to Dj, Dl will select Uk's $(c^u, 0)$ tariff. The arguments in Step A demonstrate that Ui cannot secure strictly positive profit in this case.

Now suppose $\widetilde{w}_*^i > c^u$ and $\widetilde{A}_*^i < 0$ for $* = j$ or $* = l$. D$*$ will accept Ui's $\left(\widetilde{w}_*^i, \widetilde{A}_*^i \right)$ offer and also accept Uk's $(c^u, 0)$ offer. By doing so, D$*$ can increase its profit by collecting the fixed fee from Ui and purchasing all of the input from Uk. If Ui offers such a tariff to both downstream producers, they will both collect the fixed fee from Ui and purchase the input from Uk. Consequently, Ui will secure negative profit. If Ui offers $\left(\widetilde{w}_l^i, \widetilde{A}_l^i \right)$ where $\widetilde{w}_l^i > c^u$ and $\widetilde{A}_l^i < 0$ to Dl and offers some tariff $\left(\widetilde{w}_j^i, \widetilde{A}_j^i \right) \neq (c^u, 0)$ to Dj, Dl will purchase the input from Uk, and so will operate with unit input price c^u. Ui's variable profit in this case will be the same as in Step A where Ui offers the $(c^u, 0)$ tariff to Dl and a distinct tariff to Dj. The arguments in Step A demonstrate that Ui cannot secure strictly positive profit in this case.

Next suppose $\widetilde{w}_*^i \geq c^u$ and $\widetilde{A}_*^i > 0$ for $* = j$ or $* = l$. Lemma 8 implies that D$*$ will reject Ui's $\left(\widetilde{w}_*^i, \widetilde{A}_*^i \right)$ offer and accept Uk's $(c^u, 0)$ offer. If Ui offers such a tariff to both downstream producers, they will both reject Ui's offer. Consequently, Ui will secure zero profit. If Ui offers $\left(\widetilde{w}_l^i, \widetilde{A}_l^i \right)$ where $\widetilde{w}_*^i \geq c^u$ and $\widetilde{A}_*^i > 0$ to Dl and offers some tariff $\left(\widetilde{w}_j^i, \widetilde{A}_j^i \right) \neq (c^u, 0)$ to Dj, Dl will select Uk's $(c^u, 0)$ tariff. The arguments in Step A demonstrate that Ui cannot secure strictly positive profit in this case.

These observations imply that if Ui secures strictly positive profit from tariffs $\left(\widetilde{w}_j^i, \widetilde{A}_j^i \right)$

and $\left(\widetilde{w}_l^i, \widetilde{A}_l^i\right)$ that are distinct from $(c^u, 0)$, the tariffs must satisfy the conditions in one of the following cases:

Case (i). $\widetilde{A}_j^i > 0,\ \widetilde{w}_j^i < c^u,\ \widetilde{A}_l^i < 0$, and $\widetilde{w}_l^i < c^u$.

Case (ii). $\widetilde{A}_j^i < 0,\ \widetilde{w}_j^i < c^u,\ \widetilde{A}_l^i > 0$, and $\widetilde{w}_l^i < c^u$.

Case (iii). $\widetilde{A}_j^i > 0,\ \widetilde{w}_j^i < c^u,\ \widetilde{A}_l^i > 0$, and $\widetilde{w}_l^i < c^u$.

Lemma 8 implies that Dl will accept Ui's $\left(\widetilde{w}_l^i, \widetilde{A}_l^i\right)$ offer rather than Uk's offer $(c^u, 0)$ in Case (i). (139) and (140) imply that Dj will accept Ui's $(\widetilde{w}_j^i,\ \widetilde{A}_j^i)$ offer (where $\widetilde{A}_j^i > 0$ and $\widetilde{w}_j^i < c^u$) rather than Uk's $(c^u, 0)$ offer under PC in this case if:

$$\left[1-\gamma^2\right]\left[q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right)\right]^2 - \widetilde{A}_j^i \ > \ \left[1-\gamma^2\right]\left[q_j\left(c^u, \widetilde{w}_l^i\right)\right]^2$$

$$\Leftrightarrow \ \widetilde{A}_j^i \ < \ \left[1-\gamma^2\right]\left\{\left[q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right)\right]^2 - \left[q_j\left(c^u, \widetilde{w}_l^i\right)\right]^2\right\}. \tag{145}$$

Ui can secure strictly positive profit when both its tariff offers are accepted in Case (i) if:

$$\left[\widetilde{w}_j^i - c^u\right] q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right) + \widetilde{A}_j^i + \left[\widetilde{w}_l^i - c^u\right] q_l\left(\widetilde{w}_l^i, \widetilde{w}_j^i\right) + \widetilde{A}_l^i \ > \ 0$$

$$\Leftrightarrow \ \widetilde{A}_j^i \ > \ \left[c^u - \widetilde{w}_j^i\right] q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right) + \left[c^u - \widetilde{w}_l^i\right] q_l\left(\widetilde{w}_l^i, \widetilde{w}_j^i\right) - \widetilde{A}_l^i. \tag{146}$$

$\widetilde{A}_l^i < 0$ in Case (i). Therefore:

$$\left[c^u - \widetilde{w}_j^i\right] q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right) + \left[c^u - \widetilde{w}_l^i\right] q_l\left(\widetilde{w}_l^i, \widetilde{w}_j^i\right)$$
$$< \ \left[c^u - \widetilde{w}_j^i\right] q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right) + \left[c^u - \widetilde{w}_l^i\right] q_l\left(\widetilde{w}_l^i, \widetilde{w}_j^i\right) - \widetilde{A}_l^i. \tag{147}$$

(139) implies:

$$q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right) \ > \ 0 \ \Leftrightarrow \ \frac{\left[2-\gamma^2\right]\left[\alpha_j - \widetilde{w}_j^i - c_j^d\right] - \gamma\left[\alpha_l - \widetilde{w}_l^i - c_l^d\right]}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]} \ > \ 0$$

$$\Leftrightarrow \ \left[2-\gamma^2\right]\left[\alpha_j - \widetilde{w}_j^i - c_j^d\right] - \gamma\left[\alpha_l - \widetilde{w}_l^i - c_l^d\right] \ > \ 0$$

$$\Leftrightarrow \ \left[2-\gamma^2\right] \Delta_j + \left[2-\gamma^2\right]\left[c^u - \widetilde{w}_j^i\right] - \gamma\,\Delta_l - \gamma\left[c^u - \widetilde{w}_l^i\right] \ > \ 0\,; \tag{148}$$

$$q_l\left(\widetilde{w}_l^i, \widetilde{w}_j^i\right) \ > \ 0 \ \Leftrightarrow \ \frac{\left[2-\gamma^2\right]\left[\alpha_l - \widetilde{w}_l^i - c_l^d\right] - \gamma\left[\alpha_j - \widetilde{w}_j^i - c_j^d\right]}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]} \ > \ 0$$

$$\Leftrightarrow \ \left[2-\gamma^2\right]\left[\alpha_l - \widetilde{w}_l^i - c_l^d\right] - \gamma\left[\alpha_j - \widetilde{w}_j^i - c_j^d\right] \ > \ 0$$

$$\Leftrightarrow \ \left[2-\gamma^2\right] \Delta_l + \left[2-\gamma^2\right]\left[c^u - \widetilde{w}_l^i\right] - \gamma\,\Delta_j - \gamma\left[c^u - \widetilde{w}_j^i\right] \ > \ 0\,. \tag{149}$$

(139) implies:

$$\left[\, c^u - \widetilde{w}^i_j \,\right] q_j\left(\widetilde{w}^i_j, \widetilde{w}^i_l\right) + \left[\, c^u - \widetilde{w}^i_l \,\right] q_l\left(\widetilde{w}^i_l, \widetilde{w}^i_j\right) \;>\; \left[\, 1 - \gamma^2 \,\right] \left\{ \left[\, q_j\left(\widetilde{w}^i_j, \widetilde{w}^i_l\right) \right]^2 - \left[\, q_j\left(c^u, \widetilde{w}^i_l\right) \right]^2 \right\}$$

$$\Leftrightarrow \quad \left[\, c^u - \widetilde{w}^i_j \,\right] \frac{\left[\, 2 - \gamma^2 \,\right] \left[\, \alpha_j - \widetilde{w}^i_j - c^d_j \,\right] - \gamma \left[\, \alpha_l - \widetilde{w}^i_l - c^d_l \,\right]}{\left[\, 1 - \gamma^2 \,\right] \left[\, 4 - \gamma^2 \,\right]}$$

$$+ \; \left[\, c^u - \widetilde{w}^i_l \,\right] \frac{\left[\, 2 - \gamma^2 \,\right] \left[\, \alpha_l - \widetilde{w}^i_l - c^d_l \,\right] - \gamma \left[\, \alpha_j - \widetilde{w}^i_j - c^d_j \,\right]}{\left[\, 1 - \gamma^2 \,\right] \left[\, 4 - \gamma^2 \,\right]}$$

$$> \; \left[\, 1 - \gamma^2 \,\right] \left\{ \left[\frac{\left[\, 2 - \gamma^2 \,\right] \left[\, \alpha_j - \widetilde{w}^i_j - c^d_j \,\right] - \gamma \left[\, \alpha_l - \widetilde{w}^i_l - c^d_l \,\right]}{\left[\, 1 - \gamma^2 \,\right] \left[\, 4 - \gamma^2 \,\right]} \right]^2 \right.$$

$$\left. - \; \left[\frac{\left[\, 2 - \gamma^2 \,\right] \left[\, \alpha_j - c^u - c^d_j \,\right] - \gamma \left[\, \alpha_l - \widetilde{w}^i_l - c^d_l \,\right]}{\left[\, 1 - \gamma^2 \,\right] \left[\, 4 - \gamma^2 \,\right]} \right]^2 \right\}$$

$$\Leftrightarrow \quad \left[\, c^u - \widetilde{w}^i_j \,\right] \left[\, \left(2 - \gamma^2 \right) \left(\alpha_j - \widetilde{w}^i_j - c^d_j \right) - \gamma \left(\alpha_l - \widetilde{w}^i_l - c^d_l \right) \right]$$

$$+ \; \left[\, c^u - \widetilde{w}^i_l \,\right] \left[\, \left(2 - \gamma^2 \right) \left(\alpha_l - \widetilde{w}^i_l - c^d_l \right) - \gamma \left(\alpha_j - \widetilde{w}^i_j - c^d_j \right) \right]$$

$$> \; \left[\frac{1}{4 - \gamma^2} \right] \left\{ \left[\, \left(2 - \gamma^2 \right) \left(\alpha_j - \widetilde{w}^i_j - c^d_j \right) - \gamma \left(\alpha_l - \widetilde{w}^i_l - c^d_l \right) \right]^2 \right.$$

$$\left. - \; \left[\, \left(2 - \gamma^2 \right) \left(\alpha_j - c^u - c^d_j \right) - \gamma \left(\alpha_l - \widetilde{w}^i_l - c^d_l \right) \right]^2 \right\}$$

$$\Leftrightarrow \quad \left[\, c^u - \widetilde{w}^i_j \,\right] \left[\, \left(2 - \gamma^2 \right) \Delta_j + \left(2 - \gamma^2 \right) \left(c^u - \widetilde{w}^i_j \right) - \gamma \, \Delta_l - \gamma \left(c^u - \widetilde{w}^i_l \right) \right]$$

$$+ \; \left[\, c^u - \widetilde{w}^i_l \,\right] \left[\, \left(2 - \gamma^2 \right) \Delta_l + \left(2 - \gamma^2 \right) \left(c^u - \widetilde{w}^i_l \right) - \gamma \, \Delta_j - \gamma \left(c^u - \widetilde{w}^i_j \right) \right]$$

$$> \; \left[\frac{1}{4 - \gamma^2} \right] \left\{ \left[\, \left(2 - \gamma^2 \right) \Delta_j + \left(2 - \gamma^2 \right) \left(c^u - \widetilde{w}^i_j \right) - \gamma \, \Delta_l - \gamma \left(c^u - \widetilde{w}^i_l \right) \right]^2 \right.$$

$$\left. - \; \left[\, \left(2 - \gamma^2 \right) \Delta_j - \gamma \, \Delta_l - \gamma \left(c^u - \widetilde{w}^i_l \right) \right]^2 \right\}$$

$$\Leftrightarrow \quad \left[\, c^u - \widetilde{w}^i_j \,\right] \left[\, \left(2 - \gamma^2 \right) \Delta_j - \gamma \, \Delta_l + \left(2 - \gamma^2 \right) \left(c^u - \widetilde{w}^i_j \right) - \gamma \left(c^u - \widetilde{w}^i_l \right) \right]$$

$$+ \; \left[\, c^u - \widetilde{w}^i_l \,\right] \left[\, \left(2 - \gamma^2 \right) \Delta_l - \gamma \, \Delta_j + \left(2 - \gamma^2 \right) \left(c^u - \widetilde{w}^i_l \right) - \gamma \left(c^u - \widetilde{w}^i_j \right) \right]$$

$$> \; \left[\frac{1}{4 - \gamma^2} \right] \left[\, \left(2 - \gamma^2 \right) \Delta_j + \left(2 - \gamma^2 \right) \left(c^u - \widetilde{w}^i_j \right) - \gamma \, \Delta_l - \gamma \left(c^u - \widetilde{w}^i_l \right) \right.$$

$$\left. + \; \left(2 - \gamma^2 \right) \Delta_j - \gamma \, \Delta_l - \gamma \left(c^u - \widetilde{w}^i_l \right) \right]$$

$$
\begin{aligned}
& \cdot \left[\left(2-\gamma^2 \right) \Delta_j + \left(2-\gamma^2 \right) \left(c^u - \widetilde{w}_j^i \right) - \gamma \, \Delta_l - \gamma \left(c^u - \widetilde{w}_l^i \right) \right. \\
& \qquad \qquad \left. - \; \left(2-\gamma^2 \right) \Delta_j + \gamma \, \Delta_l + \gamma \left(c^u - \widetilde{w}_l^i \right) \right] \\
\Leftrightarrow \quad & \left[c^u - \widetilde{w}_j^i \right] \left[\left(2-\gamma^2 \right) \Delta_j - \gamma \, \Delta_l + \left(2-\gamma^2 \right) \left(c^u - \widetilde{w}_j^i \right) - \gamma \left(c^u - \widetilde{w}_l^i \right) \right] \\
& + \; \left[c^u - \widetilde{w}_l^i \right] \left[\left(2-\gamma^2 \right) \Delta_l - \gamma \, \Delta_j + \left(2-\gamma^2 \right) \left(c^u - \widetilde{w}_l^i \right) - \gamma \left(c^u - \widetilde{w}_j^i \right) \right] \\
> \quad & \left[\frac{\left(2-\gamma^2 \right) \left(c^u - \widetilde{w}_j^i \right)}{4-\gamma^2} \right] \left[2 \left(2-\gamma^2 \right) \Delta_j - 2 \, \gamma \, \Delta_l + \left(2-\gamma^2 \right) \left(c^u - \widetilde{w}_j^i \right) - 2 \, \gamma \left(c^u - \widetilde{w}_l^i \right) \right] \\
\Leftrightarrow \quad & \left[4-\gamma^2 \right] \left[c^u - \widetilde{w}_j^i \right] \left[\left(2-\gamma^2 \right) \Delta_j - \gamma \, \Delta_l + \left(2-\gamma^2 \right) \left(c^u - \widetilde{w}_j^i \right) - \gamma \left(c^u - \widetilde{w}_l^i \right) \right] \\
& + \; \left[4-\gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right] \left[\left(2-\gamma^2 \right) \Delta_l - \gamma \, \Delta_j + \left(2-\gamma^2 \right) \left(c^u - \widetilde{w}_l^i \right) - \gamma \left(c^u - \widetilde{w}_j^i \right) \right] \\
> \quad & \left[2-\gamma^2 \right] \left[c^u - \widetilde{w}_j^i \right] \left[2 \left(2-\gamma^2 \right) \Delta_j - 2 \, \gamma \, \Delta_l + \left(2-\gamma^2 \right) \left(c^u - \widetilde{w}_j^i \right) - 2 \, \gamma \left(c^u - \widetilde{w}_l^i \right) \right] \\
\Leftrightarrow \quad & 0 \; < \; \left[c^u - \widetilde{w}_j^i \right] \left\{ \left[4-\gamma^2 \right] \left[\left(2-\gamma^2 \right) \Delta_j - \gamma \, \Delta_l \right] \right. \\
& \qquad + \; \left[4-\gamma^2 \right] \left[2-\gamma^2 \right] \left[c^u - \widetilde{w}_j^i \right] - \gamma \left[4-\gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right] \\
& \qquad - 2 \left[2-\gamma^2 \right] \left[\left(2-\gamma^2 \right) \Delta_j - \gamma \, \Delta_l \right] \\
& \qquad \left. - \; \left[2-\gamma^2 \right]^2 \left[c^u - \widetilde{w}_j^i \right] + 2 \, \gamma \left[2-\gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right] \right\} \\
& + \; \left[4-\gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right] \left[\left(2-\gamma^2 \right) \Delta_l - \gamma \, \Delta_j + \left(2-\gamma^2 \right) \left(c^u - \widetilde{w}_l^i \right) - \gamma \left(c^u - \widetilde{w}_j^i \right) \right] \\
\Leftrightarrow \quad & 0 \; < \; \left[c^u - \widetilde{w}_j^i \right] \left\{ \left[4-\gamma^2 - 4 + 2 \, \gamma^2 \right] \left[\left(2-\gamma^2 \right) \Delta_j - \gamma \, \Delta_l \right] \right. \\
& \qquad + \; \left[4-\gamma^2 - 2 + \gamma^2 \right] \left[2-\gamma^2 \right] \left[c^u - \widetilde{w}_j^i \right] \\
& \qquad \left. + \, \gamma \left[4 - 2 \, \gamma^2 - 4 + \gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right] \right\} \\
& + \; \left[4-\gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right] \left[\left(2-\gamma^2 \right) \Delta_l - \gamma \, \Delta_j + \left(2-\gamma^2 \right) \left(c^u - \widetilde{w}_l^i \right) - \gamma \left(c^u - \widetilde{w}_j^i \right) \right] \\
\Leftrightarrow \quad & 0 \; < \; \left[c^u - \widetilde{w}_j^i \right] \left\{ \gamma^2 \left[\left(2-\gamma^2 \right) \Delta_j - \gamma \, \Delta_l \right] + 2 \left[2-\gamma^2 \right] \left[c^u - \widetilde{w}_j^i \right] - \gamma^3 \left[c^u - \widetilde{w}_l^i \right] \right\} \\
& + \; \left[4-\gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right] \left[\left(2-\gamma^2 \right) \Delta_l - \gamma \, \Delta_j + \left(2-\gamma^2 \right) \left(c^u - \widetilde{w}_l^i \right) - \gamma \left(c^u - \widetilde{w}_j^i \right) \right] .
\end{aligned}
\tag{150}
$$

Observe that:

$$\gamma^2 \left[\left(2 - \gamma^2 \right) \Delta_j - \gamma \, \Delta_l \right] + 2 \left[2 - \gamma^2 \right] \left[c^u - \widetilde{w}^i_j \right] - \gamma^3 \left[c^u - \widetilde{w}^i_l \right] \ > \ 0$$

$$\Leftrightarrow \ \gamma^2 \left\{ \left[2 - \gamma^2 \right] \Delta_j - \gamma \, \Delta_l + \frac{2 \left[2 - \gamma^2 \right]}{\gamma^2} \left[c^u - \widetilde{w}^i_j \right] - \gamma \left[c^u - \widetilde{w}^i_l \right] \right\} \ > \ 0$$

$$\Leftrightarrow \ \left[2 - \gamma^2 \right] \Delta_j - \gamma \, \Delta_l + \frac{2 \left[2 - \gamma^2 \right]}{\gamma^2} \left[c^u - \widetilde{w}^i_j \right] - \gamma \left[c^u - \widetilde{w}^i_l \right] \ > \ 0$$

$$\Leftrightarrow \ \left[2 - \gamma^2 \right] \Delta_j - \gamma \, \Delta_l + \left[2 - \gamma^2 \right] \left[c^u - \widetilde{w}^i_j \right] - \gamma \left[c^u - \widetilde{w}^i_l \right]$$

$$+ \left[\frac{2 \left(2 - \gamma^2 \right)}{\gamma^2} - \left(2 - \gamma^2 \right) \right] \left[c^u - \widetilde{w}^i_j \right] \ > \ 0$$

$$\Leftrightarrow \ \left[2 - \gamma^2 \right] \Delta_j - \gamma \, \Delta_l + \left[2 - \gamma^2 \right] \left[c^u - \widetilde{w}^i_j \right] - \gamma \left[c^u - \widetilde{w}^i_l \right]$$

$$+ \ \left[2 - \gamma^2 \right] \left[\frac{2}{\gamma^2} - 1 \right] \left[c^u - \widetilde{w}^i_j \right] \ > \ 0 \, . \tag{151}$$

$\widetilde{w}^i_j < c^u$ in Case (i). Furthermore, $\frac{2}{\gamma^2} > 1 \Leftrightarrow \gamma^2 < 2$. Therefore, the last term in (151) is positive. Consequently, (148) implies the inequality in (151) holds. (149) and (151) imply the inequality in (150) holds when $q_j \left(\widetilde{w}^i_j, \widetilde{w}^i_l \right) > 0$ and $q_l \left(\widetilde{w}^i_l, \widetilde{w}^i_j \right) > 0$. Therefore, (147) and (150) imply:

$$\left[c^u - \widetilde{w}^i_j \right] q_j \left(\widetilde{w}^i_j, \widetilde{w}^i_l \right) + \left[c^u - \widetilde{w}^i_l \right] q_l \left(\widetilde{w}^i_l, \widetilde{w}^i_j \right) - \widetilde{A}^i_l$$

$$> \ \left[1 - \gamma^2 \right] \left\{ \left[q_j \left(\widetilde{w}^i_j, \widetilde{w}^i_l \right) \right]^2 - \left[q_j \left(c^u, \widetilde{w}^i_l \right) \right]^2 \right\} \tag{152}$$

when $q_j \left(\widetilde{w}^i_j, \widetilde{w}^i_l \right) > 0$ and $q_l \left(\widetilde{w}^i_l, \widetilde{w}^i_j \right) > 0$. Hence, (152) implies there does not exist an $\widetilde{A}^i_j$ that satisfies both (145) and (146) when $q_j \left(\widetilde{w}^i_j, \widetilde{w}^i_l \right) > 0$ and $q_l \left(\widetilde{w}^i_l, \widetilde{w}^i_j \right) > 0$. Therefore, U$i$ cannot secure strictly positive profit by offering tariffs that satisfy Case (i).

Case (ii) is symmetric to Case (i). In both cases, Ui offers a negative fixed fee ($A < 0$) and a below-cost variable charge ($w < c^u$) to one downstream producer and a positive fixed fee and a below-cost variable charge to the other downstream producer. Therefore, the arguments in Case (i) imply that Ui cannot secure strictly positive profit by offering tariffs that satisfy Case (ii).

If only one downstream supplier accepts Ui's tariff offer in Case (iii), then Ui's profit is the same as in Step A where Ui offers the $(c^u, 0)$ tariff to Dl and a distinct tariff with a positive fixed fee and a below-cost variable charge to Dj. We have shown that the maximum such profit is not strictly positive. Therefore, Ui cannot secure strictly positive profit by offering tariffs that satisfy Case (iii) if only one downstream supplier accepts Ui's offer.

Suppose both downstream suppliers accept Ui's tariff offers that satisfy Case (iii). Then (139) and (140) imply:

$$\left[1-\gamma^2\right]\left[q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right)\right]^2 - \widetilde{A}_j^i \;>\; \left[1-\gamma^2\right]\left[q_j\left(c^u, \widetilde{w}_l^i\right)\right]^2$$

$$\Leftrightarrow \;\; \widetilde{A}_j^i \;<\; \left[1-\gamma^2\right]\left\{\left[q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right)\right]^2 - \left[q_j\left(c^u, \widetilde{w}_l^i\right)\right]^2\right\}; \quad \text{and} \tag{153}$$

$$\left[1-\gamma^2\right]\left[q_l\left(\widetilde{w}_l^i, \widetilde{w}_j^i\right)\right]^2 - \widetilde{A}_l^i \;>\; \left[1-\gamma^2\right]\left[q_l\left(c^u, \widetilde{w}_j^i\right)\right]^2$$

$$\Leftrightarrow \;\; \widetilde{A}_l^i \;<\; \left[1-\gamma^2\right]\left\{\left[q_l\left(\widetilde{w}_l^i, \widetilde{w}_j^i\right)\right]^2 - \left[q_l\left(c^u, \widetilde{w}_j^i\right)\right]^2\right\}. \tag{154}$$

When (153) and (154) hold:

$$\widetilde{A}_j^i + \widetilde{A}_l^i$$
$$< \left[1-\gamma^2\right]\left\{\left[q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right)\right]^2 - \left[q_j\left(c^u, \widetilde{w}_l^i\right)\right]^2 + \left[q_l\left(\widetilde{w}_l^i, \widetilde{w}_j^i\right)\right]^2 - \left[q_l\left(c^u, \widetilde{w}_j^i\right)\right]^2\right\}. \tag{155}$$

Ui can secure strictly positive profit in Case (iii) if:

$$\left[\widetilde{w}_j^i - c^u\right] q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right) + \widetilde{A}_j^i + \left[\widetilde{w}_l^i - c^u\right] q_l\left(\widetilde{w}_l^i, \widetilde{w}_j^i\right) + \widetilde{A}_l^i \;>\; 0$$

$$\Leftrightarrow \;\; \widetilde{A}_j^i + \widetilde{A}_l^i \;>\; \left[c^u - \widetilde{w}_j^i\right] q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right) + \left[c^u - \widetilde{w}_l^i\right] q_l\left(\widetilde{w}_l^i, \widetilde{w}_j^i\right). \tag{156}$$

(155) and (156) cannot both hold if:

$$\left[c^u - \widetilde{w}_j^i\right] q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right) + \left[c^u - \widetilde{w}_l^i\right] q_l\left(\widetilde{w}_l^i, \widetilde{w}_j^i\right)$$
$$> \left[1-\gamma^2\right]\left\{\left[q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right)\right]^2 - \left[q_j\left(c^u, \widetilde{w}_l^i\right)\right]^2 + \left[q_l\left(\widetilde{w}_l^i, \widetilde{w}_j^i\right)\right]^2 - \left[q_l\left(c^u, \widetilde{w}_j^i\right)\right]^2\right\}. \tag{157}$$

(139) implies that (157) holds if and only if:

$$\left[c^u - \widetilde{w}_j^i\right] \frac{\left[2-\gamma^2\right]\left[\alpha_j - \widetilde{w}_j^i - c_j^d\right] - \gamma\left[\alpha_l - \widetilde{w}_l^i - c_l^d\right]}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]}$$

$$+ \left[c^u - \widetilde{w}_l^i\right] \frac{\left[2-\gamma^2\right]\left[\alpha_l - \widetilde{w}_l^i - c_l^d\right] - \gamma\left[\alpha_j - \widetilde{w}_j^i - c_j^d\right]}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]}$$

$$> \left[1-\gamma^2\right]\left\{\left[\frac{\left[2-\gamma^2\right]\left[\alpha_j - \widetilde{w}_j^i - c_j^d\right] - \gamma\left[\alpha_l - \widetilde{w}_l^i - c_l^d\right]}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]}\right]^2\right.$$

$$
\begin{aligned}
& - \left[\frac{[\,2-\gamma^2\,]\,[\,\alpha_j - c^u - c^d_j\,] - \gamma\,[\,\alpha_l - \widetilde{w}^i_l - c^d_l\,]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \right]^2 \\
& + \left[\frac{[\,2-\gamma^2\,]\,[\,\alpha_l - \widetilde{w}^i_l - c^d_l\,] - \gamma\,[\,\alpha_j - \widetilde{w}^i_j - c^d_j\,]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \right]^2 \\
& - \left[\frac{[\,2-\gamma^2\,]\,[\,\alpha_l - c^u - c^d_l\,] - \gamma\,[\,\alpha_j - \widetilde{w}^i_j - c^d_j\,]}{[\,1-\gamma^2\,]\,[\,4-\gamma^2\,]} \right]^2 \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow \quad & [\,4-\gamma^2\,]\,[\,c^u - \widetilde{w}^i_j\,]\,[\,(\,2-\gamma^2\,)\,(\,\alpha_j - \widetilde{w}^i_j - c^d_j\,) - \gamma\,(\,\alpha_l - \widetilde{w}^i_l - c^d_l\,)\,] \\
& + [\,4-\gamma^2\,]\,[\,c^u - \widetilde{w}^i_l\,]\,[\,(\,2-\gamma^2\,)\,(\,\alpha_l - \widetilde{w}^i_l - c^d_l\,) - \gamma\,(\,\alpha_j - \widetilde{w}^i_j - c^d_j\,)\,] \\
& > [\,(\,2-\gamma^2\,)\,(\,\alpha_j - \widetilde{w}^i_j - c^d_j\,) - \gamma\,(\,\alpha_l - \widetilde{w}^i_l - c^d_l\,)\,]^2 \\
& \quad - [\,(\,2-\gamma^2\,)\,(\,\alpha_j - c^u - c^d_j\,) - \gamma\,(\,\alpha_l - \widetilde{w}^i_l - c^d_l\,)\,]^2 \\
& \quad + [\,(\,2-\gamma^2\,)\,(\,\alpha_l - \widetilde{w}^i_l - c^d_l\,) - \gamma\,(\,\alpha_j - \widetilde{w}^i_j - c^d_j\,)\,]^2 \\
& \quad - [\,(\,2-\gamma^2\,)\,(\,\alpha_l - c^u - c^d_l\,) - \gamma\,(\,\alpha_j - \widetilde{w}^i_j - c^d_j\,)\,]^2 \\
\Leftrightarrow \quad & [\,4-\gamma^2\,]\,[\,c^u - \widetilde{w}^i_j\,]\,[\,(\,2-\gamma^2\,)\;\Delta_j + (\,2-\gamma^2\,)\,(\,c^u - \widetilde{w}^i_j\,) - \gamma\,\Delta_l - \gamma\,(\,c^u - \widetilde{w}^i_l\,)\,] \\
& + [\,4-\gamma^2\,]\,[\,c^u - \widetilde{w}^i_l\,]\,[\,(\,2-\gamma^2\,)\;\Delta_l + (\,2-\gamma^2\,)\,(\,c^u - \widetilde{w}^i_l\,) - \gamma\,\Delta_j - \gamma\,(\,c^u - \widetilde{w}^i_j\,)\,] \\
& > [\,(\,2-\gamma^2\,)\;\Delta_j + (\,2-\gamma^2\,)\,(\,c^u - \widetilde{w}^i_j\,) - \gamma\,\Delta_l - \gamma\,(\,c^u - \widetilde{w}^i_l\,)\,]^2 \\
& \quad - [\,(\,2-\gamma^2\,)\;\Delta_j - \gamma\,\Delta_l - \gamma\,(\,c^u - \widetilde{w}^i_l\,)\,]^2 \\
& \quad + [\,(\,2-\gamma^2\,)\;\Delta_l + (\,2-\gamma^2\,)\,(\,c^u - \widetilde{w}^i_l\,) - \gamma\,\Delta_j - \gamma\,(\,c^u - \widetilde{w}^i_j\,)\,]^2 \\
& - [\,(\,2-\gamma^2\,)\,\Delta_l - \gamma\,\Delta_j - \gamma\,(\,c^u - \widetilde{w}^i_j\,)\,]^2 . \qquad (158)
\end{aligned}
$$

The term to the right of the inequality in (158) can be written as:

$$
\begin{aligned}
& [\,(\,2-\gamma^2\,)\;\Delta_j + (\,2-\gamma^2\,)\,(\,c^u - \widetilde{w}^i_j\,) - \gamma\,\Delta_l - \gamma\,(\,c^u - \widetilde{w}^i_l\,) \\
& \qquad + \,(\,2-\gamma^2\,)\,\Delta_j - \gamma\,\Delta_l - \gamma\,(\,c^u - \widetilde{w}^i_l\,)\,] \\
& \cdot [\,(\,2-\gamma^2\,)\,\Delta_j + (\,2-\gamma^2\,)\,(\,c^u - \widetilde{w}^i_j\,) - \gamma\,\Delta_l - \gamma\,(\,c^u - \widetilde{w}^i_l\,) \\
& \qquad - \,(\,2-\gamma^2\,)\,\Delta_j + \gamma\,\Delta_l + \gamma\,(\,c^u - \widetilde{w}^i_l\,)\,]
\end{aligned}
$$

$$
\begin{aligned}
&+ \; \Big[\left(2-\gamma^2 \right) \, \Delta_l + \left(2-\gamma^2 \right) \left(c^u - \widetilde{w}_l^i \right) - \gamma \, \Delta_j - \gamma \left(c^u - \widetilde{w}_j^i \right) \\
&\qquad\qquad + \; \left(2-\gamma^2 \right) \Delta_l - \gamma \, \Delta_j - \gamma \left(c^u - \widetilde{w}_j^i \right) \Big] \\
&\quad \cdot \Big[\left(2-\gamma^2 \right) \Delta_l + \left(2-\gamma^2 \right) \left(c^u - \widetilde{w}_l^i \right) - \gamma \, \Delta_j - \gamma \left(c^u - \widetilde{w}_j^i \right) \\
&\qquad\qquad - \; \left(2-\gamma^2 \right) \Delta_l + \gamma \, \Delta_j + \gamma \left(c^u - \widetilde{w}_j^i \right) \Big] \\
&= \; \Big[\, 2 \left(\left[2-\gamma^2 \right] \Delta_j - \gamma \, \Delta_l \right) + \left(2-\gamma^2 \right) \left(c^u - \widetilde{w}_j^i \right) - 2 \, \gamma \left(c^u - \widetilde{w}_l^i \right) \Big] \\
&\quad \cdot \left[2-\gamma^2 \right] \left[c^u - \widetilde{w}_j^i \right] \\
&\quad + \; \Big[2 \left(\left[2-\gamma^2 \right] \Delta_l - \gamma \, \Delta_j \right) + \left(2-\gamma^2 \right) \left(c^u - \widetilde{w}_l^i \right) - 2 \, \gamma \left(c^u - \widetilde{w}_j^i \right) \Big] \\
&\qquad \cdot \left[2-\gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right] . \qquad (159)
\end{aligned}
$$

(158) and (159) imply that (157) holds if and only if:

$$
\begin{aligned}
&\left[c^u - \widetilde{w}_j^i \right] \Big\{ \left[4-\gamma^2 \right] \left[\left(2-\gamma^2 \right) \Delta_j - \gamma \, \Delta_l \right] \\
&\qquad\qquad + \; \left[4-\gamma^2 \right] \left[2-\gamma^2 \right] \left[c^u - \widetilde{w}_j^i \right] - \gamma \left[4-\gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right] \\
&\qquad\qquad - 2 \left[2-\gamma^2 \right] \left[\left(2-\gamma^2 \right) \, \Delta_j - \gamma \, \Delta_l \right] \\
&\qquad\qquad - \; \left[2-\gamma^2 \right]^2 \left[c^u - \widetilde{w}_j^i \right] + 2 \, \gamma \left[2-\gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right] \Big\} \\
&+ \; \left[c^u - \widetilde{w}_l^i \right] \Big\{ \left[4-\gamma^2 \right] \left[\left(2-\gamma^2 \right) \Delta_l - \gamma \, \Delta_j \right] \\
&\qquad\qquad + \; \left[4-\gamma^2 \right] \left[2-\gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right] - \gamma \left[4-\gamma^2 \right] \left[c^u - \widetilde{w}_j^i \right] \\
&\qquad\qquad - 2 \left[2-\gamma^2 \right] \left[\left(2-\gamma^2 \right) \Delta_l - \gamma \, \Delta_j \right] \\
&\qquad\qquad - \; \left[2-\gamma^2 \right]^2 \left[c^u - \widetilde{w}_l^i \right] + 2 \, \gamma \left[2-\gamma^2 \right] \left[c^u - \widetilde{w}_j^i \right] \Big\} \; > \; 0 \\
\Leftrightarrow \quad &\left[c^u - \widetilde{w}_j^i \right] \Big\{ \left[4-\gamma^2 - 4 + 2 \, \gamma^2 \right] \left[\left(2-\gamma^2 \right) \, \Delta_j - \gamma \, \Delta_l \right] \\
&\qquad\qquad + \; \left[4-\gamma^2 - 2 + \gamma^2 \right] \left[2-\gamma^2 \right] \left[c^u - \widetilde{w}_j^i \right] \\
&\qquad\qquad + \; \gamma \left[4 - 2 \, \gamma^2 - 4 + \gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right] \Big\} \\
&+ \; \left[c^u - \widetilde{w}_l^i \right] \Big\{ \left[4-\gamma^2 - 4 + 2 \, \gamma^2 \right] \left[\left(2-\gamma^2 \right) \Delta_l - \gamma \, \Delta_j \right] \\
&\qquad\qquad + \; \left[4-\gamma^2 - 2 + \gamma^2 \right] \left[2-\gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right]
\end{aligned}
$$

$$+ \gamma \left[4 - 2\gamma^2 - 4 + \gamma^2 \right] \left[c^u - \widetilde{w}_j^i \right] \Big\} \; > \; 0$$

$$\Leftrightarrow \quad \left[c^u - \widetilde{w}_j^i \right] \left\{ \gamma^2 \left[\left(2 - \gamma^2 \right) \Delta_j - \gamma \Delta_l \right] + 2 \left[2 - \gamma^2 \right] \left[c^u - \widetilde{w}_j^i \right] - \gamma^3 \left[c^u - \widetilde{w}_l^i \right] \right\}$$

$$+ \; \left[c^u - \widetilde{w}_l^i \right]$$

$$\cdot \left\{ \gamma^2 \left[\left(2 - \gamma^2 \right) \Delta_l - \gamma \Delta_j \right] + 2 \left[2 - \gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right] - \gamma^3 \left[c^u - \widetilde{w}_j^i \right] \right\} \; > \; 0 \, . \quad (160)$$

(151) implies:

$$\gamma^2 \left[\left(2 - \gamma^2 \right) \Delta_l - \gamma \Delta_j \right] + 2 \left[2 - \gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right] - \gamma^3 \left[c^u - \widetilde{w}_j^i \right] \; > \; 0$$

$$\Leftrightarrow \quad \left[2 - \gamma^2 \right] \Delta_l - \gamma \Delta_j + \left[2 - \gamma^2 \right] \left[c^u - \widetilde{w}_l^i \right] - \gamma \left[c^u - \widetilde{w}_j^i \right]$$

$$+ \; \left[2 - \gamma^2 \right] \left[\frac{2}{\gamma^2} - 1 \right] \left[c^u - \widetilde{w}_l^i \right] \; > \; 0 \, . \quad (161)$$

The last term in (161) is positive because $\frac{2}{\gamma^2} > 1 \Leftrightarrow \gamma^2 < 2$ and because $\widetilde{w}_j^i < c^u$ in Case (iii). Therefore, (149) implies that the inequality in (161) holds. (151) and (161) imply that the inequality in (160) holds when $q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right) > 0$ and $q_l\left(\widetilde{w}_l^i, \widetilde{w}_j^i\right) > 0$. Hence, (157) and (160) imply there does not exist $\widetilde{A}_j^i$ and $\widetilde{A}_l^i$ that satisfy both (155) and (156) when $q_j\left(\widetilde{w}_j^i, \widetilde{w}_l^i\right) > 0$ and $q_l\left(\widetilde{w}_l^i, \widetilde{w}_j^i\right) > 0$ in Case (iii). Therefore, Ui cannot secure strictly positive profit by offering tariffs that satisfy Case (iii). ■

Lemma 9. *Under VI, D2's downstream profit declines as the unit price (w_2) or the fixed fee (A_2) it faces increases. D1's downstream profit increases as the unit price its rival faces (w_2) increases.*

Proof. Under VI, D1 chooses P_1 to maximize:

$$\left[P_1 - w_1 - c_1^d \right] q_1(P_1, P_2) - A_1 + \left[w_1 - c^u \right] q_1(P_1, P_2) + A_1$$

$$= \; \left[P_1 - c^u - c_1^d \right] q_1(P_1, P_2) \, . \quad (162)$$

(162) implies that D1 effectively perceives its input price to be c^u under VI. Therefore, (138) implies that equilibrium prices are:

$$P_1\left(c^u, w_2\right) \; = \; \frac{2}{4 - \gamma^2} \left[\alpha_1 - \gamma \alpha_2 + c^u + c_1^d \right] + \frac{\gamma}{4 - \gamma^2} \left[\alpha_2 - \gamma \alpha_1 + w_2 + c_2^d \right] ; \quad (163)$$

$$P_2\left(w_2, c^u\right) \; = \; \frac{2}{4 - \gamma^2} \left[\alpha_2 - \gamma \alpha_1 + w_2 + c_2^d \right] + \frac{\gamma}{4 - \gamma^2} \left[\alpha_1 - \gamma \alpha_2 + c^u + c_1^d \right] . \quad (164)$$

(139) implies the corresponding equilibrium quantities are:

$$q_1(c^u, w_2) = \frac{[2-\gamma^2][\alpha_1 - c^u - c_1^d] - \gamma[\alpha_2 - w_2 - c_2^d]}{[1-\gamma^2][4-\gamma^2]}; \tag{165}$$

$$q_2(w_2, c^u) = \frac{[2-\gamma^2][\alpha_2 - w_2 - c_2^d] - \gamma[\alpha_1 - c^u - c_1^d]}{[1-\gamma^2][4-\gamma^2]}. \tag{166}$$

(162) implies that D1's profit-maximizing choice of P_1 is determined by:

$$q_1 + [P_1 - c^u - c_1^d]\frac{\partial q_1}{\partial P_1} = 0. \tag{167}$$

(27) and (167) imply:

$$P_1 - c^u - c_1^d = [1-\gamma^2]\, q_1. \tag{168}$$

(140), (162) and (168) imply:

$$\pi_1(c^u, w_2) = [1-\gamma^2][q_1(c^u, w_2)]^2 \quad \text{and} \tag{169}$$

$$\pi_2(w_2, c^u) = [1-\gamma^2][q_2(w_2, c^u)]^2 - A_2, \tag{170}$$

where $q_1(c^u, w_2)$ and $q_2(w_2, c^u)$ are as specified in (165) and (166).

(139), (140), (165), (166), (169), and (170) imply that when $q_1(\cdot) > 0$ and $q_2(\cdot) > 0$, equilibrium prices, quantities, and variable profits are the same under VI and under VS in the setting where D1 purchases the input from U1 under the $(c^u, 0)$ tariff. Therefore, the conclusion in the lemma follows from Lemma 8. ■

Conclusion 6. *Suppose U1 and U2 can set discriminatory two-part tariffs and cannot collude. Then under VI, there exists an equilibrium in which each upstream supplier offers tariff* $(c^u, 0)$ *to both downstream suppliers.*

Proof. The proof of Lemma 9 demonstrates that when $q_1(\cdot) > 0$ and $q_2(\cdot) > 0$, equilibrium prices, quantities, and variable profits are the same under VI and under VS in the setting where D1 purchases the input from U1 under the $(c^u, 0)$ tariff. Consequently, the Conclusion holds if there there exists an equilibrium under VS in which: (i) U2 offers $(c^u, 0)$ to both D1 and D2; and (ii) U1 offers $(c^u, 0)$ to D2. The maintained assumption that D1 purchases the input from U1 when it is indifferent between purchasing the input from U1 and U2 ensures that D1 purchases the input from U1 in this equilibrium.

The proof proceeds in three steps. Step 1 demonstrates that U1 cannot increase the sum of its upstream profit and D1's downstream profit (Π_1^v) by offering a tariff other than $(c^u, 0)$

to D2 when U2 offers $(c^u, 0)$ to both D1 and D2. Step 2 demonstrates that U2 cannot secure strictly positive profit by offering tariff $(c^u, 0)$ to Dl and a distinct tariff to Dj ($j, l \in \{1, 2\}$, $j \neq l$) when U1 offers $(c^u, 0)$ to D2. Step 3 demonstrates that U2 cannot secure strictly positive profit by offering tariffs other than $(c^u, 0)$ to both D1 and D2 when U1 offers tariff $(c^u, 0)$ to D2.

Step 1. First suppose U1 offers $\left(\widetilde{w}_2^1, \widetilde{A}_2^1\right) \neq (c^u, 0)$ to D2 while U2 offers the $(c^u, 0)$ tariff to D1 and D2.

Lemma 9 implies that Π_1^v will only increase in this case if $\widetilde{w}_2^1 > c^u$ or $\widetilde{A}_2^1 > 0$ or both.

If $\widetilde{w}_2^1 > c^u$ and $\widetilde{A}_2^1 \geq 0$, Lemma 9 implies that D2 will reject U1's $\left(\widetilde{w}_2^1, \widetilde{A}_2^1\right)$ offer and accept U2's $(c^u, 0)$ offer. Consequently, Π_1^v will not increase.

If $\widetilde{w}_2^1 > c^u$ and $\widetilde{A}_2^1 < 0$, D2 will accept U1's $\left(\widetilde{w}_2^1, \widetilde{A}_2^1\right)$ offer and also accept U2's $(c^u, 0)$ offer. By doing so, D2 can increase its profit by collecting the fixed fee from U1 and purchasing all of the input from U2. Π_1^v will decline in this case.

If $\widetilde{w}_2^1 \geq c^u$ and $\widetilde{A}_2^1 > 0$, Lemma 9 implies that D2 will reject U1's $\left(\widetilde{w}_2^1, \widetilde{A}_2^1\right)$ offer and accept U2's $(c^u, 0)$ offer. Consequently, Π_1^v will not increase.

Now suppose $\widetilde{w}_2^1 < c^u$ and $\widetilde{A}_2^1 > 0$. (142), (166), and (170) imply that D2 will accept U1's $\left(\widetilde{w}_2^1, \widetilde{A}_2^1\right)$ offer rather than U2's $(c^u, 0)$ offer if:

$$\left[1-\gamma^2\right]\left[q_2\left(\widetilde{w}_2^1, c^u\right)\right]^2 - \widetilde{A}_2^1 \;>\; \left[1-\gamma^2\right]\left[q_2\left(c^u, c^u\right)\right]^2$$

$$\Leftrightarrow \;\; \widetilde{A}_2^1 \;<\; \frac{\left(\left[2-\gamma^2\right]\Delta_2 - \gamma\,\Delta_1 + \left[2-\gamma^2\right]\left[c^u - \widetilde{w}_2^1\right]\right)^2}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]^2} - \frac{\left(\left[2-\gamma^2\right]\Delta_2 - \gamma\,\Delta_1\right)^2}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]^2}$$

$$\Leftrightarrow \;\; \widetilde{A}_2^1 \;<\; \frac{2\left[2-\gamma^2\right]\left[c^u - \widetilde{w}_2^1\right]\left[\left(2-\gamma^2\right)\Delta_2 - \gamma\,\Delta_1\right] + \left[2-\gamma^2\right]^2\left[c^u - \widetilde{w}_2^1\right]^2}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]^2}$$

$$\Leftrightarrow \;\; \widetilde{A}_2^1 \;<\; \frac{\left[2-\gamma^2\right]\left[c^u - \widetilde{w}_2^1\right]\left[2\left(\left[2-\gamma^2\right]\Delta_2 - \gamma\,\Delta_1\right) + \left(2-\gamma^2\right)\left(c^u - \widetilde{w}_2^1\right)\right]}{\left[1-\gamma^2\right]\left[4-\gamma^2\right]^2}. \tag{171}$$

(165), (166), and (169) imply that Π_1^v increases when D2 purchases the input from U1 under the $\left(\widetilde{w}_2^1, \widetilde{A}_2^1\right)$ tariff if and only if:

$$\left[1-\gamma^2\right]\left[q_1\left(c^u, \widetilde{w}_2^1\right)\right]^2 + \left[\widetilde{w}_2^1 - c^u\right] q_2\left(\widetilde{w}_2^1, c^u\right) + \widetilde{A}_2^1 \;>\; \left[1-\gamma^2\right]\left[q_1\left(c^u, c^u\right)\right]^2$$

$$\Leftrightarrow \;\; \widetilde{A}_2^1 \;>\; \left[1-\gamma^2\right]\left\{\left[q_1\left(c^u, c^u\right)\right]^2 - \left[q_1\left(c^u, \widetilde{w}_2^1\right)\right]^2\right\} + \left[c^u - \widetilde{w}_2^1\right] q_2\left(\widetilde{w}_2^1, c^u\right)$$

$$= \left[1-\gamma^2\right]\left[q_1\left(c^u, c^u\right) + q_1\left(c^u, \widetilde{w}_2^1\right)\right]\left[q_1\left(c^u, c^u\right) - q_1\left(c^u, \widetilde{w}_2^1\right)\right] + \left[c^u - \widetilde{w}_2^1\right] q_2\left(\widetilde{w}_2^1, c^u\right)$$

$$= \left[1 - \gamma^2 \right] \left[\frac{(2 - \gamma^2)\,\Delta_1 - \gamma\,\Delta_2}{(1 - \gamma^2)(4 - \gamma^2)} + \frac{(2 - \gamma^2)\,\Delta_1 - \gamma\left(\alpha_2 - \widetilde{w}_2^1 - c_2^d\right)}{(1 - \gamma^2)(4 - \gamma^2)} \right]$$

$$\cdot \left[\frac{(2 - \gamma^2)\,\Delta_1 - \gamma\,\Delta_2}{(1 - \gamma^2)(4 - \gamma^2)} - \frac{(2 - \gamma^2)\,\Delta_1 - \gamma\left(\alpha_2 - \widetilde{w}_2^1 - c_2^d\right)}{(1 - \gamma^2)(4 - \gamma^2)} \right]$$

$$+ \left[c^u - \widetilde{w}_2^1 \right] \frac{[2 - \gamma^2]\left[\alpha_2 - \widetilde{w}_2^1 - c_2^d\right] - \gamma\left[\alpha_1 - c^u - c_1^d\right]}{[1 - \gamma^2][4 - \gamma^2]}$$

$$= \left[1 - \gamma^2 \right] \left[\frac{2(2 - \gamma^2)\,\Delta_1 - \gamma\,\Delta_2 - \gamma\left(\alpha_2 - \widetilde{w}_2^1 - c_2^d\right)}{(1 - \gamma^2)(4 - \gamma^2)} \right] \left[\frac{-\gamma\,\Delta_2 + \gamma\left(\alpha_2 - \widetilde{w}_2^1 - c_2^d\right)}{(1 - \gamma^2)(4 - \gamma^2)} \right]$$

$$+ \left[c^u - \widetilde{w}_2^1 \right] \frac{[2 - \gamma^2]\left[\alpha_2 - \widetilde{w}_2^1 - c_2^d\right] - \gamma\,\Delta_1}{[1 - \gamma^2][4 - \gamma^2]}$$

$$= \left[1 - \gamma^2 \right] \left[\frac{2(2 - \gamma^2)\,\Delta_1 - \gamma\,\Delta_2 - \gamma\,\Delta_2 - \gamma(c^u - \widetilde{w}_2^1)}{(1 - \gamma^2)(4 - \gamma^2)} \right] \left[\frac{-\gamma\,\Delta_2 + \gamma\,\Delta_2 + \gamma(c^u - \widetilde{w}_2^1)}{(1 - \gamma^2)(4 - \gamma^2)} \right]$$

$$+ \left[c^u - \widetilde{w}_2^1 \right] \frac{[2 - \gamma^2]\,\Delta_2 + [2 - \gamma^2][c^u - \widetilde{w}_2^1] - \gamma\,\Delta_1}{[1 - \gamma^2][4 - \gamma^2]}$$

$$= \frac{c^u - \widetilde{w}_2^1}{[1 - \gamma^2][4 - \gamma^2]^2} \left\{ \gamma\left[2\left(2 - \gamma^2\right)\Delta_1 - 2\,\gamma\,\Delta_2 - \gamma\left(c^u - \widetilde{w}_2^1\right) \right] \right.$$

$$\left. + \left[4 - \gamma^2 \right] \left[\left(2 - \gamma^2\right)\Delta_2 - \gamma\,\Delta_1 + \left(2 - \gamma^2\right)\left(c^u - \widetilde{w}_2^1\right) \right] \right\}. \quad (172)$$

(171) and (172) cannot both hold if:

$$\frac{c^u - \widetilde{w}_2^1}{[1 - \gamma^2][4 - \gamma^2]^2} \left\{ \gamma\left[2\left(2 - \gamma^2\right)\Delta_1 - 2\,\gamma\,\Delta_2 - \gamma\left(c^u - \widetilde{w}_2^1\right) \right] \right.$$

$$\left. + \left[4 - \gamma^2 \right] \left[\left(2 - \gamma^2\right)\Delta_2 - \gamma\,\Delta_1 + \left(2 - \gamma^2\right)\left(c^u - \widetilde{w}_2^1\right) \right] \right\}$$

$$> \frac{[2 - \gamma^2]\,[2([2 - \gamma^2]\,\Delta_2 - \gamma\,\Delta_1) + (2 - \gamma^2)(c^u - \widetilde{w}_2^1)]\,[c^u - \widetilde{w}_2^1]}{[1 - \gamma^2][4 - \gamma^2]^2}$$

$$\Leftrightarrow \quad \gamma\left[2\left(2 - \gamma^2\right)\Delta_1 - 2\,\gamma\,\Delta_2 - \gamma\left(c^u - \widetilde{w}_2^1\right) \right]$$

$$+ \left[4 - \gamma^2 \right] \left[\left(2 - \gamma^2\right)\Delta_2 - \gamma\,\Delta_1 + \left(2 - \gamma^2\right)\left(c^u - \widetilde{w}_2^1\right) \right]$$

$$> \left[2 - \gamma^2 \right] \left[2\left(\left[2 - \gamma^2 \right] \Delta_2 - \gamma\,\Delta_1 \right) + \left(2 - \gamma^2\right)\left(c^u - \widetilde{w}_2^1\right) \right]$$

$$\Leftrightarrow \quad \gamma\left[2\left(2 - \gamma^2\right)\Delta_1 - 2\,\gamma\,\Delta_2 - \gamma\left(c^u - \widetilde{w}_2^1\right) \right] + \left[4 - \gamma^2 - 4 + 2\,\gamma^2 \right] \left[\left(2 - \gamma^2\right)\Delta_2 - \gamma\,\Delta_1 \right]$$

$$+ \left[4 - \gamma^2 - 2 + \gamma^2 \right] \left[2 - \gamma^2 \right] \left[c^u - \widetilde{w}_2^1 \right] > 0$$

$$\Leftrightarrow \quad \gamma \left[2 \left(2 - \gamma^2 \right) \Delta_1 - 2 \gamma \Delta_2 - \gamma \left(c^u - \widetilde{w}_2^1 \right) \right]$$

$$+ \; \gamma^2 \left[\left(2 - \gamma^2 \right) \Delta_2 - \gamma \Delta_1 \right] + 2 \left[2 - \gamma^2 \right] \left[c^u - \widetilde{w}_2^1 \right] \; > \; 0$$

$$\Leftrightarrow \; \gamma \left[2 \left(2 - \gamma^2 \right) \Delta_1 - 2 \gamma \Delta_2 \right] + \gamma^2 \left[\left(2 - \gamma^2 \right) \Delta_2 - \gamma \Delta_1 \right] + \left[4 - 2 \gamma^2 - \gamma^2 \right] \left[c^u - \widetilde{w}_2^1 \right] \; > \; 0$$

$$\Leftrightarrow \quad 2 \gamma \left[\left(2 - \gamma^2 \right) \Delta_1 - \gamma \Delta_2 \right] + \gamma^2 \left[\left(2 - \gamma^2 \right) \Delta_2 - \gamma \Delta_1 \right] + \left[4 - 3 \gamma^2 \right] \left[c^u - \widetilde{w}_2^1 \right] \; > \; 0 \, . \tag{173}$$

Because D1 and D2 both produce strictly positive output in equilibrium, (165) and (166) imply that $[2 - \gamma^2] \Delta_j - \gamma \Delta_l > 0$ for $l, j \in \{1, 2\}$ ($l \neq j$). Therefore, the inequality in (173) holds because $\widetilde{w}_2^1 < c^u$ and $4 - 3\gamma^2 > 0$. Consequently, there does not exist an $\widetilde{A}_2^1$ that satisfies both (171) and (172), so U1 cannot increase Π_1^v by offering a $\left(\widetilde{w}_2^1, \widetilde{A}_2^1 \right)$ tariff to D2 in which $\widetilde{w}_2^1 < c^u$ and $\widetilde{A}_2^1 > 0$.

Step 2. Now suppose U2 offers $(c^u, 0)$ to Dl and $\left(\widetilde{w}_j^2, \widetilde{A}_j^2 \right) \neq (c^u, 0)$ to Dj while U1 offers $(c^u, 0)$ to D2. The analysis in Step A in Conclusion 5 demonstrates that U2 cannot secure strictly positive profit in this case.

Step 3. Now suppose U2 offers $\left(\widetilde{w}_j^2, \widetilde{A}_j^2 \right) \neq (c^u, 0)$ to Dj and $\left(\widetilde{w}_l^2, \widetilde{A}_l^2 \right) \neq (c^u, 0)$ to Dl while U1 offers $(c^u, 0)$ to D2. The analysis in Step B in Conclusion 5 demonstrates that U2 cannot secure strictly positive profit in this case. ■

Implication. Lemma 2 and Conclusions 4 – 6 imply that under the equilibrium identified in Conclusions 5 and 6, D2's loss from collusion is the same under VS and under VI when U1 and U2 can offer two-part tariffs.

IV. Additional Conclusions from Numerical Solutions.

The text focuses on the extent to which collusion reduces D2's profit. Tables TA1 – TA16 below illustrate the impact of collusion on D1's profit, aggregate industry profit (i.e., the combined profit of U1, U2, D1, and D2), consumer surplus, and welfare (i.e., the sum of consumer surplus and aggregate industry profit). The equilibrium magnitudes of these variables are reported for illustrative values of the product homogeneity parameter (γ) and the relative competitive strengths of D2 and D1 ($\frac{\Delta_2}{\Delta_1}$). The tables report outcomes for combinations of γ and $\frac{\Delta_2}{\Delta_1}$ under which D1 and D2 both produce strictly positive output in equilibrium under vertical separation (VS), both in the presence of collusion and in its absence.[3]

Tables TA1 – TA4 report the proportionate difference between D1's downstream profit under collusion (π_1^C) and in the absence of collusion (π_1^N). The entries in the tables assume that D1 secures the input from U1 at unit price c^u under VI. Tables TA5 – TA8 report the proportionate difference in aggregate industry profit under collusion (Π_I^C) and in the absence of collusion (Π_I^N). Tables TA9 – TA12 report the proportionate difference in consumer surplus under collusion (CS^C) and in the absence of collusion (CS^N), where consumer surplus is as specified in (1). Tables TA13 – TA16 report the proportionate difference in welfare under collusion (W^C) and in the absence of collusion (W^N).

Tables TA1 – TA4 indicate that collusion reduces D1's profit under VS. The systematic (and fairly substantial) reduction in D1's profit reflects the higher input price that D1 faces when U1 and U2 collude.[4] In contrast, collusion often increases D1's profit under VI. The increase in D1's profit reflects the asymmetric increase in D2's input price that U1 and U2 implement when they collude.

Collusion can reduce D1's profit under VI when γ and $\frac{\Delta_2}{\Delta_1}$ are relatively large. In the presence of intense downstream competition, D1's relatively weak competitive position leaves it producing little (or no) output, with corresponding little (or no) profit.

Tables TA5 – TA8 report that collusion often reduces D2's profit by more than it increases the combined profit of U1, U2, and D1, particularly under downstream quantity competition. However, collusion can increase aggregate industry profit when downstream competition is

[3] A blank entry in a table denotes a setting in which only one downstream producer serves customers in equilibrium under VS either in the presence of collusion or in its absence (or both).

[4] The identical entries in Tables TA1 and TA3 (and in Tables TA9 and TA11 below) reflect the linear demand and cost functions in our model. Downstream profits and consumer surplus are proportional to downstream outputs under VS, and equilibrium downstream outputs vary linearly with input prices. Furthermore, input prices (w_i) under collusion and VS increase linearly with Δ_i and do not vary with γ. (Recall Lemmas 3 and 5.)

relatively intense (under price competition with substantial product homogeneity). It is especially likely to do so under VI when $\frac{\Delta_1}{\Delta_2}$ is relatively large. Under these circumstances, the asymmetric (and relatively substantial) increase in the input price that D2 experiences when U1 and U2 collude enables (the relatively strong) D1 to increase its downstream profit considerably.

Tables TA9 – TA12 indicate that collusion generally reduces consumer surplus, particularly under VS where the colluding input suppliers increase the input prices that both downstream producers face. However, collusion can increase consumer surplus under VI with downstream quantity competition if γ is sufficiently large and $\frac{\Delta_2}{\Delta_1}$ is sufficiently small. Under these circumstances, U1 and U2 increase w_2 substantially, thereby promoting expanded output by the (substantially) more efficient downstream producer, which can increase consumer surplus. Welfare can also increase because, in addition to the increased consumer surplus, D2's weakened competitive position limits downstream profit dissipation, thereby increasing aggregate industry profit.

Tables TA13 – TA16 indicate that collusion reduces welfare under VS. The systematic welfare reduction stems from the higher input prices that both downstream producers face when U1 and U2 collude. The welfare reduction under VS can be pronounced, typically exceeding 30% in the settings under consideration. The welfare reduction is often less pronounced under VI, in part because collusion does not increase D1's perceived input price.

Collusion can increase welfare under VI and downstream quantity competition when γ and $\frac{\Delta_1}{\Delta_2}$ are sufficiently large. The substantial, asymmetric increase in D2's input price that U1 and U2 implement in this case can increase welfare by shifting downstream output of the relatively homogeneous downstream product to the (substantially) more efficient producer.

$\frac{\Delta_2}{\Delta_1}$	γ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
0.75	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
1.00	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
1.25	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
1.50	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
1.75	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
2.00	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750

Table TA1. $\frac{\pi_1^C - \pi_1^N}{\pi_1^N}$ **under VS and downstream quantity competition.**

$\frac{\Delta_2}{\Delta_1}$	γ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	0.026	0.053	0.082	0.114	0.148	0.146	0.131	0.103	0.059
0.75	0.039	0.082	0.130	0.182	0.241	0.308	0.386	0.440	0.449
1.00	0.053	0.114	0.183	0.262	0.354	0.464	0.596	0.759	0.969
1.25	0.068	0.147	0.242	0.355	0.495	0.669	0.894	1.194	1.614
1.50	0.083	0.183	0.308	0.466	0.672	0.950	1.342	1.935	2.913
1.75	0.098	0.222	0.384	0.600	0.902	1.352	2.079	3.410	6.410
2.00	0.114	0.264	0.469	0.763	1.211	1.968	3.451	7.258	25.420

Table TA2. $\frac{\pi_1^C - \pi_1^N}{\pi_1^N}$ **under VI and downstream quantity competition.**

$\frac{\Delta_2}{\Delta_1}$	γ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750		
0.75	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	
1.00	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
1.25	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
1.50	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	
1.75	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750		
2.00	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750		

Table TA3. $\frac{\pi_1^C - \pi_1^N}{\pi_1^N}$ **under VS and downstream price competition.**

$\frac{\Delta_2}{\Delta_1}$	γ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	0.025	0.053	0.084	0.122	0.172	0.181	0.167		
0.75	0.038	0.079	0.127	0.185	0.263	0.375	0.553	0.759	
1.00	0.051	0.106	0.170	0.249	0.356	0.515	0.781	1.323	2.973
1.25	0.064	0.133	0.213	0.312	0.446	0.646	0.982	1.595	-1.000
1.50	0.077	0.160	0.256	0.372	0.523	0.724	0.882	-1.000	
1.75	0.090	0.187	0.296	0.423	0.562	0.605	-1.000		
2.00	0.103	0.213	0.334	0.457	0.511	-0.237	-1.000		

Table TA4. $\frac{\pi_1^C - \pi_1^N}{\pi_1^N}$ **under VI and downstream price competition.**

$\frac{\Delta_2}{\Delta_1}$	γ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	-0.231	-0.214	-0.200	-0.190	-0.184	-0.183	-0.188	-0.201	-0.221
0.75	-0.226	-0.203	-0.180	-0.159	-0.138	-0.120	-0.103	-0.090	-0.081
1.00	-0.225	-0.200	-0.175	-0.150	-0.125	-0.100	-0.075	-0.050	-0.025
1.25	-0.226	-0.202	-0.178	-0.155	-0.133	-0.112	-0.093	-0.075	-0.061
1.50	-0.227	-0.206	-0.185	-0.167	-0.150	-0.136	-0.127	-0.122	-0.123
1.75	-0.229	-0.210	-0.193	-0.179	-0.168	-0.161	-0.160	-0.165	-0.178
2.00	-0.231	-0.214	-0.200	-0.190	-0.184	-0.183	-0.188	-0.201	-0.221

Table TA5. $\frac{\Pi_I^C - \Pi_I^N}{\Pi_I^N}$ **under VS and downstream quantity competition.**

$\frac{\Delta_2}{\Delta_1}$	γ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	-0.033	-0.014	0.008	0.032	0.061	0.086	0.095	0.086	0.055
0.75	-0.074	-0.056	-0.035	-0.011	0.017	0.052	0.097	0.152	0.202
1.00	-0.112	-0.099	-0.084	-0.069	-0.052	-0.032	-0.008	0.023	0.064
1.25	-0.143	-0.135	-0.127	-0.121	-0.116	-0.113	-0.112	-0.114	-0.119
1.50	-0.167	-0.162	-0.160	-0.160	-0.165	-0.174	-0.188	-0.210	-0.240
1.75	-0.184	-0.182	-0.184	-0.189	-0.199	-0.215	-0.238	-0.268	-0.307
2.00	-0.197	-0.198	-0.202	-0.210	-0.224	-0.243	-0.269	-0.302	-0.343

Table TA6. $\frac{\Pi_I^C - \Pi_I^N}{\Pi_I^N}$ **under VI and downstream quantity competition.**

$\frac{\Delta_2}{\Delta_1}$					γ				
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	-0.228	-0.202	-0.171	-0.134	-0.088	-0.034	0.023		
0.75	-0.223	-0.191	-0.149	-0.095	-0.021	0.084	0.243	0.491	
1.00	-0.222	-0.188	-0.143	-0.083	0.000	0.125	0.333	0.750	2.000
1.25	-0.223	-0.189	-0.147	-0.090	-0.013	0.100	0.276	0.578	1.075
1.50	-0.225	-0.193	-0.154	-0.104	-0.038	0.052	0.176	0.335	
1.75	-0.226	-0.198	-0.163	-0.120	-0.065	0.005	0.089		
2.00	-0.228	-0.202	-0.171	-0.134	-0.088	-0.034	0.023		

Table TA7. $\frac{\Pi_I^C - \Pi_I^N}{\Pi_I^N}$ **under VS and downstream price competition.**

$\frac{\Delta_2}{\Delta_1}$					γ				
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	-0.032	-0.009	0.019	0.056	0.103	0.150	0.165		
0.75	-0.072	-0.047	-0.013	0.034	0.101	0.202	0.363	0.623	
1.00	-0.109	-0.087	-0.054	-0.007	0.064	0.175	0.369	0.771	2.004
1.25	-0.140	-0.120	-0.091	-0.048	0.017	0.118	0.285	0.590	0.593
1.50	-0.163	-0.146	-0.120	-0.081	-0.023	0.063	0.194	0.072	
1.75	-0.180	-0.165	-0.141	-0.106	-0.054	0.020	-0.086		
2.00	-0.193	-0.179	-0.157	-0.124	-0.077	-0.011	-0.148		

Table TA8. $\frac{\Pi_I^C - \Pi_I^N}{\Pi_I^N}$ **under VI and downstream price competition.**

$\frac{\Delta_2}{\Delta_1}$	γ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
0.75	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
1.00	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
1.25	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
1.50	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
1.75	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
2.00	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750

Table TA9. $\frac{CS^C - CS^N}{CS^N}$ **under VS and downstream quantity competition.**

$\frac{\Delta_2}{\Delta_1}$	γ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	-0.150	-0.150	-0.152	-0.155	-0.160	-0.113	-0.044	0.056	0.205
0.75	-0.270	-0.270	-0.270	-0.272	-0.276	-0.283	-0.295	-0.279	-0.211
1.00	-0.375	-0.374	-0.374	-0.374	-0.375	-0.378	-0.384	-0.393	-0.410
1.25	-0.457	-0.456	-0.455	-0.454	-0.452	-0.451	-0.451	-0.452	-0.456
1.50	-0.519	-0.518	-0.516	-0.513	-0.510	-0.507	-0.502	-0.497	-0.490
1.75	-0.565	-0.564	-0.561	-0.558	-0.554	-0.548	-0.541	-0.531	-0.517
2.00	-0.600	-0.598	-0.595	-0.592	-0.586	-0.579	-0.570	-0.557	-0.538

Table TA10. $\frac{CS^C - CS^N}{CS^N}$ **under VI and downstream quantity competition.**

$\frac{\Delta_2}{\Delta_1}$	γ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750		
0.75	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	
1.00	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
1.25	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750
1.50	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	
1.75	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750		
2.00	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750	-0.750		

Table TA11. $\frac{CS^C - CS^N}{CS^N}$ **under VS and downstream price competition.**

$\frac{\Delta_2}{\Delta_1}$	γ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	-0.189	-0.227	-0.263	-0.297	-0.330	-0.297	-0.206		
0.75	-0.317	-0.361	-0.403	-0.442	-0.480	-0.516	-0.550	-0.525	
1.00	-0.423	-0.468	-0.510	-0.549	-0.586	-0.620	-0.654	-0.686	-0.718
1.25	-0.504	-0.547	-0.586	-0.622	-0.655	-0.685	-0.712	-0.736	-0.697
1.50	-0.563	-0.603	-0.638	-0.670	-0.697	-0.721	-0.741	-0.725	
1.75	-0.606	-0.642	-0.674	-0.701	-0.723	-0.741	-0.748		
2.00	-0.638	-0.670	-0.698	-0.721	-0.739	-0.751	-0.689		

Table TA12. $\frac{CS^C - CS^N}{CS^N}$ **under VI and downstream price competition.**

$\frac{\Delta_2}{\Delta_1}$	γ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	-0.413	-0.409	-0.406	-0.404	-0.403	-0.403	-0.404	-0.406	-0.410
0.75	-0.412	-0.407	-0.403	-0.399	-0.395	-0.392	-0.389	-0.387	-0.386
1.00	-0.411	-0.406	-0.402	-0.397	-0.393	-0.389	-0.385	-0.382	-0.378
1.25	-0.411	-0.407	-0.402	-0.398	-0.394	-0.391	-0.388	-0.385	-0.383
1.50	-0.412	-0.407	-0.403	-0.400	-0.397	-0.395	-0.393	-0.392	-0.393
1.75	-0.412	-0.408	-0.405	-0.402	-0.400	-0.399	-0.399	-0.400	-0.402
2.00	-0.413	-0.409	-0.406	-0.404	-0.403	-0.403	-0.404	-0.406	-0.410

Table TA13. $\frac{W^C - W^N}{W^N}$ **under VS and downstream quantity competition.**

$\frac{\Delta_2}{\Delta_1}$	γ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	-0.074	-0.064	-0.052	-0.039	-0.025	0.009	0.042	0.074	0.109
0.75	-0.143	-0.135	-0.127	-0.117	-0.106	-0.093	-0.077	-0.042	0.014
1.00	-0.205	-0.202	-0.198	-0.194	-0.190	-0.186	-0.180	-0.174	-0.167
1.25	-0.254	-0.255	-0.255	-0.256	-0.258	-0.261	-0.264	-0.270	-0.277
1.50	-0.291	-0.294	-0.297	-0.302	-0.307	-0.314	-0.323	-0.333	-0.347
1.75	-0.318	-0.322	-0.328	-0.334	-0.341	-0.350	-0.360	-0.373	-0.389
2.00	-0.338	-0.343	-0.349	-0.356	-0.364	-0.374	-0.385	-0.398	-0.413

Table TA14. $\frac{W^C - W^N}{W^N}$ **under VI and downstream quantity competition.**

$\frac{\Delta_2}{\Delta_1}$	γ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	-0.412	-0.407	-0.401	-0.394	-0.387	-0.379	-0.372		
0.75	-0.411	-0.404	-0.397	-0.388	-0.378	-0.365	-0.351	-0.334	
1.00	-0.411	-0.404	-0.396	-0.386	-0.375	-0.361	-0.344	-0.321	-0.292
1.25	-0.411	-0.404	-0.396	-0.387	-0.377	-0.364	-0.348	-0.329	-0.310
1.50	-0.411	-0.405	-0.398	-0.390	-0.380	-0.369	-0.356	-0.344	
1.75	-0.412	-0.406	-0.399	-0.392	-0.384	-0.374	-0.365		
2.00	-0.412	-0.407	-0.401	-0.394	-0.387	-0.379	-0.372		

Table TA15. $\frac{W^C - W^N}{W^N}$ **under VS and downstream price competition.**

$\frac{\Delta_2}{\Delta_1}$	γ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.50	-0.087	-0.091	-0.093	-0.094	-0.093	-0.066	-0.025		
0.75	-0.159	-0.167	-0.174	-0.179	-0.183	-0.185	-0.183	-0.140	
1.00	-0.222	-0.234	-0.244	-0.253	-0.261	-0.267	-0.270	-0.270	-0.264
1.25	-0.270	-0.284	-0.296	-0.306	-0.314	-0.320	-0.321	-0.316	-0.386
1.50	-0.305	-0.320	-0.332	-0.341	-0.347	-0.349	-0.343	-0.427	
1.75	-0.331	-0.345	-0.356	-0.363	-0.366	-0.362	-0.444		
2.00	-0.350	-0.362	-0.372	-0.376	-0.376	-0.368	-0.424		

Table TA16. $\frac{W^C - W^N}{W^N}$ **under VI and downstream price competition.**