Technical Appendix B to Accompany

"The Impact of Vertical Integration on Losses from Collusion"

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This Technical Appendix extends the analysis in the paper to consider $n > 1$ downstream suppliers (D1,...,Dn). When the two upstream suppliers (U1 and U2) collude, they specify the input prices $(w_1, ..., w_n)$ they charge to D1 ... Dn. They also specify $f_{ji} \in [0, 1]$, the fraction of Di's total demand for the input that is supplied by Uj ($i \in \{1, ..., n\}$; $j \in \{1, 2\}$). When D1 and U1 are vertically integrated, $v_j \in [0, 1]$ is the valuation that D1 places on Uj's upstream profit ($j \in \{1, 2\}$).[1]

If a representative consumer purchases $\mathbf{q} \equiv (q_1, ..., q_n)$ at prices $\mathbf{p} \equiv (p_1, ..., p_n)$, his utility is:

$$U(\mathbf{q}, \mathbf{p}) \;=\; \sum_{i=1}^{n} \alpha_i \, q_i - \frac{1}{2} \left[\sum_{i=1}^{n} (q_i)^2 + 2\,\gamma \sum_{j \neq i} q_i \, q_j \right] - \sum_{i=1}^{n} p_i \, q_i \,. \tag{1}$$

Utility maximization entails:

$$\frac{U(\mathbf{q}, \mathbf{p})}{\partial q_i} \;=\; \alpha_i - q_i - \gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j - p_i \;=\; 0\,. \tag{2}$$

(2) implies the inverse demand curve for firm i's product is:

$$P_i(q_i, \mathbf{q}_{-i}) \;=\; \alpha_i - q_i - \gamma \sum_{j \neq i} q_j \,. \tag{3}$$

Let c_i denote firm i's constant unit cost of production (including the input price). Then (3) implies that firm i's profit is:

$$\pi_i(q_i, \mathbf{q}_{-i}) \;=\; [\, P_i(q_i, \mathbf{q}_{-i}) - c_i \,]\, q_i \,. \tag{4}$$

I. **Quantity Competition**

A. Vertical Separation. No downstream supplier is integrated with any upstream supplier.

(3) implies that $\frac{\partial P_i(\cdot)}{\partial q_i} = -1$. Therefore, (4) implies that when Di incurs unit cost c_i, its profit-maximizing choice of q_i is determined by:

$$P_i(q_i, \mathbf{q}_{-i}) - c_i - q_i \;=\; 0 \quad \Leftrightarrow \quad \alpha_i - q_i - \gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j - c_i - q_i \;=\; 0$$

[1] The analysis in the paper considers the case where $v_1 = v_2$. Conceivably, D1 might not be aware of D1's collusive arrangement with U2. $v_1 = 1$ and $v_2 = 0$ might prevail in this case.

$$\Rightarrow \quad q_i \; = \; \frac{1}{2} \left[\alpha_i - c_i - \, \gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j \right] . \tag{5}$$

Summing (5) over all firms provides:

$$\sum_{i=1}^{n} q_i \; = \; \frac{1}{2} \left[\sum_{i=1}^{n} (\alpha_i - c_i) - \, \gamma \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j \right] \; = \; \frac{1}{2} \left[\sum_{i=1}^{n} (\alpha_i - c_i) - \, \gamma \, [\, n - 1 \,] \sum_{i=1}^{n} q_i \right]$$

$$\Rightarrow \quad [\, 2 + \gamma \, (\, n - 1 \,) \,] \sum_{i=1}^{n} q_i \; = \; \sum_{i=1}^{n} (\alpha_i - c_i) \quad \Rightarrow \quad \sum_{i=1}^{n} q_i \; = \; \frac{\sum_{i=1}^{n} (\alpha_i - c_i)}{2 + \gamma \, [\, n - 1 \,]}$$

$$\Rightarrow \quad q_i \; = \; \frac{\alpha_i - c_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_j - c_j)}{2 + \gamma \, [\, n - 1 \,]} - \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j \, . \tag{6}$$

(5) implies:

$$2 \, q_i \; = \; \alpha_i - c_i - \gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_j - c_j) \quad \Rightarrow \quad \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_j - c_j) \; = \; \frac{1}{\gamma} \, [\, \alpha_i - c_i - 2 \, q_i \,] \, .$$

Therefore, (6) implies:

$$q_i \; = \; \frac{\alpha_i - c_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_j - c_j)}{2 + \gamma \, [\, n - 1 \,]} - \frac{1}{\gamma} \, [\, \alpha_i - c_i - 2 \, q_i \,]$$

$$\Rightarrow \quad q_i \left[\frac{\gamma - 2}{\gamma} \right] \; = \; \frac{\alpha_i - c_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_j - c_j)}{2 + \gamma \, [\, n - 1 \,]} - \frac{1}{\gamma} \, [\, \alpha_i - c_i \,]$$

$$\Rightarrow \quad q_i \left[\frac{\gamma - 2}{\gamma} \right] \frac{2 + \gamma \, [\, n - 1 \,]}{2 + \gamma \, [\, n - 1 \,]} \; = \; \frac{\gamma \left[\alpha_i - c_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_j - c_j) \right] - [\, \alpha_i - c_i \,] \, [\, 2 + \gamma \, (\, n - 1 \,) \,]}{\gamma \, [\, 2 + \gamma \, (\, n - 1 \,) \,]}$$

$$\Rightarrow \quad q_i \;=\; \frac{\gamma\left[\alpha_i - c_i + \sum\limits_{\substack{j=1\\ j\neq i}}^{n}(\alpha_j - c_j)\right] - [\,\alpha_i - c_i\,]\,[\,2 + \gamma\,(\,n-1\,)\,]}{[\,\gamma - 2\,]\,[\,2 + \gamma\,(\,n-1\,)\,]}$$

$$\Rightarrow \quad q_i^* \;=\; \frac{[\,2 + \gamma\,(\,n-2\,)\,]\,[\,\alpha_i - c_i\,] - \gamma \sum\limits_{\substack{j=1\\ j\neq i}}^{n}(\alpha_j - c_j)}{[\,2 - \gamma\,]\,[\,2 + \gamma\,(\,n-1\,)\,]}\,. \tag{7}$$

The first equality in (5) implies that $P_i(\cdot) = q_i^* + c_i$. Therefore, (7) implies that the equilibrium price for Di's product is:

$$p_i^* \;=\; q_i^* + c_i \;=\; \frac{[\,2 + \gamma\,(\,n-2\,)\,]\,[\,\alpha_i - c_i\,] - \gamma \sum\limits_{\substack{j=1\\ j\neq i}}^{n}(\alpha_j - c_j) + [\,2-\gamma\,]\,[\,2 + \gamma\,(\,n-1\,)\,]\,c_i}{[\,2 - \gamma\,]\,[\,2 + \gamma\,(\,n-1\,)\,]}\,. \tag{8}$$

Observe that:

$$\begin{aligned}
&[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] - [\,2+\gamma\,(\,n-2\,)\,] \\
&\quad = \; 4 + 2\,\gamma\,[\,n-1\,] - 2\,\gamma - \gamma^2\,[\,n-1\,] - 2 - \gamma\,[\,n-2\,] \\
&\quad = \; 2 + 2\,\gamma\,n - 4\,\gamma - \gamma^2\,[\,n-1\,] - \gamma\,n + 2\,\gamma \\
&\quad = \; 2 + \gamma\,n - 2\,\gamma - \gamma^2\,[\,n-1\,] \;=\; 2 + \gamma\,[\,n-2\,] - \gamma^2\,[\,n-1\,]\,.
\end{aligned} \tag{9}$$

(8) and (9) imply that under vertical separation, for $i \in \{1, ..., n\}$:

$$p_i^* \;=\; \frac{[\,2 + \gamma\,(\,n-2\,)\,]\,[\,\alpha_i + c_i\,] - \gamma^2\,[\,n-1\,]\,c_i - \gamma \sum\limits_{\substack{j=1\\ j\neq i}}^{n}(\alpha_j - c_j)}{[\,2 - \gamma\,]\,[\,2 + \gamma\,(\,n-1\,)\,]}\,. \tag{10}$$

B. Vertical Integration. D1 and U1 are vertically integrated.

(4) implies that unintegrated downstream firm Di chooses q_i to:

$$\underset{q_i}{\text{Maximize}} \;\; \pi_i(q_i, \mathbf{q}_{-i}) \;=\; \left[\,P_i(q_i, \mathbf{q}_{-i}) - w_i - c_i^d\,\right] q_i \quad \text{for} \;\; i \,\in\, \{\,2, ..., n\,\}\,. \tag{11}$$

D1 chooses q_1 to:

$$\underset{q_1}{\text{Maximize}} \;\; \pi_1(q_1, \mathbf{q}_{-1}) \;=\; \left[\,P_1(q_1, \mathbf{q}_{-1}) - w_1 - c_1^d\,\right] q_1$$

$$+\ v_1 \sum_{i=1}^{n} [\, w_i - c_1^u \,]\, f_{1i}\, q_i + v_2 \sum_{i=1}^{n} [\, w_i - c_2^u \,]\, f_{2i}\, q_i \,. \tag{12}$$

$\frac{\partial P_1(\cdot)}{\partial q_1} = -1$ from (3). Therefore, (3) and (12) imply that firm 1's profit-maximizing choice of q_1 is determined by:

$$P_1(q_1, \mathbf{q}_{-1}) - w_1 - c_1^d - q_1 + v_1 [\, w_1 - c_1^u \,]\, f_{11} + v_2 [\, w_1 - c_2^u \,]\, f_{21} \;=\; 0$$

$$\Leftrightarrow \quad \alpha_1 - q_1 - \gamma \sum_{j=2}^{n} q_j - w_1 - c_1^d - q_1 + v_1 [\, w_1 - c_1^u \,]\, f_{11} + v_2 [\, w_1 - c_2^u \,]\, f_{21} \;=\; 0$$

$$\Leftrightarrow \quad q_1 \;=\; \frac{1}{2} \left[\alpha_1 - w_1 - c_1^d - \gamma \sum_{j=2}^{n} q_j + \sum_{i=1}^{2} v_i [\, w_1 - c_i^u \,]\, f_{i1} \right]. \tag{13}$$

(5) implies that the reaction function of unintegrated firm Di is:

$$q_i \;=\; \frac{1}{2} \left[\alpha_i - c_i^d - w_i - \gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j \right] \quad \text{for } i \in \{ 2, ..., n \} \,. \tag{14}$$

(13) and (14) imply:

$$\sum_{i=1}^{n} q_i \;=\; \frac{1}{2} \left[\alpha_1 - \gamma \sum_{j=2}^{n} q_j - w_1 - c_1^d + \sum_{i=1}^{2} v_i [\, w_1 - c_i^u \,]\, f_{i1} \right]$$

$$+\ \frac{1}{2} \left[\sum_{i=2}^{n} (\, \alpha_i - c_i^d - w_i \,) - \gamma \sum_{i=2}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j \right]$$

$$=\; \frac{1}{2} \left[\sum_{i=1}^{n} (\, \alpha_i - c_i^d - w_i \,) - \gamma \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j + \sum_{i=1}^{2} v_i [\, w_1 - c_i^u \,]\, f_{i1} \right]$$

$$=\; \frac{1}{2} \left[\sum_{i=1}^{n} (\, \alpha_i - c_i^d - w_i \,) - \gamma\, [\, n-1 \,] \sum_{i=1}^{n} q_i + \sum_{i=1}^{2} v_i [\, w_1 - c_i^u \,]\, f_{i1} \right]$$

$$\Rightarrow \quad \left[\frac{2 + \gamma\, (\, n-1 \,)}{2} \right] \sum_{i=1}^{n} q_i \;=\; \frac{1}{2} \left[\sum_{i=1}^{n} (\, \alpha_i - c_i^d - w_i \,) + \sum_{i=1}^{2} v_i [\, w_1 - c_i^u \,]\, f_{i1} \right]$$

$$\Rightarrow \quad \sum_{i=1}^{n} q_i \;=\; \frac{\sum_{i=1}^{n} \left(\alpha_i - c_i^d - w_i\right) + \sum_{i=1}^{2} v_i \left[w_1 - c_i^u \right] f_{i1}}{2 + \gamma \left[n - 1 \right]} \tag{15}$$

$$\Rightarrow \quad q_1 \;=\; \frac{\sum_{i=1}^{n} \left(\alpha_i - c_i^d - w_i\right) + \sum_{i=1}^{2} v_i \left[w_1 - c_i^u \right] f_{i1}}{2 + \gamma \left[n - 1 \right]} - \sum_{j=2}^{n} q_j \,, \quad \text{and}$$

$$q_i \;=\; \frac{\sum_{j=1}^{n} \left(\alpha_j - c_j^d - w_j\right) + \sum_{j=1}^{2} v_j \left[w_1 - c_j^u \right] f_{j1}}{2 + \gamma \left[n - 1 \right]} - \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j \quad \text{for } \; i \in \{ 2, ..., n \} . \tag{16}$$

(13) and (15) imply:

$$q_1 \;=\; \frac{1}{2} \left[\alpha_1 - w_1 - c_1^d - \gamma \left(\frac{\sum_{i=1}^{n} \left(\alpha_i - c_i^d - w_i\right) + \sum_{i=1}^{2} v_i \left[w_1 - c_i^u \right] f_{i1}}{2 + \gamma \left[n - 1 \right]} - q_1 \right) + \sum_{i=1}^{2} v_i \left[w_1 - c_i^u \right] f_{i1} \right]$$

$$\Rightarrow \quad q_1 \left[2 - \gamma \right] \;=\; \alpha_1 - w_1 - c_1^d - \gamma \left[\frac{\sum_{i=1}^{n} \left(\alpha_i - c_i^d - w_i\right) + \sum_{i=1}^{2} v_i \left[w_1 - c_i^u \right] f_{i1}}{2 + \gamma \left[n - 1 \right]} \right] + \sum_{i=1}^{2} v_i \left[w_1 - c_i^u \right] f_{i1}$$

$$\Rightarrow \quad q_1 \left[2 - \gamma \right] \;=\; \alpha_1 - w_1 - c_1^d - \frac{\gamma}{2 + \gamma \left[n - 1 \right]} \sum_{i=1}^{n} \left(\alpha_i - c_i^d - w_i\right) + \left[1 - \frac{\gamma}{2 + \gamma \left[n - 1 \right]} \right] \sum_{i=1}^{2} v_i \left[w_1 - c_i^u \right] f_{i1}$$

$$\Rightarrow \quad q_1^* \;=\; \frac{1}{2 - \gamma} \left[\alpha_1 - w_1 - c_1^d \right] - \frac{\gamma}{\left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]} \sum_{i=1}^{n} \left(\alpha_i - c_i^d - w_i\right) + \frac{2 - \gamma + \gamma \left[n - 1 \right]}{\left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]} \sum_{i=1}^{2} v_i \left[w_1 - c_i^u \right] f_{i1} \,. \tag{17}$$

(14) and (15) imply that for $i \in \{2, ..., n\}$:

$$q_i = \frac{1}{2}\left[\alpha_i - c_i^d - w_i - \gamma\left(\frac{\sum_{j=1}^{n}(\alpha_j - c_j^d - w_j) + \sum_{j=1}^{2} v_j [w_1 - c_j^u] f_{j1}}{2 + \gamma [n-1]} - q_i\right)\right]$$

$$\Rightarrow \quad q_i [2 - \gamma] = \alpha_i - c_i^d - w_i - \frac{\gamma\left[\sum_{j=1}^{n}(\alpha_j - c_j^d - w_j) + \sum_{j=1}^{2} v_j [w_1 - c_j^u] f_{j1}\right]}{2 + \gamma [n-1]}$$

$$\Rightarrow \quad q_i^* = \frac{1}{2-\gamma}[\alpha_i - w_i - c_i^d] - \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]}\sum_{j=1}^{n}(\alpha_j - c_j^d - w_j)$$

$$- \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]}\sum_{j=1}^{2} v_j [w_1 - c_j^u] f_{j1} \quad \text{for } i \in \{2, ..., n\}. \qquad (18)$$

Optimal Collusive Input Prices under Vertical Separation (and Quantity Competition)

Now consider the input prices U_1 and U_2 set when they collude under vertical separation and downstream firms compete on quantities.

For $i = 1, ..., n$, Di's unit production cost is $c_i = c_i^d + w_i$. Therefore, (7) implies that the equilibrium output of Di is:

$$q_i^* = \frac{[2+\gamma(n-2)][\alpha_i - c_i^d - w_i] - \gamma \sum_{\substack{j=1 \\ j \neq i}}^{n}(\alpha_j - c_j^d - w_j)}{[2-\gamma][2+\gamma(n-1)]} \quad \text{for } i = 1, ..., n. \qquad (19)$$

Under vertical separation, U_1 and U_2 choose input prices w_i to:

$$\max_{w_i} \sum_{i=1}^{n}[w_i - c_1^u] f_{1i} q_i + \sum_{i=1}^{n}[w_i - c_2^u] f_{2i} q_i$$

$$\Leftrightarrow \quad \max_{w_i} \sum_{i=1}^{n} w_i q_i - \sum_{i=1}^{n}[c_1^u f_{1i} + c_2^u f_{2i}] q_i$$

$$\Leftrightarrow \quad \max_{w_i} \sum_{i=1}^{n}[w_i - c_1^u f_{1i} - c_2^u f_{2i}] q_i, \qquad (20)$$

where q_i is as specified in (19).

Differentiating (20) with respect to w_i provides:

$$q_i + [\, w_i - c_1^u\, f_{1i} - c_2^u\, f_{2i}\,]\ \frac{\partial q_i(\cdot)}{\partial w_i} + \sum_{\substack{j=1\\ j\neq i}}^{n} (\, w_j - c_1^u\, f_{1j} - c_2^u\, f_{2j}\,)\ \frac{\partial q_j(\cdot)}{\partial w_i} \ = \ 0 \tag{21}$$

$$\Leftrightarrow \quad q_i - \frac{1}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \{\, [\, w_i - c_1^u\, f_{1i} - c_2^u\, f_{2i}\,]\,[\,2+\gamma\,(\,n-2\,)\,]$$

$$- \ \gamma \sum_{\substack{j=1\\ j\neq i}}^{n} (\, w_j - c_1^u\, f_{1j} - c_2^u\, f_{2j}\,)\,\} \ = \ 0$$

$$\Leftrightarrow \quad \frac{[\,2+\gamma\,(\,n-2\,)\,]\,\big[\,\alpha_i - c_i^d - w_i\,\big]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} - \frac{\gamma \sum_{\substack{j=1\\ j\neq i}}^{n} \big(\,\alpha_j - c_j^d - w_j\,\big)}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$= \ \frac{[\,2+\gamma\,(\,n-2\,)\,]\,[\, w_i - c_1^u\, f_{1i} - c_2^u\, f_{2i}\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} - \frac{\gamma \sum_{\substack{j=1\\ j\neq i}}^{n} (\, w_j - c_1^u\, f_{1j} - c_2^u\, f_{2j}\,)}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$\Leftrightarrow \quad \frac{2\,[\,2+\gamma\,(\,n-2\,)\,]\ w_i}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \ = \ \frac{[\,2+\gamma\,(\,n-2\,)\,]\,\big[\,\alpha_i - c_i^d\,\big]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} - \frac{\gamma \sum_{\substack{j=1\\ j\neq i}}^{n} \big(\,\alpha_j - c_j^d - w_j\,\big)}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$+ \ \frac{[\,2+\gamma\,(\,n-2\,)\,]\,[\, c_1^u\, f_{1i} + c_2^u\, f_{2i}\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} + \frac{\gamma \sum_{\substack{j=1\\ j\neq i}}^{n} (\, w_j - c_1^u\, f_{1j} - c_2^u\, f_{2j}\,)}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$\Leftrightarrow \quad 2\,[\,2+\gamma\,(\,n-2\,)\,]\ w_i \ = \ [\,2+\gamma\,(\,n-2\,)\,]\,\big[\,\alpha_i - c_i^d\,\big] - \gamma \sum_{\substack{j=1\\ j\neq i}}^{n} \big(\,\alpha_j - c_j^d - w_j\,\big)$$

$$+ \ [\,2+\gamma\,(\,n-2\,)\,]\,[\, c_1^u\, f_{1i} + c_2^u\, f_{2i}\,]$$

$$+ \ \gamma \sum_{\substack{j=1\\ j\neq i}}^{n} (\, w_j - c_1^u\, f_{1j} - c_2^u\, f_{2j}\,)$$

$$\Leftrightarrow \quad w_i \ = \ \frac{1}{2}\,\big[\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\big] - \frac{\gamma \sum_{\substack{j=1\\ j\neq i}}^{n} \big(\,\alpha_j - c_j^d + c_1^u\, f_{1j} + c_2^u\, f_{2j} - 2\,w_j\,\big)}{2\,[\,2+\gamma\,(\,n-2\,)\,]}$$

$$\Leftrightarrow \quad w_i \;=\; \frac{1}{2}\left[\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right] - \frac{\gamma \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{n} \left(\,\alpha_j - c_j^d + c_1^u\, f_{1j} + c_2^u\, f_{2j}\,\right)}{2\left[\,2 + \gamma\,(\,n-2\,)\,\right]}$$

$$+ \frac{2\,\gamma \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{n} w_j}{2\left[\,2 + \gamma\,(\,n-2\,)\,\right]}\,. \tag{22}$$

Summing (22) provides:

$$\sum_{i=1}^{n} w_i \;=\; \frac{1}{2}\sum_{i=1}^{n}\left[\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right] - \frac{\gamma \displaystyle\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \left(\,\alpha_j - c_j^d + c_1^u\, f_{1j} + c_2^u\, f_{2j}\,\right)}{2\left[\,2 + \gamma\,(\,n-2\,)\,\right]}$$

$$+ \frac{2\,\gamma \displaystyle\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} w_j}{2\left[\,2 + \gamma\,(\,n-2\,)\,\right]}$$

$$= \; \frac{1}{2}\sum_{i=1}^{n}\left[\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right] - \frac{\gamma\left[\,n-1\,\right] \displaystyle\sum_{i=1}^{n} \left(\,\alpha_i - c_i^d + c_1^u\, f_{1j} + c_2^u\, f_{2j}\,\right)}{2\left[\,2 + \gamma\,(\,n-2\,)\,\right]}$$

$$+ \frac{2\,\gamma\left[\,n-1\,\right] \displaystyle\sum_{i=1}^{n} w_i}{2\left[\,2 + \gamma\,(\,n-2\,)\,\right]}$$

$$\Rightarrow \quad \left[1 - \frac{\gamma\,(\,n-1\,)}{2 + \gamma\left[\,n-2\,\right]}\right]\sum_{i=1}^{n} w_i \;=\; \frac{1}{2}\sum_{i=1}^{n}\left(\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right)$$

$$- \frac{\gamma\left[\,n-1\,\right] \displaystyle\sum_{i=1}^{n} \left(\,\alpha_i - c_i^d + c_1^u\, f_{1j} + c_2^u\, f_{2j}\,\right)}{2\left[\,2 + \gamma\,(\,n-2\,)\,\right]}$$

$$\Rightarrow \quad \frac{2-\gamma}{2 + \gamma\left[\,n-2\,\right]}\sum_{i=1}^{n} w_i \;=\; \frac{1}{2}\sum_{i=1}^{n}\left(\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right)$$

$$- \frac{\gamma\left[\,n-1\,\right] \displaystyle\sum_{i=1}^{n} \left(\,\alpha_i - c_i^d + c_1^u\, f_{1j} + c_2^u\, f_{2j}\,\right)}{2\left[\,2 + \gamma\,(\,n-2\,)\,\right]}$$

$$\Rightarrow \quad \sum_{i=1}^{n} w_i \;=\; \frac{2 + \gamma\left[\,n-2\,\right]}{2\,\left[\,2-\gamma\,\right]}\sum_{i=1}^{n}\left(\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right)$$

$$- \frac{\gamma\,[\,n-1\,] \sum_{i=1}^{n} \left(\,\alpha_i - c_i^d + c_1^u\, f_{1j} + c_2^u\, f_{2j}\,\right)}{2\;[\,2-\gamma\,]}$$

$$\Rightarrow \quad \sum_{i=1}^{n} w_i \;=\; \frac{2+\gamma\,[\,n-2\,]-\gamma\,[\,n-1\,]}{2\;[\,2-\gamma\,]} \sum_{i=1}^{n} \left(\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right)$$

$$\Rightarrow \quad \sum_{i=1}^{n} w_i \;=\; \frac{1}{2} \sum_{i=1}^{n} \left(\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right)$$

$$\Rightarrow \quad w_i \;=\; \frac{1}{2} \sum_{j=1}^{n} \left(\,\alpha_j - c_j^d + c_1^u\, f_{1j} + c_2^u\, f_{2j}\,\right) - \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j \tag{23}$$

$$\Rightarrow \quad \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j \;=\; \frac{1}{2} \sum_{j=1}^{n} \left(\,\alpha_j - c_j^d + c_1^u\, f_{1j} + c_2^u\, f_{2j}\,\right) - \; w_i\,. \tag{24}$$

(22) and (24) imply:

$$w_i \;=\; \frac{1}{2}\left[\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right] - \frac{\gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\,\alpha_j - c_j^d + c_1^u\, f_{1j} + c_2^u\, f_{2j}\,\right)}{2\,[\,2+\gamma\,(\,n-2\,)\,]}$$

$$+ \; \frac{\gamma \left[\,\frac{1}{2} \sum_{j=1}^{n} \left(\,\alpha_j - c_j^d + c_1^u\, f_{1j} + c_2^u\, f_{2j}\,\right) - w_i\,\right]}{2+\gamma\,[\,n-2\,]}$$

$$\Rightarrow \quad w_i \left[\,1 + \frac{\gamma}{2+\gamma\,(\,n-2\,)}\,\right] \;=\; \frac{1}{2}\left[\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right]$$

$$- \; \frac{\gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\,\alpha_j - c_j^d + c_1^u\, f_{1j} + c_2^u\, f_{2j}\,\right)}{2\,[\,2+\gamma\,(\,n-2\,)\,]}$$

$$+ \; \frac{\gamma \sum_{j=1}^{n} \left(\,\alpha_j - c_j^d + c_1^u\, f_{1j} + c_2^u\, f_{2j}\,\right)}{2\,[\,2+\gamma\,(\,n-2\,)\,]}$$

$$\Rightarrow \quad w_i \left[\,\frac{2+\gamma\,(\,n-1\,)}{2+\gamma\,(\,n-2\,)}\,\right] \;=\; \frac{1}{2}\left[\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right]$$

$$- \frac{\gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\alpha_j - c_j^d + c_1^u f_{1j} + c_2^u f_{2j} \right)}{2 \left[2 + \gamma \left(n - 2 \right) \right]}$$

$$+ \frac{\gamma \sum_{j=1}^{n} \left(\alpha_j - c_j^d + c_1^u f_{1j} + c_2^u f_{2j} \right)}{2 \left[2 + \gamma \left(n - 2 \right) \right]}$$

$$= \frac{1}{2} \left[\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right] - \frac{\gamma \left[\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right]}{2 \left[2 + \gamma \left(n - 2 \right) \right]} + \frac{\gamma \left[\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right]}{2 \left[2 + \gamma \left(n - 2 \right) \right]}$$

$$- \frac{\gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\alpha_j - c_j^d + c_1^u f_{1j} + c_2^u f_{2j} \right)}{2 \left[2 + \gamma \left(n - 2 \right) \right]} + \frac{\gamma \sum_{j=1}^{n} \left(\alpha_j - c_j^d + c_1^u f_{1j} + c_2^u f_{2j} \right)}{2 \left[2 + \gamma \left(n - 2 \right) \right]}$$

$$= \frac{1}{2} \left[\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right] + \frac{\gamma \left[\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right]}{2 \left[2 + \gamma \left(n - 2 \right) \right]}$$

$$- \frac{\gamma \sum_{j=1}^{n} \left(\alpha_j - c_j^d + c_1^u f_{1j} + c_2^u f_{2j} \right)}{2 \left[2 + \gamma \left(n - 2 \right) \right]} + \frac{\gamma \sum_{j=1}^{n} \left(\alpha_j - c_j^d + c_1^u f_{1j} + c_2^u f_{2j} \right)}{2 \left[2 + \gamma \left(n - 2 \right) \right]}$$

$$= \frac{1}{2} \left[\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right] + \frac{\gamma \left[\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right]}{2 \left[2 + \gamma \left(n - 2 \right) \right]} . \tag{25}$$

(25) implies:

$$w_i^* = \frac{2 + \gamma \left[n - 2 \right]}{2 \left[2 + \gamma \left(n - 1 \right) \right]} \left[\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right] + \frac{\gamma \left[\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right]}{2 \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$= \frac{2 + \gamma \left[n - 2 \right] + \gamma}{2 \left[2 + \gamma \left(n - 1 \right) \right]} \left[\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right] = \frac{1}{2} \left[\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right] . \tag{26}$$

(19) and (26) imply:

$$q_i^* = \frac{\left[2 + \gamma \left(n - 2 \right) \right] \left[\alpha_i - c_i^d - \frac{1}{2} \left(\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right) \right]}{\left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$- \frac{\gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\alpha_j - c_j^d - \frac{1}{2} \left[\alpha_j - c_j^d + c_1^u f_{1j} + c_2^u f_{2j} \right] \right)}{\left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$= \frac{[\,2+\gamma\,(\,n-2\,)\,]\,\left[\,\frac{1}{2}\left(\,\alpha_i - c_i^d - c_1^u\,f_{1i} - c_2^u\,f_{2i}\,\right)\,\right]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$- \frac{\gamma \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{2}\left[\,\alpha_j - c_j^d - c_1^u\,f_{1j} - c_2^u\,f_{2j}\,\right]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$= \frac{[\,2+\gamma\,(\,n-2\,)\,]\left[\,\alpha_i - c_i^d - c_1^u\,f_{1i} - c_2^u\,f_{2i}\,\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} - \frac{\gamma \sum_{\substack{j=1 \\ j\neq i}}^{n}\left(\,\alpha_j - c_j^d - c_1^u\,f_{1j} - c_2^u\,f_{2j}\,\right)}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{\gamma\left[\,\alpha_i - c_i^d - c_1^u\,f_{1i} - c_2^u\,f_{2i}\,\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} - \frac{\gamma\left[\,\alpha_i - c_i^d - c_1^u\,f_{1i} - c_2^u\,f_{2i}\,\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$= \frac{[\,2+\gamma\,(\,n-2\,)\,]\left[\,\alpha_i - c_i^d - c_1^u\,f_{1i} - c_2^u\,f_{2i}\,\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} - \frac{\gamma \sum_{j=1}^{n}\left(\,\alpha_j - c_j^d - c_1^u\,f_{1j} - c_2^u\,f_{2j}\,\right)}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{\gamma\left[\,\alpha_i - c_i^d - c_1^u\,f_{1i} - c_2^u\,f_{2i}\,\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$= \frac{[\,2+\gamma\,(\,n-1\,)\,]\left[\,\alpha_i - c_i^d - c_1^u\,f_{1i} - c_2^u\,f_{2i}\,\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} - \frac{\gamma \sum_{j=1}^{n}\left(\,\alpha_j - c_j^d - c_1^u\,f_{1j} - c_2^u\,f_{2j}\,\right)}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\,. \quad (27)$$

(10) and (26) imply:

$$p_i^* = \frac{[\,2+\gamma\,(\,n-2\,)\,]\left[\,\alpha_i + c_i^d + w_i^*\,\right] - \gamma^2\,[\,n-1\,]\left[\,c_i^d + w_i^*\,\right] - \gamma \sum_{\substack{j=1 \\ j\neq i}}^{n}\left(\,\alpha_j - c_j^d - w_j^*\,\right)}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$= \frac{[\,2+\gamma\,(\,n-2\,)\,]\left[\,\alpha_i + c_i^d\,\right] - \gamma^2\,[\,n-1\,]\;c_i^d - \gamma \sum_{\substack{j=1 \\ j\neq i}}^{n}\left(\,\alpha_j - c_j^d\,\right)}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{[\,2+\gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,]\;w_i^* + \gamma \sum_{\substack{j=1 \\ j\neq i}}^{n} w_j^*}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$= \frac{[\,2+\gamma\,(\,n-2\,)\,]\,\left[\,\alpha_i + c_i^d\,\right] - \gamma^2\,[\,n-1\,]\;c_i^d - \gamma \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{n} \left(\,\alpha_j - c_j^d\,\right)}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\;\frac{1}{2}\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{\gamma \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{2}\left[\,\alpha_j - c_j^d + c_1^u\,f_{1j} + c_2^u\,f_{2j}\,\right]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$= \frac{[\,2+\gamma\,(\,n-2\,)\,]\,\left[\,\alpha_i + c_i^d\,\right] - \gamma^2\,[\,n-1\,]\;c_i^d}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\;\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{\gamma \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{n} \left(\,\alpha_j - c_j^d + c_1^u\,f_{1j} + c_2^u\,f_{2j}\,\right) - 2\,\gamma \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{n} \left(\,\alpha_j - c_j^d\,\right)}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$= \frac{[\,2+\gamma\,(\,n-2\,)\,]\,\left[\,\alpha_i + c_i^d\,\right] - \gamma^2\,[\,n-1\,]\;c_i^d}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\;\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$- \frac{\gamma \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{n} \left(\,\alpha_j - c_j^d - c_1^u\,f_{1j} - c_2^u\,f_{2j}\,\right)}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$= \frac{[\,2+\gamma\,(\,n-2\,)\,]\,\left[\,\alpha_i + c_i^d\,\right] - \gamma^2\,[\,n-1\,]\;c_i^d}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\;\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{\gamma\;\left[\,\alpha_i - c_i^d - c_1^u\,f_{1i} - c_2^u\,f_{2i}\,\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} - \frac{\gamma\;\left[\,\alpha_i - c_i^d - c_1^u\,f_{1i} - c_2^u\,f_{2i}\,\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$- \frac{\gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\alpha_j - c_j^d - c_1^u f_{1j} - c_2^u f_{2j} \right)}{2 \left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$= \frac{\left[2 + \gamma \left(n - 2 \right) \right] \left[\alpha_i + c_i^d \right] - \gamma^2 \left[n - 1 \right] c_i^d}{\left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]} + \frac{\gamma \left[\alpha_i - c_i^d - c_1^u f_{1i} - c_2^u f_{2i} \right]}{2 \left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$+ \frac{\left[2 + \gamma \left(n - 2 \right) - \gamma^2 \left(n - 1 \right) \right] \left[\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right]}{2 \left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$- \frac{\gamma \sum_{j=1}^{n} \left(\alpha_j - c_j^d - c_1^u f_{1j} - c_2^u f_{2j} \right)}{2 \left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$= \frac{\left[2 + \gamma \left(n - 2 \right) \right] \left[\alpha_i + c_i^d \right] - \gamma^2 \left[n - 1 \right] c_i^d}{\left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$+ \frac{\left[2 + \gamma \left(n - 2 \right) - \gamma^2 \left(n - 1 \right) + \gamma \right] \left[\alpha_i - c_i^d \right]}{2 \left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$+ \frac{\left[2 + \gamma \left(n - 2 \right) - \gamma^2 \left(n - 1 \right) - \gamma \right] \left[c_1^u f_{1i} + c_2^u f_{2i} \right]}{2 \left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$- \frac{\gamma \sum_{j=1}^{n} \left(\alpha_j - c_j^d - c_1^u f_{1j} - c_2^u f_{2j} \right)}{2 \left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$= \frac{\left[2 + \gamma \left(n - 2 \right) \right] \alpha_i}{\left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]} + \frac{\left[2 + \gamma \left(n - 2 \right) - \gamma^2 \left(n - 1 \right) + \gamma \right] \alpha_i}{2 \left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$+ \frac{\left[2 + \gamma \left(n - 2 \right) - \gamma^2 \left(n - 1 \right) \right] c_i^d}{\left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]} - \frac{\left[2 + \gamma \left(n - 2 \right) - \gamma^2 \left(n - 1 \right) + \gamma \right] c_i^d}{2 \left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$+ \frac{\left[2 + \gamma \left(n - 2 \right) - \gamma^2 \left(n - 1 \right) - \gamma \right] \left[c_1^u f_{1i} + c_2^u f_{2i} \right]}{2 \left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$- \frac{\gamma \sum_{j=1}^{n} \left(\alpha_j - c_j^d - c_1^u f_{1j} - c_2^u f_{2j} \right)}{2 \left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$= \frac{\left[6 + 3 \gamma \left(n - 2 \right) - \gamma^2 \left(n - 1 \right) + \gamma \right] \alpha_i}{2 \left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]} + \frac{\left[2 + \gamma \left(n - 2 \right) - \gamma^2 \left(n - 1 \right) - \gamma \right] c_i^d}{2 \left[2 - \gamma \right] \left[2 + \gamma \left(n - 1 \right) \right]}$$

$$+ \frac{[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)-\gamma\,]\;[\,c_1^u\,f_{1i}+c_2^u\,f_{2i}\,]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$- \frac{\gamma\,\sum_{j=1}^{n}\left(\alpha_j-c_j^d-c_1^u\,f_{1j}-c_2^u\,f_{2j}\right)}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$= \frac{[\,6+3\,\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)+\gamma\,]\;\alpha_i}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}+\frac{[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)-\gamma\,]\;\alpha_i}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)-\gamma\,]\;\left[-\alpha_i+c_i^d\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)-\gamma\,]\;[\,c_1^u\,f_{1i}+c_2^u\,f_{2i}\,]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$- \frac{\gamma\,\sum_{j=1}^{n}\left[\,\alpha_j-c_j^d-c_1^u\,f_{1j}-c_2^u\,f_{2j}\,\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$= \frac{[\,6+3\,\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)+\gamma\,]\;\alpha_i}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}+\frac{[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)-\gamma\,]\;\alpha_i}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$- \frac{[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)-\gamma\,]\;\left[\,\alpha_i-c_i^d-c_1^u\,f_{1i}-c_2^u\,f_{2i}\,\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$- \frac{\gamma\,\sum_{j=1}^{n}\left(\alpha_j-c_j^d-c_1^u\,f_{1j}-c_2^u\,f_{2j}\right)}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$= \frac{[\,4+2\,\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\;\alpha_i}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$- \frac{[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)-\gamma\,]\;\left[\,\alpha_i-c_i^d-c_1^u\,f_{1i}-c_2^u\,f_{2i}\,\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$- \frac{\gamma\,\sum_{j=1}^{n}\left(\alpha_j-c_j^d-c_1^u\,f_{1j}-c_2^u\,f_{2j}\right)}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\,.$$

$$= \alpha_i-\frac{[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)-\gamma\,]\;\left[\,\alpha_i-c_i^d-c_1^u\,f_{1i}-c_2^u\,f_{2i}\,\right]}{2\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$- \frac{\gamma \sum_{j=1}^{n} \left(\alpha_j - c_j^d - c_1^u f_{1j} - c_2^u f_{2j} \right)}{2 [2 - \gamma] [2 + \gamma (n - 1)]} . \tag{28}$$

(4), (26), (27) and (28) imply that for $i = 1, ..., n$, Di's equilibrium profit is:

$$\begin{aligned}
\pi_i^* &= \left[P_i^* - c_i^d - w_i^* \right] q_i^* \\
&= \left\{ \alpha_i - c_i^d - \frac{[2 + \gamma (n - 2) - \gamma^2 (n - 1) - \gamma] \left[\alpha_i - c_i^d - c_1^u f_{1i} - c_2^u f_{2i} \right]}{2 [2 - \gamma] [2 + \gamma (n - 1)]} \right. \\
&\qquad \left. - \frac{\gamma \sum_{j=1}^{n} \left[\alpha_j - c_j^d - c_1^u f_{1j} - c_2^u f_{2j} \right]}{2 [2 - \gamma] [2 + \gamma (n - 1)]} - \frac{1}{2} \left[\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right] \right\} \\
&\quad \cdot \left\{ \frac{[2 + \gamma (n - 1)] \left[\alpha_i - c_i^d - c_1^u f_{1i} - c_2^u f_{2i} \right]}{2 [2 - \gamma] [2 + \gamma (n - 1)]} - \frac{\gamma \sum_{j=1}^{n} \left(\alpha_j - c_j^d - c_1^u f_{1j} - c_2^u f_{2j} \right)}{2 [2 - \gamma] [2 + \gamma (n - 1)]} \right\} \\
&= \left\{ - \frac{[2 + \gamma (n - 2) - \gamma^2 (n - 1) - \gamma] \left[\alpha_i - c_i^d - c_1^u f_{1i} - c_2^u f_{2i} \right]}{2 [2 - \gamma] [2 + \gamma (n - 1)]} \right. \\
&\qquad \left. - \frac{\gamma \sum_{j=1}^{n} \left[\alpha_j - c_j^d - c_1^u f_{1j} - c_2^u f_{2j} \right]}{2 [2 - \gamma] [2 + \gamma (n - 1)]} + \frac{1}{2} \left[\alpha_i - c_i^d - c_1^u f_{1i} - c_2^u f_{2i} \right] \right\} \\
&\quad \cdot \left\{ \frac{[2 + \gamma (n - 1)] \left[\alpha_i - c_i^d - c_1^u f_{1i} - c_2^u f_{2i} \right]}{2 [2 - \gamma] [2 + \gamma (n - 1)]} - \frac{\gamma \sum_{j=1}^{n} \left(\alpha_j - c_j^d - c_1^u f_{1j} - c_2^u f_{2j} \right)}{2 [2 - \gamma] [2 + \gamma (n - 1)]} \right\} .
\end{aligned} \tag{29}$$

Observe that

$$\begin{aligned}
& [2 - \gamma] [2 + \gamma (n - 1)] - \left[2 + \gamma (n - 2) - \gamma^2 (n - 1) - \gamma \right] \\
&\quad = [2 - \gamma] [2 + \gamma (n - 1)] - 2 - \gamma [n - 2] + \gamma^2 [n - 1] + \gamma \\
&\quad = 4 + 2 \gamma [n - 1] - 2 \gamma - \gamma^2 [n - 1] - 2 - \gamma [n - 2] + \gamma^2 [n - 1] + \gamma \\
&\quad = 2 - \gamma + \gamma [2n - 2 - n + 2] \; = \; 2 + \gamma [n - 1] .
\end{aligned} \tag{30}$$

(29) and (30) imply that Di's equilibrium profit is (for $i = 1, ..., n$):

$$\pi_i^* = \left\{ \frac{[2+\gamma(n-1)][\alpha_i - c_i^d - c_1^u f_{1i} - c_2^u f_{2i}]}{2[2-\gamma][2+\gamma(n-1)]} - \frac{\gamma \sum_{j=1}^{n} [\alpha_j - c_j^d - c_1^u f_{1j} - c_2^u f_{2j}]}{2[2-\gamma][2+\gamma(n-1)]} \right\}$$

$$\cdot \left\{ \frac{[2+\gamma(n-1)][\alpha_i - c_i^d - c_1^u f_{1i} - c_2^u f_{2i}]}{2[2-\gamma][2+\gamma(n-1)]} - \frac{\gamma \sum_{j=1}^{n} [\alpha_j - c_j^d - c_1^u f_{1j} - c_2^u f_{2j}]}{2[2-\gamma][2+\gamma(n-1)]} \right\}$$

$$= \left\{ \frac{[2+\gamma(n-1)][\alpha_i - c_i^d - c_1^u f_{1i} - c_2^u f_{2i}]}{2[2-\gamma][2+\gamma(n-1)]} - \frac{\gamma \sum_{j=1}^{n} [\alpha_j - c_j^d - c_1^u f_{1j} - c_2^u f_{2j}]}{2[2-\gamma][2+\gamma(n-1)]} \right\}^2 . \tag{31}$$

<u>Optimal Collusive Input Prices under Vertical Integration (and Quantity Competition)</u>

Now consider the input prices U_1 and U_2 set when they collude in the setting where downstream firm D1 is vertically integrated with upstream supplier U1 and downstream firms compete on quantities.

(3), (17), and (18) imply:

$$P_1^* = \alpha_1 - q_1^* - \gamma \sum_{i=2}^{n} q_i^*$$

$$= \alpha_1 - \frac{1}{2-\gamma}[\alpha_1 - w_1 - c_1^d] + \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]} \sum_{i=1}^{n} (\alpha_i - c_i^d - w_i)$$

$$- \frac{2-\gamma+\gamma[n-1]}{[2-\gamma][2+\gamma(n-1)]} \sum_{i=1}^{2} v_i [w_1 - c_i^u] f_{i1}$$

$$- \frac{\gamma}{2-\gamma} \sum_{i=2}^{n} (\alpha_i - w_i - c_i^d) + \frac{\gamma^2[n-1]}{[2-\gamma][2+\gamma(n-1)]} \sum_{j=1}^{n} (\alpha_j - c_j^d - w_j)$$

$$+ \frac{\gamma^2[n-1]}{[2-\gamma][2+\gamma(n-1)]} \sum_{j=1}^{2} v_j [w_1 - c_j^u] f_{j1}$$

$$= \alpha_1 - \frac{1}{2-\gamma}[\alpha_1 - w_1 - c_1^d] + \frac{\gamma + \gamma^2[n-1]}{[2-\gamma][2+\gamma(n-1)]} \sum_{i=1}^{n} (\alpha_i - c_i^d - w_i)$$

$$- \frac{2-\gamma+\gamma[n-1]-\gamma^2[n-1]}{[2-\gamma][2+\gamma(n-1)]} \sum_{i=1}^{2} v_i [w_1 - c_i^u] f_{i1}$$

$$+ \frac{\gamma}{2-\gamma}[\alpha_1 - w_1 - c_1^d] - \frac{\gamma}{2-\gamma}[\alpha_1 - w_1 - c_1^d] - \frac{\gamma}{2-\gamma} \sum_{i=2}^{n} (\alpha_i - w_i - c_i^d)$$

$$= \alpha_1 - \frac{1-\gamma}{2-\gamma}[\alpha_1 - w_1 - c_1^d] + \frac{\gamma+\gamma^2[n-1]}{[2-\gamma][2+\gamma(n-1)]} \sum_{i=1}^{n} (\alpha_i - c_i^d - w_i)$$

$$- \frac{2-\gamma+\gamma[n-1]-\gamma^2[n-1]}{[2-\gamma][2+\gamma(n-1)]} \sum_{i=1}^{2} v_i [w_1 - c_i^u] f_{i1} - \frac{\gamma}{2-\gamma} \sum_{i=1}^{n} (\alpha_i - w_i - c_i^d)$$

$$= \alpha_1 - \frac{1-\gamma}{2-\gamma}[\alpha_1 - w_1 - c_1^d] + \frac{\gamma+\gamma^2[n-1]-\gamma[2+\gamma(n-1)]}{[2-\gamma][2+\gamma(n-1)]} \sum_{i=1}^{n} (\alpha_i - c_i^d - w_i)$$

$$- \frac{2-\gamma+\gamma[n-1]-\gamma^2[n-1]}{[2-\gamma][2+\gamma(n-1)]} \sum_{i=1}^{2} v_i [w_1 - c_i^u] f_{i1}$$

$$= \alpha_1 - \frac{1-\gamma}{2-\gamma}[\alpha_1 - w_1 - c_1^d] - \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]} \sum_{i=1}^{n} (\alpha_i - c_i^d - w_i)$$

$$- \frac{2-\gamma+\gamma[n-1]-\gamma^2[n-1]}{[2-\gamma][2+\gamma(n-1)]} \sum_{i=1}^{2} v_i [w_1 - c_i^u] f_{i1} . \tag{32}$$

Under vertical integration, U_1 and U_2 choose input prices w_i (for $i = 1, ...n$) to:

$$\max_{w_i} \quad \sum_{i=1}^{n} [w_i - c_1^u] f_{1i} q_i + \sum_{i=1}^{n} [w_i - c_2^u] f_{2i} q_i + [P_1 - w_1 - c_1^d] q_1$$

$$\Leftrightarrow \quad \max_{w_i} \sum_{i=1}^{n} [w_i - c_1^u f_{1i} - c_2^u f_{2i}] q_i + [P_1 - w_1 - c_1^d] q_1 . \tag{33}$$

where P_1 is given by (32), q_1 is given by (17), and q_i is given by (18) for $i = 2, ...n$.

Differentiating (33) with respect to w_i (for $i = 2, ...n$), provides the following first-order condition:

$$q_1 \frac{\partial P_1(\cdot)}{\partial w_i} + [P_1 - w_1 - c_1^d] \frac{\partial q_1(\cdot)}{\partial w_i} + q_i + [w_i - c_1^u f_{1i} - c_2^u f_{2i}] \frac{\partial q_i(\cdot)}{\partial w_i}$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^{n} (w_j - c_1^u f_{1j} - c_2^u f_{2j}) \frac{\partial q_j(\cdot)}{\partial w_i} = 0 .$$

(17), (18), and (32) imply that for $i = 2, ...n$ and $j = 1, ..., n$ $(j \neq i)$:

$$\frac{\partial P_1(\cdot)}{\partial w_i} = \frac{\partial q_j(\cdot)}{\partial w_i} = \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]} \text{ and}$$

$$\frac{\partial q_i(\cdot)}{\partial w_i} = -\frac{1}{2-\gamma} + \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]} .$$

These derivatives and the preceding first-order condition, along with (17), (18), and (32) imply:

$$\frac{\gamma q_1}{[2-\gamma][2+\gamma(n-1)]} + \frac{\gamma \left[P_1 - w_1 - c_1^d \right]}{[2-\gamma][2+\gamma(n-1)]} + q_i$$

$$+ [w_i - c_1^u f_{1i} - c_2^u f_{2i}] \left[-\frac{1}{2-\gamma} + \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]} \right]$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^{n} (w_j - c_1^u f_{1j} - c_2^u f_{2j}) \left[\frac{\gamma}{[2-\gamma][2+\gamma(n-1)]} \right] = 0$$

$$\Rightarrow \frac{\gamma}{[2-\gamma]^2 [2+\gamma(n-1)]} [\alpha_1 - w_1 - c_1^d] - \frac{\gamma^2}{[2-\gamma]^2 [2+\gamma(n-1)]^2} \sum_{i=1}^{n} (\alpha_i - c_i^d - w_i)$$

$$+ \frac{\gamma [2-\gamma+\gamma(n-1)]}{[2-\gamma]^2 [2+\gamma(n-1)]^2} \sum_{i=1}^{2} v_i [w_1 - c_i^u] f_{i1}$$

$$+ \frac{\gamma \alpha_1}{[2-\gamma][2+\gamma(n-1)]} - \frac{\gamma [1-\gamma]}{[2-\gamma]^2 [2+\gamma(n-1)]} [\alpha_1 - w_1 - c_1^d]$$

$$- \frac{\gamma^2}{[2-\gamma]^2 [2+\gamma(n-1)]^2} \sum_{i=1}^{n} (\alpha_i - c_i^d - w_i)$$

$$- \frac{\gamma [2-\gamma+\gamma(n-1)-\gamma^2(n-1)]}{[2-\gamma]^2 [2+\gamma(n-1)]^2} \sum_{i=1}^{2} v_i [w_1 - c_i^u] f_{i1} - \frac{\gamma \left[w_1 + c_1^d \right]}{[2-\gamma][2+\gamma(n-1)]}$$

$$+ \frac{1}{2-\gamma} [\alpha_i - w_i - c_i^d] - \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]} \sum_{j=1}^{n} (\alpha_j - c_j^d - w_j)$$

$$- \frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \sum_{j=1}^{2} v_j \left[\, w_1 - c_j^u \,\right] f_{j1}$$

$$+ \,[\, w_i - c_1^u\, f_{1i} - c_2^u\, f_{2i}\,] \left[-\frac{1}{2-\gamma} + \frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \right]$$

$$+ \sum_{\substack{j=1\\ j\neq i}}^{n} (\, w_j - c_1^u\, f_{1j} - c_2^u\, f_{2j}\,) \left[\frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \right] \;=\; 0$$

$$\Rightarrow \quad \frac{\gamma - \gamma\,[\,1-\gamma\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]} \left[\, \alpha_1 - w_1 - c_1^d \,\right] - \frac{2\,\gamma^2}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \sum_{i=1}^{n} \left(\, \alpha_i - c_i^d - w_i \,\right)$$

$$+ \frac{\gamma\,[\,2-\gamma+\gamma\,(\,n-1\,)\,] - \gamma\,[\,2-\gamma+\gamma\,(\,n-1\,) - \gamma^2\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \sum_{i=1}^{2} v_i\,[\, w_1 - c_i^u \,]\, f_{i1}$$

$$+ \frac{\gamma\,\alpha_1}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} - \frac{\gamma \left[\, w_1 + c_1^d \,\right]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{1}{2-\gamma} \left[\, \alpha_i - w_i - c_i^d \,\right] - \frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \sum_{j=1}^{n} \left(\, \alpha_j - c_j^d - w_j \,\right)$$

$$- \frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \sum_{j=1}^{2} v_j \left[\, w_1 - c_j^u \,\right] f_{j1}$$

$$+ \,[\, w_i - c_1^u\, f_{1i} - c_2^u\, f_{2i}\,] \left[-\frac{1}{2-\gamma} + \frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \right]$$

$$+ \sum_{\substack{j=1\\ j\neq i}}^{n} [\, w_j - c_1^u\, f_{1j} - c_2^u\, f_{2j}\,] \left[\frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \right] \;=\; 0$$

$$\Rightarrow \quad \frac{\gamma - \gamma\,[\,1-\gamma\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]} \left[\, \alpha_1 - w_1 - c_1^d \,\right] - \frac{2\,\gamma^2}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \sum_{i=1}^{n} \left(\, \alpha_i - c_i^d - w_i \,\right)$$

$$+ \frac{\gamma^3\,[\,n-1\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \sum_{i=1}^{2} v_i\,[\, w_1 - c_i^u \,]\, f_{i1} + \frac{\gamma \left[\, \alpha_1 - w_1 - c_1^d \,\right]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{1}{2-\gamma} \left[\, \alpha_i - w_i - c_i^d \,\right] - \frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \sum_{j=1}^{n} \left(\, \alpha_j - c_j^d - w_j \,\right)$$

$$- \frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \sum_{j=1}^{2} v_j\,[\,w_1 - c_j^u\,]\,f_{j1}$$

$$+ \,[\,w_i - c_1^u\,f_{1i} - c_2^u\,f_{2i}\,]\left[-\frac{1}{2-\gamma} + \frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \right]$$

$$+ \sum_{\substack{j=1\\ j\neq i}}^{n} [\,w_j - c_1^u\,f_{1j} - c_2^u\,f_{2j}\,]\left[\frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \right] \;=\; 0$$

$$\Rightarrow \quad \frac{2\,\gamma}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]}\,[\,\alpha_1 - w_1 - c_1^d\,] + \frac{1}{2-\gamma}\,[\,\alpha_i - w_i - c_i^d\,]$$

$$- \frac{2\,\gamma^2 + \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \sum_{i=1}^{n} (\,\alpha_i - c_i^d - w_i\,)$$

$$+ \frac{\gamma^3\,[\,n-1\,] - \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \sum_{i=1}^{2} v_i\,[\,w_1 - c_i^u\,]\,f_{i1}$$

$$+ \,[\,w_i - c_1^u\,f_{1i} - c_2^u\,f_{2i}\,]\left[-\frac{1}{2-\gamma} + \frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \right]$$

$$+ \sum_{\substack{j=1\\ j\neq i}}^{n} (\,w_j - c_1^u\,f_{1j} - c_2^u\,f_{2j}\,)\left[\frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \right] \;=\; 0$$

$$\Rightarrow \quad \frac{2\,\gamma}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]}\,[\,\alpha_1 - w_1 - c_1^d\,] + \frac{1}{2-\gamma}\,[\,\alpha_i - w_i - c_i^d\,]$$

$$- \frac{2\,\gamma^2 + \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \sum_{i=1}^{n} (\,\alpha_i - c_i^d - w_i\,)$$

$$+ \frac{\gamma^3\,[\,n-1\,] - \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \sum_{i=1}^{2} v_i\,[\,w_1 - c_i^u\,]\,f_{i1}$$

$$+ \,[\,w_i - c_1^u\,f_{1i} - c_2^u\,f_{2i}\,]\left[\frac{\gamma - 2 - \gamma\,(\,n-1\,)}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \right]$$

$$+ \sum_{\substack{j=1\\ j\neq i}}^{n} (\,w_j - c_1^u\,f_{1j} - c_2^u\,f_{2j}\,)\left[\frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \right] \;=\; 0$$

$$\Rightarrow \quad \left\{ \frac{1}{2-\gamma} - \frac{2\,\gamma^2 + \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} - \frac{\gamma - 2 - \gamma\,[\,n-1\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \right\} w_i$$

$$= \frac{2\,\gamma}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]} \left[\,\alpha_1 - w_1 - c_1^d\,\right] + \frac{1}{2-\gamma}\,\left[\,\alpha_i - c_i^d\,\right]$$

$$- \frac{2\,\gamma^2 + \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2}\,\left[\,\alpha_i - c_i^d\,\right]$$

$$- \frac{2\,\gamma^2 + \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \sum_{\substack{j=1\\ j\neq i}}^{n} \left(\,\alpha_j - c_j^d - w_j\,\right)$$

$$+ \frac{\gamma^3\,[\,n-1\,] - \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \sum_{i=1}^{2} v_i\,[\,w_1 - c_i^u\,]\,f_{i1}$$

$$- [\,c_1^u\,f_{1i} + c_2^u\,f_{2i}\,] \left[\frac{\gamma - 2 - \gamma\,(\,n-1\,)}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \right]$$

$$+ \sum_{\substack{j=1\\ j\neq i}}^{n} (\,w_j - c_1^u\,f_{1j} - c_2^u\,f_{2j}\,) \left[\frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \right]. \tag{34}$$

Observe that:

$$\frac{1}{2-\gamma} - \frac{2\,\gamma^2 + \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} - \frac{\gamma - 2 - \gamma\,[\,n-1\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$= \frac{1}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \{\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2 - 2\,\gamma^2 - \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]$$

$$- [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,\gamma - 2 - \gamma\,(\,n-1\,)\,]\,\}$$

$$= \frac{1}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \{\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-1\,) - \gamma$$

$$-\,\gamma + 2 + \gamma\,(\,n-1\,)\,] \,-\, 2\,\gamma^2\,\}$$

$$= \frac{2}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2}\,\{\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,] - \gamma^2\,\}. \tag{35}$$

(34) and (35) imply that for $i = 2, ..., n$:

$$\frac{2\{[2-\gamma][2+\gamma(n-1)][2+\gamma(n-2)]-\gamma^2\}}{[2-\gamma]^2[2+\gamma(n-1)]^2}\ w_i$$

$$= \frac{2\gamma}{[2-\gamma]^2[2+\gamma(n-1)]}[\alpha_1 - w_1 - c_1^d] + \frac{1}{2-\gamma}\ [\alpha_i - c_i^d]$$

$$- \frac{2\gamma^2+\gamma[2-\gamma][2+\gamma(n-1)]}{[2-\gamma]^2[2+\gamma(n-1)]^2}\ [\alpha_i - c_i^d]$$

$$- \frac{2\gamma^2+\gamma[2-\gamma][2+\gamma(n-1)]}{[2-\gamma]^2[2+\gamma(n-1)]^2}\sum_{\substack{j=1\\ j\neq i}}^{n}(\alpha_j - c_j^d - w_j)$$

$$+ \frac{\gamma^3[n-1]-\gamma[2-\gamma][2+\gamma(n-1)]}{[2-\gamma]^2[2+\gamma(n-1)]^2}\sum_{i=1}^{2} v_i[w_1 - c_i^u] f_{i1}$$

$$- [c_1^u f_{1i} + c_2^u f_{2i}]\left[\frac{\gamma-2-\gamma(n-1)}{[2-\gamma][2+\gamma(n-1)]}\right]$$

$$+ \sum_{\substack{j=1\\ j\neq i}}^{n}(w_j - c_1^u f_{1j} - c_2^u f_{2j})\left[\frac{\gamma}{[2-\gamma][2+\gamma(n-1)]}\right]. \qquad (36)$$

Observe that:

$$\frac{1}{2-\gamma} - \frac{2\gamma^2+\gamma[2-\gamma][2+\gamma(n-1)]}{[2-\gamma]^2[2+\gamma(n-1)]^2}$$

$$= \frac{1}{[2-\gamma]^2[2+\gamma(n-1)]^2}\{[2-\gamma][2+\gamma(n-1)]^2$$

$$-\ \gamma[2-\gamma][2+\gamma(n-1)] - 2\gamma^2\}$$

$$= \frac{1}{[2-\gamma]^2[2+\gamma(n-1)]^2}\{[2-\gamma][2+\gamma(n-1)][2+\gamma(n-1)-\gamma] - 2\gamma^2\}. \qquad (37)$$

(36) and (37) imply:

$$\frac{2\{[2-\gamma][2+\gamma(n-1)][2+\gamma(n-2)]-\gamma^2\}}{[2-\gamma]^2[2+\gamma(n-1)]^2}\ w_i$$

$$= \frac{2\gamma}{[2-\gamma]^2[2+\gamma(n-1)]}[\alpha_1 - w_1 - c_1^d]$$

$$+ \; \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-1\,)-\gamma\,]-2\,\gamma^2}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2}\;[\,\alpha_i-c_i^d\,]$$

$$- \; \frac{2\,\gamma^2+\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2}\sum_{\substack{j=1\\ j\neq i}}^{n}(\,\alpha_j-c_j^d-w_j\,)$$

$$+ \; \frac{\gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2}\sum_{i=1}^{2}v_i\,[\,w_1-c_i^u\,]\,f_{i1}$$

$$- \; [\,c_1^u\,f_{1i}+c_2^u\,f_{2i}\,]\left[\frac{\gamma-2-\gamma\,(\,n-1\,)}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\right]$$

$$+ \; \sum_{\substack{j=1\\ j\neq i}}^{n}(\,w_j-c_1^u\,f_{1j}-c_2^u\,f_{2j}\,)\left[\frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\right]$$

$$\Rightarrow \quad \frac{2\,\Omega}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\;w_i$$

$$= \; \frac{2\,\gamma}{2-\gamma}\;[\,\alpha_1-w_1-c_1^d\,]$$

$$+ \; \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]-2\,\gamma^2}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\;[\,\alpha_i-c_i^d\,]$$

$$- \; \frac{2\,\gamma^2+\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\sum_{\substack{j=1\\ j\neq i}}^{n}(\,\alpha_j-c_j^d-w_j\,)$$

$$+ \; \frac{\gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\sum_{i=1}^{2}v_i\,[\,w_1-c_i^u\,]\,f_{i1}$$

$$- \; [\,c_1^u\,f_{1i}+c_2^u\,f_{2i}\,]\,[\,\gamma-2-\gamma\,(\,n-1\,)\,]+\gamma\sum_{\substack{j=1\\ j\neq i}}^{n}(\,w_j-c_1^u\,f_{1j}-c_2^u\,f_{2j}\,) \tag{38}$$

where $\Omega \;\equiv\; [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]-\gamma^2\,.$ (39)

(38) implies that for $i \,=\, 2,...,n$:

$$w_i \;=\; \frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega}\,[\,\alpha_1-w_1-c_1^d\,]$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)][2+\gamma(n-2)] - 2\gamma^2}{2\Omega} \; [\alpha_i - c_i^d]$$

$$- \frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)]}{2\Omega} \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_j - c_j^d - w_j)$$

$$+ \frac{\gamma^3[n-1] - \gamma[2-\gamma][2+\gamma(n-1)]}{2\Omega} \sum_{i=1}^{2} v_i [w_1 - c_i^u] f_{i1}$$

$$- \frac{[2-\gamma][2+\gamma(n-1)][\gamma - 2 - \gamma(n-1)]}{2\Omega} [c_1^u f_{1i} + c_2^u f_{2i}]$$

$$+ \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{2\Omega} \sum_{\substack{j=1 \\ j \neq i}}^{n} (w_j - c_1^u f_{1j} - c_2^u f_{2j}) . \tag{40}$$

Observe that:

$$\frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)]}{2\Omega} + \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{2\Omega}$$

$$= \frac{\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)]}{\Omega} . \tag{41}$$

(40) and (41) imply that for $i \in \{2, ..., n\}$:

$$w_i = \frac{\gamma[2+\gamma(n-1)]}{\Omega} [\alpha_1 - w_1 - c_1^d]$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)][2-\gamma+\gamma(n-1)] - 2\gamma^2}{2\Omega} \; [\alpha_i - c_i^d]$$

$$- \frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)]}{2\Omega} \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_j - c_j^d)$$

$$+ \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{\Omega} \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j + \frac{\gamma^2}{\Omega} \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j$$

$$+ \frac{\gamma^3[n-1] - \gamma[2-\gamma][2+\gamma(n-1)]}{2\Omega} \sum_{i=1}^{2} v_i [w_1 - c_i^u] f_{i1}$$

$$- \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{2\Omega} [c_1^u f_{1i} + c_2^u f_{2i}]$$

$$+\ \frac{[2-\gamma][2+\gamma(n-1)][2+\gamma(n-1)]}{2\,\Omega}[\,c_1^u f_{1i}+c_2^u f_{2i}\,]$$

$$-\ \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{2\,\Omega}\sum_{\substack{j=1\\ j\neq i}}^{n}(\,c_1^u f_{1j}+c_2^u f_{2j}\,)$$

$$\Rightarrow\quad w_i\ =\ \frac{\gamma[2+\gamma(n-1)]}{\Omega}\left[\alpha_1-w_1-c_1^d\right]+\frac{[2-\gamma][2+\gamma(n-1)]^2}{2\,\Omega}\left[\alpha_i-c_i^d\right]$$

$$-\ \frac{\gamma[2-\gamma][2+\gamma(n-1)]+2\,\gamma^2}{2\,\Omega}\left[\alpha_i-c_i^d\right]$$

$$-\ \frac{2\,\gamma^2+\gamma[2-\gamma][2+\gamma(n-1)]}{2\,\Omega}\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\alpha_j-c_j^d\right)$$

$$+\ \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{\Omega}\sum_{\substack{j=1\\ j\neq i}}^{n}w_j+\frac{\gamma^2}{\Omega}\sum_{\substack{j=1\\ j\neq i}}^{n}w_j$$

$$+\ \frac{\gamma^3[n-1]-\gamma[2-\gamma][2+\gamma(n-1)]}{2\,\Omega}\sum_{i=1}^{2}v_i[\,w_1-c_i^u\,]f_{i1}$$

$$+\ \frac{[2-\gamma][2+\gamma(n-1)][2+\gamma(n-1)]}{2\,\Omega}[\,c_1^u f_{1i}+c_2^u f_{2i}\,]$$

$$-\ \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{2\,\Omega}\sum_{j=1}^{n}(\,c_1^u f_{1j}+c_2^u f_{2j}\,)$$

$$\Rightarrow\quad w_i\ =\ \frac{\gamma[2+\gamma(n-1)]}{\Omega}\left[\alpha_1-w_1-c_1^d\right]+\frac{[2-\gamma][2+\gamma(n-1)]^2}{2\,\Omega}\left[\alpha_i-c_i^d\right]$$

$$-\ \frac{2\,\gamma^2+\gamma[2-\gamma][2+\gamma(n-1)]}{2\,\Omega}\sum_{j=1}^{n}\left(\alpha_j-c_j^d\right)$$

$$+\ \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{\Omega}\sum_{\substack{j=1\\ j\neq i}}^{n}w_j+\frac{\gamma^2}{\Omega}\sum_{\substack{j=1\\ j\neq i}}^{n}w_j$$

$$+\ \frac{\gamma^3[n-1]-\gamma[2-\gamma][2+\gamma(n-1)]}{2\,\Omega}\sum_{i=1}^{2}v_i[\,w_1-c_i^u\,]f_{i1}$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Omega}[c_1^u f_{1i} + c_2^u f_{2i}]$$

$$- \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{2\Omega} \sum_{j=1}^{n} (c_1^u f_{1j} + c_2^u f_{2j}) \quad \text{for } i = 2, ..., n. \tag{42}$$

Now consider U1 and U2's profit-maximizing choice of w_1 under vertical integration.[2] Differentiating (33) with respect to w_1, using (17), (18), and (32), provides:

$$q_1\left[\frac{\partial P_1(\cdot)}{\partial w_1} - 1\right] + [P_1 - w_1 - c_1^d]\frac{\partial q_1(\cdot)}{\partial w_1} + q_1 + [w_1 - c_1^u f_{11} - c_2^u f_{21}]\frac{\partial q_1(\cdot)}{\partial w_1}$$

$$+ \sum_{j=2}^{n}(w_j - c_1^u f_{1j} - c_2^u f_{2j})\frac{\partial q_j(\cdot)}{\partial w_1} = 0$$

$$\Rightarrow \quad q_1\frac{\partial P_1(\cdot)}{\partial w_1} + [P_1 - c_1^d - c_1^u f_{11} - c_2^u f_{21}]\frac{\partial q_1(\cdot)}{\partial w_1} + \sum_{j=2}^{n}(w_j - c_1^u f_{1j} - c_2^u f_{2j})\frac{\partial q_j(\cdot)}{\partial w_1} = 0$$

$$\Rightarrow \quad q_1\left[\frac{1-\gamma}{2-\gamma} + \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]} - \frac{2-\gamma+\gamma[n-1]-\gamma^2[n-1]}{[2-\gamma][2+\gamma(n-1)]}\sum_{i=1}^{2} v_i f_{i1}\right]$$

$$+ [P_1 - c_1^d - c_1^u f_{11} - c_2^u f_{21}]\Bigg[-\frac{1}{2-\gamma} + \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]}$$

$$+ \frac{2-\gamma+\gamma[n-1]}{[2-\gamma][2+\gamma(n-1)]}\sum_{i=1}^{2} v_i f_{i1}\Bigg]$$

$$+ \sum_{j=2}^{n}(w_j - c_1^u f_{1j} - c_2^u f_{2j})\Bigg[\frac{\gamma}{[2-\gamma][2+\gamma(n-1)]}$$

$$- \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]}\sum_{i=1}^{2} v_i f_{i1}\Bigg] = 0. \tag{43}$$

Observe that:

$$\frac{1-\gamma}{2-\gamma} + \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]} - \frac{2-\gamma+\gamma[n-1]-\gamma^2[n-1]}{[2-\gamma][2+\gamma(n-1)]}\sum_{i=1}^{2} v_i f_{i1}$$

[2] The choice of w_1 does not affect equilibrium outcomes when $v_1 = v_2 = 1$ and $f_{1i} = f_{2i}$ for all $i = 1, ..., n$. However. the choice of w_1 might affect equilibrium outcomes more generally.

$$= \frac{2-\gamma+\gamma[n-1]-\gamma^2[n-1]}{[2-\gamma][2+\gamma(n-1)]} - \frac{2-\gamma+\gamma[n-1]-\gamma^2[n-1]}{[2-\gamma][2+\gamma(n-1)]} \sum_{i=1}^{2} v_i f_{i1}$$

$$= \frac{2-\gamma+\gamma[n-1]-\gamma^2[n-1]}{[2-\gamma][2+\gamma(n-1)]} \left[1 - \sum_{i=1}^{2} v_i f_{i1} \right] . \qquad (44)$$

Further observe that:

$$- \frac{1}{2-\gamma} + \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]} + \frac{2-\gamma+\gamma[n-1]}{[2-\gamma][2+\gamma(n-1)]} \sum_{i=1}^{2} v_i f_{i1}$$

$$= - \frac{2-\gamma+\gamma[n-1]}{[2-\gamma][2+\gamma(n-1)]} + \frac{2-\gamma+\gamma[n-1]}{[2-\gamma][2+\gamma(n-1)]} \sum_{i=1}^{2} v_i f_{i1}$$

$$= - \frac{2-\gamma+\gamma[n-1]}{[2-\gamma][2+\gamma(n-1)]} \left[1 - \sum_{i=1}^{2} v_i f_{i1} \right] . \qquad (45)$$

In addition:

$$\frac{\gamma}{[2-\gamma][2+\gamma(n-1)]} - \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]} \sum_{i=1}^{2} v_i f_{i1}$$

$$= \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]} \left[1 - \sum_{i=1}^{2} v_i f_{i1} \right] . \qquad (46)$$

Substituting for q_1 from (17), substituting for P_1 from (32), and using (44) – (46) allows (43) to be written as:

$$\frac{2-\gamma+\gamma[n-1]-\gamma^2[n-1]}{[2-\gamma][2+\gamma(n-1)]} \left[1 - \sum_{i=1}^{2} v_i f_{i1} \right] q_1$$

$$- \frac{2-\gamma+\gamma[n-1]}{[2-\gamma][2+\gamma(n-1)]} \left[1 - \sum_{i=1}^{2} v_i f_{i1} \right] [P_1 - c_1^d - c_1^u f_{11} - c_2^u f_{21}]$$

$$+ \frac{\gamma}{[2-\gamma][2+\gamma(n-1)]} \left[1 - \sum_{i=1}^{2} v_i f_{i1} \right] \sum_{j=2}^{n} (w_j - c_1^u f_{1j} - c_2^u f_{2j}) = 0$$

$$\Rightarrow \frac{2-\gamma+\gamma[n-1]-\gamma^2[n-1]}{[2-\gamma][2+\gamma(n-1)]} \left[1 - \sum_{i=1}^{2} v_i f_{i1} \right] \frac{1}{2-\gamma} [\alpha_1 - w_1 - c_1^d]$$

$$- \frac{2-\gamma+\gamma\,[\,n-1\,]-\gamma^2\,[\,n-1\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \left[1 - \sum_{i=1}^{2} v_i\, f_{i1} \right] \gamma \sum_{i=1}^{n} \left(\alpha_i - c_i^d - w_i \right)$$

$$+ \frac{[\,2-\gamma+\gamma\,(\,n-1\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2-\gamma+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2}$$

$$\cdot \left[1 - \sum_{i=1}^{2} v_i\, f_{i1} \right] \sum_{i=1}^{2} v_i\,[\,w_1 - c_i^u\,]\, f_{i1}$$

$$- \frac{2-\gamma+\gamma\,[\,n-1\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1 - \sum_{i=1}^{2} v_i\, f_{i1} \right] \alpha_1$$

$$+ \frac{2-\gamma+\gamma\,[\,n-1\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1 - \sum_{i=1}^{2} v_i\, f_{i1} \right] \frac{1-\gamma}{2-\gamma}\,[\,\alpha_1 - w_1 - c_1^d\,]$$

$$+ \frac{2-\gamma+\gamma\,[\,n-1\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \left[1 - \sum_{i=1}^{2} v_i\, f_{i1} \right] \gamma \sum_{i=1}^{n} \left(\alpha_i - c_i^d - w_i \right)$$

$$+ \frac{[\,2-\gamma+\gamma\,(\,n-1\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2-\gamma+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2}$$

$$\cdot \left[1 - \sum_{i=1}^{2} v_i f_{i1} \right] \sum_{i=1}^{2} v_i\,[\,w_1 - c_i^u\,]\, f_{i1}$$

$$+ \frac{2-\gamma+\gamma\,[\,n-1\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1 - \sum_{i=1}^{2} v_i\, f_{i1} \right] [\,c_1^d + c_1^u\, f_{11} + c_2^u\, f_{21}\,]$$

$$+ \frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1 - \sum_{i=1}^{2} v_i\, f_{i1} \right] \sum_{j=2}^{n} (\,w_j - c_1^u\, f_{1j} - c_2^u\, f_{2j}\,) \;=\; 0$$

$$\Rightarrow \quad \frac{2-\gamma+\gamma\,[\,n-1\,]-\gamma^2\,[\,n-1\,]+[\,1-\gamma\,]\,[\,2-\gamma+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$\cdot \left[1 - \sum_{i=1}^{2} v_i f_{i1} \right] \frac{1}{2-\gamma}\,[\,\alpha_1 - w_1 - c_1^d\,]$$

$$+ \frac{\gamma^2\,[\,n-1\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \left[1 - \sum_{i=1}^{2} v_i f_{i1} \right] \gamma \sum_{i=1}^{n} \left(\alpha_i - c_i^d - w_i \right)$$

$$+ \frac{2\,[\,2-\gamma+\gamma\,(\,n-1\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2-\gamma+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2}$$

$$\cdot \left[1 - \sum_{i=1}^{2} v_i f_{i1} \right] \sum_{i=1}^{2} v_i\,[\,w_1 - c_i^u\,]\,f_{i1}$$

$$- \frac{2-\gamma+\gamma\,[\,n-1\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1 - \sum_{i=1}^{2} v_i\,f_{i1} \right] [\,\alpha_1 - c_1^d - c_1^u\,f_{11} - c_2^u\,f_{21}\,]$$

$$+ \frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1 - \sum_{i=1}^{2} v_i\,f_{i1} \right] \sum_{j=2}^{n} (\,w_j - c_1^u\,f_{1j} - c_2^u\,f_{2j}\,) \;=\; 0$$

$$\Rightarrow \quad \frac{[\,2-\gamma\,]\,[\,2-\gamma+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1 - \sum_{i=1}^{2} v_i\,f_{i1} \right] \frac{1}{2-\gamma}\,[\,\alpha_1 - w_1 - c_1^d\,]$$

$$+ \frac{\gamma^2\,[\,n-1\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \left[1 - \sum_{i=1}^{2} v_i\,f_{i1} \right] \gamma \sum_{i=1}^{n} \left(\,\alpha_i - c_i^d - w_i\,\right)$$

$$+ \frac{2\,[\,2-\gamma+\gamma\,(\,n-1\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2-\gamma+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2}$$

$$\cdot \left[1 - \sum_{i=1}^{2} v_i\,f_{i1} \right] \sum_{i=1}^{2} v_i\,[\,w_1 - c_i^u\,]\,f_{i1}$$

$$- \frac{2-\gamma+\gamma\,[\,n-1\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1 - \sum_{i=1}^{2} v_i\,f_{i1} \right] [\,\alpha_1 - c_1^d - c_1^u\,f_{11} - c_2^u\,f_{21}\,]$$

$$+ \frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1 - \sum_{i=1}^{2} v_i\,f_{i1} \right] \sum_{j=2}^{n} (\,w_j - c_1^u\,f_{1j} - c_2^u\,f_{2j}\,) \;=\; 0\,. \tag{47}$$

(47) implies:

$$\left[1 - \sum_{i=1}^{2} v_i f_{i1} \right] w_1 \left\{ \frac{[\,2-\gamma\,]\,[\,2-\gamma+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]} \right.$$

$$\left. - \frac{2\,[\,2-\gamma+\gamma\,(\,n-1\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2-\gamma+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \sum_{i=1}^{2} v_i\,f_{i1} \right\}$$

$$= \frac{[\,2-\gamma\,]\,[\,2-\gamma+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1-\sum_{i=1}^{2} v_i\, f_{i1}\right] \frac{1}{2-\gamma}\,[\,\alpha_1 - c_1^d\,]$$

$$+ \frac{\gamma^2\,[\,n-1\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \left[1-\sum_{i=1}^{2} v_i\, f_{i1}\right] \gamma \sum_{i=1}^{n} (\,\alpha_i - c_i^d - w_i\,)$$

$$- \frac{2\,[\,2-\gamma+\gamma\,(\,n-1\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2-\gamma+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \left[1-\sum_{i=1}^{2} v_i\, f_{i1}\right] \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$- \frac{2-\gamma+\gamma\,[\,n-1\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1-\sum_{i=1}^{2} v_i\, f_{i1}\right] [\,\alpha_1 - c_1^d - c_1^u\, f_{11} - c_2^u\, f_{21}\,]$$

$$+ \frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1-\sum_{i=1}^{2} v_i\, f_{i1}\right] \sum_{j=2}^{n} (\,w_j - c_1^u\, f_{1j} - c_2^u\, f_{2j}\,)$$

$$\Rightarrow \quad \left[1-\sum_{i=1}^{2} v_i\, f_{i1}\right] w_1$$

$$\cdot \Bigg\{ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-2\,)\,]-\gamma^2\,[\,n-1\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]}$$

$$- \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \sum_{i=1}^{2} v_i\, f_{i1} \Bigg\}$$

$$= \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-2\,)\,]-\gamma^2\,[\,n-1\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1-\sum_{i=1}^{2} v_i\, f_{i1}\right] [\,\alpha_1 - c_1^d\,]$$

$$+ \frac{\gamma^2\,[\,n-1\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \left[1-\sum_{i=1}^{2} v_i f_{i1}\right] \gamma \sum_{i=1}^{n} (\,\alpha_i - c_i^d - w_i\,)$$

$$- \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \left[1-\sum_{i=1}^{2} v_i\, f_{i1}\right] \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$- \frac{2+\gamma\,[\,n-2\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1-\sum_{i=1}^{2} v_i\, f_{i1}\right] [\,\alpha_1 - c_1^d\,]$$

$$+ \frac{2+\gamma\,[\,n-2\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1-\sum_{i=1}^{2} v_i\, f_{i1}\right] [\,c_1^u\, f_{11} + c_2^u\, f_{21}\,]$$

$$+ \frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[1 - \sum_{i=1}^{2} v_i\, f_{i1} \right] \sum_{j=2}^{n} (\, w_j - c_1^u\, f_{1j} - c_2^u\, f_{2j}\,)\,. \tag{48}$$

Dividing each of the terms in (48) by $1 - \sum_{i=1}^{2} v_i\, f_{i1}$ (assuming $\sum_{i=1}^{2} v_i\, f_{i1} \neq 1$) provides:

$$w_1 \left\{ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-2\,)\,] - \gamma^2\,[\,n-1\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]} - \frac{2\,[\,2+\gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \sum_{i=1}^{2} v_i f_{i1} \right\}$$

$$= \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-2\,)\,] - \gamma^2\,[\,n-1\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]} \left[\, \alpha_1 - c_1^d \,\right]$$

$$+ \frac{\gamma^2\,[\,n-1\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2}\; \gamma \sum_{i=1}^{n} \left(\, \alpha_i - c_i^d - w_i \,\right)$$

$$- \frac{2\,[\,2+\gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]^2} \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$- \frac{2+\gamma\,[\,n-2\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[\, \alpha_1 - c_1^d \,\right]$$

$$+ \frac{2+\gamma\,[\,n-2\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \left[\, c_1^u\, f_{11} + c_2^u\, f_{21} \,\right]$$

$$+ \frac{\gamma}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \sum_{j=2}^{n} (\, w_j - c_1^u\, f_{1j} - c_2^u\, f_{2j}\,)\,. \tag{49}$$

Combining the terms involving $\left[\, \alpha_1 - c_1^d \,\right]$ and then multiplying each term in (49) by $[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]$ provides:

$$w_1 \left\{ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-2\,)\,] - \gamma^2\,[\,n-1\,]}{2-\gamma} - \frac{2\,[\,2+\gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \sum_{i=1}^{2} v_i\, f_{i1} \right\}$$

$$= - \frac{\gamma^2\,[\,n-1\,]}{2-\gamma} \left[\, \alpha_1 - c_1^d \,\right] + \frac{\gamma^2\,[\,n-1\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\; \gamma \sum_{i=1}^{n} \left(\, \alpha_i - c_i^d - w_i \,\right)$$

$$- \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]} \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$+ \; [\,2+\gamma\,(\,n-2\,)\,]\;[\,c_1^u\, f_{11} + c_2^u\, f_{21}\,] + \gamma \sum_{j=2}^{n} (\,w_j - c_1^u\, f_{1j} - c_2^u\, f_{2j}\,)\,. \tag{50}$$

Multiplying each term in (50) by $[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]$ provides:

$$w_1 \{\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-1\,)\,] - \gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]$$

$$- 2\left[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\right]\,[\,2+\gamma\,(\,n-2\,)\,] \sum_{i=1}^{2} v_i\, f_{i1}\,\}$$

$$= \; - \; \gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]\left[\,\alpha_1 - c_1^d\,\right] + \; \gamma^3\,[\,n-1\,] \sum_{i=1}^{n} \left(\,\alpha_i - c_i^d - w_i\,\right)$$

$$- 2\left[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\right]\,[\,2+\gamma\,(\,n-2\,)\,] \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$+ \; [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]\;[\,c_1^u\, f_{11} + c_2^u\, f_{21}\,]$$

$$+ \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] \sum_{j=2}^{n} (\,w_j - c_1^u\, f_{1j} - c_2^u\, f_{2j}\,)$$

$$\Rightarrow \quad w_1 \left\{\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-1\,)\,] - \gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,\right\}$$

$$= \; -\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]\left[\,\alpha_1 - c_1^d\,\right] + \; \gamma^3\,[\,n-1\,] \sum_{i=1}^{n} \left(\,\alpha_i - c_i^d - w_i\,\right)$$

$$+ 2\left[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\right]\,[\,2+\gamma\,(\,n-2\,)\,] \sum_{i=1}^{2} v_i\,[\,w_1 - c_i^u\,]\, f_{i1}$$

$$+ \; [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]\;[\,c_1^u\, f_{11} + c_2^u\, f_{21}\,]$$

$$+ \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] \sum_{j=2}^{n} (\,w_j - c_1^u\, f_{1j} - c_2^u\, f_{2j}\,)$$

$$\Rightarrow \quad w_1\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-1\,)\,]$$

$$= \; - \; \gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]\left[\,\alpha_1 - w_1 - c_1^d\,\right] + \gamma^3\,[\,n-1\,] \sum_{i=1}^{n} \left(\,\alpha_i - c_i^d - w_i\,\right)$$

$$+\,2\left[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\right][\,2+\gamma\,(\,n-2\,)\,]\sum_{i=1}^{2}v_i\,[\,w_1-c_i^u\,]\,f_{i1}$$

$$+\;[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]\;[\,c_1^u\,f_{11}+c_2^u\,f_{21}\,]$$

$$+\,\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\sum_{j=2}^{n}(\,w_j-c_1^u\,f_{1j}-c_2^u\,f_{2j}\,)$$

$$\Rightarrow\quad w_1\left\{\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,\right\}$$

$$=\;-\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]\left[\,\alpha_1-w_1-c_1^d\,\right]-\gamma^2\,w_1+\gamma^3\,[\,n-1\,]\sum_{i=1}^{n}\left(\,\alpha_i-c_i^d-w_i\,\right)$$

$$+\,2\left[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\right][\,2+\gamma\,(\,n-2\,)\,]\sum_{i=1}^{2}v_i\,[\,w_1-c_i^u\,]\,f_{i1}$$

$$+\;[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]\;[\,c_1^u\,f_{11}+c_2^u\,f_{21}\,]$$

$$+\,\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\sum_{j=2}^{n}(\,w_j-c_1^u\,f_{1j}-c_2^u\,f_{2j}\,)$$

$$\Rightarrow\quad w_1\;=\;-\,\frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega}\left[\,\alpha_1-w_1-c_1^d\,\right]-\frac{\gamma^2}{\Omega}\,w_1$$

$$+\;\frac{\gamma^3\,[\,n-1\,]}{\Omega}\sum_{i=1}^{n}\left(\,\alpha_i-c_i^d-w_i\,\right)$$

$$+\;\frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Omega}\sum_{i=1}^{2}v_i\,[\,w_1-c_i^u\,]\,f_{i1}$$

$$+\;\frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Omega}\;[\,c_1^u\,f_{11}+c_2^u\,f_{21}\,]$$

$$+\;\frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega}\sum_{j=2}^{n}(\,w_j-c_1^u\,f_{1j}-c_2^u\,f_{2j}\,)$$

$$\Rightarrow\quad w_1\;=\;-\;\frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega}\left[\,\alpha_1-w_1-c_1^d\,\right]-\frac{\gamma^2}{\Omega}\,w_1$$

$$+\;\frac{\gamma^3\,[\,n-1\,]}{\Omega}\sum_{i=1}^{n}\left(\,\alpha_i-c_i^d-w_i\,\right)$$

$$+ \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Omega} \sum_{i=1}^{2} v_i\,[\,w_1 - c_i^u\,]\,f_{i1}$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,2-\gamma+\gamma\,(\,n-1\,)\,]}{\Omega}\,[\,c_1^u\,f_{11} + c_2^u\,f_{21}\,]$$

$$+ \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega} \sum_{j=2}^{n} (\,w_j - c_1^u\,f_{1j} - c_2^u\,f_{2j}\,)$$

$$\Rightarrow \quad w_1 \;=\; -\frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega}\left[\,\alpha_1 - w_1 - c_1^d\,\right] - \frac{\gamma^2}{\Omega}\,w_1$$

$$+ \frac{\gamma^3\,[\,n-1\,]}{\Omega} \sum_{i=1}^{n} \left(\,\alpha_j - c_j^d - w_j\,\right) + \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega} \sum_{j=2}^{n} w_j$$

$$+ \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Omega} \sum_{i=1}^{2} v_i\,[\,w_1 - c_i^u\,]\,f_{i1}$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{\Omega}\,[\,c_1^u\,f_{11} + c_2^u\,f_{21}\,]$$

$$- \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega}\,[\,c_1^u\,f_{11} + c_2^u\,f_{21}\,]$$

$$- \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega} \sum_{j=2}^{n} (\,c_1^u\,f_{1j} + c_2^u\,f_{2j}\,)$$

$$\Rightarrow \quad w_1 \;=\; -\frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega}\left[\,\alpha_1 - w_1 - c_1^d\,\right] + \frac{\gamma^3\,[\,n-1\,]}{\Omega} \sum_{j=1}^{n} \left(\,\alpha_j - c_j^d - w_j\,\right)$$

$$+ \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega} \sum_{j=2}^{n} w_j - \frac{\gamma^2}{\Omega}\,w_1$$

$$+ \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Omega} \sum_{i=1}^{2} v_i\,[\,w_1 - c_i^u\,]\,f_{i1}$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{\Omega}\,[\,c_1^u\,f_{11} + c_2^u\,f_{21}\,]$$

$$- \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega} \sum_{j=1}^{n} (\,c_1^u\,f_{1j} + c_2^u\,f_{2j}\,)\,. \tag{51}$$

Summing (42) and (51) provides:

$$
\begin{aligned}
\sum_{i=1}^{n} w_i \;=\; & \frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]}{\Omega}\,[\,\alpha_1 - w_1 - c_1^d\,] \\
& + \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Omega}\sum_{i=2}^{n}(\,\alpha_i - c_i^d\,) \\
& - \frac{2\,\gamma^2 + \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\,\Omega}\,[\,n-1\,]\sum_{j=1}^{n}(\,\alpha_j - c_j^d\,) \\
& + \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega}\sum_{i=2}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} w_j + \frac{\gamma^2}{\Omega}\sum_{i=2}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} w_j \\
& + \frac{\gamma^3\,[\,n-1\,] - \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\,\Omega}\,[\,n-1\,]\sum_{i=1}^{2} v_i\,[\,w_1 - c_i^u\,]\,f_{i1} \\
& + \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Omega}\sum_{i=2}^{n}(\,c_1^u\,f_{1i} + c_2^u\,f_{2i}\,) \\
& - \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]}{2\,\Omega}\sum_{j=1}^{n}(\,c_1^u\,f_{1j} + c_2^u\,f_{2j}\,) \\
& - \frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega}\,[\,\alpha_1 - w_1 - c_1^d\,] + \frac{\gamma^3\,[\,n-1\,]}{\Omega}\sum_{j=1}^{n}\left(\,\alpha_j - c_j^d - w_j\,\right) \\
& + \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega}\sum_{j=2}^{n} w_j - \frac{\gamma^2}{\Omega}\,w_1 \\
& + \frac{2\,[\,2+\gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Omega}\sum_{i=1}^{2} v_i\,[\,w_1 - c_i^u\,]\,f_{i1} \\
& + \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{\Omega}\,[\,c_1^u\,f_{11} + c_2^u\,f_{21}\,] \\
& - \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega}\sum_{j=1}^{n}(\,c_1^u\,f_{1j} + c_2^u\,f_{2j}\,) \\
=\; & \frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]\,[\,1-\gamma\,]}{\Omega}\,[\,\alpha_1 - w_1 - c_1^d\,]
\end{aligned}
$$

$$+ \frac{\gamma^3 [\, n-1 \,]}{\Omega} \sum_{j=1}^{n} (\, \alpha_j - c_j^d \,) - \frac{\gamma^3 [\, n-1 \,]}{\Omega} \sum_{i=1}^{n} w_i$$

$$+ \frac{[\, 2-\gamma \,] [\, 2+\gamma (\, n-1 \,) \,]^2}{2\,\Omega} \sum_{i=2}^{n} (\, \alpha_i - c_i^d \,)$$

$$- \frac{2\,\gamma^2 + \gamma [\, 2-\gamma \,] [\, 2+\gamma (\, n-1 \,) \,]}{2\,\Omega} [\, n-1 \,] \sum_{j=1}^{n} (\, \alpha_j - c_j^d \,)$$

$$+ \frac{\gamma [\, 2-\gamma \,] [\, 2+\gamma (\, n-1 \,) \,]}{\Omega} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j + \frac{\gamma^2}{\Omega} \sum_{i=2}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j - \frac{\gamma^2}{\Omega} w_1$$

$$+ \frac{\gamma^3 [\, n-1 \,] - \gamma [\, 2-\gamma \,] [\, 2+\gamma (\, n-1 \,) \,]}{2\,\Omega} [\, n-1 \,] \sum_{i=1}^{2} v_i [\, w_1 - c_i^u \,] f_{i1}$$

$$+ \frac{2 [\, 2+\gamma (\, n-2 \,) - \gamma^2 (\, n-1 \,) \,] [\, 2+\gamma (\, n-2 \,) \,]}{\Omega} \sum_{i=1}^{2} v_i [\, w_1 - c_i^u \,] f_{i1}$$

$$+ \frac{[\, 2-\gamma \,] [\, 2+\gamma (\, n-1 \,) \,]^2}{2\,\Omega} \sum_{i=1}^{n} (\, c_1^u f_{1i} + c_2^u f_{2i} \,)$$

$$+ \frac{[\, 2-\gamma \,] [\, 2+\gamma (\, n-1 \,) \,]^2}{2\,\Omega} [\, c_1^u f_{11} + c_2^u f_{21} \,]$$

$$- \frac{\gamma [\, n+1 \,] [\, 2-\gamma \,] [\, 2+\gamma (\, n-1 \,) \,]}{2\,\Omega} \sum_{j=1}^{n} (\, c_1^u f_{1j} + c_2^u f_{2j} \,) \, .$$

The first term in the fifth line of the expression immediately above is the sum of the first term in the fourth line and the first term in the ninth line of the preceding expression. This is the case because:

$$\sum_{i=2}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j + \sum_{j=2}^{n} w_j \;=\; \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j \, .$$

The last term immediately above is the sum of the seventh and the twelfth lines in the preceding expression. The two terms before the last immediately above are the sum of the sixth and the eleventh lines in the preceding expression.

Continuing:

$$\sum_{i=1}^{n} w_i \;=\; \frac{\gamma [\, 2+\gamma (\, n-1 \,) \,] [\, n-1 \,] [\, 1-\gamma \,]}{\Omega} [\, \alpha_1 - w_1 - c_1^d \,] - \frac{\gamma^3 [\, n-1 \,]}{\Omega} \sum_{i=1}^{n} w_i$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Omega} \sum_{i=2}^{n} (\alpha_i - c_i^d)$$

$$- \frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)] - 2\gamma^3}{2\Omega} [n-1] \sum_{j=1}^{n} (\alpha_j - c_j^d)$$

$$+ \frac{\gamma[2-\gamma][2+\gamma(n-1)][n-1]}{\Omega} \sum_{i=1}^{n} w_i$$

$$+ \frac{\gamma^2}{\Omega} \sum_{i=2}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j - \frac{\gamma^2}{\Omega} w_1$$

$$+ \frac{\gamma^3[n-1] - \gamma[2-\gamma][2+\gamma(n-1)]}{2\Omega} [n-1] \sum_{i=1}^{2} v_i [w_1 - c_i^u] f_{i1}$$

$$+ \frac{2[2+\gamma(n-2) - \gamma^2(n-1)][2+\gamma(n-2)]}{\Omega} \sum_{i=1}^{2} v_i [w_1 - c_i^u] f_{i1}$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)][2+\gamma(n-1) - \gamma(n+1)]}{2\Omega} \sum_{i=1}^{n} (c_1^u f_{1i} + c_2^u f_{2i})$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Omega} [c_1^u f_{11} + c_2^u f_{21}]$$

$$= \frac{\gamma[2+\gamma(n-1)][n-1][1-\gamma]}{\Omega} [\alpha_1 - w_1 - c_1^d] - \frac{\gamma^3[n-1]}{\Omega} \sum_{i=1}^{n} w_i$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Omega} \sum_{i=2}^{n} (\alpha_i - c_i^d)$$

$$- \frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)] - 2\gamma^3}{2\Omega} [n-1] \sum_{j=1}^{n} (\alpha_j - c_j^d)$$

$$+ \frac{\gamma[2-\gamma][2+\gamma(n-1)][n-1]}{\Omega} \sum_{i=1}^{n} w_i$$

$$+ \frac{\gamma^2}{\Omega} \sum_{i=2}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j - \frac{\gamma^2}{\Omega} \left[\sum_{i=1}^{n} w_i - \sum_{j=2}^{n} w_j \right]$$

$$+ \frac{\gamma^3 [\, n-1\,] - \gamma\,[\, 2-\gamma\,]\,[\, 2+\gamma\,(\, n-1\,)\,]}{2\,\Omega}\,[\, n-1\,] \sum_{i=1}^{2} v_i\,[\, w_1 - c_i^u\,]\, f_{i1}$$

$$+ \frac{2\,[\, 2+\gamma\,(\, n-2\,) - \gamma^2\,(\, n-1\,)\,]\,[\, 2+\gamma\,(\, n-2\,)\,]}{\Omega} \sum_{i=1}^{2} v_i\,[\, w_1 - c_i^u\,]\, f_{i1}$$

$$+ \frac{[\, 2-\gamma\,]\,[\, 2+\gamma\,(\, n-1\,)\,]\,[\, 2+\gamma\,(\, n-1\,) - \gamma\,(\, n+1\,)\,]}{2\,\Omega} \sum_{i=1}^{n} (\, c_1^u\, f_{1i} + c_2^u\, f_{2i}\,)$$

$$+ \frac{[\, 2-\gamma\,]\,[\, 2+\gamma\,(\, n-1\,)\,]^2}{2\,\Omega}\,[\, c_1^u\, f_{11} + c_2^u\, f_{21}\,]$$

$$= \frac{\gamma\,[\, 2+\gamma\,(\, n-1\,)\,]\,[\, n-1\,]\,[\, 1-\gamma\,]}{\Omega}\,\left[\, \alpha_1 - w_1 - c_1^d\,\right] - \frac{\gamma^3\,[\, n-1\,]}{\Omega} \sum_{i=1}^{n} w_i$$

$$+ \frac{[\, 2-\gamma\,]\,[\, 2+\gamma\,(\, n-1\,)\,]^2}{2\,\Omega} \sum_{i=2}^{n} \left(\, \alpha_i - c_i^d\,\right)$$

$$- \frac{2\,\gamma^2 + \gamma\,[\, 2-\gamma\,]\,[\, 2+\gamma\,(\, n-1\,)\,] - 2\,\gamma^3}{2\,\Omega}\,[\, n-1\,] \sum_{j=1}^{n} \left(\, \alpha_j - c_j^d\,\right)$$

$$+ \frac{\gamma\,[\, 2-\gamma\,]\,[\, 2+\gamma\,(\, n-1\,)\,]\,[\, n-1\,]}{\Omega} \sum_{i=1}^{n} w_i$$

$$+ \frac{\gamma^2}{\Omega} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j - \frac{\gamma^2}{\Omega} \sum_{i=1}^{n} w_i$$

$$+ \frac{\gamma^3 [\, n-1\,] - \gamma\,[\, 2-\gamma\,]\,[\, 2+\gamma\,(\, n-1\,)\,]}{2\,\Omega}\,[\, n-1\,] \sum_{i=1}^{2} v_i\,[\, w_1 - c_i^u\,]\, f_{i1}$$

$$+ \frac{2\,[\, 2+\gamma\,(\, n-2\,) - \gamma^2\,(\, n-1\,)\,]\,[\, 2+\gamma\,(\, n-2\,)\,]}{\Omega} \sum_{i=1}^{2} v_i\,[\, w_1 - c_i^u\,]\, f_{i1}$$

$$+ \frac{[\, 2-\gamma\,]\,[\, 2+\gamma\,(\, n-1\,)\,]\,[\, 2+\gamma\,(\, n-1\,) - \gamma\,(\, n+1\,)\,]}{2\,\Omega} \sum_{i=1}^{n} (\, c_1^u\, f_{1i} + c_2^u\, f_{2i}\,)$$

$$+ \frac{[\, 2-\gamma\,]\,[\, 2+\gamma\,(\, n-1\,)\,]^2}{2\,\Omega}\,[\, c_1^u\, f_{11} + c_2^u\, f_{21}\,]$$

$$= \frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]\,[\,1-\gamma\,]}{\Omega}\,\left[\,\alpha_1 - w_1 - c_1^d\,\right] - \frac{\gamma^3\,[\,n-1\,]}{\Omega}\,\sum_{i=1}^{n} w_i$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Omega}\,\sum_{i=2}^{n}\left(\,\alpha_i - c_i^d\,\right)$$

$$- \frac{2\,\gamma^2 + \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] - 2\,\gamma^3}{2\,\Omega}\,[\,n-1\,]\,\sum_{j=1}^{n}\left(\,\alpha_j - c_j^d\,\right)$$

$$+ \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]}{\Omega}\,\sum_{i=1}^{n} w_i$$

$$+ \frac{\gamma^2}{\Omega}\,[\,n-1\,]\,\sum_{i=1}^{n} w_i - \frac{\gamma^2}{\Omega}\,\sum_{i=1}^{n} w_i$$

$$+ \frac{\gamma^3\,[\,n-1\,] - \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\,\Omega}\,[\,n-1\,]\,\sum_{i=1}^{2} v_i\,[\,w_1 - c_i^u\,]\,f_{i1}$$

$$+ \frac{2\,[\,2+\gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Omega}\,\sum_{i=1}^{2} v_i\,[\,w_1 - c_i^u\,]\,f_{i1}$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-1\,) - \gamma\,(\,n+1\,)\,]}{2\,\Omega}\,\sum_{i=1}^{n}\left(\,c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right)$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Omega}\,[\,c_1^u\,f_{11} + c_2^u\,f_{21}\,]$$

$$= \frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]\,[\,1-\gamma\,]}{\Omega}\,\left[\,\alpha_1 - w_1 - c_1^d\,\right] - \frac{\gamma^3\,[\,n-1\,]}{\Omega}\,\sum_{i=1}^{n} w_i$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Omega}\,\sum_{i=2}^{n}\left(\,\alpha_i - c_i^d\,\right)$$

$$- \frac{2\,\gamma^2 + \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] - 2\gamma^3}{2\,\Omega}\,[\,n-1\,]\,\sum_{j=1}^{n}\left(\,\alpha_j - c_j^d\,\right)$$

$$+ \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]}{\Omega}\,\sum_{i=1}^{n} w_i \;+ \frac{\gamma^2}{\Omega}\,[\,n-2\,]\,\sum_{i=1}^{n} w_i$$

$$+ \frac{\gamma^3 [\, n-1\,] - \gamma [\, 2-\gamma\,] [\, 2+\gamma\,(\, n-1\,)\,]}{2\,\Omega} \; [\, n-1\,] \sum_{i=1}^{2} v_i \, [\, w_1 - c_i^u \,] \, f_{i1}$$

$$+ \frac{2\,[\, 2+\gamma\,(\, n-2\,) - \gamma^2\,(\, n-1\,)\,]\,[\, 2+\gamma\,(\, n-2\,)\,]}{\Omega} \sum_{i=1}^{2} v_i \, [\, w_1 - c_i^u \,] \, f_{i1}$$

$$+ \frac{[\, 2-\gamma\,]\,[\, 2+\gamma\,(\, n-1\,)\,]\,[\, 2+\gamma\,(\, n-1\,) - \gamma\,(\, n+1\,)\,]}{2\,\Omega} \sum_{i=1}^{n} (\, c_1^u \, f_{1i} + c_2^u \, f_{2i}\,)$$

$$+ \frac{[\, 2-\gamma\,]\,[\, 2+\gamma\,(\, n-1\,)\,]^2}{2\,\Omega} \; [\, c_1^u \, f_{11} + c_2^u \, f_{21}\,] \,. \tag{52}$$

(51) implies:

$$w_1 \;=\; - \frac{\gamma^2 [\, n-1\,]\,[\, 2+\gamma\,(\, n-1\,)\,]}{\Omega} \left[\, \alpha_1 - w_1 - c_1^d \,\right] + \frac{\gamma^3 [\, n-1\,]}{\Omega} \sum_{j=2}^{n} \left(\, \alpha_j - c_j^d - w_j \,\right)$$

$$+ \frac{\gamma^3 [\, n-1\,]}{\Omega} \left[\, \alpha_1 - c_1^d \,\right] - \frac{\gamma^3 [\, n-1\,]}{\Omega} \; w_1$$

$$+ \frac{\gamma\,[\, 2-\gamma\,]\,[\, 2+\gamma\,(\, n-1\,)\,]}{\Omega} \sum_{j=2}^{n} w_j - \frac{\gamma^2}{\Omega} \; w_1$$

$$+ \frac{2\,[\, 2+\gamma\,(\, n-2\,) - \gamma^2\,(\, n-1\,)\,]\,[\, 2+\gamma\,(\, n-2\,)\,]}{\Omega} \sum_{i=1}^{2} v_i \, [\, w_1 - c_i^u \,] \, f_{i1}$$

$$+ \frac{[\, 2-\gamma\,]\,[\, 2+\gamma\,(\, n-1\,)\,]^2}{\Omega} \; [\, c_1^u \, f_{11} + c_2^u \, f_{21}\,]$$

$$- \frac{\gamma\,[\, 2-\gamma\,]\,[\, 2+\gamma\,(\, n-1\,)\,]}{\Omega} \sum_{j=1}^{n} (\, c_1^u \, f_{1j} + c_2^u \, f_{2j}\,)$$

$$= \; - \frac{\gamma^2 [\, n-1\,]\,[\, 2+\gamma\,(\, n-1\,)\,]}{\Omega} \; \left[\, \alpha_1 - c_1^d \,\right]$$

$$+ \frac{\gamma^3 [\, n-1\,]}{\Omega} \sum_{j=2}^{n} \left(\, \alpha_j - c_j^d \,\right) + \frac{\gamma^3 [\, n-1\,]}{\Omega} \; \left[\, \alpha_1 - c_1^d \,\right]$$

$$+ \frac{\gamma^2 [\, n-1\,]\,[\, 2+\gamma\,(\, n-1\,)\,]}{\Omega} \; w_1 - \frac{\gamma^3 [\, n-1\,]}{\Omega} \; w_1 - \frac{\gamma^2}{\Omega} \; w_1$$

$$+ \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Omega} \sum_{i=1}^{2} v_i\, f_{i1}\, w_1$$

$$+ \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^3\,[\,n-1\,]}{\Omega} \sum_{j=2}^{n} w_j$$

$$- \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Omega} \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{\Omega}\,[\,c_1^u\, f_{11} + c_2^u\, f_{21}\,]$$

$$- \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega} \sum_{j=1}^{n} (\,c_1^u\, f_{1j} + c_2^u\, f_{2j}\,)$$

$$= -\frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega}\left[\,\alpha_1 - c_1^d\,\right] + \frac{\gamma^3\,[\,n-1\,]}{\Omega} \sum_{j=1}^{n} \left(\,\alpha_j - c_j^d\,\right)$$

$$+ \frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega}\, w_1 - \frac{\gamma^3\,[\,n-1\,]}{\Omega}\, w_1 - \frac{\gamma^2}{\Omega}\, w_1$$

$$+ \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Omega} \sum_{i=1}^{2} v_i\, f_{i1}\, w_1$$

$$+ \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^3\,[\,n-1\,]}{\Omega} \sum_{j=2}^{n} w_j$$

$$- \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Omega} \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{\Omega}\,[\,c_1^u\, f_{11} + c_2^u\, f_{21}\,]$$

$$- \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega} \sum_{j=1}^{n} (\,c_1^u\, f_{1j} + c_2^u\, f_{2j}\,)$$

$$\Rightarrow \quad \frac{\gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega} \sum_{j=2}^{n} w_j$$

$$= \; - \, \frac{\gamma^2 \, [\, n-1 \,] \, [\, 2 + \gamma \, (\, n-1 \,) \,]}{\Omega} \; \left[\, \alpha_1 - c_1^d \,\right] + \frac{\gamma^3 \, [\, n-1 \,]}{\Omega} \sum_{j\,=\,1}^{n} \left(\, \alpha_j - c_j^d \,\right)$$

$$+ \; \frac{\gamma^2 \, [\, n-1 \,] \, [\, 2 + \gamma \, (\, n-1 \,) \,]}{\Omega} \, w_1 - \frac{\gamma^3 \, [\, n-1 \,]}{\Omega} \, w_1 - \frac{\gamma^2}{\Omega} \, w_1 - w_1$$

$$+ \; \frac{2 \, [\, 2 + \gamma \, (\, n-2 \,) - \gamma^2 \, (\, n-1 \,) \,] \, [\, 2 + \gamma \, (\, n-2 \,) \,]}{\Omega} \sum_{i\,=\,1}^{2} v_i \, f_{i1} \, w_1$$

$$- \; \frac{2 \, [\, 2 + \gamma \, (\, n-2 \,) - \gamma^2 \, (\, n-1 \,) \,] \, [\, 2 + \gamma \, (\, n-2 \,) \,]}{\Omega} \sum_{i\,=\,1}^{2} v_i \, c_i^u \, f_{i1}$$

$$+ \; \frac{[\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n-1 \,) \,]^2}{\Omega} \; [\, c_1^u \, f_{11} + c_2^u \, f_{21} \,]$$

$$- \; \frac{\gamma \, [\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n-1 \,) \,]}{\Omega} \sum_{j\,=\,1}^{n} (\, c_1^u \, f_{1j} + c_2^u \, f_{2j} \,)$$

$$\Rightarrow \quad \sum_{j\,=\,2}^{n} w_j \; = \; - \, \frac{\gamma^2 \, [\, n-1 \,] \, [\, 2 + \gamma \, (\, n-1 \,) \,]}{\gamma^3 \, [\, n-1 \,] - \gamma \, [\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n-1 \,) \,]} \, [\, \alpha_1 - c_1^d \,]$$

$$+ \; \frac{\gamma^3 \, [\, n-1 \,]}{\gamma^3 \, [\, n-1 \,] - \gamma \, [\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n-1 \,) \,]} \sum_{j\,=\,1}^{n} \left(\, \alpha_j - c_j^d \,\right)$$

$$+ \; \frac{\gamma^2 \, [\, n-1 \,] \, [\, 2 + \gamma \, (\, n-1 \,) \,] - \gamma^3 \, [\, n-1 \,] - \gamma^2 - \Omega}{\gamma^3 \, [\, n-1 \,] - \gamma \, [\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n-1 \,) \,]} \; w_1$$

$$+ \; \frac{2 \, [\, 2 + \gamma \, (\, n-2 \,) - \gamma^2 \, (\, n-1 \,) \,] \, [\, 2 + \gamma \, (\, n-2 \,) \,]}{\gamma^3 \, [\, n-1 \,] - \gamma \, [\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n-1 \,) \,]} \sum_{i\,=\,1}^{2} v_i \, f_{i1} \, w_1$$

$$- \; \frac{2 \, [\, 2 + \gamma \, (\, n-2 \,) - \gamma^2 \, (\, n-1 \,) \,] \, [\, 2 + \gamma \, (\, n-2 \,) \,]}{\gamma^3 \, [\, n-1 \,] - \gamma \, [\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n-1 \,) \,]} \sum_{i\,=\,1}^{2} v_i \, c_i^u \, f_{i1}$$

$$+ \; \frac{[\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n-1 \,) \,]^2}{\gamma^3 \, [\, n-1 \,] - \gamma \, [\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n-1 \,) \,]} \; [\, c_1^u \, f_{11} + c_2^u \, f_{21} \,]$$

$$- \; \frac{\gamma \, [\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n-1 \,) \,]}{\gamma^3 \, [\, n-1 \,] - \gamma \, [\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n-1 \,) \,]} \sum_{j\,=\,1}^{n} (\, c_1^u \, f_{1j} + c_2^u \, f_{2j} \,) \, . \tag{53}$$

Observe that:

$$\sum_{j=2}^{n} w_j \;=\; \sum_{i=1}^{n} w_i - w_1 \,.$$

Using (52) to substitute for $\sum_{i=1}^{n} w_i$ in this equation, and then expressing the $\sum_{i=1}^{n} w_i$ terms in (52) as $\sum_{j=2}^{n} w_j + w_1$ and bringing the $\sum_{j=2}^{n} w_j$ terms to the left-hand side of the equation above provides:

$$\left\{ 1 + \frac{\gamma^3 [\, n-1 \,]}{\Omega} - \frac{\gamma [\, 2-\gamma \,] [\, 2+\gamma (\, n-1 \,) \,] [\, n-1 \,]}{\Omega} - \frac{\gamma^2}{\Omega} [\, n-2 \,] \right\} \sum_{j=2}^{n} w_j$$

$$= \frac{\gamma [\, 2+\gamma (\, n-1 \,) \,] [\, n-1 \,] [\, 1-\gamma \,]}{\Omega} \left[\alpha_1 - w_1 - c_1^d \right] - \frac{\gamma^3 [\, n-1 \,]}{\Omega} \, w_1 - w_1$$

$$+ \frac{[\, 2-\gamma \,] [\, 2+\gamma (\, n-1 \,) \,]^2}{2\,\Omega} \sum_{i=2}^{n} (\, \alpha_i - c_i^d \,)$$

$$- \frac{2\,\gamma^2 + \gamma [\, 2-\gamma \,] [\, 2+\gamma (\, n-1 \,) \,] - 2\,\gamma^3}{2\,\Omega} [\, n-1 \,] \sum_{j=1}^{n} (\, \alpha_j - c_j^d \,)$$

$$+ \frac{\gamma [\, 2-\gamma \,] [\, 2+\gamma (\, n-1 \,) \,] [\, n-1 \,]}{\Omega} \, w_1 + \frac{\gamma^2}{\Omega} [\, n-2 \,] \, w_1$$

$$+ \frac{\gamma^3 [\, n-1 \,] - \gamma [\, 2-\gamma \,] [\, 2+\gamma (\, n-1 \,) \,]}{2\,\Omega} [\, n-1 \,] \sum_{i=1}^{2} v_i [\, w_1 - c_i^u \,] \, f_{i1}$$

$$+ \frac{2 [\, 2+\gamma (\, n-2 \,) - \gamma^2 (\, n-1 \,) \,] [\, 2+\gamma (\, n-2 \,) \,]}{\Omega} \sum_{i=1}^{2} v_i [\, w_1 - c_i^u \,] \, f_{i1}$$

$$+ \frac{[\, 1-\gamma \,] [\, 2-\gamma \,] [\, 2+\gamma (\, n-1 \,) \,]}{\Omega} \sum_{i=1}^{n} (\, c_1^u \, f_{1i} + c_2^u \, f_{2i} \,)$$

$$+ \frac{[\, 2-\gamma \,] [\, 2+\gamma (\, n-1 \,) \,]^2}{2\,\Omega} [\, c_1^u \, f_{11} + c_2^u \, f_{21} \,]$$

$$= -\frac{\gamma^3 [\, n-1 \,] + \Omega}{\Omega} \, w_1 - \frac{\gamma [\, 2+\gamma (\, n-1 \,) \,] [\, n-1 \,] [\, 1-\gamma \,]}{\Omega} \, w_1$$

$$+ \frac{\gamma [\, 2-\gamma \,] [\, 2+\gamma (\, n-1 \,) \,] [\, n-1 \,] + \gamma^2 [\, n-2 \,]}{\Omega} \, w_1$$

$$+ \frac{\gamma^3 [n-1] - \gamma [2-\gamma][2+\gamma(n-1)]}{2\Omega} [n-1] \sum_{i=1}^{2} v_i w_1 f_{i1}$$

$$+ \frac{2[2+\gamma(n-2) - \gamma^2(n-1)][2+\gamma(n-2)]}{\Omega} \sum_{i=1}^{2} v_i w_1 f_{i1}$$

$$+ \frac{\gamma[2+\gamma(n-1)][n-1][1-\gamma]}{\Omega} [\alpha_1 - c_1^d]$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Omega} \sum_{i=2}^{n} (\alpha_i - c_i^d)$$

$$- \frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)] - 2\gamma^3}{2\Omega} [n-1] \sum_{j=1}^{n} (\alpha_j - c_j^d)$$

$$- \frac{\gamma^3 [n-1] - \gamma [2-\gamma][2+\gamma(n-1)]}{2\Omega} [n-1] \sum_{i=1}^{2} v_i c_i^u f_{i1}$$

$$- \frac{2[2+\gamma(n-2) - \gamma^2(n-1)][2+\gamma(n-2)]}{\Omega} \sum_{i=1}^{2} v_i c_i^u f_{i1}$$

$$+ \frac{[1-\gamma][2-\gamma][2+\gamma(n-1)]}{\Omega} \sum_{i=1}^{n} (c_1^u f_{1i} + c_2^u f_{2i})$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Omega} [c_1^u f_{11} + c_2^u f_{21}]. \tag{54}$$

Observe that the numerators of the second and third terms in this expression involve:

$$\gamma[2+\gamma(n-1)][n-1][2-\gamma-(1-\gamma)] = \gamma[2+\gamma(n-1)][n-1]. \tag{55}$$

(54) and (55) imply:

$$\left\{ \frac{\gamma^3 [n-1] - \gamma[2-\gamma][2+\gamma(n-1)][n-1] - \gamma^2 [n-2] + \Omega}{\Omega} \right\} \sum_{j=2}^{n} w_j$$

$$= \frac{\gamma[2+\gamma(n-1)][n-1] + \gamma^2[n-2] - \gamma^3[n-1] - \Omega}{\Omega} w_1$$

$$+ \frac{\gamma^3 [n-1] - \gamma [2-\gamma][2+\gamma(n-1)]}{2\Omega} [n-1] \sum_{i=1}^{2} v_i w_1 f_{i1}$$

$$+ \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Omega} \sum_{i=1}^{2} v_i\, w_1\, f_{i1}$$

$$+ \frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]\,[\,1-\gamma\,]}{\Omega}\,[\,\alpha_1 - c_1^d\,]$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Omega} \sum_{i=2}^{n} (\,\alpha_i - c_i^d\,)$$

$$- \frac{2\,\gamma^2+\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]-2\,\gamma^3}{2\,\Omega}\,[\,n-1\,] \sum_{j=1}^{n} (\,\alpha_j - c_j^d\,)$$

$$- \frac{\gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\,\Omega}\,[\,n-1\,] \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$- \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Omega} \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$+ \frac{[\,1-\gamma\,]\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega} \sum_{i=1}^{n} (\,c_1^u\, f_{1i} + c_2^u\, f_{2i}\,)$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Omega}\,[\,c_1^u\, f_{11} + c_2^u\, f_{21}\,]\,. \tag{56}$$

(39) implies that the numerator of the coefficient of $\sum_{j=2}^{n} w_j$ in (56) is:

$$\begin{aligned}
&\gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]-\gamma^2\,[\,n-2\,]+\Omega \\
&= \gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]-\gamma^2\,[\,n-2\,] \\
&\quad + [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]-\gamma^2 \\
&= \gamma^3\,[\,n-1\,]-\gamma^2\,[\,n-1\,]+[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)-\gamma\,(\,n-1\,)\,] \\
&= [\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]\,.
\end{aligned} \tag{57}$$

(56) and (57) imply:

$$\sum_{j=2}^{n} w_j = \frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]+\gamma^2\,[\,n-2\,]-\gamma^3\,[\,n-1\,]-\Omega}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]}\, w_1$$

$$+ \frac{\gamma^3 [n-1] - \gamma [2-\gamma] [2+\gamma(n-1)]}{2\left\{ [2-\gamma]^2 [2+\gamma(n-1)] - \gamma^2 [n-1][1-\gamma] \right\}} [n-1] \sum_{i=1}^{2} v_i w_1 f_{i1}$$

$$+ \frac{2[2+\gamma(n-2) - \gamma^2(n-1)][2+\gamma(n-2)]}{[2-\gamma]^2 [2+\gamma(n-1)] - \gamma^2 [n-1][1-\gamma]} \sum_{i=1}^{2} v_i w_1 f_{i1}$$

$$+ \frac{\gamma [2+\gamma(n-1)][n-1][1-\gamma]}{[2-\gamma]^2 [2+\gamma(n-1)] - \gamma^2 [n-1][1-\gamma]} [\alpha_1 - c_1^d]$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\left\{ [2-\gamma]^2 [2+\gamma(n-1)] - \gamma^2 [n-1][1-\gamma] \right\}} \sum_{i=2}^{n} (\alpha_i - c_i^d)$$

$$- \frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)] - 2\gamma^3}{2\left\{ [2-\gamma]^2 [2+\gamma(n-1)] - \gamma^2 [n-1][1-\gamma] \right\}} [n-1] \sum_{j=1}^{n} (\alpha_j - c_j^d)$$

$$- \frac{\gamma^3 [n-1] - \gamma [2-\gamma] [2+\gamma(n-1)]}{2\left\{ [2-\gamma]^2 [2+\gamma(n-1)] - \gamma^2 [n-1][1-\gamma] \right\}} [n-1] \sum_{i=1}^{2} v_i c_i^u f_{i1}$$

$$- \frac{2[2+\gamma(n-2) - \gamma^2(n-1)][2+\gamma(n-2)]}{[2-\gamma]^2 [2+\gamma(n-1)] - \gamma^2 [n-1][1-\gamma]} \sum_{i=1}^{2} v_i c_i^u f_{i1}$$

$$+ \frac{[1-\gamma][2-\gamma][2+\gamma(n-1)]}{[2-\gamma]^2 [2+\gamma(n-1)] - \gamma^2 [n-1][1-\gamma]} \sum_{i=1}^{n} (c_1^u f_{1i} + c_2^u f_{2i})$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\left\{ [2-\gamma]^2 [2+\gamma(n-1)] - \gamma^2 [n-1][1-\gamma] \right\}} [c_1^u f_{11} + c_2^u f_{21}]. \tag{58}$$

(53) and (58) imply:

$$- \frac{\gamma^2 [n-1][2+\gamma(n-1)]}{\gamma^3 [n-1] - \gamma [2-\gamma][2+\gamma(n-1)]} [\alpha_1 - c_1^d]$$

$$+ \frac{\gamma^3 [n-1]}{\gamma^3 [n-1] - \gamma [2-\gamma][2+\gamma(n-1)]} \sum_{j=1}^{n} (\alpha_j - c_j^d)$$

$$+ \frac{\gamma^2 [n-1][2+\gamma(n-1)] - \gamma^3 [n-1] - \gamma^2 - \Omega}{\gamma^3 [n-1] - \gamma [2-\gamma][2+\gamma(n-1)]} w_1$$

$$+ \frac{2[2+\gamma(n-2) - \gamma^2(n-1)][2+\gamma(n-2)]}{\gamma^3 [n-1] - \gamma [2-\gamma][2+\gamma(n-1)]} \sum_{i=1}^{2} v_i f_{i1} w_1$$

$$
\begin{aligned}
& - \frac{2\,[\,2 + \gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,]\,[\,2 + \gamma\,(\,n-2\,)\,]}{\gamma^3\,[\,n-1\,] - \gamma\,[\,2-\gamma\,]\,[\,2 + \gamma\,(\,n-1\,)\,]} \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1} \\
& + \frac{[\,2-\gamma\,]\,[\,2 + \gamma\,(\,n-1\,)\,]^2}{\gamma^3\,[\,n-1\,] - \gamma\,[\,2-\gamma\,]\,[\,2 + \gamma\,(\,n-1\,)\,]}\,[\,c_1^u\, f_{11} + c_2^u\, f_{21}\,] \\
& - \frac{\gamma\,[\,2-\gamma\,]\,[\,2 + \gamma\,(\,n-1\,)\,]}{\gamma^3\,[\,n-1\,] - \gamma\,[\,2-\gamma\,]\,[\,2 + \gamma\,(\,n-1\,)\,]} \sum_{j=1}^{n} (\,c_1^u\, f_{1j} + c_2^u\, f_{2j}\,)
\end{aligned}
$$

$$
\begin{aligned}
= \;& \frac{\gamma\,[\,2 + \gamma\,(\,n-1\,)\,]\,[\,n-1\,] + \gamma^2\,[\,n-2\,] - \gamma^3\,[\,n-1\,] - \Omega}{[\,2-\gamma\,]^2\,[\,2 + \gamma\,(\,n-1\,)\,] - \gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]}\; w_1 \\
& + \frac{\gamma^3\,[\,n-1\,] - \gamma\,[\,2-\gamma\,]\,[\,2 + \gamma\,(\,n-1\,)\,]}{2\left\{\,[\,2-\gamma\,]^2\,[\,2 + \gamma\,(\,n-1\,)\,] - \gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]\,\right\}}\,[\,n-1\,] \sum_{i=1}^{2} v_i\, w_1\, f_{i1} \\
& + \frac{2\,[\,2 + \gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,]\,[\,2 + \gamma\,(\,n-2\,)\,]}{[\,2-\gamma\,]^2\,[\,2 + \gamma\,(\,n-1\,)\,] - \gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]} \sum_{i=1}^{2} v_i\, w_1\, f_{i1} \\
& + \frac{\gamma\,[\,2 + \gamma\,(\,n-1\,)\,]\,[\,n-1\,]\,[\,1-\gamma\,]}{[\,2-\gamma\,]^2\,[\,2 + \gamma\,(\,n-1\,)\,] - \gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]}\,[\,\alpha_1 - c_1^d\,] \\
& + \frac{[\,2-\gamma\,]\,[\,2 + \gamma\,(\,n-1\,)\,]^2}{2\left\{\,[\,2-\gamma\,]^2\,[\,2 + \gamma\,(\,n-1\,)\,] - \gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]\,\right\}} \sum_{i=2}^{n} (\,\alpha_i - c_i^d\,) \\
& - \frac{2\,\gamma^2 + \gamma\,[\,2-\gamma\,]\,[\,2 + \gamma\,(\,n-1\,)\,] - 2\gamma^3}{2\left\{\,[\,2-\gamma\,]^2\,[\,2 + \gamma\,(\,n-1\,)\,] - \gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]\,\right\}}\,[\,n-1\,] \sum_{j=1}^{n} (\,\alpha_j - c_j^d\,) \\
& - \frac{\gamma^3\,[\,n-1\,] - \gamma\,[\,2-\gamma\,]\,[\,2 + \gamma\,(\,n-1\,)\,]}{2\left\{\,[\,2-\gamma\,]^2\,[\,2 + \gamma\,(\,n-1\,)\,] - \gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]\,\right\}}\,[\,n-1\,] \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1} \\
& - \frac{2\,[\,2 + \gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,]\,[\,2 + \gamma\,(\,n-2\,)\,]}{[\,2-\gamma\,]^2\,[\,2 + \gamma\,(\,n-1\,)\,] - \gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]} \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1} \\
& + \frac{[\,1-\gamma\,]\,[\,2-\gamma\,]\,[\,2 + \gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2 + \gamma\,(\,n-1\,)\,] - \gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]} \sum_{i=1}^{n} (\,c_1^u\, f_{1i} + c_2^u\, f_{2i}\,) \\
& + \frac{[\,2-\gamma\,]\,[\,2 + \gamma\,(\,n-1\,)\,]^2}{2\left\{\,[\,2-\gamma\,]^2\,[\,2 + \gamma\,(\,n-1\,)\,] - \gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]\,\right\}}\,[\,c_1^u\, f_{11} + c_2^u\, f_{21}\,]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad & \frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^3\,[\,n-1\,]-\gamma^2-\Omega}{\gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\;w_1 \\
& -\frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]+\gamma^2\,[\,n-2\,]-\gamma^3\,[\,n-1\,]-\Omega}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]}\,w_1 \\
& +\frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\;\sum_{i=1}^{2} v_i\,f_{i1}\,w_1 \\
& -\frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]}\;\sum_{i=1}^{2} v_i\,w_1\,f_{i1} \\
& -\frac{\gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\left\{\,[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]\,\right\}}\;[\,n-1\,]\;\sum_{i=1}^{2} v_i\,w_1\,f_{i1} \\
=\quad & \frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\;[\,\alpha_1-c_1^d\,] \\
& +\frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]\,[\,1-\gamma\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]}\;[\,\alpha_1-c_1^d\,] \\
& -\frac{\gamma^3\,[\,n-1\,]}{\gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\;\sum_{j=1}^{n}(\,\alpha_j-c_j^d\,) \\
& -\frac{2\,\gamma^2+\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]-2\,\gamma^3}{2\left\{\,[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]\,\right\}}\;[\,n-1\,]\;\sum_{j=1}^{n}(\,\alpha_j-c_j^d\,) \\
& +\frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\left\{\,[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]\,\right\}}\;\sum_{i=2}^{n}(\,\alpha_i-c_i^d\,) \\
& +\frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\;\sum_{i=1}^{2} v_i\,c_i^u\,f_{i1} \\
& -\frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{\gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\;[\,c_1^u\,f_{11}+c_2^u\,f_{21}\,] \\
& +\frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\left\{\,[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]\,\right\}}\;[\,c_1^u\,f_{11}+c_2^u\,f_{21}\,] \\
& +\frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}\;\sum_{j=1}^{n}(\,c_1^u\,f_{1j}+c_2^u\,f_{2j}\,)
\end{aligned}
$$

$$+\ \frac{[\,1-\gamma\,]\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]}\ \sum_{i=1}^{n}(\,c_1^u\,f_{1i}+c_2^u\,f_{2i}\,)$$

$$-\ \frac{\gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\left\{\,[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]\,\right\}}\ [\,n-1\,]\sum_{i=1}^{2}v_i\,c_i^u\,f_{i1}$$

$$-\ \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]}\ \sum_{i=1}^{2}v_i\,c_i^u\,f_{i1}\,. \tag{59}$$

Define:

$$\Phi_1\ \equiv\ \gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,;$$

$$\Phi_2\ \equiv\ [\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]\,. \tag{60}$$

(60) implies that (59) can be written as:

$$\frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^3\,[\,n-1\,]-\gamma^2-\Omega}{\Phi_1}\ w_1$$

$$-\ \frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]+\gamma^2\,[\,n-2\,]-\gamma^3\,[\,n-1\,]-\Omega}{\Phi_2}\ w_1$$

$$+\ \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Phi_1}\ \sum_{i=1}^{2}v_i\,f_{i1}\,w_1$$

$$-\ \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Phi_2}\ \sum_{i=1}^{2}v_i\,w_1\,f_{i1}$$

$$-\ \frac{\gamma^3\,[\,n-1\,]-\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\,\Phi_2}\ [\,n-1\,]\ \sum_{i=1}^{2}v_i\,w_1\,f_{i1}$$

$$=\ \left\{\frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_1}+\frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]\,[\,1-\gamma\,]}{\Phi_2}\right\}[\,\alpha_1-c_1^d\,]$$

$$-\ \left\{\frac{\gamma^3}{\Phi_1}+\frac{2\,\gamma^2+\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]-2\,\gamma^3}{2\,\Phi_2}\right\}[\,n-1\,]\ \sum_{j=1}^{n}\left(\,\alpha_j-c_j^d\,\right)$$

$$+\ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2}\ \sum_{i=2}^{n}\left(\,\alpha_i-c_i^d\,\right)$$

$$+\ \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Phi_1}\ \sum_{i=1}^{2}v_i\,c_i^u\,f_{i1}$$

$$+ \left\{ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} - \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{\Phi_1} \right\} [\,c_1^u\, f_{11} + c_2^u\, f_{21}\,]$$

$$+ \left\{ \frac{[\,1-\gamma\,]\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_2} \right.$$

$$\left. + \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_1} \right\} \sum_{i=1}^{n} (\,c_1^u\, f_{1i} + c_2^u\, f_{2i}\,)$$

$$- \left\{ 4\left[\,2+\gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,\right]\,[\,2+\gamma\,(\,n-2\,)\,] + \gamma^3\,[\,n-1\,]^2 \right.$$

$$\left. - \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,] \right\} \frac{\sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}}{2\,\Phi_2}\,. \qquad (61)$$

(53) and (60) imply:

$$\sum_{j=1}^{n} w_j = \sum_{j=2}^{n} w_j + w_1$$

$$= w_1 - \frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_1} \left[\,\alpha_1 - c_1^d\,\right] + \frac{\gamma^3\,[\,n-1\,]}{\Phi_1} \sum_{j=1}^{n} \left(\,\alpha_j - c_j^d\,\right)$$

$$+ \frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,] - \gamma^3\,[\,n-1\,] - \gamma^2 - \Omega}{\Phi_1}\, w_1$$

$$+ \frac{2\,[\,2+\gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Phi_1} \sum_{i=1}^{2} v_i\, f_{i1}\, w_1$$

$$- \frac{2\,[\,2+\gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Phi_1} \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{\Phi_1}\, [\,c_1^u\, f_{11} + c_2^u\, f_{21}\,]$$

$$- \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_1} \sum_{j=1}^{n} (\,c_1^u\, f_{1j} + c_2^u\, f_{2j}\,)$$

$$\Rightarrow \sum_{\substack{j=1 \\ j\neq i}}^{n} w_j = -w_i + w_1 - \frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_1} \left[\,\alpha_1 - c_1^d\,\right] + \frac{\gamma^3\,[\,n-1\,]}{\Phi_1} \sum_{j=1}^{n} \left(\,\alpha_j - c_j^d\,\right)$$

$$+ \frac{\gamma^2 [n-1][2+\gamma(n-1)] - \gamma^3 [n-1] - \gamma^2 - \Omega}{\Phi_1} w_1$$

$$+ \frac{2[2+\gamma(n-2) - \gamma^2(n-1)][2+\gamma(n-2)]}{\Phi_1} \sum_{i=1}^{2} v_i f_{i1} w_1$$

$$- \frac{2[2+\gamma(n-2) - \gamma^2(n-1)][2+\gamma(n-2)]}{\Phi_1} \sum_{i=1}^{2} v_i c_i^u f_{i1}$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{\Phi_1} [c_1^u f_{11} + c_2^u f_{21}]$$

$$- \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{\Phi_1} \sum_{j=1}^{n} (c_1^u f_{1j} + c_2^u f_{2j}). \qquad (62)$$

(42), (60), and (62) imply that for $i \in \{2, ..., n\}$:

$$w_i = \frac{\gamma[2+\gamma(n-1)]}{\Omega} [\alpha_1 - w_1 - c_1^d] + \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Omega} [\alpha_i - c_i^d]$$

$$- \frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)]}{2\Omega} \sum_{j=1}^{n} (\alpha_j - c_j^d)$$

$$+ \frac{\gamma^3 [n-1] - \gamma[2-\gamma][2+\gamma(n-1)]}{2\Omega} \sum_{i=1}^{2} v_i [w_1 - c_i^u] f_{i1}$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Omega} [c_1^u f_{1i} + c_2^u f_{2i}]$$

$$- \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{2\Omega} \sum_{j=1}^{n} (c_1^u f_{1j} + c_2^u f_{2j})$$

$$+ \frac{\gamma[2-\gamma][2+\gamma(n-1)] + \gamma^2}{\Omega}$$

$$\cdot \Bigg\{ -w_i + w_1 - \frac{\gamma^2 [n-1][2+\gamma(n-1)]}{\Phi_1} [\alpha_1 - c_1^d] + \frac{\gamma^3 [n-1]}{\Phi_1} \sum_{j=1}^{n} (\alpha_j - c_j^d)$$

$$+ \frac{\gamma^2 [n-1][2+\gamma(n-1)] - \gamma^3 [n-1] - \gamma^2 - \Omega}{\Phi_1} w_1$$

$$+ \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Phi_1} \sum_{i=1}^{2} v_i\, f_{i1}\, w_1$$

$$- \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Phi_1} \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{\Phi_1} \left[\, c_1^u\, f_{11} + c_2^u\, f_{21}\,\right]$$

$$- \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_1} \sum_{j=1}^{n} (\, c_1^u\, f_{1j} + c_2^u\, f_{2j}\,) \Bigg\}$$

$$\Rightarrow \quad \left[1 + \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]+\gamma^2}{\Omega} \right] w_i$$

$$= \; - \frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega}\, w_1 + \frac{\Phi_1}{2\,\Omega} \sum_{i=1}^{2} v_i\, w_1\, f_{i1}$$

$$+ \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]+\gamma^2}{\Omega}$$

$$\cdot \left\{ 1 + \frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^3\,[\,n-1\,]-\gamma^2-\Omega}{\Phi_1} \right.$$

$$\left. + \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Phi_1} \sum_{i=1}^{2} v_i\, f_{i1} \right\} w_1$$

$$+ \frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]}{\Omega} \left[\, \alpha_1 - c_1^d\,\right] + \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Omega} \left[\, \alpha_i - c_i^d\,\right]$$

$$- \frac{2\,\gamma^2 + \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\,\Omega} \sum_{j=1}^{n} \left(\, \alpha_j - c_j^d\,\right) - \frac{\Phi_1}{2\,\Omega} \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Omega} \left[\, c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right]$$

$$- \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\,\Omega} \sum_{j=1}^{n} (\, c_1^u\, f_{1j} + c_2^u\, f_{2j}\,)$$

$$+ \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]+\gamma^2}{\Omega}$$

$$\cdot \left\{ - \frac{\gamma^2 [n-1][2+\gamma(n-1)]}{\Phi_1} [\alpha_1 - c_1^d] + \frac{\gamma^3 [n-1]}{\Phi_1} \sum_{j=1}^{n} (\alpha_j - c_j^d) \right.$$

$$- \frac{2[2+\gamma(n-2)-\gamma^2(n-1)][2+\gamma(n-2)]}{\Phi_1} \sum_{i=1}^{2} v_i c_i^u f_{i1}$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{\Phi_1} [c_1^u f_{11} + c_2^u f_{21}]$$

$$\left. - \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{\Phi_1} \sum_{j=1}^{n} (c_1^u f_{1j} + c_2^u f_{2j}) \right\}. \tag{63}$$

(39) implies that the numerator of the term multiplying w_i in (63) is:

$$\begin{aligned} &\Omega + \gamma[2-\gamma][2+\gamma(n-1)] + \gamma^2 \\ &= [2-\gamma][2+\gamma(n-1)][2+\gamma(n-2)] - \gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)] + \gamma^2 \\ &= [2-\gamma][2+\gamma(n-1)][2+\gamma(n-2)+\gamma] = [2-\gamma][2+\gamma(n-1)]^2 . \end{aligned} \tag{64}$$

(63) and (64) imply that for $i \in \{2, ..., n\}$:

$$[2-\gamma][2+\gamma(n-1)]^2 w_i$$

$$= -\gamma[2+\gamma(n-1)] w_1 + \frac{\Phi_1}{2} \sum_{i=1}^{2} v_i w_1 f_{i1}$$

$$+ \{\gamma[2-\gamma][2+\gamma(n-1)] + \gamma^2\}$$

$$\cdot \left\{ 1 + \frac{\gamma^2[n-1][2+\gamma(n-1)] - \gamma^3[n-1] - \gamma^2 - \Omega}{\Phi_1} \right.$$

$$\left. + \frac{2[2+\gamma(n-2)-\gamma^2(n-1)][2+\gamma(n-2)]}{\Phi_1} \sum_{i=1}^{2} v_i f_{i1} \right\} w_1$$

$$+ \gamma[2+\gamma(n-1)][\alpha_1 - c_1^d] + \frac{[2-\gamma][2+\gamma(n-1)]^2}{2} [\alpha_i - c_i^d]$$

$$- \frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)]}{2} \sum_{j=1}^{n} (\alpha_j - c_j^d) - \frac{\Phi_1}{2} \sum_{i=1}^{2} v_i c_i^u f_{i1}$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2} [c_1^u f_{1i} + c_2^u f_{2i}]$$

$$- \frac{\gamma [2-\gamma][2+\gamma(n-1)]}{2} \sum_{j=1}^{n} (c_1^u f_{1j} + c_2^u f_{2j})$$

$$+ \left\{ \gamma [2-\gamma][2+\gamma(n-1)] + \gamma^2 \right\}$$

$$\cdot \Bigg\{ - \frac{\gamma^2 [n-1][2+\gamma(n-1)]}{\Phi_1} \left[\alpha_1 - c_1^d \right] + \frac{\gamma^3 [n-1]}{\Phi_1} \sum_{j=1}^{n} \left(\alpha_j - c_j^d \right)$$

$$- \frac{2[2+\gamma(n-2)-\gamma^2(n-1)][2+\gamma(n-2)]}{\Phi_1} \sum_{i=1}^{2} v_i c_i^u f_{i1}$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{\Phi_1} [c_1^u f_{11} + c_2^u f_{21}]$$

$$- \frac{\gamma [2-\gamma][2+\gamma(n-1)]}{\Phi_1} \sum_{j=1}^{n} (c_1^u f_{1j} + c_2^u f_{2j}) \Bigg\} . \tag{65}$$

Define

$$\Phi_3 \equiv -2 \left[2+\gamma(n-2) - \gamma^2 (n-1) \right] [2+\gamma(n-2)] ; \tag{66}$$

$$\Phi_4 \equiv \gamma [2-\gamma][2+\gamma(n-1)] + \gamma^2 . \tag{67}$$

To simplify the terms involving w_1 in (65), first observe that (39) and (66) imply:

$$\gamma^2 [n-1][2+\gamma(n-1)] - \gamma^3 [n-1] - \gamma^2 - \Omega$$

$$= \gamma^2 [n-1][2+\gamma(n-1)] - \gamma^3 [n-1] - [2-\gamma][2+\gamma(n-1)][2+\gamma(n-2)]$$

$$= \gamma^2 [n-1][2+\gamma(n-2)] - [2-\gamma][2+\gamma(n-1)][2+\gamma(n-2)]$$

$$= [2+\gamma(n-2)] \left[\gamma^2 (n-1) - (2-\gamma)(2+\gamma[n-1]) \right]$$

$$= [2+\gamma(n-2)] \left[\gamma^2 (n-1) - 4 - 2\gamma(n-1) + 2\gamma + \gamma^2 (n-1) \right]$$

$$= [2+\gamma(n-2)] \left[2\gamma^2 (n-1) - 4 - 2\gamma(n-2) \right]$$

$$= [2+\gamma(n-2)] \, 2 \left[\gamma^2 (n-1) - 2 - \gamma(n-2) \right]$$

$$= -2 \left[2+\gamma(n-2) - \gamma^2 (n-1) \right] [2+\gamma(n-2)] = \Phi_3 . \tag{68}$$

Further observe that (60) implies:

$$\gamma [2-\gamma][2+\gamma(n-1)] + \gamma^2 - \gamma [2+\gamma(n-1)] + \frac{\Phi_1}{2} \sum_{i=1}^{2} v_i f_{i1}$$

$$= \frac{1}{2}\left[2\,\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] + 2\,\gamma^2 - 2\,\gamma\,[\,2+\gamma\,(\,n-1\,)\,] + \Phi_1 \sum_{i=1}^{2} v_i\, f_{i1} \right]$$

$$= \frac{1}{2}\left\{ \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] + \Phi_1 \sum_{i=1}^{2} v_i\, f_{i1} \right.$$

$$\left. + \,\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] + 2\,\gamma^2 - 2\,\gamma\,[\,2+\gamma\,(\,n-1\,)\,] \right\}$$

$$= \frac{1}{2}\left[\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] + \Phi_1 \sum_{i=1}^{2} v_i\, f_{i1} - \gamma^2\,[\,2+\gamma\,(\,n-1\,)\,] + 2\,\gamma^2 \right]$$

$$= \frac{1}{2}\left[\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] + \Phi_1 \sum_{i=1}^{2} v_i\, f_{i1} - \gamma^3\,[\,n-1\,] \right]$$

$$= \frac{1}{2}\left[-\,\Phi_1 + \Phi_1 \sum_{i=1}^{2} v_i\, f_{i1} \right] \;=\; -\,\frac{\Phi_1}{2}\left[1 - \sum_{i=1}^{2} v_i\, f_{i1} \right]. \tag{69}$$

(66), (68), and (69) imply that the terms involving w_1 in (65) are:

$$-\,\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,w_1 + \frac{\Phi_1}{2} \sum_{i=1}^{2} v_i\, w_1\, f_{i1}$$

$$+ \;\left\{ \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] + \gamma^2 \right\}$$

$$\cdot \left\{ 1 + \frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,] - \gamma^3\,[\,n-1\,] - \gamma^2 - \Omega}{\Phi_1} \right.$$

$$\left. + \;\frac{2\,[\,2+\gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Phi_1} \sum_{i=1}^{2} v_i\, f_{i1} \right\} w_1$$

$$= \;-\,\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,w_1 + \frac{\Phi_1}{2} \sum_{i=1}^{2} v_i\, w_1\, f_{i1}$$

$$+ \;\left\{ \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] + \gamma^2 \right\} \left[1 + \frac{\Phi_3}{\Phi_1} - \frac{\Phi_3}{\Phi_1} \sum_{i=1}^{2} v_i\, f_{i1} \right] w_1$$

$$= \;-\,\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,w_1 + \frac{\Phi_1}{2} \sum_{i=1}^{2} v_i\, w_1\, f_{i1}$$

$$+ \;\left\{ \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] + \gamma^2 \right\} \left[1 + \frac{\Phi_3}{\Phi_1} \left(1 - \sum_{i=1}^{2} v_i\, f_{i1} \right) \right] w_1$$

$$= \; - \, \gamma \, [\, 2 + \gamma \, (\, n - 1 \,) \,] \, w_1 + \left\{ \, \gamma \, [\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n - 1 \,) \,] + \gamma^2 \, \right\} \, w_1 + \frac{\Phi_1}{2} \, \sum_{i=1}^{2} v_i \, w_1 \, f_{i1}$$

$$+ \; \left\{ \, \gamma \, [\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n - 1 \,) \,] + \gamma^2 \, \right\} \, \frac{\Phi_3}{\Phi_1} \left[1 - \sum_{i=1}^{2} v_i \, f_{i1} \right] w_1 \, . \tag{70}$$

(67) and (69) imply that (70) can be written as:

$$- \, \gamma \, [\, 2 + \gamma \, (\, n - 1 \,) \,] \, w_1 + \frac{\Phi_1}{2} \, \sum_{i=1}^{2} v_i \, w_1 \, f_{i1}$$

$$+ \; \left\{ \, \gamma \, [\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n - 1 \,) \,] + \gamma^2 \, \right\}$$

$$\cdot \left\{ 1 + \frac{\gamma^2 \, [\, n - 1 \,] \, [\, 2 + \gamma \, (\, n - 1 \,) \,] - \gamma^3 \, [\, n - 1 \,] - \gamma^2 - \Omega}{\Phi_1} \right.$$

$$\left. + \; \frac{2 \, [\, 2 + \gamma \, (\, n - 2 \,) - \gamma^2 \, (\, n - 1 \,) \,] \, [\, 2 + \gamma \, (\, n - 2 \,) \,]}{\Phi_1} \, \sum_{i=1}^{2} v_i \, f_{i1} \right\} w_1$$

$$= \; - \, \frac{\Phi_1}{2} \left[1 - \sum_{i=1}^{2} v_i \, f_{i1} \right] w_1$$

$$+ \; \left\{ \, \gamma \, [\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n - 1 \,) \,] + \gamma^2 \, \right\} \, \frac{\Phi_3}{\Phi_1} \left[1 - \sum_{i=1}^{2} v_i \, f_{i1} \right] w_1$$

$$= \; - \, \frac{\Phi_1}{2} \left[1 - \sum_{i=1}^{2} v_i \, f_{i1} \right] w_1 + \Phi_4 \, \frac{\Phi_3}{\Phi_1} \left[1 - \sum_{i=1}^{2} v_i \, f_{i1} \right] w_1$$

$$= \; \Phi_4 \left[1 - \sum_{i=1}^{2} v_i \, f_{i1} \right] \left[\frac{\Phi_3}{\Phi_1} - \frac{\Phi_1}{2 \, \Phi_4} \right] \, w_1 . \tag{71}$$

(71) implies that (65) can be written as:

$$[\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n - 1 \,) \,]^2 \; w_i$$

$$= \; \Phi_4 \left[1 - \sum_{i=1}^{2} v_i \, f_{i1} \right] \left[\frac{\Phi_3}{\Phi_1} - \frac{\Phi_1}{2 \, \Phi_4} \right] \, w_1$$

$$+ \, \gamma \, [\, 2 + \gamma \, (\, n - 1 \,) \,] \, [\, \alpha_1 - c_1^d \,] + \frac{[\, 2 - \gamma \,] \, [\, 2 + \gamma \, (\, n - 1 \,) \,]^2}{2} \; [\, \alpha_i - c_i^d \,]$$

$$- \frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)]}{2} \sum_{j=1}^{n} (\alpha_j - c_j^d) - \frac{\Phi_1}{2} \sum_{i=1}^{2} v_i \, c_i^u \, f_{i1}$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2} [c_1^u f_{1i} + c_2^u f_{2i}]$$

$$- \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{2} \sum_{j=1}^{n} (c_1^u f_{1j} + c_2^u f_{2j})$$

$$+ \{\gamma[2-\gamma][2+\gamma(n-1)] + \gamma^2\}$$

$$\cdot \Bigg\{ - \frac{\gamma^2[n-1][2+\gamma(n-1)]}{\Phi_1} [\alpha_1 - c_1^d] + \frac{\gamma^3[n-1]}{\Phi_1} \sum_{j=1}^{n} (\alpha_j - c_j^d)$$

$$- \frac{2[2+\gamma(n-2)-\gamma^2(n-1)][2+\gamma(n-2)]}{\Phi_1} \sum_{i=1}^{2} v_i \, c_i^u \, f_{i1}$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{\Phi_1} [c_1^u f_{11} + c_2^u f_{21}]$$

$$- \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{\Phi_1} \sum_{j=1}^{n} (c_1^u f_{1j} + c_2^u f_{2j}) \Bigg\}. \qquad (72)$$

Now we simplify the coefficient on w_1 in (61). First consider the terms with Φ_1 in the denominator. (39) implies:

$$\frac{\gamma^2[n-1][2+\gamma(n-1)] - \gamma^3[n-1] - \gamma^2 - \Omega}{\Phi_1}$$

$$+ \frac{2[2+\gamma(n-2)-\gamma^2(n-1)][2+\gamma(n-2)]}{\Phi_1} \sum_{i=1}^{2} v_i \, f_{i1}$$

$$= \frac{1}{\Phi_1} \{ \gamma^2[n-1][2+\gamma(n-1)] - \gamma^3[n-1] - [2-\gamma][2+\gamma(n-1)][2+\gamma(n-2)]$$

$$+ 2\left[2+\gamma(n-2)-\gamma^2(n-1)\right][2+\gamma(n-2)] \sum_{i=1}^{2} v_i \, f_{i1} \}$$

$$= \frac{1}{\Phi_1} \{ \gamma^2[n-1][2+\gamma(n-1)-\gamma] - [2-\gamma][2+\gamma(n-1)][2+\gamma(n-2)]$$

$$+ 2\left[2+\gamma(n-2)-\gamma^2(n-1)\right][2+\gamma(n-2)] \sum_{i=1}^{2} v_i \, f_{i1} \}$$

$$= \frac{1}{\Phi_1}\{\gamma^2[n-1][2+\gamma(n-2)]-[2-\gamma][2+\gamma(n-1)][2+\gamma(n-2)]$$

$$+2\left[2+\gamma(n-2)-\gamma^2(n-1)\right][2+\gamma(n-2)]\sum_{i=1}^{2} v_i f_{i1}\}$$

$$= \frac{1}{\Phi_1}\{\left[\gamma^2(n-1)-(2-\gamma)(2+\gamma[n-1])\right][2+\gamma(n-2)]$$

$$+2\left[2+\gamma(n-2)-\gamma^2(n-1)\right][2+\gamma(n-2)]\sum_{i=1}^{2} v_i f_{i1}\}$$

$$= \frac{2+\gamma[n-2]}{\Phi_1}\{\gamma^2[n-1]-[2-\gamma][2+\gamma(n-1)]$$

$$+2\left[2+\gamma(n-2)-\gamma^2(n-1)\right]\sum_{i=1}^{2} v_i f_{i1}\}$$

$$= \frac{2+\gamma[n-2]}{\Phi_1}\{2\gamma^2[n-1]-\gamma^2[n-1]-[2-\gamma][2+\gamma(n-1)]$$

$$+2\left[2+\gamma(n-2)-\gamma^2(n-1)\right]\sum_{i=1}^{2} v_i f_{i1}\}$$

$$= \frac{2+\gamma[n-2]}{\Phi_1}\{2\gamma^2[n-1]-\gamma^2[n-1]-2[2+\gamma(n-1)]+\gamma[2+\gamma(n-1)]$$

$$+2\left[2+\gamma(n-2)-\gamma^2(n-1)\right]\sum_{i=1}^{2} v_i f_{i1}\}$$

$$= \frac{2+\gamma[n-2]}{\Phi_1}\{2\gamma^2[n-1]-\gamma^2[n-1]-2[2+\gamma(n-1)]+2\gamma+\gamma^2[n-1]$$

$$+2\left[2+\gamma(n-2)-\gamma^2(n-1)\right]\sum_{i=1}^{2} v_i f_{i1}\}$$

$$= \frac{2+\gamma[n-2]}{\Phi_1}\{\left[2\gamma^2(n-1)-2(2+\gamma[n-1])+2\gamma\right]$$

$$+2\left[2+\gamma(n-2)-\gamma^2(n-1)\right]\sum_{i=1}^{2} v_i f_{i1}\}$$

$$= \frac{2+\gamma[n-2]}{\Phi_1}\{2\left[\gamma^2(n-1)-(2+\gamma[n-1])+\gamma\right]$$

$$- \, 2 \left[\gamma^2 (n-1) - 2 - \gamma (n-2) \right] \sum_{i=1}^{2} v_i \, f_{i1} \, \}$$

$$= \; \frac{2 \left[2 + \gamma (n-2) \right] \left[\gamma^2 (n-1) - 2 - \gamma (n-2) \right]}{\Phi_1} \left[1 - \sum_{i=1}^{2} v_i \, f_{i1} \right]$$

$$= \; - \, \frac{2 \left[2 + \gamma (n-2) \right] \left[2 + \gamma (n-2) - \gamma^2 (n-1) \right]}{\Phi_1} \left[1 - \sum_{i=1}^{2} v_i \, f_{i1} \right]$$

$$= \; \frac{\Phi_3}{\Phi_1} \left[1 - \sum_{i=1}^{2} v_i \, f_{i1} \right] . \tag{73}$$

The last equality in (73) reflects (66).

Now consider the terms with Φ_2 in the denominator that multiply w_1 in (61). (60) and (66) imply:

$$- \; \frac{\gamma \left[2 + \gamma (n-1) \right] \left[n-1 \right] + \gamma^2 \left[n-2 \right] - \gamma^3 \left[n-1 \right] - \Omega}{\Phi_2}$$

$$- \; \frac{2 \left[2 + \gamma (n-2) - \gamma^2 (n-1) \right] \left[2 + \gamma (n-2) \right]}{\Phi_2} \sum_{i=1}^{2} v_i \, f_{i1}$$

$$- \; \frac{\gamma^3 \left[n-1 \right] - \gamma \left[2 - \gamma \right] \left[2 + \gamma (n-1) \right]}{2 \, \Phi_2} \left[n-1 \right] \sum_{i=1}^{2} v_i \, f_{i1}$$

$$= \; - \; \frac{\gamma \left[2 + \gamma (n-1) \right] \left[n-1 \right] + \gamma^2 \left[n-2 \right] - \gamma^3 \left[n-1 \right] - \Omega}{\Phi_2}$$

$$+ \; \left[\frac{\Phi_3}{\Phi_2} - \frac{(n-1) \, \Phi_1}{2 \, \Phi_2} \right] \sum_{i=1}^{2} v_i \, f_{i1} \, . \tag{74}$$

(39) and (60) imply:

$$\gamma \left[2 + \gamma (n-1) \right] \left[n-1 \right] + \gamma^2 \left[n-2 \right] - \gamma^3 \left[n-1 \right] - \Omega$$

$$= \; \gamma \left[2 + \gamma (n-1) \right] \left[n-1 \right] + \gamma^2 \left[n-2 \right] - \gamma^3 \left[n-1 \right]$$

$$- \; \left[2 - \gamma \right] \left[2 + \gamma (n-1) \right] \left[2 + \gamma (n-2) \right] + \gamma^2$$

$$= \; \gamma \left[2 + \gamma (n-1) \right] \left[n-1 \right] + \gamma^2 \left[n-1 \right] - \gamma^2 - \gamma^3 \left[n-1 \right]$$

$$- \; \left[2 - \gamma \right] \left[2 + \gamma (n-1) \right] \left[2 + \gamma (n-2) \right] + \gamma^2$$

$$= \gamma [2 + \gamma(n-1)][n-1] + \gamma^2 [n-1] - \gamma^3 [n-1]$$

$$- [2-\gamma][2+\gamma(n-1)][2+\gamma(n-2)]$$

$$= \gamma [2 + \gamma(n-1)][n-1] + \gamma^2 [n-1][1-\gamma]$$

$$- [2-\gamma][2+\gamma(n-1)][2+\gamma(n-2)]$$

$$= [2+\gamma(n-1)][\gamma(n-1) - (2-\gamma)(2+\gamma[n-2])] + \gamma^2 [n-1][1-\gamma]$$

$$= [2+\gamma(n-1)][\gamma(n-1) - (2-\gamma)(2-\gamma+\gamma[n-1])] + \gamma^2 [n-1][1-\gamma]$$

$$= [2+\gamma(n-1)]\left[\gamma(n-1) - (2-\gamma)^2 - (2-\gamma)\gamma(n-1)\right] + \gamma^2 [n-1][1-\gamma]$$

$$= [2+\gamma(n-1)]\left[-(1-\gamma)\gamma(n-1) - (2-\gamma)^2\right] + \gamma^2 [n-1][1-\gamma]$$

$$= -[1-\gamma]\gamma[n-1][2+\gamma(n-1)] - [2-\gamma]^2[2+\gamma(n-1)] + \gamma^2[n-1][1-\gamma]$$

$$= -[1-\gamma]\gamma[n-1][2+\gamma(n-1)] - \Phi_2 . \tag{75}$$

(74) and (75) imply:

$$- \frac{\gamma[2+\gamma(n-1)][n-1] + \gamma^2[n-2] - \gamma^3[n-1] - \Omega}{\Phi_2}$$

$$- \frac{2[2+\gamma(n-2) - \gamma^2(n-1)][2+\gamma(n-2)]}{\Phi_2} \sum_{i=1}^{2} v_i f_{i1}$$

$$- \frac{\gamma^3[n-1] - \gamma[2-\gamma][2+\gamma(n-1)]}{2\,\Phi_2} [n-1] \sum_{i=1}^{2} v_i f_{i1}$$

$$= \frac{[1-\gamma]\gamma[n-1][2+\gamma(n-1)] + \Phi_2}{\Phi_2} + \left[\frac{\Phi_3}{\Phi_2} - \frac{(n-1)\Phi_1}{2\,\Phi_2}\right] \sum_{i=1}^{2} v_i f_{i1}$$

$$= \frac{[1-\gamma]\gamma[n-1][2+\gamma(n-1)] + \Phi_2}{\Phi_2} + \left[\frac{2\,\Phi_3 - (n-1)\Phi_1}{2\,\Phi_2}\right] \sum_{i=1}^{2} v_i f_{i1}. \tag{76}$$

Observe that:

$$\frac{[1-\gamma]\gamma[n-1][2+\gamma(n-1)] + \Phi_2}{\Phi_2} = -\left[\frac{2\,\Phi_3 - (n-1)\Phi_1}{2\,\Phi_2}\right]$$

$$\Leftrightarrow \quad [1-\gamma]\gamma[n-1][2+\gamma(n-1)] + \Phi_2 = -\left[\frac{2\,\Phi_3 - (n-1)\Phi_1}{2}\right]$$

$$\Leftrightarrow \quad 2\,[\,1-\gamma\,]\,\gamma\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]+2\,\Phi_2 \;=\; -\,2\,\Phi_3+[\,n-1\,]\,\Phi_1$$

$$\Leftrightarrow \quad 2\,[\,1-\gamma\,]\,\gamma\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]+2\,\Phi_2+2\,\Phi_3-[\,n-1\,]\,\Phi_1 \;=\; 0. \tag{77}$$

(77) holds because (60) and (66) imply:

$$\begin{aligned}
&2\,[\,1-\gamma\,]\,\gamma\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]+2\,\Phi_2+2\,\Phi_3-[\,n-1\,]\,\Phi_1\\
&= 2\,[\,1-\gamma\,]\,\gamma\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]+2\,[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]-2\,\gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]\\
&\quad -4\left[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,\right][\,2+\gamma\,(\,n-2\,)\,]\\
&\quad +2\,[\,n-1\,]\,\gamma\left[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,\right].
\end{aligned} \tag{78}$$

The last term in (78) holds because, from (60):

$$\begin{aligned}
-\,[\,n-1\,]\,\Phi_1 &= -\,\gamma^3\,[\,n-1\,]^2+\gamma\,[\,2-\gamma\,]\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]\\
&= \gamma\,[\,n-1\,]\left\{\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,n-1\,]\,\right\}\\
&= \gamma\,[\,n-1\,]\left\{\,4+2\,\gamma\,[\,n-1\,]-2\,\gamma-2\,\gamma^2\,[\,n-1\,]\,\right\}\\
&= \gamma\,[\,n-1\,]\left\{\,4+2\,\gamma\,[\,n-2\,]-2\,\gamma^2\,[\,n-1\,]\,\right\}\\
&= 2\,[\,n-1\,]\,\gamma\left[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,\right].
\end{aligned} \tag{79}$$

Observe that the sum of the last two terms in (78) is:

$$\begin{aligned}
&2\,[\,n-1\,]\,\gamma\left[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,\right]-4\left[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,\right][\,2+\gamma\,(\,n-2\,)\,]\\
&\qquad = \left[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,\right][\,2\,\gamma\,(\,n-1\,)-4\,(\,2+\gamma\,[\,n-2\,]\,)\,].
\end{aligned}$$

Therefore, (78) and (79) imply:

$$\begin{aligned}
&2\,[\,1-\gamma\,]\,\gamma\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]+2\,\Phi_2+2\,\Phi_3-[\,n-1\,]\,\Phi_1\\
&= 2\,[\,1-\gamma\,]\,\gamma\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]+2\,[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]-2\,\gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]\\
&\quad +\;[\,2\,(\,n-1\,)\,\gamma-4\,(\,2+\gamma\,[\,n-2\,]\,)\,]\left[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,\right]\\
&= 2\,[\,1-\gamma\,]\,\gamma\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)-\gamma\,]+2\,[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]\\
&\quad +\;[\,2\,(\,n-1\,)\,\gamma-\;4\,(\,2+\gamma\,[\,n-2\,]\,)\,]\left[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,\right]\\
&= 2\,[\,1-\gamma\,]\,\gamma\,[\,n-1\,]\,[\,2+\gamma\,(\,n-2\,)\,]+2\,[\,2-\gamma\,]^2\,[\,2+\gamma\,(\,n-1\,)\,]\\
&\quad +\;[\,2\,(\,n-1\,)\,\gamma-\;4\,(\,2+\gamma\,[\,n-2\,]\,)\,]\left[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,\right]
\end{aligned}$$

$$
\begin{aligned}
&= 2[1-\gamma]\gamma[n-1][2+\gamma(n-2)]+2[2-\gamma]^2[2+\gamma(n-1)] \\
&\quad + [2(n-1)\gamma - 4(2+\gamma[n-1]-\gamma)]\left[2+\gamma(n-2)-\gamma^2(n-1)\right] \\
&= 2[1-\gamma]\gamma[n-1][2+\gamma(n-2)]+2[2-\gamma]^2[2+\gamma(n-1)] \\
&\quad + [2(n-1)\gamma - 4(2+\gamma[n-1]-\gamma)][2+\gamma(n-2)] \\
&\quad - [2(n-1)\gamma - 4(2+\gamma[n-1]-\gamma)]\gamma^2[n-1] \\
&= 2[1-\gamma]\gamma[n-1][2+\gamma(n-2)]+2[2-\gamma]^2[2+\gamma(n-1)] \\
&\quad + [-2(n-1)\gamma - 4(2-\gamma)][2+\gamma(n-2)] \\
&\quad - [2(n-1)\gamma - 4(2+\gamma[n-1]-\gamma)]\gamma^2[n-1] \\
&= [2(1-\gamma)\gamma(n-1)-2(n-1)\gamma - 4(2-\gamma)][2+\gamma(n-2)] \\
&\quad - [2(n-1)\gamma - 4(2+\gamma[n-1]-\gamma)]\gamma^2[n-1]+2[2-\gamma]^2[2+\gamma(n-1)] \\
&= \left[-2\gamma^2(n-1) - 4(2-\gamma)\right][2+\gamma(n-2)] \\
&\quad - [2(n-1)\gamma - 4(2+\gamma[n-1]-\gamma)]\gamma^2[n-1]+2[2-\gamma]^2[2+\gamma(n-1)] \\
&= \left[-2\gamma^2(n-1) - 4(2-\gamma)\right][2+\gamma(n-2)] \\
&\quad - [2(n-1)\gamma - 4(2+\gamma[n-2])]\gamma^2[n-1]+2[2-\gamma]^2[2+\gamma(n-1)] \\
&= \left[-2\gamma^2(n-1) - 4(2-\gamma)\right][2+\gamma(n-2)]+4[2+\gamma(n-2)]\gamma^2[n-1] \\
&\quad - [2\,(n-1)\gamma]\gamma^2[n-1]+2[2-\gamma]^2[2+\gamma(n-1)] \\
&= \left[-2\gamma^2(n-1) - 4(2-\gamma)+4\gamma^2(n-1)\right][2+\gamma(n-2)] \\
&\quad - [2(n-1)\gamma]\gamma^2[n-1]+2[2-\gamma]^2[2+\gamma(n-1)] \\
&= \left[2\gamma^2(n-1) - 4(2-\gamma)\right][2+\gamma(n-2)] \\
&\quad - [2(n-1)\gamma]\gamma^2[n-1]+2[2-\gamma]^2[2+\gamma(n-1)] \\
&= \left[2\gamma^2(n-1) - 4(2-\gamma)\right][2-\gamma+\gamma(n-1)] \\
&\quad - [2(n-1)\gamma]\gamma^2[n-1]+2[2-\gamma]^2[2+\gamma(n-1)] \\
&= \left[2\gamma^2(n-1)\right][2-\gamma+\gamma(n-1)]-4[2-\gamma]^2-4[2-\gamma]\gamma[n-1] \\
&\quad - [2(n-1)\gamma]\gamma^2[n-1]+4[2-\gamma]^2+2[2-\gamma]^2\gamma[n-1]
\end{aligned}
$$

$$= \left[2\gamma^2 (n-1) \right] [2-\gamma+\gamma(n-1)] - 4[2-\gamma]\gamma[n-1]$$

$$- [2(n-1)\gamma]\gamma^2[n-1] + 2[2-\gamma]^2\gamma[n-1]$$

$$= 2\gamma^2[n-1][2-\gamma+\gamma(n-1)] + 2\left[(2-\gamma)^2 - 2(2-\gamma) - \gamma^2(n-1)\right]\gamma[n-1]$$

$$= 2\gamma^2[n-1][2-\gamma+\gamma(n-1)] + 2\left[(2-\gamma)(2-\gamma-2) - \gamma^2(n-1)\right]\gamma[n-1]$$

$$= 2\gamma^2[n-1][2-\gamma+\gamma(n-1)] - 2\gamma[2-\gamma+\gamma(n-1)]\gamma[n-1]$$

$$= 2\gamma^2[n-1][2-\gamma+\gamma(n-1)] - 2\gamma^2[n-1][2-\gamma+\gamma(n-1)] = 0. \tag{80}$$

(76), (77), and (80) imply:

$$- \frac{\gamma[2+\gamma(n-1)][n-1] + \gamma^2[n-2] - \gamma^3[n-1] - \Omega}{\Phi_2}$$

$$- \frac{2[2+\gamma(n-2) - \gamma^2(n-1)][2+\gamma(n-2)]}{\Phi_2} \sum_{i=1}^{2} v_i f_{i1}$$

$$- \frac{\gamma^3[n-1] - \gamma[2-\gamma][2+\gamma(n-1)]}{2\Phi_2} [n-1] \sum_{i=1}^{2} v_i f_{i1}$$

$$= - \left[\frac{2\Phi_3 - (n-1)\Phi_1}{2\Phi_2} \right] \left[1 - \sum_{i=1}^{2} v_i f_{i1} \right]. \tag{81}$$

(73) and (81) imply that the coefficient on w_1 in (61) can be written as:

$$\frac{\gamma^2[n-1][2+\gamma(n-1)] - \gamma^3[n-1] - \gamma^2 - \Omega}{\Phi_1}$$

$$+ \frac{2[2+\gamma(n-2) - \gamma^2(n-1)][2+\gamma(n-2)]}{\Phi_1} \sum_{i=1}^{2} v_i f_{i1}$$

$$- \frac{\gamma[2+\gamma(n-1)][n-1] + \gamma^2[n-2] - \gamma^3[n-1] - \Omega}{\Phi_2}$$

$$- \frac{2[2+\gamma(n-2) - \gamma^2(n-1)][2+\gamma(n-2)]}{\Phi_2} \sum_{i=1}^{2} v_i f_{i1}$$

$$- \frac{\gamma^3[n-1] - \gamma[2-\gamma][2+\gamma(n-1)]}{2\Phi_2} [n-1] \sum_{i=1}^{2} v_i f_{i1}$$

$$= \frac{\Phi_3}{\Phi_1} \left[1 - \sum_{i=1}^{2} v_i \, f_{i1} \right] - \left[\frac{2\,\Phi_3 - (\,n-1\,)\,\Phi_1}{2\,\Phi_2} \right] \left[1 - \sum_{i=1}^{2} v_i \, f_{i1} \right]$$

$$= \left[\frac{\Phi_3}{\Phi_1} - \frac{2\,\Phi_3 - (\,n-1\,)\,\Phi_1}{2\,\Phi_2} \right] \left[1 - \sum_{i=1}^{2} v_i \, f_{i1} \right] . \tag{82}$$

(61) and (82) imply:

$$\left[\frac{\Phi_3}{\Phi_1} - \frac{2\,\Phi_3 - (\,n-1\,)\,\Phi_1}{2\,\Phi_2} \right] \left[1 - \sum_{i=1}^{2} v_i \, f_{i1} \right] w_1$$

$$\begin{aligned}
= & \left\{ \frac{\gamma^2 \,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_1} + \frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]\,[\,1-\gamma\,]}{\Phi_2} \right\} [\,\alpha_1 - c_1^d\,] \\
& - \left\{ \frac{\gamma^3}{\Phi_1} + \frac{2\,\gamma^2 + \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] - 2\,\gamma^3}{2\,\Phi_2} \right\} [\,n-1\,] \sum_{j=1}^{n} (\,\alpha_j - c_j^d\,) \\
& + \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} \sum_{i=2}^{n} (\,\alpha_i - c_i^d\,) \\
& + \frac{2\,[\,2+\gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Phi_1} \sum_{i=1}^{2} v_i \, c_i^u \, f_{i1} \\
& + \left\{ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} - \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{\Phi_1} \right\} [\,c_1^u \, f_{11} + c_2^u \, f_{21}\,] \\
& + \left\{ \frac{[\,1-\gamma\,]\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_2} \right. \\
& \qquad \left. + \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_1} \right\} \sum_{i=1}^{n} (\,c_1^u \, f_{1i} + c_2^u \, f_{2i}\,) \\
& - \{\,4\left[\,2+\gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,\right]\,[\,2+\gamma\,(\,n-2\,)\,] + \gamma^3\,[\,n-1\,]^2 \\
& \qquad - \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]\,\} \frac{\sum_{i=1}^{2} v_i \, c_i^u \, f_{i1}}{2\,\Phi_2} .
\end{aligned} \tag{83}$$

The first and third terms to the right of the equality in (83) can be written as:

$$
\begin{aligned}
& \left\{ \frac{\gamma^2 [\, n-1\,]\, [\, 2+\gamma\, (\, n-1\,)\,]}{\Phi_1} + \frac{\gamma\, [\, 2+\gamma\, (\, n-1\,)\,]\, [\, n-1\,]\, [\, 1-\gamma\,]}{\Phi_2} \right\} [\, \alpha_1 - c_1^d\,] \\
& \quad + \frac{[\, 2-\gamma\,]\, [\, 2+\gamma\, (\, n-1\,)\,]^2}{2\, \Phi_2} \sum_{i=2}^{n} (\, \alpha_i - c_i^d\,) \\
= & \left\{ \frac{\gamma^2 [\, n-1\,]\, [\, 2+\gamma\, (\, n-1\,)\,]}{\Phi_1} + \frac{\gamma\, [\, 2+\gamma\, (\, n-1\,)\,]\, [\, n-1\,]\, [\, 1-\gamma\,]}{\Phi_2} \right\} [\, \alpha_1 - c_1^d\,] \\
& \quad + \frac{[\, 2-\gamma\,]\, [\, 2+\gamma\, (\, n-1\,)\,]^2}{2\, \Phi_2} \sum_{i=2}^{n} (\, \alpha_i - c_i^d\,) + \frac{[\, 2-\gamma\,]\, [\, 2+\gamma\, (\, n-1\,)\,]^2}{2\, \Phi_2} [\, \alpha_1 - c_1^d\,] \\
& \quad - \frac{[\, 2-\gamma\,]\, [\, 2+\gamma\, (\, n-1\,)\,]^2}{2\, \Phi_2} [\, \alpha_1 - c_1^d\,] \\
= & \left\{ \frac{\gamma^2 [\, n-1\,]\, [\, 2+\gamma\, (\, n-1\,)\,]}{\Phi_1} + \frac{\gamma\, [\, 2+\gamma\, (\, n-1\,)\,]\, [\, n-1\,]\, [\, 1-\gamma\,]}{\Phi_2} \right\} [\, \alpha_1 - c_1^d\,] \\
& \quad + \frac{[\, 2-\gamma\,]\, [\, 2+\gamma\, (\, n-1\,)\,]^2}{2\, \Phi_2} \sum_{i=1}^{n} (\, \alpha_i - c_i^d\,) - \frac{[\, 2-\gamma\,]\, [\, 2+\gamma\, (\, n-1\,)\,]^2}{2\, \Phi_2} [\, \alpha_1 - c_1^d\,] \\
= & \left\{ \frac{\gamma^2 [\, n-1\,]\, [\, 2+\gamma\, (\, n-1\,)\,]}{\Phi_1} + \frac{\gamma\, [\, 2+\gamma\, (\, n-1\,)\,]\, [\, n-1\,]\, [\, 1-\gamma\,]}{\Phi_2} \right. \\
& \qquad \qquad \left. - \frac{[\, 2-\gamma\,]\, [\, 2+\gamma\, (\, n-1\,)\,]^2}{2\, \Phi_2} \right\} [\, \alpha_1 - c_1^d\,] \\
& \quad + \frac{[\, 2-\gamma\,]\, [\, 2+\gamma\, (\, n-1\,)\,]^2}{2\, \Phi_2} \sum_{i=1}^{n} (\, \alpha_i - c_i^d\,) \\
= & \, [\, 2+\gamma\, (\, n-1\,)\,]\, [\, \alpha_1 - c_1^d\,] \\
& \cdot \left\{ \frac{\gamma^2 [\, n-1\,]}{\Phi_1} + \frac{2\, \gamma\, [\, n-1\,]\, [\, 1-\gamma\,] - [\, 2-\gamma\,]\, [\, 2+\gamma\, (\, n-1\,)\,]}{2\, \Phi_2} \right\} \\
& + \frac{[\, 2-\gamma\,]\, [\, 2+\gamma\, (\, n-1\,)\,]^2}{2\, \Phi_2} \sum_{i=1}^{n} (\, \alpha_i - c_i^d\,) \, . \qquad (84)
\end{aligned}
$$

(83) and (84) imply:

$$\left[\frac{\Phi_3}{\Phi_1} - \frac{2\,\Phi_3 - (\,n-1\,)\,\Phi_1}{2\,\Phi_2} \right] \left[1 - \sum_{i=1}^{2} v_i\, f_{i1} \right] w_1$$

$$= \; [\,2 + \gamma\,(\,n-1\,)\,]\,\left[\,\alpha_1 - c_1^d\,\right]$$

$$\cdot \left\{ \frac{\gamma^2\,[\,n-1\,]}{\Phi_1} + \frac{2\,\gamma\,[\,n-1\,]\,[\,1-\gamma\,] - [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\,\Phi_2} \right\}$$

$$+ \; \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} \sum_{i=1}^{n} \left(\,\alpha_i - c_i^d\,\right)$$

$$- \; \left\{ \frac{\gamma^3}{\Phi_1} + \frac{2\,\gamma^2 + \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] - 2\,\gamma^3}{2\,\Phi_2} \right\} [\,n-1\,] \sum_{j=1}^{n} \left(\,\alpha_j - c_j^d\,\right)$$

$$+ \; \frac{2\,[\,2+\gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Phi_1} \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$+ \; \left\{ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} - \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{\Phi_1} \right\} [\,c_1^u\, f_{11} + c_2^u\, f_{21}\,]$$

$$+ \; \left\{ \frac{[\,1-\gamma\,]\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_2} \right.$$

$$\left. + \; \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_1} \right\} \sum_{i=1}^{n} (\,c_1^u\, f_{1i} + c_2^u\, f_{2i}\,)$$

$$- \; \{\,4\left[\,2+\gamma\,(\,n-2\,) - \gamma^2\,(\,n-1\,)\,\right]\,[\,2+\gamma\,(\,n-2\,)\,] + \gamma^3\,[\,n-1\,]^2$$

$$- \; \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]\,\}\; \frac{\sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}}{2\,\Phi_2}\,. \tag{85}$$

(60), (66), (67), and (85) imply:

$$\left[\frac{\Phi_3}{\Phi_1} - \frac{2\,\Phi_3 - (\,n-1\,)\,\Phi_1}{2\,\Phi_2} \right] \left[1 - \sum_{i=1}^{2} v_i\, f_{i1} \right] w_1$$

$$= \; [\,2 + \gamma\,(\,n-1\,)\,]\,\left[\,\alpha_1 - c_1^d\,\right]$$

$$\cdot \left\{ \frac{\gamma^2 [n-1]}{\Phi_1} + \frac{2\gamma [n-1][1-\gamma] - [2-\gamma][2+\gamma(n-1)]}{2\Phi_2} \right\}$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Phi_2} \sum_{i=1}^{n} (\alpha_i - c_i^d)$$

$$- \left[\frac{\gamma^3}{\Phi_1} + \frac{\Phi_4 + \gamma^2 - 2\gamma^3}{2\Phi_2} \right] [n-1] \sum_{j=1}^{n} (\alpha_j - c_j^d) - \frac{\Phi_3}{\Phi_1} \sum_{i=1}^{2} v_i c_i^u f_{i1}$$

$$+ \left\{ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Phi_2} - \frac{[2-\gamma][2+\gamma(n-1)]^2}{\Phi_1} \right\} [c_1^u f_{11} + c_2^u f_{21}]$$

$$+ \left\{ \frac{[1-\gamma][2-\gamma][2+\gamma(n-1)]}{\Phi_2} \right.$$

$$\left. + \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{\Phi_1} \right\} \sum_{i=1}^{n} (c_1^u f_{1i} + c_2^u f_{2i})$$

$$+ [2\Phi_3 - \Phi_1(n-1)] \frac{\sum_{i=1}^{2} v_i c_i^u f_{i1}}{2\Phi_2}$$

$$= [2+\gamma(n-1)][\alpha_1 - c_1^d]$$

$$\cdot \left\{ \frac{\gamma^2 [n-1]}{\Phi_1} + \frac{2\gamma [n-1][1-\gamma] - [2-\gamma][2+\gamma(n-1)]}{2\Phi_2} \right\}$$

$$+ \left\{ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Phi_2} - \frac{\gamma^3 [n-1]}{\Phi_1} \right.$$

$$\left. - \frac{[\Phi_4 + \gamma^2(1-2\gamma)][n-1]}{2\Phi_2} \right\} \sum_{j=1}^{n} (\alpha_j - c_j^d) - \frac{\Phi_3}{\Phi_1} \sum_{i=1}^{2} v_i c_i^u f_{i1}$$

$$+ [2-\gamma][2+\gamma(n-1)]^2 \left[\frac{1}{2\Phi_2} - \frac{1}{\Phi_1} \right] [c_1^u f_{11} + c_2^u f_{21}]$$

$$+ \left\{ \frac{[1-\gamma][2-\gamma][2+\gamma(n-1)]}{\Phi_2} + \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{\Phi_1} \right\} \sum_{i=1}^{n} (c_1^u f_{1i} + c_2^u f_{2i})$$

$$+ [2\Phi_3 - \Phi_1(n-1)] \frac{\sum_{i=1}^{2} v_i c_i^u f_{i1}}{2\Phi_2}$$

$$= \; [\, 2 + \gamma\,(\, n-1\,)\,]\,[\,\alpha_1 - c_1^d\,]$$

$$\cdot \left\{ \frac{\gamma^2\,[\,n-1\,]}{\Phi_1} + \frac{2\,\gamma\,[\,n-1\,]\,[\,1-\gamma\,] - [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\,\Phi_2} \right\}$$

$$+ \left\{ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} - \frac{\gamma^3\,[\,n-1\,]}{\Phi_1} \right.$$

$$\left. - \frac{[\,\Phi_4 + \gamma^2\,(\,1-2\,\gamma\,)\,]\,[\,n-1\,]}{2\,\Phi_2} \right\} \sum_{j=1}^{n} (\,\alpha_j - c_j^d\,)$$

$$+ \left[\frac{2\,\Phi_3 - \Phi_1\,(\,n-1\,)}{2\,\Phi_2} - \frac{\Phi_3}{\Phi_1} \right] \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$+ \; [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2 \left[\frac{1}{2\,\Phi_2} - \frac{1}{\Phi_1} \right] [\,c_1^u\, f_{11} + c_2^u\, f_{21}\,]$$

$$+ \; [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] \left[\frac{2\,(\,1-\gamma\,)}{2\,\Phi_2} + \frac{\gamma}{\Phi_1} \right] \sum_{i=1}^{n} (\,c_1^u\, f_{1i} + c_2^u\, f_{2i}\,)\,. \tag{86}$$

(67) implies:

$$\begin{aligned} \Phi_4 + \gamma^2\,[\,1-2\,\gamma\,] \;&=\; \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] + \gamma^2 + \gamma^2\,[\,1-2\,\gamma\,] \\ &=\; \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] + 2\,\gamma^2\,[\,1-\gamma\,]\,. \end{aligned} \tag{87}$$

(87) implies that the coefficient on $\sum_{j=1}^{n} \left(\,\alpha_j - c_j^d\,\right)$ in (86) is:

$$\frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} - \frac{\gamma^3\,[\,n-1\,]}{\Phi_1} - \frac{[\,\Phi_4 + \gamma^2\,(\,1-2\,\gamma\,)\,]\,[\,n-1\,]}{2\,\Phi_2}$$

$$= \; \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2 - [\,\Phi_4 + \gamma^2\,(\,1-2\,\gamma\,)\,]\,[\,n-1\,]}{2\,\Phi_2} - \frac{\gamma^3\,[\,n-1\,]}{\Phi_1}$$

$$= \; \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2 - \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,] - 2\,\gamma^2\,[\,1-\gamma\,]\,[\,n-1\,]}{2\,\Phi_2}$$

$$- \; \frac{\gamma^3\,[\,n-1\,]}{\Phi_1}$$

$$= \; \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-1\,) - \gamma\,(\,n-1\,)\,] - 2\,\gamma^2\,[\,1-\gamma\,]\,[\,n-1\,]}{2\,\Phi_2} - \frac{\gamma^3\,[\,n-1\,]}{\Phi_1}$$

$$= \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,1-\gamma\,]\,[\,n-1\,]}{\Phi_2} - \frac{\gamma^3\,[\,n-1\,]}{\Phi_1}\,. \tag{88}$$

(86) and (88) imply:

$$\left[\frac{\Phi_3}{\Phi_1} - \frac{2\,\Phi_3-(\,n-1\,)\,\Phi_1}{2\,\Phi_2}\right]\left[1-\sum_{i=1}^{2} v_i\,f_{i1}\right] w_1$$

$$= [\,2+\gamma\,(\,n-1\,)\,]\,[\,\alpha_1 - c_1^d\,]$$

$$\cdot\left\{\frac{\gamma^2\,[\,n-1\,]}{\Phi_1} + \frac{2\,\gamma\,[\,n-1\,]\,[\,1-\gamma\,]-[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\,\Phi_2}\right\}$$

$$+ \left\{\frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,1-\gamma\,]\,[\,n-1\,]}{\Phi_2} - \frac{\gamma^3\,[\,n-1\,]}{\Phi_1}\right\}\sum_{j=1}^{n}(\,\alpha_j - c_j^d\,)$$

$$+ \left[\frac{2\,\Phi_3-\Phi_1\,(\,n-1\,)}{2\,\Phi_2} - \frac{\Phi_3}{\Phi_1}\right]\sum_{i=1}^{2} v_i\,c_i^u\,f_{i1}$$

$$+ [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2\left[\frac{1}{2\,\Phi_2} - \frac{1}{\Phi_1}\right][\,c_1^u\,f_{11} + c_2^u\,f_{21}\,]$$

$$+ [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\left[\frac{2\,(\,1-\gamma\,)}{2\,\Phi_2} + \frac{\gamma}{\Phi_1}\right]\sum_{i=1}^{n}(\,c_1^u\,f_{1i} + c_2^u\,f_{2i}\,)\,. \tag{89}$$

<u>Definition</u>. $\Psi \equiv \left[\frac{\Phi_3}{\Phi_1} - \frac{2\,\Phi_3-(\,n-1\,)\,\Phi_1}{2\,\Phi_2}\right]\left[1-\sum_{i=1}^{2} v_i\,f_{i1}\right].$ (90)

(89) implies that if $\Psi \neq 0$:

$$w_1 = \frac{2+\gamma\,[\,n-1\,]}{\Psi}\,[\,\alpha_1 - c_1^d\,]$$

$$\cdot\left\{\frac{\gamma^2\,[\,n-1\,]}{\Phi_1} + \frac{2\,\gamma\,[\,n-1\,]\,[\,1-\gamma\,]-[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\,\Phi_2}\right\}$$

$$+ \frac{1}{\Psi}\left\{\frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]-\gamma^2\,[\,1-\gamma\,]\,[\,n-1\,]}{\Phi_2} - \frac{\gamma^3\,[\,n-1\,]}{\Phi_1}\right\}\sum_{j=1}^{n}(\,\alpha_j - c_j^d\,)$$

$$- \frac{1}{1-\sum_{i=1}^{2} v_i\,f_{i1}}\sum_{i=1}^{2} v_i\,c_i^u\,f_{i1}$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{\Psi} \left[\frac{1}{2\Phi_2} - \frac{1}{\Phi_1} \right] [c_1^u f_{11} + c_2^u f_{21}]$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]}{\Psi} \left[\frac{2(1-\gamma)}{2\Phi_2} + \frac{\gamma}{\Phi_1} \right] \sum_{i=1}^{n} (c_1^u f_{1i} + c_2^u f_{2i}) . \tag{91}$$

Definition. $$\varphi \equiv \frac{2\Phi_2\Phi_3\Phi_4 - \Phi_2\Phi_1^2}{2\Phi_2\Phi_3 - 2\Phi_3\Phi_1 + [n-1]\Phi_1^2} . \tag{92}$$

Observe that:

$$\frac{\Phi_4 \left[\frac{\Phi_3}{\Phi_1} - \frac{\Phi_1}{2\Phi_4} \right]}{\frac{\Phi_3}{\Phi_1} - \frac{2\Phi_3 - [n-1]\Phi_1}{2\Phi_2}} = \frac{\frac{\Phi_4\Phi_3}{\Phi_1} - \frac{\Phi_1}{2}}{\frac{2\Phi_2\Phi_3 - 2\Phi_3\Phi_1 + [n-1]\Phi_1^2}{2\Phi_1\Phi_2}} = \frac{\frac{2\Phi_4\Phi_3 - \Phi_1^2}{2\Phi_1}}{\frac{2\Phi_2\Phi_3 - 2\Phi_3\Phi_1 + [n-1]\Phi_1^2}{2\Phi_1\Phi_2}}$$

$$= \frac{2\Phi_2\Phi_3\Phi_4 - \Phi_2\Phi_1^2}{2\Phi_2\Phi_3 - 2\Phi_3\Phi_1 + [n-1]\Phi_1^2} = \varphi . \tag{93}$$

(60) implies:

$$\begin{aligned}
\Phi_1 &= \gamma^3 [n-1] + \left[\gamma^2 - 2\gamma\right] [2 + \gamma(n-1)] \\
&= \gamma^3 [n-1] + 2\gamma^2 + \gamma^3 [n-1] - 4\gamma - 2\gamma^2 [n-1] \\
&= 2\gamma^3 [n-1] - 2\gamma^2 [n-2] - 4\gamma = 2\gamma \left[\gamma^2 (n-1) - \gamma(n-2) - 2\right] \\
&= -2\gamma \left[2 + \gamma(n-2) - \gamma^2 (n-1)\right] .
\end{aligned} \tag{94}$$

To simplify the denominator of φ in (92), observe that (66) and (94) imply:

$$\begin{aligned}
&-2\Phi_3\Phi_1 + [n-1]\Phi_1^2 \\
&= -8\gamma[2+\gamma(n-2)] \left[2+\gamma(n-2) - \gamma^2(n-1)\right]^2 \\
&\qquad + 4[n-1]\gamma^2 \left[2+\gamma(n-2) - \gamma^2(n-1)\right]^2 \\
&= \left[2+\gamma(n-2) - \gamma^2(n-1)\right]^2 \left\{ 4[n-1]\gamma^2 - 8\gamma[2+\gamma(n-2)] \right\} \\
&= \left[2+\gamma(n-2) - \gamma^2(n-1)\right]^2 \left\{ 4[n-1]\gamma^2 - 8\gamma[2-\gamma+\gamma(n-1)] \right\} \\
&= \left[2+\gamma(n-2) - \gamma^2(n-1)\right]^2 \left[4\gamma^2(n-1) - 8\gamma^2(n-1) - 8\gamma(2-\gamma) \right] \\
&= \left[2+\gamma(n-2) - \gamma^2(n-1)\right]^2 \left[-4\gamma^2(n-1) - 8\gamma(2-\gamma) \right]
\end{aligned}$$

$$= \; - 4 \gamma [\gamma (n - 1) + 2 (2 - \gamma)] \left[2 + \gamma (n - 2) - \gamma^2 (n - 1) \right]^2 . \tag{95}$$

(60), (66), and (95) imply that the denominator of φ in (92) can be written as:

$$\begin{aligned} & 2 \Phi_2 \Phi_3 - 2 \Phi_3 \Phi_1 + [n - 1] \Phi_1^2 \\ = & \; - 4 \left\{ [2 - \gamma]^2 [2 + \gamma (n - 1)] - \gamma^2 [n - 1] [1 - \gamma] \right\} \\ & \quad \cdot \left[2 + \gamma (n - 2) - \gamma^2 (n - 1) \right] [2 + \gamma (n - 2)] \\ & - 4 \gamma [\gamma (n - 1) + 2 (2 - \gamma)] \left[2 + \gamma (n - 2) - \gamma^2 (n - 1) \right]^2 \\ = & \; - 4 \left[2 + \gamma (n - 2) - \gamma^2 (n - 1) \right] \\ & \quad \cdot \left\{ [2 - \gamma]^2 [2 + \gamma (n - 1)] [2 + \gamma (n - 2)] - \gamma^2 [n - 1] [1 - \gamma] [2 + \gamma (n - 2)] \right. \\ & \qquad \left. + \gamma [\gamma (n - 1) + 2 (2 - \gamma)] \left[2 + \gamma (n - 2) - \gamma^2 (n - 1) \right] \right\} . \end{aligned} \tag{96}$$

(60), (66), (67), and (94) imply that the numerator of φ in (92) can be written as:

$$\begin{aligned} & 2 \Phi_2 \Phi_3 \Phi_4 - \Phi_2 \Phi_1^2 \\ = & \; - 4 \left[2 + \gamma (n - 2) - \gamma^2 (n - 1) \right] [2 + \gamma (n - 2)] \\ & \quad \cdot \left\{ [2 - \gamma]^2 [2 + \gamma (n - 1)] - \gamma^2 [n - 1] [1 - \gamma] \right\} \left\{ \gamma [2 - \gamma] [2 + \gamma (n - 1)] + \gamma^2 \right\} \\ & - 4 \gamma^2 \left[2 + \gamma (n - 2) - \gamma^2 (n - 1) \right]^2 \\ & \qquad \cdot \left\{ [2 - \gamma]^2 [2 + \gamma (n - 1)] - \gamma^2 [n - 1] [1 - \gamma] \right\} . \end{aligned} \tag{97}$$

(60), (67), (94), (96), and (97) imply that after canceling out $- 4 [2 + \gamma (n - 2) - \gamma^2 (n - 1)]$, the common term in the numerator and denominator of φ, φ can be written as:

$$\varphi \; = \; \frac{\varphi_1}{\varphi_2} \tag{98}$$

where:

$$\begin{aligned} \varphi_1 \; \equiv & \; [2 + \gamma (n - 2)] \\ & \quad \cdot \left\{ [2 - \gamma]^2 [2 + \gamma (n - 1)] - \gamma^2 [n - 1] [1 - \gamma] \right\} \left\{ \gamma [2 - \gamma] [2 + \gamma (n - 1)] + \gamma^2 \right\} \\ & \quad + \gamma^2 \left[2 + \gamma (n - 2) - \gamma^2 (n - 1) \right] \left\{ [2 - \gamma]^2 [2 + \gamma (n - 1)] - \gamma^2 [n - 1] [1 - \gamma] \right\} \\ = & \; \left\{ [2 - \gamma]^2 [2 + \gamma (n - 1)] - \gamma^2 [n - 1] [1 - \gamma] \right\} \\ & \quad \cdot \left\{ [2 + \gamma (n - 2)] \left[\gamma (2 - \gamma) (2 + \gamma [n - 1]) + \gamma^2 \right] \right. \end{aligned}$$

$$+ \gamma^2 \left[2 + \gamma (n - 2) - \gamma^2 (n - 1) \right] \}$$

$$= \Phi_2 \left\{ [2 + \gamma (n - 2)] \Phi_4 - \frac{\gamma \Phi_1}{2} \right\}; \tag{99}$$

$$\varphi_2 \equiv [2-\gamma]^2 [2+\gamma(n-1)][2+\gamma(n-2)] - \gamma^2 [n-1][1-\gamma][2+\gamma(n-2)]$$
$$+ \gamma [\gamma(n-1) + 2(2-\gamma)] \left[2+\gamma(n-2) - \gamma^2(n-1)\right]$$
$$= \left\{ [2-\gamma]^2 [2+\gamma(n-1)] - \gamma^2 [n-1][1-\gamma] \right\} [2+\gamma(n-2)]$$
$$+ \gamma [\gamma(n-1) + 2(2-\gamma)] \left[2+\gamma(n-2) - \gamma^2(n-1)\right]$$
$$= [2+\gamma(n-2)] \Phi_2 - [\gamma(n-1) + 2(2-\gamma)] \frac{\Phi_1}{2}. \tag{100}$$

Using (83) to derive an expression for w_1, then substituting this expression into (72), using (92) and (93), implies that for $i \in \{2, ..., n\}$:

$$[2-\gamma][2+\gamma(n-1)]^2 \, w_i$$
$$= \varphi \left\{ \frac{\gamma^2 [n-1][2+\gamma(n-1)]}{\Phi_1} + \frac{\gamma [2+\gamma(n-1)][n-1][1-\gamma]}{\Phi_2} \right\} [\alpha_1 - c_1^d]$$
$$- \varphi \left\{ \frac{\gamma^3}{\Phi_1} + \frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)] - 2\gamma^3}{2\Phi_2} \right\} [n-1] \sum_{j=1}^{n} (\alpha_j - c_j^d)$$
$$+ \varphi \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Phi_2} \sum_{i=2}^{n} (\alpha_i - c_i^d)$$
$$+ \varphi \frac{2[2+\gamma(n-2) - \gamma^2(n-1)][2+\gamma(n-2)]}{\Phi_1} \sum_{i=1}^{2} v_i \, c_i^u \, f_{i1}$$
$$+ \varphi \left\{ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Phi_2} - \frac{[2-\gamma][2+\gamma(n-1)]^2}{\Phi_1} \right\} [c_1^u f_{11} + c_2^u f_{21}]$$
$$+ \varphi \left\{ \frac{[1-\gamma][2-\gamma][2+\gamma(n-1)]}{\Phi_2} \right.$$
$$\left. + \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{\Phi_1} \right\} \sum_{i=1}^{n} (c_1^u f_{1i} + c_2^u f_{2i})$$
$$- \varphi \left\{ 4 \left[2+\gamma(n-2) - \gamma^2(n-1)\right][2+\gamma(n-2)] + \gamma^3 [n-1]^2 \right.$$

$$- \ \gamma [2 - \gamma] [2 + \gamma (n-1)] [n-1] \} \frac{\sum_{i=1}^{2} v_i \, c_i^u \, f_{i1}}{2 \, \Phi_2}$$

$$+ \gamma [2 + \gamma (n-1)] [\alpha_1 - c_1^d] + \frac{[2-\gamma][2+\gamma(n-1)]^2}{2} [\alpha_i - c_i^d]$$

$$- \frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)]}{2} \sum_{j=1}^{n} (\alpha_j - c_j^d) - \frac{\Phi_1}{2} \sum_{i=1}^{2} v_i \, c_i^u \, f_{i1}$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{2} [c_1^u \, f_{1i} + c_2^u \, f_{2i}]$$

$$- \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{2} \sum_{j=1}^{n} (c_1^u \, f_{1j} + c_2^u \, f_{2j})$$

$$+ \{ \gamma [2-\gamma][2+\gamma(n-1)] + \gamma^2 \}$$

$$\cdot \Bigg\{ - \frac{\gamma^2 [n-1][2+\gamma(n-1)]}{\Phi_1} [\alpha_1 - c_1^d] + \frac{\gamma^3 [n-1]}{\Phi_1} \sum_{j=1}^{n} (\alpha_j - c_j^d)$$

$$- \frac{2[2+\gamma(n-2) - \gamma^2(n-1)][2+\gamma(n-2)]}{\Phi_1} \sum_{i=1}^{2} v_i \, c_i^u \, f_{i1}$$

$$+ \frac{[2-\gamma][2+\gamma(n-1)]^2}{\Phi_1} [c_1^u \, f_{11} + c_2^u \, f_{21}]$$

$$- \frac{\gamma[2-\gamma][2+\gamma(n-1)]}{\Phi_1} \sum_{j=1}^{n} (c_1^u \, f_{1j} + c_2^u \, f_{2j}) \Bigg\} . \tag{101}$$

Observe that the first and third terms to the right of the equality in (101) can be written as:

$$\varphi \left\{ \frac{\gamma^2 [n-1][2+\gamma(n-1)]}{\Phi_1} + \frac{\gamma[2+\gamma(n-1)][n-1][1-\gamma]}{\Phi_2} \right\} [\alpha_1 - c_1^d]$$

$$+ \varphi \, \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\,\Phi_2} \sum_{i=2}^{n} (\alpha_i - c_i^d)$$

$$= \ \varphi \left\{ \frac{\gamma^2 [n-1][2+\gamma(n-1)]}{\Phi_1} + \frac{\gamma[2+\gamma(n-1)][n-1][1-\gamma]}{\Phi_2} \right\} [\alpha_1 - c_1^d]$$

$$+ \varphi \, \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} \sum_{i=2}^{n} (\,\alpha_i - c_i^d\,) + \varphi \, \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} \, [\,\alpha_1 - c_1^d\,]$$

$$- \varphi \, \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} \, [\,\alpha_1 - c_1^d\,]$$

$$= \varphi \left\{ \frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_1} + \frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]\,[\,1-\gamma\,]}{\Phi_2} \right\} [\,\alpha_1 - c_1^d\,]$$

$$+ \varphi \, \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} \sum_{i=1}^{n} (\,\alpha_i - c_i^d\,) - \varphi \, \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} \, [\,\alpha_1 - c_1^d\,]$$

$$= \varphi \left\{ \frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_1} + \frac{\gamma\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,]\,[\,1-\gamma\,]}{\Phi_2} \right.$$

$$\left. - \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} \right\} [\,\alpha_1 - c_1^d\,]$$

$$+ \varphi \, \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} \sum_{i=1}^{n} (\,\alpha_i - c_i^d\,)$$

$$= \varphi\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,\alpha_1 - c_1^d\,]$$

$$\cdot \left\{ \frac{\gamma^2\,[\,n-1\,]}{\Phi_1} + \frac{2\,\gamma\,[\,n-1\,]\,[\,1-\gamma\,] - [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\,\Phi_2} \right\}$$

$$+ \varphi \, \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} \sum_{i=1}^{n} (\,\alpha_i - c_i^d\,)\,. \qquad (102)$$

(101) and (102) imply that for $i \in \{2, ..., n\}$:

$$[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2 \; w_i$$

$$= \varphi\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,\alpha_1 - c_1^d\,]$$

$$\cdot \left\{ \frac{\gamma^2\,[\,n-1\,]}{\Phi_1} + \frac{2\,\gamma\,[\,n-1\,]\,[\,1-\gamma\,] - [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\,\Phi_2} \right\}$$

$$+ \varphi \, \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} \sum_{i=1}^{n} (\,\alpha_i - c_i^d\,)$$

$$- \varphi \left\{ \frac{\gamma^3}{\Phi_1} + \frac{2\,\gamma^2 + \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] - 2\,\gamma^3}{2\,\Phi_2} \right\} [\,n-1\,] \sum_{j=1}^{n} (\,\alpha_j - c_j^d\,)$$

$$+ \varphi \, \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Phi_1} \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$+ \varphi \left\{ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} - \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{\Phi_1} \right\} [\,c_1^u\, f_{11} + c_2^u\, f_{21}\,]$$

$$+ \varphi \left\{ \frac{[\,1-\gamma\,]\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_2} \right.$$

$$\left. + \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_1} \right\} \sum_{i=1}^{n} (\,c_1^u\, f_{1i} + c_2^u\, f_{2i}\,)$$

$$- \varphi \left\{ 4 \left[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,\right] [\,2+\gamma\,(\,n-2\,)\,] + \gamma^3\,[\,n-1\,]^2 \right.$$

$$\left. - \; \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\,[\,n-1\,] \right\} \frac{\sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}}{2\,\Phi_2}$$

$$+ \gamma\,[\,2+\gamma\,(\,n-1\,)\,] \left[\,\alpha_1 - c_1^d\,\right] + \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2} \left[\,\alpha_i - c_i^d\,\right]$$

$$- \frac{2\,\gamma^2 + \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2} \sum_{j=1}^{n} \left(\,\alpha_j - c_j^d\,\right) - \frac{\Phi_1}{2} \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2} \,[\,c_1^u\, f_{1i} + c_2^u\, f_{2i}\,]$$

$$- \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2} \sum_{j=1}^{n} (\,c_1^u\, f_{1j} + c_2^u\, f_{2j}\,)$$

$$+ \left\{ \gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] + \gamma^2 \right\}$$

$$\cdot \left\{ - \frac{\gamma^2\,[\,n-1\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_1} \left[\,\alpha_1 - c_1^d\,\right] + \frac{\gamma^3\,[\,n-1\,]}{\Phi_1} \sum_{j=1}^{n} \left(\,\alpha_j - c_j^d\,\right) \right.$$

$$- \frac{2\,[\,2+\gamma\,(\,n-2\,)-\gamma^2\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\Phi_1} \sum_{i=1}^{2} v_i\, c_i^u\, f_{i1}$$

$$+ \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{\Phi_1} \,[\,c_1^u\, f_{11} + c_2^u\, f_{21}\,]$$

$$- \frac{\gamma [2-\gamma][2+\gamma(n-1)]}{\Phi_1} \sum_{j=1}^{n} (c_1^u f_{1j} + c_2^u f_{2j}) \Bigg\} . \tag{103}$$

(67) implies that the coefficient of $\left[\alpha_1 - c_1^d \right]$ in (103) is:

$$\varphi [2+\gamma(n-1)] \left\{ \frac{\gamma^2 [n-1]}{\Phi_1} + \frac{2\gamma [n-1][1-\gamma] - [2-\gamma][2+\gamma(n-1)]}{2\Phi_2} \right\}$$

$$+ \gamma [2+\gamma(n-1)] - \left\{ \gamma [2-\gamma][2+\gamma(n-1)] + \gamma^2 \right\} \frac{\gamma^2 [n-1][2+\gamma(n-1)]}{\Phi_1}$$

$$= [2+\gamma(n-1)]$$

$$\cdot \left\{ \varphi \frac{\gamma^2 [n-1]}{\Phi_1} + \varphi \frac{2\gamma [n-1][1-\gamma] - [2-\gamma][2+\gamma(n-1)]}{2\Phi_2} + \gamma \right.$$

$$\left. - \left\{ \gamma [2-\gamma][2+\gamma(n-1)] + \gamma^2 \right\} \frac{\gamma^2 [n-1]}{\Phi_1} \right\}$$

$$= [2+\gamma(n-1)]$$

$$\cdot \left\{ \varphi \frac{2\gamma [n-1][1-\gamma] - [2-\gamma][2+\gamma(n-1)]}{2\Phi_2} + \gamma \right.$$

$$\left. + \left\{ \varphi - \gamma [2-\gamma][2+\gamma(n-1)] - \gamma^2 \right\} \frac{\gamma^2 [n-1]}{\Phi_1} \right\}$$

$$= [2+\gamma(n-1)]$$

$$\cdot \left\{ \varphi \frac{2\gamma [n-1][1-\gamma] - [2-\gamma][2+\gamma(n-1)]}{2\Phi_2} + \gamma + \frac{\gamma^2 [n-1][\varphi - \Phi_4]}{\Phi_1} \right\} . \tag{104}$$

(67) implies that the coefficient of $\sum_{i=1}^{n} \left(\alpha_i - c_i^d \right)$ in (103) is:

$$\varphi \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Phi_2} - \varphi \left\{ \frac{\gamma^3}{\Phi_1} + \frac{2\gamma^2 + \gamma [2-\gamma][2+\gamma(n-1)] - 2\gamma^3}{2\Phi_2} \right\} [n-1]$$

$$- \frac{2\gamma^2 + \gamma [2-\gamma][2+\gamma(n-1)]}{2} + \left\{ \gamma [2-\gamma][2+\gamma(n-1)] + \gamma^2 \right\} \frac{\gamma^3 [n-1]}{\Phi_1}$$

$$= \varphi \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Phi_2} - \varphi \frac{\gamma^3 [n-1]}{\Phi_1}$$

$$
\begin{aligned}
&- \varphi \, \frac{\{ 2\gamma^2 [1-\gamma] + \gamma [2-\gamma][2+\gamma(n-1)] \} [n-1]}{2\Phi_2} \\
&- \frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)]}{2} + \frac{\Phi_4 \gamma^3 [n-1]}{\Phi_1} \\
= \; & \varphi \, \frac{[2-\gamma][2+\gamma(n-1)]^2}{2\Phi_2} - \varphi \, \frac{\{ 2\gamma^2 [1-\gamma] + \gamma[2-\gamma][2+\gamma(n-1)] \} [n-1]}{2\Phi_2} \\
&- \frac{\gamma^3 [n-1][\varphi - \Phi_4]}{\Phi_1} - \frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)]}{2} \\
= \; & \varphi \, \frac{[2-\gamma][2+\gamma(n-1)]^2 - 2\gamma^2[1-\gamma][n-1] - \gamma[2-\gamma][2+\gamma(n-1)][n-1]}{2\Phi_2} \\
&- \frac{\gamma^3 [n-1][\varphi - \Phi_4]}{\Phi_1} - \frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)]}{2} \\
= \; & \varphi \, \frac{[2-\gamma][2+\gamma(n-1)][2+\gamma(n-1)-\gamma(n-1)] - 2\gamma^2[1-\gamma][n-1]}{2\Phi_2} \\
&- \frac{\gamma^3 [n-1][\varphi - \Phi_4]}{\Phi_1} - \frac{2\gamma^2 + \gamma[2-\gamma][2+\gamma(n-1)]}{2} \\
= \; & \varphi \, \frac{[2-\gamma][2+\gamma(n-1)] - \gamma^2[1-\gamma][n-1]}{\Phi_2} - \frac{\gamma^3[n-1][\varphi-\Phi_4]}{\Phi_1} - \frac{\Phi_4 + \gamma^2}{2} \, .
\end{aligned}
\tag{105}
$$

(66) and (67) imply that the coefficient of $\sum_{i=1}^{2} v_i \, c_i^u \, f_{i1}$ in (103) is:

$$
\begin{aligned}
&\varphi \, \frac{2[2+\gamma(n-2)-\gamma^2(n-1)][2+\gamma(n-2)]}{\Phi_1} - \frac{\Phi_1}{2} \\
&- \frac{\varphi}{2\Phi_2} \{ 4 \left[2+\gamma(n-2) - \gamma^2(n-1) \right] [2+\gamma(n-2)] + \gamma^3 [n-1]^2 \\
&\qquad\qquad - \gamma[2-\gamma][2+\gamma(n-1)][n-1] \} \\
&- \left\{ \gamma[2-\gamma][2+\gamma(n-1)] + \gamma^2 \right\} \frac{2[2+\gamma(n-2)-\gamma^2(n-1)][2+\gamma(n-2)]}{\Phi_1} \\
= \; & - \varphi \frac{\Phi_3}{\Phi_1} - \frac{\Phi_1}{2} - \frac{\varphi}{2\Phi_2} \{ -2\Phi_3 + \gamma^3[n-1]^2 - \gamma[2-\gamma][2+\gamma(n-1)][n-1] \} + \Phi_4 \frac{\Phi_3}{\Phi_1} \\
= \; & - \varphi \frac{\Phi_3}{\Phi_1} - \frac{\Phi_1}{2} - \frac{\varphi}{2\Phi_2} \left[-2\Phi_3 + \gamma^3(n-1)^2 - \left(\Phi_4 - \gamma^2 \right)(n-1) \right] + \Phi_4 \frac{\Phi_3}{\Phi_1}
\end{aligned}
$$

$$= \; [\,\Phi_4 - \varphi\,]\,\frac{\Phi_3}{\Phi_1} - \frac{\Phi_1}{2} - \frac{\varphi}{2\,\Phi_2}\,\big[-2\,\Phi_3 + \gamma^3\,(\,n-1\,)^2 - \big(\,\Phi_4 - \gamma^2\,\big)\,(\,n-1\,)\big]\,. \tag{106}$$

(67) implies that the coefficient of $[\,c_1^u\,f_{11} + c_2^u\,f_{21}\,]$ in (103) is:

$$\varphi\left\{\frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{2\,\Phi_2} - \frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{\Phi_1}\right\}$$

$$+\;\big\{\,\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] + \gamma^2\,\big\}\;\frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2}{\Phi_1}$$

$$= \; [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2\left\{\varphi\left[\frac{1}{2\,\Phi_2} - \frac{1}{\Phi_1}\right] + \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] + \gamma^2}{\Phi_1}\right\}$$

$$= \; [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2\left[\varphi\left(\frac{1}{2\,\Phi_2} - \frac{1}{\Phi_1}\right) + \frac{\Phi_4}{\Phi_1}\right]. \tag{107}$$

(67) implies that the coefficient of $\sum_{i=1}^{n}(\,c_1^u\,f_{1i} + c_2^u\,f_{2i}\,)$ in (103) is:

$$\varphi\left\{\frac{[\,1-\gamma\,]\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_2} + \frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_1}\right\}$$

$$-\;\frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2} - \big\{\,\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] + \gamma^2\,\big\}\,\frac{\gamma\,[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{\Phi_1}$$

$$= \; [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]\left[\varphi\left(\frac{1-\gamma}{\Phi_2} + \frac{\gamma}{\Phi_1}\right) - \frac{\gamma}{2} - \frac{\gamma\,\Phi_4}{\Phi_1}\right]. \tag{108}$$

(103) – (108) imply that for $i \in \{\,2, ..., n\,\}$:

$$[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]^2\;w_i$$

$$= \; [\,2+\gamma\,(\,n-1\,)\,]\,\big[\,\alpha_1 - c_1^d\,\big]$$

$$\cdot\left\{\varphi\;\frac{2\,\gamma\,[\,n-1\,]\,[\,1-\gamma\,] - [\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,]}{2\,\Phi_2} + \gamma + \frac{\gamma^2\,[\,n-1\,]\,[\,\varphi-\Phi_4\,]}{\Phi_1}\right\}$$

$$+\;\sum_{i=1}^{n}\big(\,\alpha_i - c_i^d\,\big)$$

$$\cdot\left\{\varphi\;\frac{[\,2-\gamma\,]\,[\,2+\gamma\,(\,n-1\,)\,] - \gamma^2\,[\,1-\gamma\,]\,[\,n-1\,]}{\Phi_2} - \frac{\gamma^3\,[\,n-1\,]\,[\,\varphi-\Phi_4\,]}{\Phi_1} - \frac{\Phi_4+\gamma^2}{2}\right\}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{2} v_i \, c_i^u \, f_{i1} \left\{ \frac{\Phi_3 \left[\Phi_4 - \varphi \right]}{\Phi_1} - \frac{\Phi_1}{2} - \frac{\varphi \left[-2 \, \Phi_3 + \gamma^3 \, (n - 1)^2 - (\Phi_4 - \gamma^2) (n - 1) \right]}{2 \, \Phi_2} \right\} \\
&+ \left[c_1^u \, f_{11} + c_2^u \, f_{21} \right] \left[2 - \gamma \right] \left[2 + \gamma \, (n - 1) \right]^2 \left\{ \varphi \left[\frac{1}{2 \, \Phi_2} - \frac{1}{\Phi_1} \right] + \frac{\Phi_4}{\Phi_1} \right\} \\
&+ \frac{\left[2 - \gamma \right] \left[2 + \gamma \, (n - 1) \right]^2}{2} \left[c_1^u \, f_{1i} + c_2^u \, f_{2i} \right] \\
&+ \sum_{i=1}^{n} (c_1^u \, f_{1i} + c_2^u \, f_{2i}) \left[2 - \gamma \right] \left[2 + \gamma \, (n - 1) \right] \left\{ \varphi \left[\frac{1 - \gamma}{\Phi_2} + \frac{\gamma}{\Phi_1} \right] - \frac{\gamma}{2} - \frac{\gamma \, \Phi_4}{\Phi_1} \right\} \\
&+ \frac{\left[2 - \gamma \right] \left[2 + \gamma \, (n - 1) \right]^2}{2} \left[\alpha_i - c_i^d \right] . \qquad (109)
\end{aligned}
$$

Dividing all terms in (109) by $[\,2 - \gamma\,]\,[\,2 + \gamma\,(\,n - 1\,)\,]^2$ implies that for $i \in \{2, ..., n\}$:

$$
\begin{aligned}
w_i \;=\; & \frac{1}{\left[2 - \gamma \right] \left[2 + \gamma \, (n - 1) \right]} \left[\alpha_1 - c_1^d \right] \\
& \cdot \left\{ \varphi \, \frac{2 \, \gamma \left[n - 1 \right] \left[1 - \gamma \right] - \left[2 - \gamma \right] \left[2 + \gamma \, (n - 1) \right]}{2 \, \Phi_2} + \gamma + \frac{\gamma^2 \left[n - 1 \right] \left[\varphi - \Phi_4 \right]}{\Phi_1} \right\} \\
&+ \frac{1}{\left[2 - \gamma \right] \left[2 + \gamma \, (n - 1) \right]^2} \sum_{i=1}^{n} (\alpha_i - c_i^d) \\
& \cdot \left\{ \varphi \, \frac{\left[2 - \gamma \right] \left[2 + \gamma \, (n - 1) \right] - \gamma^2 \left[1 - \gamma \right] \left[n - 1 \right]}{\Phi_2} - \frac{\gamma^3 \left[n - 1 \right] \left[\varphi - \Phi_4 \right]}{\Phi_1} - \frac{\Phi_4 + \gamma^2}{2} \right\} \\
&+ \frac{1}{\left[2 - \gamma \right] \left[2 + \gamma \, (n - 1) \right]^2} \sum_{i=1}^{2} v_i \, c_i^u \, f_{i1} \\
& \cdot \left\{ \frac{\Phi_3 \left[\Phi_4 - \varphi \right]}{\Phi_1} - \frac{\Phi_1}{2} - \frac{\varphi \left[- 2 \, \Phi_3 + \gamma^3 \, (n - 1)^2 - (\Phi_4 - \gamma^2) (n - 1) \right]}{2 \, \Phi_2} \right\} \\
&+ \left[c_1^u \, f_{11} + c_2^u \, f_{21} \right] \left[\varphi \left(\frac{1}{2 \, \Phi_2} - \frac{1}{\Phi_1} \right) + \frac{\Phi_4}{\Phi_1} \right] + \frac{1}{2} \left[c_1^u \, f_{1i} + c_2^u \, f_{2i} \right] \\
&+ \frac{1}{2 + \gamma \left[n - 1 \right]} \sum_{i=1}^{n} (c_1^u \, f_{1i} + c_2^u \, f_{2i}) \left[\varphi \left(\frac{1 - \gamma}{\Phi_2} + \frac{\gamma}{\Phi_1} \right) - \frac{\gamma}{2} - \frac{\gamma \, \Phi_4}{\Phi_1} \right] + \frac{\alpha_i - c_i^d}{2} . \qquad (110)
\end{aligned}
$$

II. Price Competition

Now consider the setting where downstream firms compete on price.

Summing (2) for all firms provides:

$$\sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} q_i - \gamma \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j - \sum_{i=1}^{n} p_i \;=\; 0$$

$$\Leftrightarrow \quad \sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} q_i - \gamma \, [\, n-1 \,] \sum_{i=1}^{n} q_i - \sum_{i=1}^{n} p_i \;=\; 0$$

$$\Leftrightarrow \quad \alpha_i - q_i - p_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j - \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j - \gamma \, [\, n-1 \,] \, q_i - \gamma \, [\, n-1 \,] \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j - \sum_{\substack{j=1 \\ j \neq i}}^{n} p_j \;=\; 0 \, . \qquad (111)$$

(2) and (111) imply:

$$\gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j + \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j - \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j - \gamma \, [\, n-1 \,] \, q_i - \gamma \, [\, n-1 \,] \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j - \sum_{\substack{j=1 \\ j \neq i}}^{n} p_j \;=\; 0$$

$$\Leftrightarrow \quad \gamma \, [\, n-1 \,] \, q_i \;=\; \sum_{\substack{j=1 \\ j \neq i}}^{n} (\, \alpha_j - p_j \,) - [\, 1 + \gamma \, (\, n-2 \,) \,] \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j \, . \qquad (112)$$

(2) and (112) imply that for $i = 1, ..., n$:

$$\gamma \, [\, n-1 \,] \, q_i \;=\; \sum_{\substack{j=1 \\ j \neq i}}^{n} (\, \alpha_j - p_j \,) - [\, 1 + \gamma \, (\, n-2 \,) \,] \, \frac{1}{\gamma} \, [\, \alpha_i - q_i - p_i \,]$$

$$\Leftrightarrow \quad \left[\gamma \, (\, n-1 \,) - \frac{1 + \gamma \, (\, n-2 \,)}{\gamma} \right] q_i \;=\; \sum_{\substack{j=1 \\ j \neq i}}^{n} (\, \alpha_j - p_j \,) - \frac{1 + \gamma \, [\, n-2 \,]}{\gamma} \, [\, \alpha_i - p_i \,]$$

$$\Leftrightarrow \quad \left[\frac{\gamma^2 \, (\, n-1 \,) - 1 - \gamma \, (\, n-2 \,)}{\gamma} \right] q_i \;=\; \frac{1}{\gamma} \left[\gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} (\, \alpha_j - p_j \,) - (\, 1 + \gamma \, [\, n-2 \,] \,) \, (\, \alpha_i - p_i \,) \right]$$

$$\Leftrightarrow \quad q_i^* \;=\; \frac{[\,\alpha_i - p_i\,]\,[\,1 + \gamma\,(\,n-2\,)\,] - \gamma \sum\limits_{\substack{j=1\\ j\neq i}}^{n} (\,\alpha_j - p_j\,)}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}\,. \tag{113}$$

(113) identifies Di's output, given prices $(p_i, \mathbf{p}_{-i})$.

A. Vertical Separation. No downstream supplier is integrated with any upstream supplier.

(4) implies that firm i's profit is:

$$\pi_i(p_i, \mathbf{p}_{-i}) \;=\; [\,p_i - c_i\,]\,q_i^*\,. \tag{114}$$

where q_i^* is specified in (113).

Differentiating (114) with respect to p_i, using (113), provides:

$$\frac{[\,\alpha_i - p_i\,]\,[\,1 + \gamma\,(\,n-2\,)\,] - \gamma \sum\limits_{\substack{j=1\\ j\neq i}}^{n} (\,\alpha_j - p_j\,)}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]} - \frac{[\,1+\gamma\,(\,n-2\,)\,]\,[\,p_i - c_i\,]}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]} \;=\; 0$$

$$\Leftrightarrow \quad [\,\alpha_i - p_i\,]\,[\,1 + \gamma\,(\,n-2\,)\,] - \gamma \sum_{\substack{j=1\\ j\neq i}}^{n} (\,\alpha_j - p_j\,) \;=\; [\,1+\gamma\,(\,n-2\,)\,]\,[\,p_i - c_i\,]$$

$$\Leftrightarrow \quad [\,\alpha_i + c_i\,]\,[\,1 + \gamma\,(\,n-2\,)\,] - \gamma \sum_{\substack{j=1\\ j\neq i}}^{n} (\,\alpha_j - p_j\,) \;=\; 2\,[\,1+\gamma\,(\,n-2\,)\,]\,p_i$$

$$\Leftrightarrow \quad p_i \;=\; \frac{1}{2}\,[\,\alpha_i + c_i\,] - \frac{\gamma \sum\limits_{\substack{j=1\\ j\neq i}}^{n} (\,\alpha_j - p_j\,)}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\,. \tag{115}$$

(115) implies:

$$2\,[\,1+\gamma\,(\,n-2\,)\,]\,p_i \;=\; [\,1+\gamma\,(\,n-2\,)\,]\,[\,\alpha_i + c_i\,] - \gamma \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j + \gamma \sum_{\substack{j=1\\ j\neq i}}^{n} p_j$$

$$\Rightarrow \quad \sum_{\substack{j=1\\ j\neq i}}^{n} p_j \;=\; \frac{2}{\gamma}\,[\,1+\gamma\,(\,n-2\,)\,]\,p_i - \frac{1}{\gamma}\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,\alpha_i + c_i\,] + \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j\,. \tag{116}$$

Summing (115) for all firms provides:

$$\sum_{i=1}^{n} p_i \;=\; \frac{1}{2}\sum_{i=1}^{n}(\alpha_i + c_i) - \frac{\gamma\,[\,n-1\,]\,\sum_{i=1}^{n}(\alpha_i - p_i)}{2\,[\,1+\gamma\,(\,n-2\,)\,]}$$

$$\Rightarrow\;\; p_i + \sum_{\substack{j=1\\ j\neq i}}^{n} p_j \;=\; \frac{1}{2}\,[\,\alpha_i + c_i\,] + \frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{n}(\alpha_j + c_j) - \frac{\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\,[\,\alpha_i - p_i\,]$$

$$-\;\frac{\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j + \frac{\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\sum_{\substack{j=1\\ j\neq i}}^{n} p_j$$

$$\Rightarrow\;\; \left[1 - \frac{\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\right] p_i \;=\; \left[\frac{1}{2} - \frac{\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\right]\alpha_i + \frac{1}{2}\,c_i$$

$$+\;\left[\frac{1}{2} - \frac{\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\right]\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j + \frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{n} c_j$$

$$+\;\left[\frac{\gamma\,[\,n-1\,] - 2\,[\,1+\gamma\,(\,n-2\,)\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\right]\sum_{\substack{j=1\\ j\neq i}}^{n} p_j$$

$$\Rightarrow\;\; \{\,2\,[\,1+\gamma\,(\,n-2\,)\,] - \gamma\,[\,n-1\,]\,\}\,p_i \;=\; \{\,1+\gamma\,[\,n-2\,] - \gamma\,[\,n-1\,]\,\}\,\alpha_i$$

$$+\;[\,1+\gamma\,(\,n-2\,)\,]\,c_i + \{\,1+\gamma\,[\,n-2\,] - \gamma\,[\,n-1\,]\,\}\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j$$

$$+\;[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n} c_j + \{\,\gamma\,[\,n-1\,] - 2\,[\,1+\gamma\,(\,n-2\,)\,]\,\}\sum_{\substack{j=1\\ j\neq i}}^{n} p_j\,. \qquad (117)$$

Observe that:

$$2\,[\,1+\gamma\,(\,n-2\,)\,] - \gamma\,[\,n-1\,] \;=\; 2 + 2\,\gamma\,n - 4\,\gamma - \gamma\,n + \gamma \;=\; 2 + \gamma\,[\,n-3\,]\,; \text{ and}$$

$$1 + \gamma\,[\,n-2\,] - \gamma\,[\,n-1\,] \;=\; 1 - \gamma\,. \qquad (118)$$

(116), (117), and (118) imply:

$[\,2+\gamma\,(\,n-3\,)\,]\,p_i$

$$= \; [\,1-\gamma\,]\,\alpha_i + \,[\,1+\gamma\,(\,n-2\,)\,]\,c_i + [\,1-\gamma\,]\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j + \,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}c_j$$

$$-\,[\,2+\gamma\,(\,n-3\,)\,]\left\{\frac{2}{\gamma}\,[\,1+\gamma\,(\,n-2\,)\,]\,p_i - \frac{1}{\gamma}\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,\alpha_i + c_i\,] + \sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j\right\}$$

$$\Rightarrow \;\; [\,2+\gamma\,(\,n-3\,)\,]\left\{1+\frac{2}{\gamma}\,[\,1+\gamma\,(\,n-2\,)\,]\right\}p_i$$

$$= \;\; \left\{1-\gamma+\frac{1}{\gamma}\,[\,2+\gamma\,(\,n-3\,)\,]\,[\,1+\gamma\,(\,n-2\,)\,]\right\}\alpha_i$$

$$+\,[\,1+\gamma\,(\,n-2\,)\,]\left[1+\frac{1}{\gamma}\,(\,2+\gamma\,[\,n-3\,]\,)\right]c_i$$

$$+\,[\,1-\gamma-2-\gamma\,(\,n-3\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j + [\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}c_j$$

$$\Rightarrow \;\; \frac{1}{\gamma}\,[\,2+\gamma\,(\,n-3\,)\,]\,[\,\gamma+2+2\,\gamma\,(\,n-2\,)\,]\,p_i$$

$$= \;\; \frac{1}{\gamma}\,\{\,\gamma-\gamma^2+[\,2+\gamma\,(\,n-3\,)\,]\,[\,1+\gamma\,(\,n-2\,)\,]\,\}\,\alpha_i$$

$$+\;\frac{1}{\gamma}\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,\gamma+2+\gamma\,(\,n-3\,)\,]\,c_i$$

$$-\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j + [\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}c_j\,. \tag{119}$$

Observe that:

$$\gamma-\gamma^2+[\,2+\gamma\,(\,n-3\,)\,]\,[\,1+\gamma\,(\,n-2\,)\,]$$

$$= \;\gamma-\gamma^2+2+2\,\gamma\,n-4\,\gamma+\gamma\,n-3\,\gamma+\gamma^2\left[n^2-5\,n+6\right]$$

$$= \;2+3\,\gamma\,n-6\,\gamma+\gamma^2\left[n^2-5\,n+5\right] \;=\; 2+3\,\gamma\,[\,n-2\,]+\gamma^2\left[n^2-5\,n+5\right]. \tag{120}$$

(119) and (120) imply:

$$p_i \;=\; \frac{1}{[\,2+\gamma\,(\,n-3\,)\,]\,[\,2+\gamma\,(\,2\,n-3\,)\,]}$$

$$\cdot \left\{ \left\{ 2 + 3\gamma [n-2] + \gamma^2 \left[n^2 - 5n + 5 \right] \right\} \alpha_i - \gamma [1 + \gamma (n-2)] \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j \right.$$

$$\left. + [1 + \gamma (n-2)] [2 + \gamma (n-2)] c_i + \gamma [1 + \gamma (n-2)] \sum_{\substack{j=1 \\ j \neq i}}^{n} c_j \right\}. \qquad (121)$$

Observe that:

$$[1 + \gamma (n-2)] [2 + \gamma (n-2)] - [2 + \gamma (n-3)] [2 + \gamma (2n-3)]$$

$$= 2 + \gamma [n-2] + 2\gamma [n-2] + \gamma^2 [n-2]^2 - 4 - 2\gamma [2n-3]$$

$$- 2\gamma [n-3] - \gamma^2 [2n-3] [n-3]$$

$$= -2 + 3\gamma [n-2] - 2\gamma [3n-6] + \gamma^2 \left[n^2 - 4n + 4 - 2n^2 + 9n - 9 \right]$$

$$= -2 - 3\gamma n + 6\gamma - \gamma^2 \left[n^2 - 5n + 5 \right]$$

$$= -2 - 3\gamma [n-2] - \gamma^2 \left[n^2 - 5n + 5 \right]. \qquad (122)$$

(121) and (122) imply:

$$p_i - c_i = \frac{1}{[2 + \gamma (n-3)] [2 + \gamma (2n-3)]}$$

$$\cdot \left\{ \left\{ 2 + 3\gamma [n-2] + \gamma^2 \left[n^2 - 5n + 5 \right] \right\} \alpha_i - \gamma [1 + \gamma (n-2)] \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j \right.$$

$$\left. - \left[2 + 3\gamma (n-2) + \gamma^2 \left(n^2 - 5n + 5 \right) \right] c_i + \gamma [1 + \gamma (n-2)] \sum_{\substack{j=1 \\ j \neq i}}^{n} c_j \right\}. \quad (123)$$

(113) implies:

$$\frac{\partial q_i^*}{\partial p_i} = - \frac{1 + \gamma [n-2]}{[1 - \gamma] [1 + \gamma (n-1)]}. \qquad (124)$$

(114) implies that Di's profit-maximizing price is determined by:

$$[p_i - c_i] \frac{\partial q_i^*}{\partial p_i} + q_i^* = 0 \quad \Leftrightarrow \quad q_i^* = - [p_i - c_i] \frac{\partial q_i^*}{\partial p_i}. \qquad (125)$$

(123), (124), and (125) imply that for $i = 1, ..., n$:

$$q_i^* \;=\; \frac{1 + \gamma\,[\,n-2\,]}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,]\,[\,2+\gamma\,(\,2\,n-3\,)\,]}$$

$$\cdot \Bigg\{ \left[\,2 + 3\,\gamma\,(\,n-2\,) + \gamma^2\left(n^2 - 5\,n + 5\right)\right]\,\alpha_i - \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j$$

$$- \left[\,2 + 3\,\gamma\,(\,n-2\,) + \gamma^2\left(n^2 - 5\,n + 5\right)\right]c_i + \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n} c_j \Bigg\}. \quad (126)$$

B. <u>Vertical Integration</u>. D1 and U1 are vertically integrated.

(114) implies that unintegrated downstream firm Di's problem is:

$$\underset{p_i}{\text{Maximize}}\;\; \pi_i(p_i, \mathbf{p}_{-i}) \;=\; \left[\,p_i - w_i - c_i^d\,\right] q_i^* \quad \text{for } i \in \{2, ..., n\}, \quad (127)$$

where q_i^* is given by (113).

D1's problem is:

$$\underset{p_1}{\text{Maximize}}\;\; \pi_1(p_1, \mathbf{p}_{-1}) \;=\; \left[\,p_1 - w_1 - c_1^d\,\right] q_1^*$$

$$+\;\; v_1 \sum_{i=1}^{n}[\,w_i - c_1^u\,]\,f_{1i}\,q_i^* + v_2 \sum_{i=1}^{n}[\,w_i - c_2^u\,]\,f_{2i}\,q_i^*\,, \quad (128)$$

where q_i^* is given by (113).

(128) implies that firm 1's profit-maximizing choice of p_1 is determined by:

$$q_1^* + \left[\,p_1 - w_1 - c_1^d\,\right]\frac{\partial q_1^*}{\partial p_1} + v_1\,[\,w_1 - c_1^u\,]\,f_{11}\,\frac{\partial q_1^*}{\partial p_1} + v_2\,[\,w_1 - c_2^u\,]\,f_{21}\,\frac{\partial q_1^*}{\partial p_1}$$

$$+\; v_1 \sum_{i=2}^{n}[\,w_i - c_1^u\,]\,f_{1i}\,\frac{\partial q_i^*}{\partial p_1} + v_2 \sum_{i=2}^{n}[\,w_i - c_2^u\,]\,f_{2i}\,\frac{\partial q_i^*}{\partial p_1} \;=\; 0$$

$$\Rightarrow\quad q_1^* + \left[\,p_1 - w_1 - c_1^d + v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,\right]\frac{\partial q_1^*}{\partial p_1}$$

$$+\; \sum_{i=2}^{n}\left[\,v_1\,(\,w_i - c_1^u\,)\,f_{1i} + v_2\,(\,w_i - c_2^u\,)\,f_{2i}\,\right]\frac{\partial q_i^*}{\partial p_1} \;=\; 0\,. \quad (129)$$

(113) and (129) imply:

$$\frac{[\,\alpha_1 - p_1\,]\,[\,1 + \gamma\,(\,n-2\,)\,] - \gamma \sum_{j=2}^{n} (\,\alpha_j - p_j\,)}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}$$

$$- \left[\,p_1 - w_1 - c_1^d + v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,\right] \frac{1+\gamma\,[\,n-2\,]}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}$$

$$+ \sum_{i=2}^{n} [\,v_1\,(\,w_i - c_1^u\,)\,f_{1i} + v_2\,(\,w_i - c_2^u\,)\,f_{2i}\,] \frac{\gamma}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]} \;=\; 0$$

$$\Rightarrow \left\{ \frac{1+\gamma\,[\,n-2\,]}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]} + \frac{1+\gamma\,[\,n-2\,]}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]} \right\} p_1$$

$$= \frac{\alpha_1\,[\,1 + \gamma\,(\,n-2\,)\,] - \gamma \sum_{j=2}^{n} (\,\alpha_j - p_j\,)}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}$$

$$- \left[\,-\left(w_1 + c_1^d\right) + v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,\right] \frac{1+\gamma\,[\,n-2\,]}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{\gamma}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]} \sum_{i=2}^{n} [\,v_1\,(\,w_i - c_1^u\,)\,f_{1i} + v_2\,(\,w_i - c_2^u\,)\,f_{2i}\,]$$

$$\Rightarrow\; 2\,[\,1+\gamma\,(\,n-2\,)\,]\,p_1$$

$$= \alpha_1\,[\,1 + \gamma\,(\,n-2\,)\,] - \gamma \sum_{j=2}^{n} (\,\alpha_j - p_j\,)$$

$$- \left[\,-\left(w_1 + c_1^d\right) + v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,\right] [\,1+\gamma\,(\,n-2\,)\,]$$

$$+ \gamma \sum_{i=2}^{n} [\,v_1\,(\,w_i - c_1^u\,)\,f_{1i} + v_2\,(\,w_i - c_2^u\,)\,f_{2i}\,]$$

$$\Rightarrow\; p_1 \;=\; \frac{\alpha_1}{2} - \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \sum_{j=2}^{n} (\,\alpha_j - p_j\,)$$

$$- \frac{1}{2}\left[\,-\left(w_1 + c_1^d\right) + v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,\right]$$

$$+ \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \sum_{i=2}^{n} [\,v_1\,(\,w_i - c_1^u\,)\,f_{1i} + v_2\,(\,w_i - c_2^u\,)\,f_{2i}\,]\,. \qquad (130)$$

(115) implies that firm i's reaction function is:

$$p_i \;=\; \frac{1}{2}\left[\,\alpha_i + w_i + c_i^d\,\right] - \frac{\gamma \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{n} (\,\alpha_j - p_j\,)}{2\left[\,1 + \gamma\,(\,n-2\,)\,\right]} \quad \text{for } \; i \,\in\, \{\,2, ..., n\,\}\,. \tag{131}$$

Summing (130) and (131) over all firms provides:

$$\begin{aligned}
\sum_{i=1}^{n} p_i \;=\; & \frac{\alpha_1}{2} - \frac{\gamma}{2\left[\,1+\gamma\,(\,n-2\,)\,\right]} \sum_{j=2}^{n} (\,\alpha_j - p_j\,) \\
& - \frac{1}{2}\left[\,-\left(\,w_1 + c_1^d\,\right) + v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,\right] \\
& + \frac{\gamma}{2\left[\,1+\gamma\,(\,n-2\,)\,\right]} \sum_{i=2}^{n} \left[\,v_1\,(\,w_i - c_1^u\,)\,f_{1i} + v_2\,(\,w_i - c_2^u\,)\,f_{2i}\,\right] \\
& + \sum_{i=2}^{n} \frac{1}{2}\left[\,\alpha_i + w_i + c_i^d\,\right] - \frac{\gamma}{2\left[\,1+\gamma\,(\,n-2\,)\,\right]} \sum_{i=2}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} (\,\alpha_j - p_j\,) \\
=\; & \frac{\alpha_1}{2} + \sum_{i=2}^{n} \frac{1}{2}\left[\,\alpha_i + w_i + c_i^d\,\right] + \frac{1}{2}\left[\,w_1 + c_1^d\,\right] \\
& - \frac{\gamma}{2\left[\,1+\gamma\,(\,n-2\,)\,\right]} \sum_{j \neq 1} (\,\alpha_j - p_j\,) - \frac{\gamma}{2\left[\,1+\gamma\,(\,n-2\,)\,\right]} \sum_{i=2}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} (\,\alpha_j - p_j\,) \\
& - \frac{1}{2}\left[\,v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,\right] \\
& + \frac{\gamma}{2\left[\,1+\gamma\,(\,n-2\,)\,\right]} \sum_{i=2}^{n} \left[\,v_1\,(\,w_i - c_1^u\,)\,f_{1i} + v_2\,(\,w_i - c_2^u\,)\,f_{2i}\,\right] \\
=\; & \sum_{i=1}^{n} \frac{1}{2}\left[\,\alpha_i + w_i + c_i^d\,\right] - \frac{\gamma}{2\left[\,1+\gamma\,(\,n-2\,)\,\right]} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} (\,\alpha_j - p_j\,) \\
& - \frac{1}{2}\left[\,v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,\right] \\
& + \frac{\gamma}{2\left[\,1+\gamma\,(\,n-2\,)\,\right]} \sum_{i=2}^{n} \left[\,v_1\,(\,w_i - c_1^u\,)\,f_{1i} + v_2\,(\,w_i - c_2^u\,)\,f_{2i}\,\right]
\end{aligned}$$

$$= \sum_{i=1}^{n} \frac{1}{2} [\, \alpha_i + w_i + c_i^d \,] - \frac{\gamma [\, n-1 \,]}{2 [\, 1 + \gamma (n-2) \,]} \sum_{i=1}^{n} (\alpha_i - p_i)$$

$$- \frac{1}{2} [\, v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21} \,]$$

$$+ \frac{\gamma}{2 [\, 1 + \gamma (n-2) \,]} \sum_{i=2}^{n} [\, v_1 (w_i - c_1^u) f_{1i} + v_2 (w_i - c_2^u) f_{2i} \,]$$

$$= \sum_{i=1}^{n} \frac{1}{2} [\, \alpha_i + w_i + c_i^d \,] - \frac{\gamma [\, n-1 \,]}{2 [\, 1 + \gamma (n-2) \,]} \sum_{i=1}^{n} \alpha_i + \frac{\gamma [\, n-1 \,]}{2 [\, 1 + \gamma (n-2) \,]} \sum_{i=1}^{n} p_i$$

$$- \frac{1}{2} [\, v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21} \,]$$

$$+ \frac{\gamma}{2 [\, 1 + \gamma (n-2) \,]} \sum_{i=2}^{n} [\, v_1 (w_i - c_1^u) f_{1i} + v_2 (w_i - c_2^u) f_{2i} \,]$$

$$\Rightarrow \left[1 - \frac{\gamma [\, n-1 \,]}{2 [\, 1 + \gamma (n-2) \,]} \right] \sum_{i=1}^{n} p_i$$

$$= \sum_{i=1}^{n} \frac{1}{2} [\, \alpha_i + w_i + c_i^d \,] - \frac{\gamma [\, n-1 \,]}{2 [\, 1 + \gamma (n-2) \,]} \sum_{i=1}^{n} \alpha_i$$

$$- \frac{1}{2} [\, v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21} \,]$$

$$+ \frac{\gamma}{2 [\, 1 + \gamma (n-2) \,]} \sum_{i=2}^{n} [\, v_1 (w_i - c_1^u) f_{1i} + v_2 (w_i - c_2^u) f_{2i} \,] . \tag{132}$$

The coefficient of $\sum_{i=1}^{n} p_i$ in (132) is:

$$1 - \frac{\gamma [\, n-1 \,]}{2 [\, 1 + \gamma (n-2) \,]} = \frac{2 [\, 1 + \gamma (n-2) \,] - \gamma [\, n-1 \,]}{2 [\, 1 + \gamma (n-2) \,]}$$

$$= \frac{2 [\, 1 - \gamma + \gamma (n-1) \,] - \gamma [\, n-1 \,]}{2 [\, 1 + \gamma (n-2) \,]} = \frac{2 [\, 1 - \gamma \,] + \gamma [\, n-1 \,]}{2 [\, 1 + \gamma (n-2) \,]} . \tag{133}$$

(132) and (133) imply:

$$\frac{2 [\, 1 - \gamma \,] + \gamma [\, n-1 \,]}{2 [\, 1 + \gamma (n-2) \,]} \sum_{i=1}^{n} p_i$$

$$= \sum_{i=1}^{n} \frac{1}{2} \left[\alpha_i + w_i + c_i^d \right] - \frac{\gamma \left[n-1 \right]}{2 \left[1 + \gamma \left(n-2 \right) \right]} \sum_{i=1}^{n} \alpha_i$$

$$- \frac{1}{2} \left[v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right]$$

$$+ \frac{\gamma}{2 \left[1 + \gamma \left(n-2 \right) \right]} \sum_{i=2}^{n} \left[v_1 \left(w_i - c_1^u \right) f_{1i} + v_2 \left(w_i - c_2^u \right) f_{2i} \right]$$

$$\Rightarrow \sum_{i=1}^{n} p_i = \frac{1 + \gamma \left[n-2 \right]}{2 \left[1-\gamma \right] + \gamma \left[n-1 \right]} \sum_{i=1}^{n} \left(\alpha_i + w_i + c_i^d \right) - \frac{\gamma \left[n-1 \right]}{2 \left[1-\gamma \right] + \gamma \left[n-1 \right]} \sum_{i=1}^{n} \alpha_i$$

$$- \frac{1 + \gamma \left[n-2 \right]}{2 \left[1-\gamma \right] + \gamma \left[n-1 \right]} \left[v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right]$$

$$+ \frac{\gamma}{2 \left[1-\gamma \right] + \gamma \left[n-1 \right]} \sum_{i=2}^{n} \left[v_1 \left(w_i - c_1^u \right) f_{1i} + v_2 \left(w_i - c_2^u \right) f_{2i} \right]$$

$$\Rightarrow \sum_{j=2}^{n} p_j = -p_1 + \frac{1 + \gamma \left[n-2 \right]}{2 \left[1-\gamma \right] + \gamma \left[n-1 \right]} \sum_{i=1}^{n} \left(\alpha_i + w_i + c_i^d \right) - \frac{\gamma \left[n-1 \right]}{2 \left[1-\gamma \right] + \gamma \left[n-1 \right]} \sum_{i=1}^{n} \alpha_i$$

$$- \frac{1 + \gamma \left[n-2 \right]}{2 \left[1-\gamma \right] + \gamma \left[n-1 \right]} \left[v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right]$$

$$+ \frac{\gamma}{2 \left[1-\gamma \right] + \gamma \left[n-1 \right]} \sum_{i=2}^{n} \left[v_1 \left(w_i - c_1^u \right) f_{1i} + v_2 \left(w_i - c_2^u \right) f_{2i} \right] \qquad (134)$$

and for $i \in \{2, ..., n\}$:

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} p_j = -p_i + \frac{1 + \gamma \left[n-2 \right]}{2 \left[1-\gamma \right] + \gamma \left[n-1 \right]} \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d \right) - \frac{\gamma \left[n-1 \right]}{2 \left[1-\gamma \right] + \gamma \left[n-1 \right]} \sum_{j=1}^{n} \alpha_j$$

$$- \frac{1 + \gamma \left[n-2 \right]}{2 \left[1-\gamma \right] + \gamma \left[n-1 \right]} \left[v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right]$$

$$+ \frac{\gamma}{2 \left[1-\gamma \right] + \gamma \left[n-1 \right]} \sum_{j=2}^{n} \left[v_1 \left(w_j - c_1^u \right) f_{1j} + v_2 \left(w_j - c_2^u \right) f_{2j} \right]. \qquad (135)$$

(130) and (134) imply:

$$p_1 = \frac{\gamma}{2 \left[1 + \gamma \left(n-2 \right) \right]} \sum_{j=2}^{n} p_j + \frac{\alpha_1}{2} - \frac{\gamma}{2 \left[1 + \gamma \left(n-2 \right) \right]} \sum_{j=2}^{n} \alpha_j$$

$$- \frac{1}{2} \left[- \left(w_1 + c_1^d \right) + v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right]$$

$$+ \frac{\gamma}{2 \left[1 + \gamma \left(n - 2 \right) \right]} \sum_{i=2}^{n} \left[v_1 \left(w_i - c_1^u \right) f_{1i} + v_2 \left(w_i - c_2^u \right) f_{2i} \right]$$

$$= - \frac{\gamma}{2 \left[1 + \gamma \left(n - 2 \right) \right]} p_1 + \frac{\gamma}{2 \left[1 + \gamma \left(n - 2 \right) \right]} \frac{1 + \gamma \left[n - 2 \right]}{2 \left[1 - \gamma \right] + \gamma \left[n - 1 \right]} \sum_{i=1}^{n} \left(\alpha_i + w_i + c_i^d \right)$$

$$- \frac{\gamma}{2 \left[1 + \gamma \left(n - 2 \right) \right]} \frac{\gamma \left[n - 1 \right]}{2 \left[1 - \gamma \right] + \gamma \left[n - 1 \right]} \sum_{i=1}^{n} \alpha_i$$

$$- \frac{\gamma}{2 \left[1 + \gamma \left(n - 2 \right) \right]} \frac{1 + \gamma \left[n - 2 \right]}{2 \left[1 - \gamma \right] + \gamma \left[n - 1 \right]} \left[v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right]$$

$$+ \frac{\gamma}{2 \left[1 + \gamma \left(n - 2 \right) \right]} \frac{\gamma}{2 \left[1 - \gamma \right] + \gamma \left[n - 1 \right]} \sum_{i=2}^{n} \left[v_1 \left(w_i - c_1^u \right) f_{1i} + v_2 \left(w_i - c_2^u \right) f_{2i} \right]$$

$$+ \frac{\alpha_1}{2} - \frac{\gamma}{2 \left[1 + \gamma \left(n - 2 \right) \right]} \sum_{j=2}^{n} \alpha_j$$

$$- \frac{1}{2} \left[- \left(w_1 + c_1^d \right) + v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right]$$

$$+ \frac{\gamma}{2 \left[1 + \gamma \left(n - 2 \right) \right]} \sum_{i=2}^{n} \left[v_1 \left(w_i - c_1^u \right) f_{1i} + v_2 \left(w_i - c_2^u \right) f_{2i} \right]$$

$$= - \frac{\gamma}{2 \left[1 + \gamma \left(n - 2 \right) \right]} p_1 + \frac{\gamma}{2 \left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right]} \sum_{i=1}^{n} \left(\alpha_i + w_i + c_i^d \right)$$

$$- \frac{\gamma}{2 \left[1 + \gamma \left(n - 2 \right) \right]} \frac{\gamma \left[n - 1 \right]}{2 \left[1 - \gamma \right] + \gamma \left[n - 1 \right]} \sum_{i=1}^{n} \alpha_i$$

$$- \frac{\gamma}{2 \left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right]} \left[v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right]$$

$$+ \frac{\gamma}{2 \left[1 + \gamma \left(n - 2 \right) \right]} \frac{\gamma}{2 \left[1 - \gamma \right] + \gamma \left[n - 1 \right]} \sum_{i=2}^{n} \left[v_1 \left(w_i - c_1^u \right) f_{1i} + v_2 \left(w_i - c_2^u \right) f_{2i} \right]$$

$$+ \frac{\alpha_1}{2} + \frac{1}{2} \left[w_1 + c_1^d \right] - \frac{\gamma}{2 \left[1 + \gamma \left(n - 2 \right) \right]} \sum_{j=2}^{n} \alpha_j$$

$$- \frac{1}{2} \left[v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right]$$

$$+ \frac{\gamma}{2[1+\gamma(n-2)]} \sum_{i=2}^{n} [v_1(w_i - c_1^u) f_{1i} + v_2(w_i - c_2^u) f_{2i}]$$

$$= -\frac{\gamma}{2[1+\gamma(n-2)]} p_1 + \frac{\gamma}{2[2(1-\gamma)+\gamma(n-1)]} \sum_{i=1}^{n} (\alpha_i + w_i + c_i^d) + \frac{1}{2} [\alpha_1 + w_1 + c_1^d]$$

$$- \frac{\gamma}{2[1+\gamma(n-2)]} \frac{\gamma[n-1]}{2[1-\gamma]+\gamma[n-1]} \sum_{i=1}^{n} \alpha_i - \frac{\gamma}{2[1+\gamma(n-2)]} \sum_{j=2}^{n} \alpha_j$$

$$- \left[\frac{\gamma}{2[2(1-\gamma)+\gamma(n-1)]} + \frac{1}{2} \right] [v_1(w_1 - c_1^u) f_{11} + v_2(w_1 - c_2^u) f_{21}]$$

$$+ \frac{\gamma}{2[1+\gamma(n-2)]} \left[1 + \frac{\gamma}{2(1-\gamma)+\gamma(n-1)} \right]$$

$$\cdot \sum_{i=2}^{n} [v_1(w_i - c_1^u) f_{1i} + v_2(w_i - c_2^u) f_{2i}]. \qquad (136)$$

Observe that:

$$\frac{\gamma}{2[2(1-\gamma)+\gamma(n-1)]} + \frac{1}{2} = \frac{\gamma + 2[1-\gamma] + \gamma[n-1]}{2[2(1-\gamma)+\gamma(n-1)]}$$

$$= \frac{2+\gamma[n-2]}{2[2(1-\gamma)+\gamma(n-1)]}; \text{ and}$$

$$1 + \frac{\gamma}{2[1-\gamma]+\gamma[n-1]} = \frac{2[1-\gamma]+\gamma[n-1]+\gamma}{2[1-\gamma]+\gamma[n-1]} = \frac{2+\gamma[n-2]}{2[1-\gamma]+\gamma[n-1]}. \qquad (137)$$

(136) and (137) imply:

$$\left[1 + \frac{\gamma}{2[1+\gamma(n-2)]} \right] p_1$$

$$= \frac{\gamma}{2[2(1-\gamma)+\gamma(n-1)]} \sum_{i=1}^{n} (\alpha_i + w_i + c_i^d) + \frac{1}{2} [\alpha_1 + w_1 + c_1^d]$$

$$- \frac{\gamma}{2[1+\gamma(n-2)]} \frac{\gamma[n-1]}{2[1-\gamma]+\gamma[n-1]} \sum_{i=1}^{n} \alpha_i - \frac{\gamma}{2[1+\gamma(n-2)]} \sum_{j=2}^{n} \alpha_j$$

$$- \frac{2+\gamma[n-2]}{2[2(1-\gamma)+\gamma(n-1)]} [v_1(w_1 - c_1^u) f_{11} + v_2(w_1 - c_2^u) f_{21}]$$

$$+ \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\,\frac{2+\gamma\,[\,n-2\,]}{2\,[\,1-\gamma\,]+\gamma\,[\,n-1\,]}\,\sum_{i=2}^{n}[\,v_1\,(\,w_i-c_1^u\,)\,f_{1i}+v_2\,(\,w_i-c_2^u\,)\,f_{2i}\,]\,. \tag{138}$$

The coefficient of p_1 in (138) is:

$$1+\frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \;=\; \frac{2\,[\,1+\gamma\,(\,n-2\,)\,]+\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \;=\; \frac{2-\gamma+2\,\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\,. \tag{139}$$

(138) and (139) imply:

$$\begin{aligned}
&\frac{2-\gamma+2\,\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\;p_1 \\
&= \frac{\gamma}{2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]}\sum_{i=1}^{n}\left(\,\alpha_i+w_i+c_i^d\,\right)+\frac{1}{2}\,[\,\alpha_1+w_1+c_1^d\,] \\
&\quad - \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\,\frac{\gamma\,[\,n-1\,]}{2\,[\,1-\gamma\,]+\gamma\,[\,n-1\,]}\sum_{i=1}^{n}\alpha_i-\frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\sum_{j=2}^{n}\alpha_j \\
&\quad - \frac{2+\gamma\,[\,n-2\,]}{2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]}\;[\,v_1\,(\,w_1-c_1^u\,)\,f_{11}+v_2\,(\,w_1-c_2^u\,)\,f_{21}\,] \\
&\quad + \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\,\frac{2+\gamma\,[\,n-2\,]}{2\,[\,1-\gamma\,]+\gamma\,[\,n-1\,]}\sum_{i=2}^{n}[\,v_1\,(\,w_i-c_1^u\,)\,f_{1i}+v_2\,(\,w_i-c_2^u\,)\,f_{2i}\,] \\
&= \frac{1}{2\,[\,1-\gamma\,]+\gamma\,[\,n-1\,]}\left\{\frac{\gamma}{2}\sum_{i=1}^{n}\left(\,\alpha_i+w_i+c_i^d\,\right)+\frac{1}{2}\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,[\,\alpha_1+w_1+c_1^d\,]\right. \\
&\qquad - \frac{\gamma^2\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\sum_{i=1}^{n}\alpha_i-\frac{\gamma\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\sum_{j=2}^{n}\alpha_j \\
&\qquad - \frac{2+\gamma\,[\,n-2\,]}{2}\;[\,v_1\,(\,w_1-c_1^u\,)\,f_{11}+v_2\,(\,w_1-c_2^u\,)\,f_{21}\,] \\
&\qquad \left. + \frac{\gamma\,[\,2+\gamma\,(\,n-2\,)\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\sum_{i=2}^{n}[\,v_1\,(\,w_i-c_1^u\,)\,f_{1i}+v_2\,(\,w_i-c_2^u\,)\,f_{2i}\,]\right\}
\end{aligned}$$

$$\Rightarrow \quad p_1^* \;=\; \frac{1}{[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,[\,2-\gamma+2\,\gamma\,(\,n-1\,)\,]}$$

$$\cdot \Bigg\{ \gamma\,[\,1+\gamma\,(\,n-2\,)\,] \sum_{i=1}^{n} \big(\,\alpha_i + w_i + c_i^d\,\big)$$

$$+ \,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,\big[\,\alpha_1 + w_1 + c_1^d\,\big]$$

$$-\,\gamma^2\,[\,n-1\,] \sum_{i=1}^{n} \alpha_i - \gamma\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,] \sum_{j=2}^{n} \alpha_j$$

$$-\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]\,[\,v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,]$$

$$+\,\gamma\,[\,2+\gamma\,(\,n-2\,)\,] \sum_{i=2}^{n} [\,v_1\,(\,w_i - c_1^u\,)\,f_{1i} + v_2\,(\,w_i - c_2^u\,)\,f_{2i}\,] \Bigg\}. \tag{140}$$

(131) and (135) imply that for $i \in \{2, ..., n\}$:

$$p_i \;=\; \frac{1}{2}\,[\,\alpha_i + w_i + c_i^d\,] - \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \sum_{\substack{j=1 \\ j\neq i}}^{n} \alpha_j + \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \sum_{\substack{j=1 \\ j\neq i}}^{n} p_j$$

$$=\; \frac{1}{2}\,[\,\alpha_i + w_i + c_i^d\,] - \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \sum_{\substack{j=1 \\ j\neq i}}^{n} \alpha_j$$

$$-\;\frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\,p_i - \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\;\frac{\gamma\,[\,n-1\,]}{2\,[\,1-\gamma\,]+\gamma\,[\,n-1\,]} \sum_{j=1}^{n} \alpha_j$$

$$+\;\frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\;\frac{1+\gamma\,[\,n-2\,]}{2\,[\,1-\gamma\,]+\gamma\,[\,n-1\,]} \sum_{j=1}^{n} \big(\,\alpha_j + w_j + c_j^d\,\big)$$

$$-\;\frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\;\frac{1+\gamma\,[\,n-2\,]}{2\,[\,1-\gamma\,]+\gamma\,[\,n-1\,]}\;[\,v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,]$$

$$+\;\frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\;\frac{\gamma}{2\,[\,1-\gamma\,]+\gamma\,[\,n-1\,]} \sum_{j=2}^{n} [\,v_1\,(\,w_j - c_1^u\,)\,f_{1j} + v_2\,(\,w_j - c_2^u\,)\,f_{2j}\,]$$

$$=\; \frac{1}{2}\,[\,\alpha_i + w_i + c_i^d\,] - \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \sum_{\substack{j=1 \\ j\neq i}}^{n} \alpha_j$$

$$-\;\frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\,p_i - \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\;\frac{\gamma\,[\,n-1\,]}{2\,[\,1-\gamma\,]+\gamma\,[\,n-1\,]} \sum_{j=1}^{n} \alpha_j$$

$$+ \frac{\gamma}{2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]} \sum_{j=1}^{n} \left(\,\alpha_j + w_j + c_j^d\,\right)$$

$$- \frac{\gamma}{2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]} \left[\,v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,\right]$$

$$+ \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \; \frac{\gamma}{2\,[\,1-\gamma\,]+\gamma\,[\,n-1\,]} \sum_{j=2}^{n} \left[\,v_1\,(\,w_j - c_1^u\,)\,f_{1j} + v_2\,(\,w_j - c_2^u\,)\,f_{2j}\,\right]$$

$$\Rightarrow \quad \left[\,1 + \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\,\right] p_i$$

$$= \frac{1}{2}\left[\,\alpha_i + w_i + c_i^d\,\right] - \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \sum_{j \neq i} \alpha_j$$

$$- \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \; \frac{\gamma\,[\,n-1\,]}{2\,[\,1-\gamma\,]+\gamma\,[\,n-1\,]} \sum_{j=1}^{n} \alpha_j$$

$$+ \frac{\gamma}{2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]} \sum_{j=1}^{n} \left(\,\alpha_j + w_j + c_j^d\,\right)$$

$$- \frac{\gamma}{2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]} \left[\,v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,\right]$$

$$+ \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \; \frac{\gamma}{2\,[\,1-\gamma\,]+\gamma\,[\,n-1\,]} \sum_{j=2}^{n} \left[\,v_1\,(\,w_j - c_1^u\,)\,f_{1j} + v_2\,(\,w_j - c_2^u\,)\,f_{2j}\,\right]. \tag{141}$$

(139) and (141) imply that for $i \in \{2, ..., n\}$:

$$\frac{2-\gamma+2\,\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\, p_i$$

$$= \frac{1}{2}\left[\,\alpha_i + w_i + c_i^d\,\right] - \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j$$

$$- \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \; \frac{\gamma\,[\,n-1\,]}{2\,[\,1-\gamma\,]+\gamma\,[\,n-1\,]} \sum_{j=1}^{n} \alpha_j$$

$$+ \frac{\gamma}{2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]} \sum_{j=1}^{n} \left(\,\alpha_j + w_j + c_j^d\,\right)$$

$$- \frac{\gamma}{2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]} \left[\,v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,\right]$$

$$+ \frac{\gamma}{2[1+\gamma(n-2)]} \frac{\gamma}{2[1-\gamma]+\gamma[n-1]} \sum_{j=2}^{n} [v_1(w_j - c_1^u) f_{1j} + v_2(w_j - c_2^u) f_{2j}]$$

$$= \frac{1}{2[1-\gamma]+\gamma[n-1]} \Bigg\{ \frac{1}{2} [2(1-\gamma)+\gamma(n-1)] \left[\alpha_i + w_i + c_i^d\right]$$

$$- \frac{\gamma[2(1-\gamma)+\gamma(n-1)]}{2[1+\gamma(n-2)]} \sum_{\substack{j=1 \\ j\neq i}}^{n} \alpha_j$$

$$- \frac{\gamma^2[n-1]}{2[1+\gamma(n-2)]} \sum_{j=1}^{n} \alpha_j + \frac{\gamma}{2} \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d\right)$$

$$- \frac{\gamma}{2} [v_1(w_1 - c_1^u) f_{11} + v_2(w_1 - c_2^u) f_{21}]$$

$$+ \frac{\gamma^2}{2[1+\gamma(n-2)]} \sum_{j=2}^{n} [v_1(w_j - c_1^u) f_{1j} + v_2(w_j - c_2^u) f_{2j}] \Bigg\}$$

$$\Rightarrow \quad p_i^* = \frac{1}{[2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]}$$

$$\cdot \Bigg\{ \gamma[1+\gamma(n-2)] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d\right)$$

$$+ [1+\gamma(n-2)][2(1-\gamma)+\gamma(n-1)] \left[\alpha_i + w_i + c_i^d\right]$$

$$- \gamma^2[n-1] \sum_{j=1}^{n} \alpha_j - \gamma[2(1-\gamma)+\gamma(n-1)] \sum_{\substack{j=1 \\ j\neq i}}^{n} \alpha_j$$

$$- \gamma[1+\gamma(n-2)] [v_1(w_1 - c_1^u) f_{11} + v_2(w_1 - c_2^u) f_{21}]$$

$$+ \gamma^2 \sum_{j=2}^{n} [v_1(w_j - c_1^u) f_{1j} + v_2(w_j - c_2^u) f_{2j}] \Bigg\}. \quad (142)$$

Optimal Collusive Input Prices under Vertical Separation (and Price Competition)

Now consider the input prices U_1 and U_2 set when they collude under vertical separation and downstream firms compete on prices.

(126) implies:

$$q_i^* \;=\; \frac{1+\gamma\,[\,n-2\,]}{\Xi}\left\{\begin{array}{l}\left[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\left(n^2-5\,n+5\right)\right]\,\alpha_i-\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\displaystyle\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j \\ -\left[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\left(n^2-5\,n+5\right)\right]\left[\,c_i^d+w_i\,\right] \\ +\,\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\displaystyle\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\,c_j^d+w_j\,\right]\end{array}\right\} \tag{143}$$

where

$$\Xi \;\equiv\; [\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,]\,[\,2+\gamma\,(\,2\,n-3\,)\,]\,. \tag{144}$$

When they collude under VS, U_1 and U_2 choose input prices w_i to:

$$\max_{w_i}\;\sum_{i=1}^{n}[\,w_i-c_1^u\,]\,f_{1i}\,q_i+\sum_{i=1}^{n}[\,w_i-c_2^u\,]\,f_{2i}\,q_i$$

$$\Leftrightarrow\quad \max_{w_i}\;\sum_{i=1}^{n}[\,w_i-\,c_1^u\,f_{1i}-c_2^u\,f_{2i}\,]\,q_i\,, \tag{145}$$

where q_i is as specified in (143).

Differentiating (145) with respect to w_i provides:

$$q_i+[\,w_i-c_1^u\,f_{1i}-c_2^u\,f_{2i}\,]\;\frac{\partial q_i(\cdot)}{\partial w_i}+\sum_{\substack{j=1\\ j\neq i}}^{n}(\,w_j-c_1^u\,f_{1j}-c_2^u\,f_{2j}\,)\;\frac{\partial q_j(\cdot)}{\partial w_i}\;=\;0 \tag{146}$$

$$\Leftrightarrow\quad q_i-\frac{[\,1+\gamma\,(\,n-2\,)\,]}{\Xi}\left\{\,[\,w_i-c_1^u\,f_{1i}-c_2^u\,f_{2i}\,]\left[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\left(n^2-5\,n+5\right)\right]\right.$$

$$\left.-\,\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}(\,w_j-c_1^u\,f_{1j}-c_2^u\,f_{2j}\,)\right\}\;=\;0$$

$$\Leftrightarrow\quad \frac{1+\gamma\,[\,n-2\,]}{\Xi}\left\{\begin{array}{l}\left[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\left(n^2-5\,n+5\right)\right]\alpha_i-\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\displaystyle\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j \\ -\left[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\left(n^2-5\,n+5\right)\right]\left[\,c_i^d+w_i\,\right]\end{array}\right.$$

$$+\ \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j\,=\,1\\ j\,\neq\, i}}^{n}[\,c_j^d+w_j\,]\ \Bigg\}$$

$$=\ \frac{[\,1+\gamma\,(\,n-2\,)\,]}{\Xi}\Big\{\,[\,w_i-c_1^u\,f_{1i}-c_2^u\,f_{2i}\,]\,\big[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\big(\,n^2-5\,n+5\,\big)\,\big]$$

$$-\,\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j\,=\,1\\ j\,\neq\, i}}^{n}(\,w_j-c_1^u\,f_{1j}-c_2^u\,f_{2j}\,)\,\Big\}$$

$$\Leftrightarrow\quad \big[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\big(\,n^2-5\,n+5\,\big)\,\big]\,\alpha_i-\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j\,=\,1\\ j\,\neq\, i}}^{n}\alpha_j$$

$$-\ \big[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\big(\,n^2-5\,n+5\,\big)\,\big]\,\big[\,c_i^d+w_i\,\big]$$

$$+\ \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j\,=\,1\\ j\,\neq\, i}}^{n}[\,c_j^d+w_j\,]$$

$$=\ [\,w_i-c_1^u\,f_{1i}-c_2^u\,f_{2i}\,]\,\big[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\big(\,n^2-5\,n+5\,\big)\,\big]$$

$$-\ \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j\,=\,1\\ j\,\neq\, i}}^{n}(\,w_j-c_1^u\,f_{1j}-c_2^u\,f_{2j}\,)$$

$$\Leftrightarrow\quad \big[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\big(\,n^2-5\,n+5\,\big)\,\big]\,\alpha_i-\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j\,=\,1\\ j\,\neq\, i}}^{n}\alpha_j$$

$$-\ \big[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\big(\,n^2-5\,n+5\,\big)\,\big]\,c_i^d+\ \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j\,=\,1\\ j\,\neq\, i}}^{n}c_j^d$$

$$-\ \big[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\big(\,n^2-5\,n+5\,\big)\,\big]\ w_i+\ \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j\,=\,1\\ j\,\neq\, i}}^{n}w_j$$

$$=\ w_i\,\big[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\big(\,n^2-5\,n+5\,\big)\,\big]-\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j\,=\,1\\ j\,\neq\, i}}^{n}w_j$$

$$-\ [\,c_1^u\,f_{1i}+c_2^u\,f_{2i}\,]\,\big[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\big(\,n^2-5\,n+5\,\big)\,\big]$$

$$+\,\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}(\,c_1^u\,f_{1j}+c_2^u\,f_{2j}\,)$$

$$\Leftrightarrow\quad \Big[2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\Big(n^2-5\,n+5\Big)\Big]\;\alpha_i-\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j$$

$$-\;\Big[2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\Big(n^2-5\,n+5\Big)\Big]\,c_i^d+\;\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}c_j^d$$

$$=\;2\Big[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\Big(n^2-5\,n+5\Big)\Big]\,w_i-2\,\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}w_j$$

$$-\;[\,c_1^u\,f_{1i}+c_2^u\,f_{2i}\,]\Big[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\Big(n^2-5\,n+5\Big)\Big]$$

$$+\,\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}(\,c_1^u\,f_{1j}+c_2^u\,f_{2j}\,)$$

$$\Leftrightarrow\quad 2\Big[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\Big(n^2-5\,n+5\Big)\Big]\,w_i$$

$$=\;\Big[2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\Big(n^2-5\,n+5\Big)\Big]\;\alpha_i-\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j$$

$$-\;\Big[2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\Big(n^2-5\,n+5\Big)\Big]\,c_i^d+\;\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}c_j^d$$

$$+\;[\,c_1^u\,f_{1i}+c_2^u\,f_{2i}\,]\Big[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,\Big(n^2-5\,n+5\Big)\Big]$$

$$-\,\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}(\,c_1^u\,f_{1j}+c_2^u\,f_{2j}\,)+2\,\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}w_j$$

$$\Leftrightarrow\quad w_i\;=\;\frac{1}{2}\,[\,\alpha_i-c_i^d+c_1^u\,f_{1i}+c_2^u\,f_{2i}\,]-\frac{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,\sum_{j\neq i}\Big(\alpha_j-c_j^d+c_1^u\,f_{1j}+c_2^u\,f_{2j}\Big)}{2\,[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,]}$$

$$+\;\frac{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]}{[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,]}\sum_{\substack{j=1\\ j\neq i}}^{n}w_j\,. \tag{147}$$

Summing (147) provides:

$$\sum_{i=1}^{n} w_i \;=\; \frac{1}{2}\sum_{i=1}^{n}\left[\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right]$$

$$-\;\frac{\gamma\left[\,1+\gamma\,(\,n-2\,)\,\right]\sum_{i=1}^{n}\sum_{j\neq i}\left(\,\alpha_j - c_j^d + c_1^u\, f_{1j} + c_2^u\, f_{2j}\,\right)}{2\left[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,\right]}$$

$$+\;\frac{2\,\gamma\left[\,1+\gamma\,(\,n-2\,)\,\right]}{2\left[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,\right]}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} w_j$$

$$\Rightarrow\quad \sum_{i=1}^{n} w_i \;=\; \frac{1}{2}\sum_{i=1}^{n}\left[\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right]$$

$$-\;\frac{\gamma\left[\,1+\gamma\,(\,n-2\,)\,\right]\left[\,n-1\,\right]\sum_{j=1}^{n}\left(\,\alpha_j - c_j^d + c_1^u\, f_{1j} + c_2^u\, f_{2j}\,\right)}{2\left[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,\right]}$$

$$+\;\frac{2\,\gamma\left[\,1+\gamma\,(\,n-2\,)\,\right]\left[\,n-1\,\right]}{2\left[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,\right]}\sum_{i=1}^{n} w_i$$

$$\Rightarrow\quad \left\{1-\frac{\gamma\left[\,1+\gamma\,(\,n-2\,)\,\right]\left[\,n-1\,\right]}{\left[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,\right]}\right\}\sum_{i=1}^{n} w_i$$

$$=\;\frac{1}{2}\left\{1-\frac{\gamma\left[\,1+\gamma\,(\,n-2\,)\,\right]\left[\,n-1\,\right]}{\left[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,\right]}\right\}\sum_{i=1}^{n}\left[\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right]$$

$$\Rightarrow\quad \sum_{i=1}^{n} w_i \;=\; \frac{1}{2}\sum_{i=1}^{n}\left[\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right]$$

$$\Rightarrow\quad \sum_{j\neq i} w_j \;=\; \frac{1}{2}\sum_{i=1}^{n}\left[\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right] - w_i\,. \tag{148}$$

(147) and (148) imply:

$$w_i \;=\; \frac{1}{2}\left[\,\alpha_i - c_i^d + c_1^u\, f_{1i} + c_2^u\, f_{2i}\,\right] - \frac{\gamma\left[\,1+\gamma\,(\,n-2\,)\,\right]\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\,\alpha_j - c_j^d + c_1^u\, f_{1j} + c_2^u\, f_{2j}\,\right)}{2\left[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,\right]}$$

$$+ \frac{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\left[\frac{1}{2}\sum_{i=1}^{n}\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right] - w_i\right]}{2+3\,\gamma\,[\,n-2\,]+\gamma^2\,[\,n^2-5\,n+5\,]}$$

$$\Rightarrow \quad w_i\left[1+\frac{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]}{2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)}\right]$$

$$= \frac{1}{2}\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right] - \frac{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{j\neq i}\left(\,\alpha_j - c_j^d + c_1^u\,f_{1j} + c_2^u\,f_{2j}\,\right)}{2\,[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,]}$$

$$+ \frac{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{i=1}^{n}\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right]}{2\,[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,]}$$

$$= \frac{1}{2}\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right] - \frac{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\,\alpha_j - c_j^d + c_1^u\,f_{1j} + c_2^u\,f_{2j}\,\right)}{2\,[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,]}$$

$$- \frac{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right]}{2\,[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,]}$$

$$+ \frac{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right]}{2\,[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,]}$$

$$+ \frac{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{i=1}^{n}\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right]}{2\,[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,]}$$

$$= \frac{1}{2}\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right] - \frac{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{j=1}^{n}\left(\,\alpha_j - c_j^d + c_1^u\,f_{1j} + c_2^u\,f_{2j}\,\right)}{2\,[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,]}$$

$$+ \frac{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right]}{2\,[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,]}$$

$$+ \frac{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{i=1}^{n}\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right]}{2\,[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,]}$$

$$= \frac{1}{2}\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right] + \frac{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\left[\,\alpha_i - c_i^d + c_1^u\,f_{1i} + c_2^u\,f_{2i}\,\right]}{2\,[\,2+3\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n^2-5\,n+5\,)\,]}$$

$$= \frac{1}{2} \left\{ 1 + \frac{\gamma \left[1 + \gamma \left(n - 2 \right) \right]}{\left[2 + 3 \gamma \left(n - 2 \right) + \gamma^2 \left(n^2 - 5 n + 5 \right) \right]} \right\} \left[\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right]$$

$$\Rightarrow \quad w_i^* = \frac{1}{2} \left[\alpha_i - c_i^d + c_1^u f_{1i} + c_2^u f_{2i} \right] . \tag{149}$$

(26) and (149) imply that under vertical separation, the input prices U_1 and U_2 set when they collude are the same under price competition and quantity competition.

Optimal Collusive Input Prices under Vertical Integration (and Price Competition)

Now consider the input prices U_1 and U_2 set when they collude in the setting where downstream firm D1 is vertically integrated with upstream supplier U1 and downstream firms compete on prices.

(142) implies:

$$\begin{aligned} \sum_{i=2}^{n} p_i^* = \; & \frac{1}{\left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right] \left[2 - \gamma + 2 \gamma \left(n - 1 \right) \right]} \\ & \cdot \left\{ \gamma \left[n - 1 \right] \left[1 + \gamma \left(n - 2 \right) \right] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d \right) \right. \\ & + \left[1 + \gamma \left(n - 2 \right) \right] \left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right] \sum_{i=2}^{n} \left[\alpha_i + w_i + c_i^d \right] \\ & - \gamma^2 \left[n - 1 \right]^2 \sum_{j=1}^{n} \alpha_j - \gamma \left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right] \sum_{i=2}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j \\ & - \gamma \left[n - 1 \right] \left[1 + \gamma \left(n - 2 \right) \right] \left[v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right] \\ & \left. + \gamma^2 \left[n - 1 \right] \sum_{j=2}^{n} \left[v_1 \left(w_j - c_1^u \right) f_{1j} + v_2 \left(w_j - c_2^u \right) f_{2j} \right] \right\} , \end{aligned} \tag{150}$$

and for $i \in \{2, ..., n\}$:

$$\begin{aligned} \sum_{\substack{k=2 \\ k \neq i}}^{n} p_k^* = \; & \frac{1}{\left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right] \left[2 - \gamma + 2 \gamma \left(n - 1 \right) \right]} \\ & \cdot \left\{ \gamma \left[n - 2 \right] \left[1 + \gamma \left(n - 2 \right) \right] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d \right) \right. \end{aligned}$$

$$+ [1+\gamma(n-2)][2(1-\gamma)+\gamma(n-1)] \sum_{\substack{k=2 \\ k\neq i}}^{n} [\alpha_k + w_k + c_k^d]$$

$$- \gamma^2 [n-1][n-2] \sum_{j=1}^{n} \alpha_j \; - \gamma [2(1-\gamma)+\gamma(n-1)] \sum_{\substack{k=2 \\ k\neq i}}^{n} \sum_{\substack{j=1 \\ j\neq k}}^{n} \alpha_j$$

$$- \gamma [n-2][1+\gamma(n-2)] \, [v_1(w_1 - c_1^u) f_{11} + v_2(w_1 - c_2^u) f_{21}]$$

$$+ \gamma^2 [n-2] \sum_{j=2}^{n} [v_1(w_j - c_1^u) f_{1j} + v_2(w_j - c_2^u) f_{2j}] \Bigg\} . \qquad (151)$$

(113), (140), and (150) imply:

$$q_1^* = \frac{\alpha_1 [1+\gamma(n-2)] - \gamma \sum_{j=2}^{n} \alpha_j}{[1-\gamma][1+\gamma(n-1)]} - \frac{1+\gamma[n-2]}{[1-\gamma][1+\gamma(n-1)]} \, p_1^*$$

$$+ \frac{\gamma}{[1-\gamma][1+\gamma(n-1)]} \sum_{j=2}^{n} p_j^*$$

$$= \frac{\alpha_1 [1+\gamma(n-2)] - \gamma \sum_{j=2}^{n} \alpha_j}{[1-\gamma][1+\gamma(n-1)]}$$

$$- \frac{1+\gamma[n-2]}{[1-\gamma][1+\gamma(n-1)][2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]}$$

$$\cdot \Bigg\{ \gamma [1+\gamma(n-2)] \sum_{i=1}^{n} (\alpha_i + w_i + c_i^d)$$

$$+ [1+\gamma(n-2)][2(1-\gamma)+\gamma(n-1)] [\alpha_1 + w_1 + c_1^d]$$

$$- \gamma^2 [n-1] \sum_{i=1}^{n} \alpha_i - \gamma [2(1-\gamma)+\gamma(n-1)] \sum_{j=2}^{n} \alpha_j$$

$$- [1+\gamma(n-2)][2+\gamma(n-2)][v_1(w_1 - c_1^u) f_{11} + v_2(w_1 - c_2^u) f_{21}]$$

$$+ \gamma [2+\gamma(n-2)] \sum_{i=2}^{n} [v_1(w_i - c_1^u) f_{1i} + v_2(w_i - c_2^u) f_{2i}] \Bigg\}$$

$$+ \frac{\gamma}{[1-\gamma][1+\gamma(n-1)][2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]}$$

$$\cdot \Bigg\{ \gamma\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\,\sum_{j=1}^{n}\left(\,\alpha_j + w_j + c_j^d\,\right)$$

$$+\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,\sum_{i=2}^{n}\left(\,\alpha_i + w_i + c_i^d\,\right)$$

$$-\,\gamma^2\,[\,n-1\,]^2\,\sum_{j=1}^{n}\alpha_j \;-\gamma\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,\sum_{i=2}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j$$

$$-\,\gamma\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,v_1\,(\,w_1-c_1^u\,)\,f_{11}+v_2\,(\,w_1-c_2^u\,)\,f_{21}\,]$$

$$+\,\gamma^2\,[\,n-1\,]\,\sum_{j=2}^{n}[\,v_1\,(\,w_j-c_1^u\,)\,f_{1j}+v_2\,(\,w_j-c_2^u\,)\,f_{2j}\,]\Bigg\}$$

$$= \frac{\alpha_1\,[\,1+\gamma\,(\,n-2\,)\,]-\gamma\,\sum_{j=2}^{n}\alpha_j}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}$$

$$+\,\frac{1}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,[\,2-\gamma+2\,\gamma\,(\,n-1\,)\,]}$$

$$\cdot \Bigg\{ \gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\,\sum_{j=1}^{n}\left(\,\alpha_j + w_j + c_j^d\,\right)$$

$$+\,\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\sum_{i=2}^{n}\left(\,\alpha_i + w_i + c_i^d\,\right)$$

$$-\,\gamma^3\,[\,n-1\,]^2\,\sum_{j=1}^{n}\alpha_j - \gamma^2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,\sum_{i=2}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j$$

$$-\,\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,v_1\,(\,w_1-c_1^u\,)\,f_{11}+v_2\,(\,w_1-c_2^u\,)\,f_{21}\,]$$

$$+\,\gamma^3\,[\,n-1\,]\,\sum_{j=2}^{n}[\,v_1\,(\,w_j-c_1^u\,)\,f_{1j}+v_2\,(\,w_j-c_2^u\,)\,f_{2j}\,]$$

$$-\,\gamma\,[\,1+\gamma\,(\,n-2\,)\,]^2\,\sum_{i=1}^{n}\left(\,\alpha_i + w_i + c_i^d\,\right)$$

$$-\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\left[\,\alpha_1 + w_1 + c_1^d\,\right]$$

$$+\,\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\,\sum_{i=1}^{n}\alpha_i$$

$$+ \gamma [1 + \gamma (n-2)] [2(1-\gamma) + \gamma (n-1)] \sum_{j=2}^{n} \alpha_j$$

$$+ [1 + \gamma (n-2)]^2 [2 + \gamma (n-2)] [v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$- \gamma [1 + \gamma (n-2)] [2 + \gamma (n-2)] \sum_{i=2}^{n} [v_1 (w_i - c_1^u) f_{1i} + v_2 (w_i - c_2^u) f_{2i}] \Bigg\}$$

$$= \frac{\alpha_1 [1 + \gamma (n-2)] - \gamma \sum_{j=2}^{n} \alpha_j}{[1-\gamma][1 + \gamma (n-1)]}$$

$$+ \frac{1}{[1-\gamma][1+\gamma(n-1)][2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]}$$

$$\cdot \Bigg\{ \gamma [1 + \gamma (n-2)] [\gamma (n-1) - (1 + \gamma [n-2])] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d \right)$$

$$+ \gamma [1 + \gamma (n-2)] [2(1-\gamma) + \gamma (n-1)] \sum_{i=2}^{n} \left(\alpha_i + w_i + c_i^d \right)$$

$$- [1 + \gamma (n-2)]^2 [2(1-\gamma) + \gamma (n-1)] \left[\alpha_1 + w_1 + c_1^d \right]$$

$$+ \gamma^2 [n-1] [1 + \gamma (n-2) - \gamma (n-1)] \sum_{j=1}^{n} \alpha_j$$

$$+ [1 + \gamma (n-2)] \left[(1 + \gamma [n-2]) (2 + \gamma [n-2]) - \gamma^2 (n-1) \right]$$

$$\cdot [v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$+ \gamma \left[\gamma^2 (n-1) - (1 + \gamma [n-2]) (2 + \gamma [n-2]) \right]$$

$$\cdot \sum_{j=2}^{n} [v_1 (w_j - c_1^u) f_{1j} + v_2 (w_j - c_2^u) f_{2j}]$$

$$- \gamma^2 [2(1-\gamma) + \gamma (n-1)] \sum_{i=2}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j$$

$$+ \gamma [1 + \gamma (n-2)] [2(1-\gamma) + \gamma (n-1)] \sum_{j=2}^{n} \alpha_j \Bigg\}$$

$$
\begin{aligned}
&= \frac{\alpha_1\,[\,1+\gamma\,(\,n-2\,)\,]-\gamma\,\sum_{j=2}^{n}\alpha_j}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]} \\
&\quad + \frac{1}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,[\,2-\gamma+2\,\gamma\,(\,n-1\,)\,]} \\
&\quad \cdot \Bigg\{ \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,\gamma-1\,]\sum_{j=1}^{n}\big(\,\alpha_j+w_j+c_j^d\,\big) \\
&\qquad + \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\sum_{i=2}^{n}\big(\,\alpha_i+w_i+c_i^d\,\big) \\
&\qquad - [\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,\big[\,\alpha_1+w_1+c_1^d\,\big] \\
&\qquad + \gamma^2\,[\,n-1\,]\,[\,1-\gamma\,]\sum_{j=1}^{n}\alpha_j \\
&\qquad + [\,1+\gamma\,(\,n-2\,)\,]\,\big[\,(\,1+\gamma\,[\,n-2\,]\,)\,(\,2+\gamma\,[\,n-2\,]\,)-\gamma^2\,(\,n-1\,)\,\big] \\
&\qquad\quad \cdot [\,v_1\,(\,w_1-c_1^u\,)\,f_{11}+v_2\,(\,w_1-c_2^u\,)\,f_{21}\,] \\
&\qquad + \gamma\,\big[\,\gamma^2\,(\,n-1\,)-(\,1+\gamma\,[\,n-2\,]\,)\,(\,2+\gamma\,[\,n-2\,]\,)\,\big] \\
&\qquad\quad \cdot \sum_{j=2}^{n}[\,v_1\,(\,w_j-c_1^u\,)\,f_{1j}+v_2\,(\,w_j-c_2^u\,)\,f_{2j}\,] \\
&\qquad - \gamma^2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\sum_{i=2}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j \\
&\qquad + \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\sum_{j=2}^{n}\alpha_j \Bigg\}. \qquad (152)
\end{aligned}
$$

Observe that the last two terms in $\{\cdot\}$ in (152) can be written as:

$$
\begin{aligned}
&-\gamma^2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\sum_{i=2}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j+\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\sum_{j=2}^{n}\alpha_j \\
&= -\gamma^2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\sum_{i=2}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_j-\gamma^2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\sum_{j=2}^{n}\alpha_j
\end{aligned}
$$

$$+ \gamma^2 [\, 2\,(\,1-\gamma\,) + \gamma\,(\,n-1\,)\,] \sum_{j=2}^{n} \alpha_j + \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,] \sum_{j=2}^{n} \alpha_j$$

$$= \; -\gamma^2 [\, 2\,(\,1-\gamma\,) + \gamma\,(\,n-1\,)\,] \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j$$

$$+ \gamma\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,[\,\gamma+1+\gamma\,(\,n-2\,)\,] \sum_{j=2}^{n} \alpha_j$$

$$= \; -\gamma^2 [\, 2\,(\,1-\gamma\,) + \gamma\,(\,n-1\,)\,]\,[\,n-1\,] \sum_{j=1}^{n} \alpha_j$$

$$+ \gamma\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,[\,\gamma+1+\gamma\,(\,n-2\,)\,] \sum_{j=2}^{n} \alpha_j \,. \tag{153}$$

(144), (152) and (153) imply:

$$q_1^* \; = \; \frac{\alpha_1\,[\,1+\gamma\,(\,n-2\,)\,] - \gamma \sum_{j=2}^{n} \alpha_j}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}$$

$$+ \frac{1}{\Xi} \Bigg\{ \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,\gamma-1\,] \sum_{j=1}^{n} \big(\,\alpha_j + w_j + c_j^d\,\big)$$

$$+ \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,] \sum_{i=2}^{n} \big(\,\alpha_i + w_i + c_i^d\,\big)$$

$$- \,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,\big[\,\alpha_1 + w_1 + c_1^d\,\big]$$

$$+ \gamma^2\,[\,n-1\,]\,[\,1-\gamma\,] \sum_{j=1}^{n} \alpha_j$$

$$+ \,[\,1+\gamma\,(\,n-2\,)\,]\,\big[\,(\,1+\gamma\,[\,n-2\,]\,)\,(\,2+\gamma\,[\,n-2\,]\,) - \gamma^2\,(\,n-1\,)\,\big]$$

$$\cdot\,[\,v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,]$$

$$+ \gamma\,\big[\,\gamma^2\,(\,n-1\,) - (\,1+\gamma\,[\,n-2\,]\,)\,(\,2+\gamma\,[\,n-2\,]\,)\,\big]$$

$$\cdot \sum_{j=2}^{n} [\,v_1\,(\,w_j - c_1^u\,)\,f_{1j} + v_2\,(\,w_j - c_2^u\,)\,f_{2j}\,]$$

$$- \gamma^2 [\, 2\,(\, 1 - \gamma \,) + \gamma\,(\, n - 1 \,)\,]\,[\, n - 1 \,] \sum_{j=1}^{n} \alpha_j$$

$$+ \gamma\,[\, 2\,(\, 1 - \gamma \,) + \gamma\,(\, n - 1 \,)\,]\,[\, \gamma + 1 + \gamma\,(\, n - 2 \,)\,] \sum_{j=2} \alpha_j \Bigg\}$$

$$= \frac{\alpha_1\,[\, 1 + \gamma\,(\, n - 2 \,)\,] - \gamma \sum_{j=2}^{n} \alpha_j}{[\, 1 - \gamma \,]\,[\, 1 + \gamma\,(\, n - 1 \,)\,]}$$

$$+ \frac{1}{\Xi} \Bigg\{ \gamma\,[\, 1 + \gamma\,(\, n - 2 \,)\,]\,[\, \gamma - 1 \,] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d \right)$$

$$+ \gamma\,[\, 1 + \gamma\,(\, n - 2 \,)\,]\,[\, 2\,(\, 1 - \gamma \,) + \gamma\,(\, n - 1 \,)\,] \sum_{i=2}^{n} \left(\alpha_i + w_i + c_i^d \right)$$

$$- [\, 1 + \gamma\,(\, n - 2 \,)\,]^2\,[\, 2\,(\, 1 - \gamma \,) + \gamma\,(\, n - 1 \,)\,] \left[\alpha_1 + w_1 + c_1^d \right]$$

$$+ \gamma^2\,[\, n - 1 \,]\,[\, 1 - \gamma - (\, 2\,[\, 1 - \gamma \,] + \gamma\,[\, n - 1 \,]\,)\,] \sum_{j=1}^{n} \alpha_j$$

$$+ [\, 1 + \gamma\,(\, n - 2 \,)\,] \left[(\, 1 + \gamma\,[\, n - 2 \,]\,)\,(\, 2 + \gamma\,[\, n - 2 \,]\,) - \gamma^2\,(\, n - 1 \,) \right]$$

$$\cdot [\, v_1\,(\, w_1 - c_1^u \,)\, f_{11} + v_2\,(\, w_1 - c_2^u \,)\, f_{21} \,]$$

$$+ \gamma \left[\gamma^2\,(\, n - 1 \,) - (\, 1 + \gamma\,[\, n - 2 \,]\,)\,(\, 2 + \gamma\,[\, n - 2 \,]\,) \right]$$

$$\cdot \sum_{j=2}^{n} [\, v_1\,(\, w_j - c_1^u \,)\, f_{1j} + v_2\,(\, w_j - c_2^u \,)\, f_{2j} \,]$$

$$+ \gamma\,[\, 2\,(\, 1 - \gamma \,) + \gamma\,(\, n - 1 \,)\,]\,[\, \gamma + 1 + \gamma\,(\, n - 2 \,)\,] \sum_{j=2}^{n} \alpha_j \Bigg\}$$

$$= \frac{\alpha_1\,[\, 1 + \gamma\,(\, n - 2 \,)\,] - \gamma \sum_{j=2}^{n} \alpha_j}{[\, 1 - \gamma \,]\,[\, 1 + \gamma\,(\, n - 1 \,)\,]}$$

$$+ \frac{1}{\Xi} \Bigg\{ \gamma\,[\, 1 + \gamma\,(\, n - 2 \,)\,]\,[\, \gamma - 1 \,] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d \right)$$

$$+ \gamma\,[\, 1 + \gamma\,(\, n - 2 \,)\,]\,[\, 2\,(\, 1 - \gamma \,) + \gamma\,(\, n - 1 \,)\,] \sum_{i=2}^{n} \left[\alpha_i + w_i + c_i^d \right]$$

$$- \left[1 + \gamma (n - 2) \right]^2 \left[2 (1 - \gamma) + \gamma (n - 1) \right] \left[\alpha_1 + w_1 + c_1^d \right]$$

$$- \gamma^2 \left[n - 1 \right] \left[1 + \gamma (n - 2) \right] \sum_{j=1}^{n} \alpha_j$$

$$+ \left[1 + \gamma (n - 2) \right] \Big[(1 + \gamma [n - 2]) (2 + \gamma [n - 2]) - \gamma^2 (n - 1) \Big]$$

$$\cdot \left[v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21} \right]$$

$$+ \gamma \Big[\gamma^2 (n - 1) - (1 + \gamma [n - 2]) (2 + \gamma [n - 2]) \Big]$$

$$\cdot \sum_{j=2}^{n} \left[v_1 (w_j - c_1^u) f_{1j} + v_2 (w_j - c_2^u) f_{2j} \right]$$

$$+ \gamma \left[2 (1 - \gamma) + \gamma (n - 1) \right] \left[\gamma + 1 + \gamma (n - 2) \right] \sum_{j=2}^{n} \alpha_j \Bigg\} . \tag{154}$$

The first three terms in $\{\cdot\}$ in (154) can be written as:

$$\gamma \left[1 + \gamma (n - 2) \right] \left[\gamma - 1 \right] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d \right)$$

$$+ \gamma \left[1 + \gamma (n - 2) \right] \left[2 (1 - \gamma) + \gamma (n - 1) \right] \sum_{i=2}^{n} \left[\alpha_i + w_i + c_i^d \right]$$

$$- \left[1 + \gamma (n - 2) \right]^2 \left[2 (1 - \gamma) + \gamma (n - 1) \right] \left[\alpha_1 + w_1 + c_1^d \right]$$

$$= \gamma \left[1 + \gamma (n - 2) \right] \left[\gamma - 1 \right] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d \right)$$

$$+ \gamma \left[1 + \gamma (n - 2) \right] \left[2 (1 - \gamma) + \gamma (n - 1) \right] \sum_{i=2}^{n} \left(\alpha_i + w_i + c_i^d \right)$$

$$+ \gamma \left[1 + \gamma (n - 2) \right] \left[2 (1 - \gamma) + \gamma (n - 1) \right] \left[\alpha_1 + w_1 + c_1^d \right]$$

$$- \gamma \left[1 + \gamma (n - 2) \right] \left[2 (1 - \gamma) + \gamma (n - 1) \right] \left[\alpha_1 + w_1 + c_1^d \right]$$

$$- \left[1 + \gamma (n - 2) \right]^2 \left[2 (1 - \gamma) + \gamma (n - 1) \right] \left[\alpha_1 + w_1 + c_1^d \right]$$

$$= \gamma \left[1 + \gamma (n - 2) \right] \left[\gamma - 1 \right] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d \right)$$

$$
\begin{aligned}
&+ \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,\sum_{i=1}^{n}\left(\,\alpha_i+w_i+c_i^d\,\right) \\
&- \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\left[\,\alpha_1+w_1+c_1^d\,\right] \\
&- [\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\left[\,\alpha_1+w_1+c_1^d\,\right] \\
&= \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,\gamma-1+2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,\sum_{j=1}^{n}\left(\,\alpha_j+w_j+c_j^d\,\right) \\
&\quad - [\,1+\gamma\,(\,n-2\,)\,]\,[\,1+\gamma\,(\,n-2\,)+\gamma\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\left[\,\alpha_1+w_1+c_1^d\,\right] \\
&= \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,1-\gamma+\gamma\,(\,n-1\,)\,]\,\sum_{j=1}^{n}\left(\,\alpha_j+w_j+c_j^d\,\right) \\
&\quad - [\,1+\gamma\,(\,n-2\,)\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\left[\,\alpha_1+w_1+c_1^d\,\right] \\
&= \gamma\,[\,1+\gamma\,(\,n-2\,)\,]^2\,\sum_{j=1}^{n}\left(\,\alpha_j+w_j+c_j^d\,\right) \\
&\quad - [\,1+\gamma\,(\,n-2\,)\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,]\left[\,\alpha_1+w_1+c_1^d\,\right]. \qquad (155)
\end{aligned}
$$

The fourth and last terms in $\{\cdot\}$ in (154) can be written as:

$$
\begin{aligned}
&\gamma\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,[\,\gamma+1+\gamma\,(\,n-2\,)\,]\,\sum_{j=2}^{n}\alpha_j \\
&\quad - \gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\,\sum_{j=1}^{n}\alpha_j \\
&= \gamma\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,[\,\gamma+1+\gamma\,(\,n-2\,)\,]\,\sum_{j=2}^{n}\alpha_j \\
&\quad + \gamma\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,[\,\gamma+1+\gamma\,(\,n-2\,)\,]\,\alpha_1 \\
&\quad - \gamma\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,[\,\gamma+1+\gamma\,(\,n-2\,)\,]\,\alpha_1 \\
&\quad - \gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\,\sum_{j=1}^{n}\alpha_j \\
&= \gamma\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,[\,\gamma+1+\gamma\,(\,n-2\,)\,]\,\sum_{j=1}^{n}\alpha_j
\end{aligned}
$$

$$- \gamma [2 (1 - \gamma) + \gamma (n - 1)] [\gamma + 1 + \gamma (n - 2)] \alpha_1$$

$$- \gamma^2 [n - 1] [1 + \gamma (n - 2)] \sum_{j=1}^{n} \alpha_j$$

$$= \gamma \{ [2 + \gamma (n - 3)] [1 + \gamma (n - 1)] - \gamma [n - 1] [1 + \gamma (n - 2)] \} \sum_{j=1}^{n} \alpha_j$$

$$- \gamma [2 + \gamma (n - 3)] [1 + \gamma (n - 1)] \alpha_1 . \qquad (156)$$

The fifth and sixth terms in $\{\cdot\}$ in (154) can be written as:

$$[1 + \gamma (n - 2)] \left[(1 + \gamma [n - 2]) (2 + \gamma [n - 2]) - \gamma^2 (n - 1) \right]$$

$$\cdot [v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$+ \gamma \left[\gamma^2 (n - 1) - (1 + \gamma [n - 2]) (2 + \gamma [n - 2]) \right]$$

$$\cdot \sum_{j=2}^{n} [v_1 (w_j - c_1^u) f_{1j} + v_2 (w_j - c_2^u) f_{2j}]$$

$$= [1 + \gamma (n - 2)] \left[(1 + \gamma [n - 2]) (2 + \gamma [n - 2]) - \gamma^2 (n - 1) \right]$$

$$\cdot [v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$+ \gamma \left[\gamma^2 (n - 1) - (1 + \gamma [n - 2]) (2 + \gamma [n - 2]) \right]$$

$$\cdot \sum_{j=2}^{n} [v_1 (w_j - c_1^u) f_{1j} + v_2 (w_j - c_2^u) f_{2j}]$$

$$+ \gamma \left[\gamma^2 (n - 1) - (1 + \gamma [n - 2]) (2 + \gamma [n - 2]) \right]$$

$$\cdot [v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$- \gamma \left[\gamma^2 (n - 1) - (1 + \gamma [n - 2]) (2 + \gamma [n - 2]) \right]$$

$$\cdot [v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$= [1 + \gamma (n - 2)] \left[(1 + \gamma [n - 2]) (2 + \gamma [n - 2]) - \gamma^2 (n - 1) \right]$$

$$\cdot [v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$+ \gamma \left[\gamma^2 (n - 1) - (1 + \gamma [n - 2]) (2 + \gamma [n - 2]) \right]$$

$$\cdot \sum_{j=1}^{n} [v_1 (w_j - c_1^u) f_{1j} + v_2 (w_j - c_2^u) f_{2j}]$$

$$- \gamma \left[\gamma^2 (n-1) - (1+\gamma[n-2])(2+\gamma[n-2]) \right]$$

$$\cdot [v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$= \Big\{ [1+\gamma(n-2)] \left[(1+\gamma[n-2])(2+\gamma[n-2]) - \gamma^2 (n-1) \right]$$

$$- \gamma \left[\gamma^2 (n-1) - (1+\gamma[n-2])(2+\gamma[n-2]) \right] \Big\}$$

$$\cdot [v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$+ \gamma \left[\gamma^2 (n-1) - (1+\gamma[n-2])(2+\gamma[n-2]) \right]$$

$$\cdot \sum_{j=1}^{n} [v_1 (w_j - c_1^u) f_{1j} + v_2 (w_j - c_2^u) f_{2j}]$$

$$= \Big\{ [1+\gamma(n-2)]^2 [2+\gamma(n-2)] - \gamma^2 [n-1][1+\gamma(n-2)]$$

$$- \gamma^3 [n-1] + \gamma [1+\gamma(n-2)][2+\gamma(n-2)] \Big\}$$

$$\cdot [v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$+ \gamma \left[\gamma^2 (n-1) - (1+\gamma[n-2])(2+\gamma[n-2]) \right]$$

$$\cdot \sum_{j=1}^{n} [v_1 (w_j - c_1^u) f_{1j} + v_2 (w_j - c_2^u) f_{2j}]$$

$$= \{ [1+\gamma(n-2)][2+\gamma(n-2)][1+\gamma(n-2)+\gamma]$$

$$- \gamma^2 [n-1][1+\gamma(n-2)+\gamma] \}$$

$$\cdot [v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$+ \gamma \left[\gamma^2 (n-1) - (1+\gamma[n-2])(2+\gamma[n-2]) \right]$$

$$\cdot \sum_{j=1}^{n} [v_1 (w_j - c_1^u) f_{1j} + v_2 (w_j - c_2^u) f_{2j}]$$

$$= \{ [1+\gamma(n-2)][2+\gamma(n-2)][1+\gamma(n-1)]$$

$$- \gamma^2 [n-1][1+\gamma(n-1)] \}$$

$$\cdot [v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$+ \gamma \left[\gamma^2 (n-1) - (1+\gamma[n-2])(2+\gamma[n-2]) \right]$$

$$\cdot \sum_{j=1}^{n} [v_1 (w_j - c_1^u) f_{1j} + v_2 (w_j - c_2^u) f_{2j}]$$

$$= \; [\,1+\gamma\,(\,n-1\,)\,]\,\big\{\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]-\gamma^2\,[\,n-1\,]\,\big\}$$

$$\cdot\,[\,v_1\,(\,w_1-c_1^u\,)\,f_{11}+v_2\,(\,w_1-c_2^u\,)\,f_{21}\,]$$

$$+\;\gamma\,\big[\,\gamma^2\,(\,n-1\,)-(\,1+\gamma\,[\,n-2\,]\,)\,(\,2+\gamma\,[\,n-2\,]\,)\,\big]$$

$$\cdot\sum_{j=1}^{n}[\,v_1\,(\,w_j-c_1^u\,)\,f_{1j}+v_2\,(\,w_j-c_2^u\,)\,f_{2j}\,]\,. \tag{157}$$

(154), (155), (156), and (157) imply:

$$q_1^* \;=\; \frac{\alpha_1\,[\,1+\gamma\,(\,n-2\,)\,]-\gamma\,\sum_{j=2}^{n}\alpha_j}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}$$

$$+\;\frac{1}{\Xi}\Bigg\{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]^2\sum_{j=1}^{n}\big(\,\alpha_j+w_j+c_j^d\,\big)$$

$$-\;[\,1+\gamma\,(\,n-2\,)\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,]\,\big[\,\alpha_1+w_1+c_1^d\,\big]$$

$$+\,\gamma\,\{\,[\,2+\gamma\,(\,n-3\,)\,]\,[\,1+\gamma\,(\,n-1\,)\,]-\gamma\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\,\}\sum_{j=1}^{n}\alpha_j$$

$$-\;\gamma\,[\,2+\gamma\,(\,n-3\,)\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,\alpha_1$$

$$+\;[\,1+\gamma\,(\,n-1\,)\,]\,\big\{\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]-\gamma^2\,[\,n-1\,]\,\big\}$$

$$\cdot\,[\,v_1\,(\,w_1-c_1^u\,)\,f_{11}+v_2\,(\,w_1-c_2^u\,)\,f_{21}\,]$$

$$+\;\gamma\,\big[\,\gamma^2\,(\,n-1\,)-(\,1+\gamma\,[\,n-2\,]\,)\,(\,2+\gamma\,[\,n-2\,]\,)\,\big]$$

$$\cdot\sum_{j=1}^{n}[\,v_1\,(\,w_j-c_1^u\,)\,f_{1j}+v_2\,(\,w_j-c_2^u\,)\,f_{2j}\,]\Bigg\}. \tag{158}$$

(113), (140), (142), (144), and (151) imply that for $i \in \{2, ..., n\}$:

$$q_i^* \;=\; \frac{\alpha_i\,[\,1+\gamma\,(\,n-2\,)\,]-\gamma\,\sum_{j=1,\,j\neq i}^{n}\alpha_j}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}-\frac{1+\gamma\,[\,n-2\,]}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}\;p_i^*$$

$$+\;\frac{\gamma}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}\;\sum_{\substack{k=1\\k\neq i}}^{n}p_k^*$$

$$= \frac{\alpha_i \left[1 + \gamma \left(n - 2 \right) \right] - \gamma \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j}{\left[1 - \gamma \right] \left[1 + \gamma \left(n - 1 \right) \right]} - \frac{1 + \gamma \left[n - 2 \right]}{\left[1 - \gamma \right] \left[1 + \gamma \left(n - 1 \right) \right]} \, p_i^*$$

$$+ \frac{\gamma}{\left[1 - \gamma \right] \left[1 + \gamma \left(n - 1 \right) \right]} \sum_{\substack{k=2 \\ k \neq i}}^{n} p_k^* + \frac{\gamma}{\left[1 - \gamma \right] \left[1 + \gamma \left(n - 1 \right) \right]} \, p_1^*$$

$$= \frac{\alpha_i \left[1 + \gamma \left(n - 2 \right) \right] - \gamma \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j}{\left[1 - \gamma \right] \left[1 + \gamma \left(n - 1 \right) \right]}$$

$$- \frac{1 + \gamma \left[n - 2 \right]}{\left[1 - \gamma \right] \left[1 + \gamma \left(n - 1 \right) \right] \left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right] \left[2 - \gamma + 2 \gamma \left(n - 1 \right) \right]}$$

$$\cdot \Bigg\{ \gamma \left[1 + \gamma \left(n - 2 \right) \right] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d \right)$$

$$+ \left[1 + \gamma \left(n - 2 \right) \right] \left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right] \left[\alpha_i + w_i + c_i^d \right]$$

$$- \gamma^2 \left[n - 1 \right] \sum_{j=1}^{n} \alpha_j - \gamma \left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right] \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j$$

$$- \gamma \left[1 + \gamma \left(n - 2 \right) \right] \left[v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right]$$

$$+ \gamma^2 \sum_{j=2}^{n} \left[v_1 \left(w_j - c_1^u \right) f_{1j} + v_2 \left(w_j - c_2^u \right) f_{2j} \right] \Bigg\}$$

$$+ \frac{\gamma}{\left[1 - \gamma \right] \left[1 + \gamma \left(n - 1 \right) \right] \left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right] \left[2 - \gamma + 2 \gamma \left(n - 1 \right) \right]}$$

$$\cdot \Bigg\{ \gamma \left[n - 2 \right] \left[1 + \gamma \left(n - 2 \right) \right] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d \right)$$

$$+ \left[1 + \gamma \left(n - 2 \right) \right] \left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right] \sum_{\substack{k=2 \\ k \neq i}}^{n} \left[\alpha_k + w_k + c_k^d \right]$$

$$- \gamma^2 \left[n - 1 \right] \left[n - 2 \right] \sum_{j=1}^{n} \alpha_j - \gamma \left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right] \sum_{\substack{k=2 \\ k \neq i}}^{n} \sum_{\substack{j=1 \\ j \neq k}}^{n} \alpha_j$$

$$- \gamma \left[n - 2 \right] \left[1 + \gamma \left(n - 2 \right) \right] \left[v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right]$$

$$+ \gamma^2 [n-2] \sum_{j=2}^{n} [v_1(w_j - c_1^u) f_{1j} + v_2(w_j - c_2^u) f_{2j}] \Bigg\}$$

$$+ \frac{\gamma}{[1-\gamma][1+\gamma(n-1)][2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]}$$

$$\cdot \Bigg\{ \gamma [1+\gamma(n-2)] \sum_{i=1}^{n} \left(\alpha_i + w_i + c_i^d \right)$$

$$+ [1+\gamma(n-2)][2(1-\gamma)+\gamma(n-1)] \left[\alpha_1 + w_1 + c_1^d \right]$$

$$- \gamma^2 [n-1] \sum_{i=1}^{n} \alpha_i - \gamma [2(1-\gamma)+\gamma(n-1)] \sum_{j=2}^{n} \alpha_j$$

$$- [1+\gamma(n-2)][2+\gamma(n-2)][v_1(w_1 - c_1^u) f_{11} + v_2(w_1 - c_2^u) f_{21}]$$

$$+ \gamma [2+\gamma(n-2)] \sum_{i=2}^{n} [v_1(w_i - c_1^u) f_{1i} + v_2(w_i - c_2^u) f_{2i}] \Bigg\}$$

$$= \frac{\alpha_i [1+\gamma(n-2)] - \gamma \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j}{[1-\gamma][1+\gamma(n-1)]}$$

$$- \frac{1+\gamma[n-2]}{[1-\gamma][1+\gamma(n-1)][2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]}$$

$$\cdot \Bigg\{ \gamma [1+\gamma(n-2)] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d \right)$$

$$+ [1+\gamma(n-2)][2(1-\gamma)+\gamma(n-1)] \left[\alpha_i + w_i + c_i^d \right]$$

$$- \gamma^2 [n-1] \sum_{j=1}^{n} \alpha_j - \gamma [2(1-\gamma)+\gamma(n-1)] \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j$$

$$- \gamma [1+\gamma(n-2)] [v_1(w_1 - c_1^u) f_{11} + v_2(w_1 - c_2^u) f_{21}]$$

$$+ \gamma^2 \sum_{j=2}^{n} [v_1(w_j - c_1^u) f_{1j} + v_2(w_j - c_2^u) f_{2j}] \Bigg\}$$

$$+ \frac{\gamma}{[1-\gamma][1+\gamma(n-1)][2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]}$$

$$\cdot \Bigg\{ \gamma\,[\,n-2\,]\,[\,1+\gamma\,(\,n-2\,)\,]\,\sum_{j=1}^{n}\big(\,\alpha_j + w_j + c_j^d\,\big)$$

$$+\ \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,\sum_{i=1}^{n}\big(\,\alpha_i + w_i + c_i^d\,\big)$$

$$+\ [\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,\sum_{\substack{k=2\\k\neq i}}^{n}\big[\,\alpha_k + w_k + c_k^d\,\big]$$

$$+\ [\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,\big[\,\alpha_1 + w_1 + c_1^d\,\big]$$

$$-\ \gamma^2\,[\,n-1\,]\,[\,n-2\,]\,\sum_{j=1}^{n}\alpha_j\ -\gamma\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,\sum_{\substack{k=2\\k\neq i}}^{n}\ \sum_{\substack{j=1\\j\neq k}}^{n}\alpha_j$$

$$-\ \gamma^2\,[\,n-1\,]\,\sum_{i=1}^{n}\alpha_i - \gamma\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,\sum_{j=2}^{n}\alpha_j$$

$$-\ \gamma\,[\,n-2\,]\,[\,1+\gamma\,(\,n-2\,)\,]\,\big[\,v_1\,(\,w_1-c_1^u\,)\,f_{11} + v_2\,(\,w_1-c_2^u\,)\,f_{21}\,\big]$$

$$-\ [\,1+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]\,\big[\,v_1\,(\,w_1-c_1^u\,)\,f_{11} + v_2\,(\,w_1-c_2^u\,)\,f_{21}\,\big]$$

$$+\ \gamma^2\,[\,n-2\,]\,\sum_{j=2}^{n}\big[\,v_1\,(\,w_j-c_1^u\,)\,f_{1j} + v_2\,(\,w_j-c_2^u\,)\,f_{2j}\,\big]$$

$$+\ \gamma\,[\,2+\gamma\,(\,n-2\,)\,]\,\sum_{i=2}^{n}\big[\,v_1\,(\,w_i-c_1^u\,)\,f_{1i} + v_2\,(\,w_i-c_2^u\,)\,f_{2i}\,\big]\Bigg\}$$

$$=\ \frac{\alpha_i\,[\,1+\gamma\,(\,n-2\,)\,] - \gamma\,\sum\limits_{\substack{j=1\\j\neq i}}^{n}\alpha_j}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}$$

$$-\ \frac{1+\gamma\,[\,n-2\,]}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,[\,2-\gamma+2\,\gamma\,(\,n-1\,)\,]}$$

$$\cdot \Bigg\{ \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,\sum_{j=1}^{n}\big(\,\alpha_j + w_j + c_j^d\,\big)$$

$$+\ [\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,\big[\,\alpha_i + w_i + c_i^d\,\big]$$

$$-\ \gamma^2\,[\,n-1\,]\,\sum_{j=1}^{n}\alpha_j\ -\gamma\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,\sum_{\substack{j=1\\j\neq i}}^{n}\alpha_j$$

$$- \gamma \, [\, 1 + \gamma \, (\, n - 2\,)\,] \; [\, v_1 \, (\, w_1 - c_1^u \,)\, f_{11} + v_2 \, (\, w_1 - c_2^u \,)\, f_{21} \,]$$

$$+ \, \gamma^2 \sum_{j\,=\,2}^{n} [\, v_1 \, (\, w_j - c_1^u \,)\, f_{1j} + v_2 \, (\, w_j - c_2^u \,)\, f_{2j} \,] \Bigg\}$$

$$+ \, \frac{\gamma}{[\, 1 - \gamma \,]\, [\, 1 + \gamma \, (\, n - 1\,)\,]\, [\, 2 \, (\, 1 - \gamma \,) + \gamma \, (\, n - 1\,)\,]\, [\, 2 - \gamma + 2 \, \gamma \, (\, n - 1\,)\,]}$$

$$\cdot \Bigg\{ \gamma \, [\, n - 1 \,]\, [\, 1 + \gamma \, (\, n - 2\,)\,] \sum_{j\,=\,1}^{n} \big(\, \alpha_j + w_j + c_j^d \,\big)$$

$$+ \, [\, 1 + \gamma \, (\, n - 2\,)\,]\, [\, 2 \, (\, 1 - \gamma \,) + \gamma \, (\, n - 1\,)\,] \sum_{\substack{k\,=\,1 \\ k\,\neq\, i}}^{n} \big[\, \alpha_k + w_k + c_k^d \,\big]$$

$$- \, \gamma^2 \, [\, n - 1 \,]\, [\, n - 1 \,] \sum_{j\,=\,1}^{n} \alpha_j \;\; - \gamma \, [\, 2 \, (\, 1 - \gamma \,) + \gamma \, (\, n - 1\,)\,] \sum_{\substack{k\,=\,2 \\ k\,\neq\, i}}^{n} \; \sum_{\substack{j\,=\,1 \\ j\,\neq\, k}}^{n} \alpha_j$$

$$- \, \gamma \, [\, 2 \, (\, 1 - \gamma \,) + \gamma \, (\, n - 1\,)\,] \sum_{j\,=\,2}^{n} \alpha_j$$

$$- \, [\, 1 + \gamma \, (\, n - 2\,)\,]\, [\, 2 + 2 \, \gamma \, (\, n - 2\,)\,]\, [\, v_1 \, (\, w_1 - c_1^u \,)\, f_{11} + v_2 \, (\, w_1 - c_2^u \,)\, f_{21} \,]$$

$$+ \, \gamma \, [\, 2 + 2 \, \gamma \, (\, n - 2\,)\,] \sum_{j\,=\,2}^{n} [\, v_1 \, (\, w_j - c_1^u \,)\, f_{1j} + v_2 \, (\, w_j - c_2^u \,)\, f_{2j} \,] \Bigg\}$$

$$= \; \frac{\alpha_i \, [\, 1 + \gamma \, (\, n - 2\,)\,] - \gamma \sum\limits_{\substack{j\,=\,1 \\ j\,\neq\, i}}^{n} \alpha_j}{[\, 1 - \gamma \,]\, [\, 1 + \gamma \, (\, n - 1\,)\,]}$$

$$- \, \frac{1}{[\, 1 - \gamma \,]\, [\, 1 + \gamma \, (\, n - 1\,)\,]\, [\, 2 \, (\, 1 - \gamma \,) + \gamma \, (\, n - 1\,)\,]\, [\, 2 - \gamma + 2 \, \gamma \, (\, n - 1\,)\,]}$$

$$\cdot \Bigg\{ \gamma \, [\, 1 + \gamma \, (\, n - 2\,)\,]^2 \sum_{j\,=\,1}^{n} \big(\, \alpha_j + w_j + c_j^d \,\big)$$

$$+ \, [\, 1 + \gamma \, (\, n - 2\,)\,]^2 \, [\, 2 \, (\, 1 - \gamma \,) + \gamma \, (\, n - 1\,)\,] \, \big[\, \alpha_i + w_i + c_i^d \,\big]$$

$$- \, \gamma^2 \, [\, n - 1 \,]\, [\, 1 + \gamma \, (\, n - 2\,)\,] \sum_{j\,=\,1}^{n} \alpha_j$$

$$- \gamma [2(1-\gamma) + \gamma(n-1)][1+\gamma(n-2)] \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j$$

$$- \gamma [1+\gamma(n-2)]^2 \, [v_1(w_1 - c_1^u) f_{11} + v_2(w_1 - c_2^u) f_{21}]$$

$$+ \gamma^2 [1+\gamma(n-2)] \sum_{j=2}^{n} [v_1(w_j - c_1^u) f_{1j} + v_2(w_j - c_2^u) f_{2j}]$$

$$- \gamma^2 [n-1][1+\gamma(n-2)] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d \right)$$

$$- \gamma [1+\gamma(n-2)][2(1-\gamma) + \gamma(n-1)] \sum_{\substack{k=1 \\ k \neq i}}^{n} [\alpha_k + w_k + c_k^d]$$

$$+ \gamma^3 [n-1][n-1] \sum_{j=1}^{n} \alpha_j \; + \gamma^2 [2(1-\gamma) + \gamma(n-1)] \sum_{\substack{k=2 \\ k \neq i}}^{n} \sum_{\substack{j=1 \\ j \neq k}}^{n} \alpha_j$$

$$+ \gamma^2 [2(1-\gamma) + \gamma(n-1)] \sum_{j=2}^{n} \alpha_j$$

$$+ \gamma [1+\gamma(n-2)][2 + 2\gamma(n-2)][v_1(w_1 - c_1^u) f_{11} + v_2(w_1 - c_2^u) f_{21}]$$

$$\left. - \gamma^2 [2 + 2\gamma(n-2)] \sum_{j=2}^{n} [v_1(w_j - c_1^u) f_{1j} + v_2(w_j - c_2^u) f_{2j}] \right\}$$

$$= \frac{\alpha_i [1+\gamma(n-2)] - \gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j}{[1-\gamma][1+\gamma(n-1)]}$$

$$- \frac{1}{[1-\gamma][1+\gamma(n-1)][2(1-\gamma) + \gamma(n-1)][2 - \gamma + 2\gamma(n-1)]}$$

$$\cdot \left\{ \gamma [1+\gamma(n-2)][1+\gamma(n-2) - \gamma(n-1)] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d \right) \right.$$

$$+ [1+\gamma(n-2)]^2 [2(1-\gamma) + \gamma(n-1)] \left[\alpha_i + w_i + c_i^d \right]$$

$$- \gamma [1+\gamma(n-2)][2(1-\gamma) + \gamma(n-1)] \sum_{\substack{k=1 \\ k \neq i}}^{n} [\alpha_k + w_k + c_k^d]$$

$$- \gamma^2 [n-1][1+\gamma(n-2)-\gamma(n-1)] \sum_{j=1}^{n} \alpha_j$$

$$- \gamma [2(1-\gamma)+\gamma(n-1)][1+\gamma(n-2)] \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j$$

$$+ \gamma^2 [2(1-\gamma)+\gamma(n-1)] \sum_{\substack{k=2 \\ k \neq i}}^{n} \sum_{\substack{j=1 \\ j \neq k}}^{n} \alpha_j + \gamma^2 [2(1-\gamma)+\gamma(n-1)] \sum_{j=2}^{n} \alpha_j$$

$$+ \gamma [1+\gamma(n-2)][1+\gamma(n-2)][v_1(w_1-c_1^u) f_{11} + v_2(w_1-c_2^u) f_{21}]$$

$$- \gamma^2 [1+\gamma(n-2)] \sum_{j=2}^{n} [v_1(w_j-c_1^u) f_{1j} + v_2(w_j-c_2^u) f_{2j}] \Bigg\} . \qquad (159)$$

The first three terms in $\{\cdot\}$ in (159) can be written as:

$$\gamma [1+\gamma(n-2)][1+\gamma(n-2)-\gamma(n-1)] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d\right)$$

$$+ [1+\gamma(n-2)]^2 [2(1-\gamma)+\gamma(n-1)] \left[\alpha_i + w_i + c_i^d\right]$$

$$- \gamma [1+\gamma(n-2)][2(1-\gamma)+\gamma(n-1)] \sum_{\substack{k=1 \\ k \neq i}}^{n} \left[\alpha_k + w_k + c_k^d\right]$$

$$= \gamma [1+\gamma(n-2)][1+\gamma(n-2)-\gamma(n-1)] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d\right)$$

$$+ [1+\gamma(n-2)]^2 [2(1-\gamma)+\gamma(n-1)] \left[\alpha_i + w_i + c_i^d\right]$$

$$+ \gamma [1+\gamma(n-2)][2(1-\gamma)+\gamma(n-1)] \left[\alpha_i + w_i + c_i^d\right]$$

$$- \gamma [1+\gamma(n-2)][2(1-\gamma)+\gamma(n-1)] \left[\alpha_i + w_i + c_i^d\right]$$

$$- \gamma [1+\gamma(n-2)][2(1-\gamma)+\gamma(n-1)] \sum_{\substack{k=1 \\ k \neq i}}^{n} \left[\alpha_k + w_k + c_k^d\right]$$

$$= \gamma [1+\gamma(n-2)][1+\gamma(n-2)-\gamma(n-1)] \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d\right)$$

$$+ [1+\gamma(n-2)]^2 [2(1-\gamma)+\gamma(n-1)] \left[\alpha_i + w_i + c_i^d\right]$$

$$+\,\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,\left[\,\alpha_i+w_i+c_i^d\,\right]$$

$$-\,\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\sum_{k=1}^{n}\left[\,\alpha_k+w_k+c_k^d\,\right]$$

$$=\;\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,\{\,1+\gamma\,[\,n-2\,]-\gamma\,[\,n-1\,]-2\,[\,1-\gamma\,]-\gamma\,[\,n-1\,]\,\}$$

$$\cdot\sum_{j=1}^{n}\left(\,\alpha_j+w_j+c_j^d\,\right)$$

$$+\;[\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,[\,1+\gamma\,(\,n-2\,)+\gamma\,]$$

$$\cdot\left[\,\alpha_i+w_i+c_i^d\,\right]$$

$$=\;\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,-1+\gamma\,(\,n-2-n+1+2-n+1\,)\,]\sum_{j=1}^{n}\left(\,\alpha_j+w_j+c_j^d\,\right)$$

$$+\;[\,1+\gamma\,(\,n-2\,)\,]\,[\,2\;+\gamma\,(\,n-3\,)\,]\,[\,1+\gamma\,(\,n-1\,)\,]\left[\,\alpha_i+w_i+c_i^d\,\right]$$

$$=\;-\,\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{j=1}^{n}\left(\,\alpha_j+w_j+c_j^d\,\right)$$

$$+\;[\,1+\gamma\,(\,n-2\,)\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,]\left[\,\alpha_i+w_i+c_i^d\,\right].$$

Define $\alpha_{-h}\equiv\sum_{\substack{j=1\\j\neq h}}^{n}\alpha_j$. Then for $i\in\{2,...,n\}$:

$$\sum_{\substack{k=2\\k\neq i}}^{n}\sum_{\substack{j=1\\j\neq k}}^{n}\alpha_j\;=\;\alpha_{-2}+\alpha_{-3}+...+\alpha_{-(i-1)}+\alpha_{-(i+1)}+...+\alpha_{-n}$$

$$=\;[\,n-2\,]\,\alpha_1+[\,n-2\,]\,\alpha_i+[\,n-3\,]\sum_{\substack{j=2\\j\neq i}}^{n}\alpha_j\;=\;\alpha_1+\alpha_i+[\,n-3\,]\sum_{j=1}^{n}\alpha_j\,.\qquad(160)$$

(160) implies that the fourth to seventh terms in $\{\cdot\}$ in (159) can be written as:

$$-\,\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)-\gamma\,(\,n-1\,)\,]\sum_{j=1}^{n}\alpha_j$$

$$- \gamma [2(1-\gamma)+\gamma(n-1)][1+\gamma(n-2)] \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j$$

$$+ \gamma^2 [2(1-\gamma)+\gamma(n-1)] \left[\alpha_1 + \alpha_i + [n-3] \sum_{j=1}^{n} \alpha_j \right]$$

$$+ \gamma^2 [2(1-\gamma)+\gamma(n-1)] \sum_{j=2}^{n} \alpha_j$$

$$= - \gamma^2 [n-1][1+\gamma(n-2)-\gamma(n-1)] \sum_{j=1}^{n} \alpha_j$$

$$- \gamma [2(1-\gamma)+\gamma(n-1)][1+\gamma(n-2)] \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j$$

$$- \gamma [2(1-\gamma)+\gamma(n-1)][1+\gamma(n-2)] \alpha_i$$

$$+ \gamma [2(1-\gamma)+\gamma(n-1)][1+\gamma(n-2)] \alpha_i$$

$$+ \gamma^2 [2(1-\gamma)+\gamma(n-1)] \left[\alpha_1 + \alpha_i + [n-3] \sum_{j=1}^{n} \alpha_j \right]$$

$$+ \gamma^2 [2(1-\gamma)+\gamma(n-1)] \sum_{j=2}^{n} \alpha_j$$

$$+ \gamma^2 [2(1-\gamma)+\gamma(n-1)] \alpha_1$$

$$- \gamma^2 [2(1-\gamma)+\gamma(n-1)] \alpha_1$$

$$= - \gamma^2 [n-1][1+\gamma(n-2)-\gamma(n-1)] \sum_{j=1}^{n} \alpha_j$$

$$- \gamma [2(1-\gamma)+\gamma(n-1)][1+\gamma(n-2)] \sum_{j=1}^{n} \alpha_j$$

$$+ \gamma [2(1-\gamma)+\gamma(n-1)][1+\gamma(n-2)] \alpha_i$$

$$+ \gamma^2 [2(1-\gamma)+\gamma(n-1)] \left[\alpha_1 + \alpha_i + [n-3] \sum_{j=1}^{n} \alpha_j \right]$$

$$+ \gamma^2 [2(1-\gamma) + \gamma(n-1)] \sum_{j=1}^{n} \alpha_j$$

$$- \gamma^2 [2(1-\gamma) + \gamma(n-1)] \alpha_1$$

$$= - \gamma^2 [n-1][1 + \gamma(n-2) - \gamma(n-1)] \sum_{j=1}^{n} \alpha_j$$

$$- \gamma [2(1-\gamma) + \gamma(n-1)][1 + \gamma(n-2) - \gamma] \sum_{j=1}^{n} \alpha_j$$

$$+ \gamma [2(1-\gamma) + \gamma(n-1)][1 + \gamma(n-2)] \alpha_i$$

$$+ \gamma^2 [2(1-\gamma) + \gamma(n-1)] \left[\alpha_i + [n-3] \sum_{j=1}^{n} \alpha_j \right]$$

$$= - \gamma^2 [n-1][1 + \gamma(n-2) - \gamma(n-1)] \sum_{j=1}^{n} \alpha_j$$

$$- \gamma [2(1-\gamma) + \gamma(n-1)][1 + \gamma(n-3)] \sum_{j=1}^{n} \alpha_j$$

$$+ \gamma [2(1-\gamma) + \gamma(n-1)][1 + \gamma(n-2)] \alpha_i$$

$$+ \gamma^2 [2(1-\gamma) + \gamma(n-1)] \alpha_i$$

$$+ \gamma^2 [2(1-\gamma) + \gamma(n-1)] [n-3] \sum_{j=1}^{n} \alpha_j$$

$$= - \gamma^2 [n-1][1 + \gamma(n-2) - \gamma(n-1)] \sum_{j=1}^{n} \alpha_j$$

$$- \gamma [2(1-\gamma) + \gamma(n-1)][1 + \gamma(n-3) - \gamma(n-3)] \sum_{j=1}^{n} \alpha_j$$

$$+ \gamma [2(1-\gamma) + \gamma(n-1)][1 + \gamma(n-2) + \gamma] \alpha_i$$

$$= - \gamma^2 [n-1][1-\gamma] \sum_{j=1}^{n} \alpha_j - \gamma [2 + \gamma(n-3)] \sum_{j=1}^{n} \alpha_j$$

$$+ \gamma [2 + \gamma(n-3)][1 + \gamma(n-1)] \alpha_i$$

$$= \; - \gamma [\gamma (n-1)(1-\gamma) + 2 + \gamma (n-3)] \sum_{j=1}^{n} \alpha_j$$

$$+ \gamma [2 + \gamma (n-3)][1 + \gamma (n-1)] \alpha_i .$$

The last two terms in $\{\cdot\}$ in (159) can be written as:

$$\gamma [1 + \gamma (n-2)][1 + \gamma (n-2)][v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$+ \gamma^2 [1 + \gamma (n-2)] [v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$- \gamma^2 [1 + \gamma (n-2)] [v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$- \gamma^2 [1 + \gamma (n-2)] \sum_{j=2}^{n} [v_1 (w_j - c_1^u) f_{1j} + v_2 (w_j - c_2^u) f_{2j}]$$

$$= \; \gamma [1 + \gamma (n-2)][1 + \gamma (n-2) + \gamma][v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$- \gamma^2 [1 + \gamma (n-2)] \sum_{j=1}^{n} [v_1 (w_j - c_1^u) f_{1j} + v_2 (w_j - c_2^u) f_{2j}]$$

$$= \; \gamma [1 + \gamma (n-2)][1 + \gamma (n-1)][v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}]$$

$$- \gamma^2 [1 + \gamma (n-2)] \sum_{j=1}^{n} [v_1 (w_j - c_1^u) f_{1j} + v_2 (w_j - c_2^u) f_{2j}] .$$

These conclusions and (159) imply that for $i \in \{2, ..., n\}$:

$$q_i^* \; = \; \frac{\alpha_i [1 + \gamma (n-2)] - \gamma \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j}{[1 - \gamma][1 + \gamma (n-1)]}$$

$$- \frac{1}{\Xi} \Bigg\{ - \gamma [1 + \gamma (n-2)][1 + \gamma (n-2)] \sum_{j=1}^{n} (\alpha_j + w_j + c_j^d)$$

$$+ [1 + \gamma (n-2)][1 + \gamma (n-1)][2 + \gamma (n-3)] \left[\alpha_i + w_i + c_i^d \right]$$

$$- \gamma [\gamma (n-1)(1-\gamma) + 2 + \gamma (n-3)] \sum_{j=1}^{n} \alpha_j$$

$$+ \gamma [2 + \gamma (n-3)][1 + \gamma (n-1)] \alpha_i$$

$$+\ \gamma\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,[\,v_1\,(\,w_1-c_1^u\,)\,f_{11}+v_2\,(\,w_1-c_2^u\,)\,f_{21}\,]$$

$$-\ \gamma^2\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{j=1}^{n}[\,v_1\,(\,w_j-c_1^u\,)\,f_{1j}+v_2\,(\,w_j-c_2^u\,)\,f_{2j}\,]\Bigg\}\,. \qquad (161)$$

U_1 and U_2 choose input prices w_i (for $i=1,2,...n$) to:

$$\max_{w_i}\ \sum_{i=1}^{n}[\,w_i-c_1^u\,]\,f_{1i}\,q_i+\sum_{i=1}^{n}[\,w_i-c_2^u\,]\,f_{2i}\,q_i+\left[\,P_1-w_1-c_1^d\,\right]q_1$$

$$\Leftrightarrow\quad \max_{w_i}\ \sum_{i=1}^{n}[\,w_i-c_1^u\,f_{1i}-c_2^u\,f_{2i}\,]\,q_i+\left[\,P_1-w_1-c_1^d\,\right]q_1$$

$$\Leftrightarrow\quad \max_{w_i}\ \sum_{i=2}^{n}[\,w_i-c_1^u\,f_{1i}-c_2^u\,f_{2i}\,]\,q_i+\left[\,P_1-c_1^d-c_1^u\,f_{11}-c_2^u\,f_{21}\,\right]q_1\,. \qquad (162)$$

where P_1 is as specified in (140), q_1 is as specified in (158), and q_i is as specified in (161) for $i=2,...n$.

(140) implies that p_1^* can be written as:

$$p_1^*\ =\ \frac{1}{[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,[\,2-\gamma+2\,\gamma\,(\,n-1\,)\,]}$$

$$\cdot\Bigg\{\gamma\,[\,1+\gamma\,(\,n-2\,)\,]\sum_{i=1}^{n}\left(\,\alpha_i+w_i+c_i^d\,\right)$$

$$+\ [\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\left[\,\alpha_1+w_1+c_1^d\,\right]$$

$$-\ \gamma^2\,[\,n-1\,]\sum_{i=1}^{n}\alpha_i-\gamma\,[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\sum_{j=2}^{n}\alpha_j$$

$$-\ [\,1+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]\,[\,v_1\,(\,w_1-c_1^u\,)\,f_{11}+v_2\,(\,w_1-c_2^u\,)\,f_{21}\,]$$

$$-\ \gamma\,[\,2+\gamma\,(\,n-2\,)\,]\,[\,v_1\,(\,w_1-c_1^u\,)\,f_{11}+v_2\,(\,w_1-c_2^u\,)\,f_{21}\,]$$

$$+\ \gamma\,[\,2+\gamma\,(\,n-2\,)\,]\,[\,v_1\,(\,w_1-c_1^u\,)\,f_{11}+v_2\,(\,w_1-c_2^u\,)\,f_{21}\,]$$

$$+\ \gamma\,[\,2+\gamma\,(\,n-2\,)\,]\sum_{i=2}^{n}[\,v_1\,(\,w_i-c_1^u\,)\,f_{1i}+v_2\,(\,w_i-c_2^u\,)\,f_{2i}\,]\Bigg\}$$

$$=\ \frac{1}{[\,2\,(\,1-\gamma\,)+\gamma\,(\,n-1\,)\,]\,[\,2-\gamma+2\,\gamma\,(\,n-1\,)\,]}$$

$$\cdot \left\{ \gamma \left[1 + \gamma \left(n - 2 \right) \right] \sum_{i=1}^{n} \left(\alpha_i + w_i + c_i^d \right) \right.$$

$$+ \left[1 + \gamma \left(n - 2 \right) \right] \left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right] \left[\alpha_1 + w_1 + c_1^d \right]$$

$$- \gamma^2 \left[n - 1 \right] \sum_{i=1}^{n} \alpha_i - \gamma \left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right] \sum_{j=2}^{n} \alpha_j$$

$$- \left[1 + \gamma \left(n - 1 \right) \right] \left[2 + \gamma \left(n - 2 \right) \right] \left[v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right]$$

$$\left. + \gamma \left[2 + \gamma \left(n - 2 \right) \right] \sum_{i=1}^{n} \left[v_1 \left(w_i - c_1^u \right) f_{1i} + v_2 \left(w_i - c_2^u \right) f_{2i} \right] \right\} . \tag{163}$$

Differentiating (162) with respect to w_i provides, for $i = 2, ...n$:

$$q_1 \frac{\partial P_1(\cdot)}{\partial w_i} + \left[P_1 - c_1^d - c_1^u f_{11} - c_2^u f_{21} \right] \frac{\partial q_1(\cdot)}{\partial w_i} + q_i + \left[w_i - c_1^u f_{1i} - c_2^u f_{2i} \right] \frac{\partial q_i(\cdot)}{\partial w_i}$$

$$+ \sum_{\substack{j=2 \\ j \neq i}}^{n} \left(w_j - c_1^u f_{1j} - c_2^u f_{2j} \right) \frac{\partial q_j(\cdot)}{\partial w_i} = 0 . \tag{164}$$

(144), (158), (161), and (163) imply that for $i, j \in \{ 2, ...n \}$ $(j \neq i)$:

$$\frac{\partial P_1(\cdot)}{\partial w_i} = \frac{\gamma \left[1 + \gamma \left(n - 2 \right) \right] + \gamma \left[2 + \gamma \left(n - 2 \right) \right] \left[v_1 f_{1i} + v_2 f_{2i} \right]}{\left[2 + \gamma \left(n - 3 \right) \right] \left[2 + \gamma \left(2n - 3 \right) \right]} ;$$

$$\frac{\partial q_1(\cdot)}{\partial w_i} = \frac{1}{\Xi} \left\{ \gamma \left[1 + \gamma \left(n - 2 \right) \right]^2 \right.$$

$$\left. + \gamma \left[\gamma^2 \left(n - 1 \right) - \left(1 + \gamma \left[n - 2 \right] \right) \left(2 + \gamma \left[n - 2 \right] \right) \right] \left[v_1 f_{1i} + v_2 f_{2i} \right] \right\} ;$$

$$\frac{\partial q_i(\cdot)}{\partial w_i} = - \frac{1}{\Xi} \left\{ - \gamma \left[1 + \gamma \left(n - 2 \right) \right]^2 - \gamma^2 \left[1 + \gamma \left(n - 2 \right) \right] \left[v_1 f_{1i} + v_2 f_{2i} \right] \right.$$

$$\left. + \left[1 + \gamma \left(n - 2 \right) \right] \left[1 + \gamma \left(n - 1 \right) \right] \left[2 + \gamma \left(n - 3 \right) \right] \right\} ;$$

$$\frac{\partial q_j(\cdot)}{\partial w_i} = - \frac{1}{\Xi} \left\{ - \gamma \left[1 + \gamma \left(n - 2 \right) \right]^2 - \gamma^2 \left[1 + \gamma \left(n - 2 \right) \right] \left[v_1 f_{1i} + v_2 f_{2i} \right] \right\}$$

$$= \frac{\gamma \left[1 + \gamma \left(n - 2 \right) \right]}{\Xi} \left[1 + \gamma \left(n - 2 \right) + \gamma \left(v_1 f_{1i} + v_2 f_{2i} \right) \right] . \tag{165}$$

(144), (158), (161), (163), and (164) imply that for $i \in \{2, ..., n\}$:

$$
\begin{aligned}
0 \;=\; & \frac{\partial P_1(\cdot)}{\partial w_i} \, \frac{\alpha_1 [\, 1 + \gamma \,(\, n - 2 \,) \,] - \gamma \sum\limits_{j=2}^{n} \alpha_j}{[\, 1 - \gamma \,] \, [\, 1 + \gamma \,(\, n - 1 \,) \,]} \\
& + \frac{\partial P_1(\cdot)}{\partial w_i} \, \frac{1}{\Xi} \Bigg\{ \gamma \, [\, 1 + \gamma \,(\, n - 2 \,) \,]^2 \sum_{j=1}^{n} \left(\alpha_j + w_j + c_j^d \right) \\
& \qquad - [\, 1 + \gamma \,(\, n - 2 \,) \,] \, [\, 1 + \gamma \,(\, n - 1 \,) \,] \, [\, 2 + \gamma \,(\, n - 3 \,) \,] \left[\alpha_1 + w_1 + c_1^d \right] \\
& \qquad + \gamma \, \{ \, [\, 2 + \gamma \,(\, n - 3 \,) \,] \, [\, 1 + \gamma \,(\, n - 1 \,) \,] - \gamma \, [\, n - 1 \,] \, [\, 1 + \gamma \,(\, n - 2 \,) \,] \, \} \sum_{j=1}^{n} \alpha_j \\
& \qquad - \gamma \, [\, 2 + \gamma \,(\, n - 3 \,) \,] \, [\, 1 + \gamma \,(\, n - 1 \,) \,] \, \alpha_1 \\
& \qquad + [\, 1 + \gamma \,(\, n - 1 \,) \,] \left\{ [\, 1 + \gamma \,(\, n - 2 \,) \,] \, [\, 2 + \gamma \,(\, n - 2 \,) \,] - \gamma^2 \, [\, n - 1 \,] \right\} \\
& \qquad \quad \cdot [\, v_1 \,(\, w_1 - c_1^u \,) \, f_{11} + v_2 \,(\, w_1 - c_2^u \,) \, f_{21} \,] \\
& \qquad + \gamma \left[\gamma^2 \,(\, n - 1 \,) - (\, 1 + \gamma \, [\, n - 2 \,] \,) \,(\, 2 + \gamma \, [\, n - 2 \,] \,) \right] \\
& \qquad \quad \cdot \sum_{j=1}^{n} [\, v_1 \,(\, w_j - c_1^u \,) \, f_{1j} + v_2 \,(\, w_j - c_2^u \,) \, f_{2j} \,] \Bigg\} \\
& + \frac{\partial q_1(\cdot)}{\partial w_i} \, \frac{1}{[\, 2 \,(\, 1 - \gamma \,) + \gamma \,(\, n - 1 \,) \,] \, [\, 2 - \gamma + 2 \, \gamma \,(\, n - 1 \,) \,]} \\
& \quad \cdot \Bigg\{ \gamma \, [\, 1 + \gamma \,(\, n - 2 \,) \,] \sum_{i=1}^{n} \left(\alpha_i + w_i + c_i^d \right) \\
& \qquad + [\, 1 + \gamma \,(\, n - 2 \,) \,] \, [\, 2 \,(\, 1 - \gamma \,) + \gamma \,(\, n - 1 \,) \,] \left[\alpha_1 + w_1 + c_1^d \right] \\
& \qquad - \gamma^2 \, [\, n - 1 \,] \sum_{i=1}^{n} \alpha_i - \gamma \, [\, 2 \,(\, 1 - \gamma \,) + \gamma \,(\, n - 1 \,) \,] \sum_{j=2}^{n} \alpha_j \\
& \qquad - [\, 1 + \gamma \,(\, n - 1 \,) \,] \, [\, 2 + \gamma \,(\, n - 2 \,) \,] \, [\, v_1 \,(\, w_1 - c_1^u \,) \, f_{11} + v_2 \,(\, w_1 - c_2^u \,) \, f_{21} \,] \\
& \qquad + \gamma \, [\, 2 + \gamma \,(\, n - 2 \,) \,] \sum_{i=1}^{n} [\, v_1 \,(\, w_i - c_1^u \,) \, f_{1i} + v_2 \,(\, w_i - c_2^u \,) \, f_{2i} \,] \Bigg\} \\
& + \frac{\alpha_i \, [\, 1 + \gamma \,(\, n - 2 \,) \,] - \gamma \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j}{[\, 1 - \gamma \,] \, [\, 1 + \gamma \,(\, n - 1 \,) \,]}
\end{aligned}
$$

$$- \frac{1}{\Xi} \Bigg\{ - \gamma \, [\, 1 + \gamma \, (\, n - 2 \,) \,]^2 \sum_{j=1}^{n} \big(\, \alpha_j + w_j + c_j^d \, \big)$$

$$+ \, [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, 1 + \gamma \, (\, n - 1 \,) \,] \, [\, 2 + \gamma \, (\, n - 3 \,) \,] \, \big[\, \alpha_i + w_i + c_i^d \, \big]$$

$$- \, \gamma \, [\, \gamma \, (\, n - 1 \,) \, (\, 1 - \gamma \,) + 2 + \gamma \, (\, n - 3 \,) \,] \sum_{j=1}^{n} \alpha_j$$

$$+ \, \gamma \, [\, 2 + \gamma \, (\, n - 3 \,) \,] \, [\, 1 + \gamma \, (\, n - 1 \,) \,] \; \alpha_i$$

$$+ \, \gamma \, [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, 1 + \gamma \, (\, n - 1 \,) \,] \, [\, v_1 \, (\, w_1 - c_1^u \,) \, f_{11} + v_2 \, (\, w_1 - c_2^u \,) \, f_{21} \,]$$

$$- \, \gamma^2 \, [\, 1 + \gamma \, (\, n - 2 \,) \,] \sum_{j=1}^{n} [\, v_1 \, (\, w_j - c_1^u \,) \, f_{1j} + v_2 \, (\, w_j - c_2^u \,) \, f_{2j} \,] \Bigg\}$$

$$- \, \big[\, c_1^d + c_1^u \, f_{11} + c_2^u \, f_{21} \, \big] \, \frac{\partial q_1(\cdot)}{\partial w_i} + [\, w_i - c_1^u \, f_{1i} - c_2^u \, f_{2i} \,] \, \frac{\partial q_i(\cdot)}{\partial w_i}$$

$$+ \sum_{\substack{j=2 \\ j \neq i}}^{n} (\, w_j - c_1^u \, f_{1j} - c_2^u \, f_{2j} \,) \, \frac{\partial q_j(\cdot)}{\partial w_i} \, . \tag{166}$$

For $i \in \{2, ..., n\}$, (166) can be written as:

$$0 \;=\; \frac{\partial P_1(\cdot)}{\partial w_i} \, \frac{\alpha_1 \, [\, 1 + \gamma \, (\, n - 2 \,) \,] - \gamma \sum_{j=2}^{n} \alpha_j}{[\, 1 - \gamma \,] \, [\, 1 + \gamma \, (\, n - 1 \,) \,]}$$

$$+ \, \frac{\partial P_1(\cdot)}{\partial w_i} \, \frac{1}{\Xi} \Bigg\{ \gamma \, [\, 1 + \gamma \, (\, n - 2 \,) \,]^2 \sum_{\substack{j=2 \\ j \neq i}}^{n} \big(\, \alpha_j + w_j + c_j^d \, \big)$$

$$+ \gamma \, [\, 1 + \gamma \, (\, n - 2 \,) \,]^2 \, \big[\, \alpha_1 + w_1 + c_1^d \, \big] + \gamma \, [\, 1 + \gamma \, (\, n - 2 \,) \,]^2 \, \big[\, \alpha_i + w_i + c_i^d \, \big]$$

$$- \, [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, 1 + \gamma \, (\, n - 1 \,) \,] \, [\, 2 + \gamma \, (\, n - 3 \,) \,] \, \big[\, \alpha_1 + w_1 + c_1^d \, \big]$$

$$+ \gamma \, \{ \, [\, 2 + \gamma \, (\, n - 3 \,) \,] \, [\, 1 + \gamma \, (\, n - 1 \,) \,] - \gamma \, [\, n - 1 \,] \, [\, 1 + \gamma \, (\, n - 2 \,) \,] \, \} \sum_{j=1}^{n} \alpha_j$$

$$- \, \gamma \, [\, 2 + \gamma \, (\, n - 3 \,) \,] \, [\, 1 + \gamma \, (\, n - 1 \,) \,] \, \alpha_1$$

$$+ \, [\, 1 + \gamma \, (\, n - 1 \,) \,] \, \big\{ \, [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, 2 + \gamma \, (\, n - 2 \,) \,] - \gamma^2 \, [\, n - 1 \,] \, \big\}$$

$$\cdot \, [\, v_1 \, (\, w_1 - c_1^u \,) \, f_{11} + v_2 \, (\, w_1 - c_2^u \,) \, f_{21} \,]$$

$$
\begin{aligned}
&+ \gamma \left[\gamma^2 (n-1) - (1 + \gamma [n-2]) (2 + \gamma [n-2]) \right] \\
&\qquad \cdot [v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}] \\
&+ \gamma \left[\gamma^2 (n-1) - (1 + \gamma [n-2]) (2 + \gamma [n-2]) \right] \\
&\qquad \cdot [v_1 (w_i - c_1^u) f_{1i} + v_2 (w_i - c_2^u) f_{2i}] \\
&+ \gamma \left[\gamma^2 (n-1) - (1 + \gamma [n-2]) (2 + \gamma [n-2]) \right] \\
&\qquad \left. \cdot \sum_{\substack{j=2 \\ j \neq i}}^{n} [v_1 (w_j - c_1^u) f_{1j} + v_2 (w_j - c_2^u) f_{2j}] \right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\partial q_1(\cdot)}{\partial w_i} \frac{1}{[2(1-\gamma) + \gamma(n-1)][2 - \gamma + 2\gamma(n-1)]} \\
&\quad \cdot \left\{ \gamma [1 + \gamma(n-2)] \sum_{\substack{j=2 \\ j \neq i}}^{n} \left(\alpha_j + w_j + c_j^d \right) \right. \\
&\quad + \gamma [1 + \gamma(n-2)] \left[\alpha_1 + w_1 + c_1^d \right] + \gamma [1 + \gamma(n-2)] \left[\alpha_i + w_i + c_i^d \right] \\
&\quad + [1 + \gamma(n-2)][2(1-\gamma) + \gamma(n-1)] \left[\alpha_1 + w_1 + c_1^d \right] \\
&\quad - \gamma^2 [n-1] \sum_{i=1}^{n} \alpha_i - \gamma [2(1-\gamma) + \gamma(n-1)] \sum_{j=2}^{n} \alpha_j \\
&\quad - [1 + \gamma(n-1)][2 + \gamma(n-2)][v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}] \\
&\quad + \gamma [2 + \gamma(n-2)][v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21}] \\
&\quad + \gamma [2 + \gamma(n-2)][v_1 (w_i - c_1^u) f_{1i} + v_2 (w_i - c_2^u) f_{2i}] \\
&\quad \left. + \gamma [2 + \gamma(n-2)] \sum_{\substack{j=2 \\ j \neq i}}^{n} [v_1 (w_j - c_1^u) f_{1i} + v_2 (w_j - c_2^u) f_{2i}] \right\}
\end{aligned}
$$

$$
+ \frac{\alpha_i [1 + \gamma(n-2)] - \gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j}{[1-\gamma][1 + \gamma(n-1)]}
$$

$$- \frac{1}{\Xi} \left\{ - \gamma \, [\, 1 + \gamma \, (\, n - 2 \,) \,]^2 \sum_{\substack{j\,=\,2 \\ j\,\neq\,i}}^{n} \left(\, \alpha_j + w_j + c^d_j \, \right) \right.$$

$$- \, \gamma \, [\, 1 + \gamma \, (\, n - 2 \,) \,]^2 \left[\, \alpha_1 + w_1 + c^d_1 \, \right] - \gamma \, [\, 1 + \gamma \, (\, n - 2 \,) \,]^2 \left[\, \alpha_i + w_i + c^d_i \, \right]$$

$$+ \, [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, 1 + \gamma \, (\, n - 1 \,) \,] \, [\, 2 + \gamma \, (\, n - 3 \,) \,] \left[\, \alpha_i + w_i + c^d_i \, \right]$$

$$- \, \gamma \, [\, \gamma \, (\, n - 1 \,) \, (\, 1 - \gamma \,) + 2 + \gamma \, (\, n - 3 \,) \,] \sum_{j\,=\,1}^{n} \alpha_j$$

$$+ \, \gamma \, [\, 2 + \gamma \, (\, n - 3 \,) \,] \, [\, 1 + \gamma \, (\, n - 1 \,) \,] \; \alpha_i$$

$$+ \, \gamma \, [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, 1 + \gamma \, (\, n - 1 \,) \,] \, [\, v_1 \, (\, w_1 - c^u_1 \,) \, f_{11} + v_2 \, (\, w_1 - c^u_2 \,) \, f_{21} \,]$$

$$- \, \gamma^2 \, [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, v_1 \, (\, w_1 - c^u_1 \,) \, f_{11} + v_2 \, (\, w_1 - c^u_2 \,) \, f_{21} \,]$$

$$- \, \gamma^2 \, [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, v_1 \, (\, w_i - c^u_1 \,) \, f_{1i} + v_2 \, (\, w_i - c^u_2 \,) \, f_{2i} \,]$$

$$\left. - \, \gamma^2 \, [\, 1 + \gamma \, (\, n - 2 \,) \,] \sum_{\substack{j\,=\,2 \\ j\,\neq\,i}}^{n} [\, v_1 \, (\, w_j - c^u_1 \,) \, f_{1j} + v_2 \, (\, w_j - c^u_2 \,) \, f_{2j} \,] \right\}$$

$$- \; \left[\, c^d_1 + c^u_1 \, f_{11} + c^u_2 \, f_{21} \, \right] \frac{\partial q_1(\cdot)}{\partial w_i} + [\, w_i - c^u_1 \, f_{1i} - c^u_2 \, f_{2i} \,] \, \frac{\partial q_i(\cdot)}{\partial w_i}$$

$$+ \sum_{\substack{j\,=\,2 \\ j\,\neq\,i}}^{n} (\, w_j - c^u_1 \, f_{1j} - c^u_2 \, f_{2j} \,) \, \frac{\partial q_j(\cdot)}{\partial w_i} \, , \qquad (167)$$

where $\frac{\partial P_1(\cdot)}{\partial w_i}$, $\frac{\partial q_1(\cdot)}{\partial w_i}$, $\frac{\partial q_i(\cdot)}{\partial w_i}$, and $\frac{\partial q_j(\cdot)}{\partial w_i}$ are as specified in (165).

For $i \in \{2, ..., n\}$, (167) can be written as:

$$[\, \kappa_1 + \kappa_2 \, (\, v_1 \, f_{11} + v_2 \, f_{21} \,) \,] \, w_1 + [\, \kappa_3 + \kappa_4 \, (\, v_1 \, f_{1i} + v_2 \, f_{2i} \,) \,] \, w_i$$

$$+ \sum_{\substack{j\,=\,2 \\ j\,\neq\,i}}^{n} [\, \kappa_5 + \kappa_6 \, (\, v_1 \, f_{1j} + v_2 \, f_{2j} \,) \,] \, w_j + \xi_1 \; = \; 0 \qquad (168)$$

where:

$$\kappa_1 \; = \; \frac{\partial P_1(\cdot)}{\partial w_i} \, \frac{\gamma \, [\, 1 + \gamma \, (\, n - 2 \,) \,]^2 - [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, 1 + \gamma \, (\, n - 1 \,) \,] \, [\, 2 + \gamma \, (\, n - 3 \,) \,]}{\Xi}$$

$$+ \frac{\partial q_1(\cdot)}{\partial w_i} \frac{\gamma [1+\gamma(n-2)] + [1+\gamma(n-2)][2(1-\gamma)+\gamma(n-1)]}{[2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]}$$

$$+ \frac{\gamma [1+\gamma(n-2)]^2}{\Xi};$$

$$\kappa_2 = \frac{\partial P_1(\cdot)}{\partial w_i} \frac{1}{\Xi} \{ [1+\gamma(n-1)] \big([1+\gamma(n-2)][2+\gamma(n-2)] - \gamma^2 [n-1] \big)$$

$$+ \gamma \left[\gamma^2 (n-1) - (1+\gamma[n-2])(2+\gamma[n-2]) \right] \}$$

$$+ \frac{\partial q_1(\cdot)}{\partial w_i} \frac{-[1+\gamma(n-1)][2+\gamma(n-2)] + \gamma[2+\gamma(n-2)]}{[2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]}$$

$$- \frac{\gamma[1+\gamma(n-2)][1+\gamma(n-1)] - \gamma^2[1+\gamma(n-2)]}{\Xi};$$

$$\kappa_3 = \frac{\partial P_1(\cdot)}{\partial w_i} \frac{\gamma[1+\gamma(n-2)]^2}{\Xi} + \frac{\partial q_i(\cdot)}{\partial w_i}$$

$$+ \frac{\partial q_1(\cdot)}{\partial w_i} \frac{\gamma[1+\gamma(n-2)]}{[2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]}$$

$$- \frac{1}{\Xi} \left\{ -\gamma[1+\gamma(n-2)]^2 + [1+\gamma(n-2)][1+\gamma(n-1)][2+\gamma(n-3)] \right\};$$

$$\kappa_4 = \frac{\partial P_1(\cdot)}{\partial w_i} \frac{\gamma[\gamma^2(n-1) - (1+\gamma[n-2])(2+\gamma[n-2])]}{\Xi}$$

$$+ \frac{\partial q_1(\cdot)}{\partial w_i} \frac{\gamma[2+\gamma(n-2)]}{[2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]} + \frac{\gamma^2[1+\gamma(n-2)]}{\Xi};$$

$$\kappa_5 = \frac{\partial P_1(\cdot)}{\partial w_i} \frac{\gamma[1+\gamma(n-2)]^2}{\Xi} + \frac{\partial q_1(\cdot)}{\partial w_i} \frac{\gamma[1+\gamma(n-2)]}{[2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]}$$

$$+ \frac{\gamma[1+\gamma(n-2)]^2}{\Xi} + \frac{\partial q_j(\cdot)}{\partial w_i};$$

$$\kappa_6 = \frac{\partial P_1(\cdot)}{\partial w_i} \frac{\gamma[\gamma^2(n-1) - (1+\gamma[n-2])(2+\gamma[n-2])]}{\Xi}$$

$$+ \frac{\partial q_1(\cdot)}{\partial w_i} \frac{\gamma[2+\gamma(n-2)]}{[2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]} + \frac{\gamma^2[1+\gamma(n-2)]}{\Xi};$$

$$\xi_1 = \frac{\partial P_1(\cdot)}{\partial w_i} \frac{\alpha_1 [1 + \gamma (n-2)] - \gamma \sum_{j=2}^{n} \alpha_j}{[1-\gamma][1+\gamma(n-1)]}$$

$$+ \frac{\partial P_1(\cdot)}{\partial w_i} \frac{1}{\Xi} \left\{ \gamma [1 + \gamma (n-2)]^2 \sum_{\substack{j=2 \\ j \neq i}}^{n} \left(\alpha_j + c_j^d \right) \right.$$

$$+ \gamma [1 + \gamma (n-2)]^2 \left[\alpha_1 + c_1^d \right] + \gamma [1 + \gamma (n-2)]^2 \left[\alpha_i + c_i^d \right]$$

$$- [1 + \gamma (n-2)][1 + \gamma (n-1)][2 + \gamma (n-3)] \left[\alpha_1 + c_1^d \right]$$

$$+ \gamma \{ [2 + \gamma (n-3)][1 + \gamma (n-1)] - \gamma [n-1][1 + \gamma (n-2)] \} \sum_{j=1}^{n} \alpha_j$$

$$- \gamma [2 + \gamma (n-3)][1 + \gamma (n-1)] \alpha_1$$

$$+ [1 + \gamma (n-1)] \left\{ [1 + \gamma (n-2)][2 + \gamma (n-2)] - \gamma^2 [n-1] \right\}$$

$$\cdot [v_1(-c_1^u) f_{11} + v_2(-c_2^u) f_{21}]$$

$$+ \gamma \left[\gamma^2 (n-1) - (1 + \gamma [n-2])(2 + \gamma [n-2]) \right]$$

$$\cdot [v_1(-c_1^u) f_{11} + v_2(-c_2^u) f_{21}]$$

$$+ \gamma \left[\gamma^2 (n-1) - (1 + \gamma [n-2])(2 + \gamma [n-2]) \right]$$

$$\cdot [v_1(-c_1^u) f_{1i} + v_2(-c_2^u) f_{2i}]$$

$$+ \gamma \left[\gamma^2 (n-1) - (1 + \gamma [n-2])(2 + \gamma [n-2]) \right]$$

$$\left. \cdot \sum_{\substack{j=2 \\ j \neq i}}^{n} [v_1(-c_1^u) f_{1j} + v_2(-c_2^u) f_{2j}] \right\}$$

$$+ \frac{\partial q_1(\cdot)}{\partial w_i} \frac{1}{[2(1-\gamma) + \gamma (n-1)][2 - \gamma + 2\gamma (n-1)]}$$

$$\cdot \left\{ \gamma [1 + \gamma (n-2)] \sum_{\substack{j=2 \\ j \neq i}}^{n} \left(\alpha_j + c_j^d \right) \right.$$

$$+ \gamma [1 + \gamma (n-2)] \left[\alpha_1 + c_1^d \right] + \gamma [1 + \gamma (n-2)] \left[\alpha_i + c_i^d \right]$$

$$+ [1 + \gamma (n-2)][2(1-\gamma) + \gamma (n-1)] \left[\alpha_1 + c_1^d \right]$$

$$- \gamma^2 [n - 1] \sum_{i=1}^{n} \alpha_i - \gamma [2 (1 - \gamma) + \gamma (n - 1)] \sum_{j=2}^{n} \alpha_j$$

$$- [1 + \gamma (n - 1)] [2 + \gamma (n - 2)] [v_1 (- c_1^u) f_{11} + v_2 (- c_2^u) f_{21}]$$

$$+ \gamma [2 + \gamma (n - 2)] [v_1 (- c_1^u) f_{11} + v_2 (- c_2^u) f_{21}]$$

$$+ \gamma [2 + \gamma (n - 2)] [v_1 (- c_1^u) f_{1i} + v_2 (- c_2^u) f_{2i}]$$

$$\left. + \gamma [2 + \gamma (n - 2)] \sum_{\substack{j=2 \\ j \neq i}}^{n} [v_1 (- c_1^u) f_{1i} + v_2 (- c_2^u) f_{2i}] \right\}$$

$$+ \frac{\alpha_i [1 + \gamma (n - 2)] - \gamma \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j}{[1 - \gamma] [1 + \gamma (n - 1)]}$$

$$- \frac{1}{\Xi} \left\{ - \gamma [1 + \gamma (n - 2)]^2 \sum_{\substack{j=2 \\ j \neq i}}^{n} \left(\alpha_j + c_j^d \right) \right.$$

$$- \gamma [1 + \gamma (n - 2)]^2 \left[\alpha_1 + c_1^d \right] - \gamma [1 + \gamma (n - 2)]^2 \left[\alpha_i + c_i^d \right]$$

$$+ [1 + \gamma (n - 2)] [1 + \gamma (n - 1)] [2 + \gamma (n - 3)] \left[\alpha_i + c_i^d \right]$$

$$- \gamma [\gamma (n - 1) (1 - \gamma) + 2 + \gamma (n - 3)] \sum_{j=1}^{n} \alpha_j$$

$$+ \gamma [2 + \gamma (n - 3)] [1 + \gamma (n - 1)] \, \alpha_i$$

$$+ \gamma [1 + \gamma (n - 2)] [1 + \gamma (n - 1)] [v_1 (- c_1^u) f_{11} + v_2 (- c_2^u) f_{21}]$$

$$- \gamma^2 [1 + \gamma (n - 2)] [v_1 (- c_1^u) f_{11} + v_2 (- c_2^u) f_{21}]$$

$$- \gamma^2 [1 + \gamma (n - 2)] [v_1 (- c_1^u) f_{1i} + v_2 (- c_2^u) f_{2i}]$$

$$\left. - \gamma^2 [1 + \gamma (n - 2)] \sum_{\substack{j=2 \\ j \neq i}}^{n} [v_1 (- c_1^u) f_{1j} + v_2 (- c_2^u) f_{2j}] \right\}$$

$$- \left[c_1^d + c_1^u f_{11} + c_2^u f_{21} \right] \frac{\partial q_1(\cdot)}{\partial w_i} + [- c_1^u f_{1i} - c_2^u f_{2i}] \frac{\partial q_i(\cdot)}{\partial w_i}$$

$$+ \sum_{\substack{j=2 \\ j \neq i}}^{n} (- c_1^u \, f_{1j} - c_2^u \, f_{2j}) \; \frac{\partial q_j(\cdot)}{\partial w_i} \, . \tag{169}$$

Differentiating (162) with respect to w_1 provides:

$$\sum_{i=2}^{n} [\, w_i - c_1^u \, f_{1i} - c_2^u \, f_{2i} \,] \, \frac{\partial q_i \, (\cdot)}{\partial w_1} + \left[\, P_1 - c_1^d - c_1^u \, f_{11} - c_2^u \, f_{21} \, \right] \frac{\partial q_1 \, (\cdot)}{\partial w_1} + \frac{\partial P_1 \, (\cdot)}{\partial w_1} \, q_1 \; = \; 0 \, . \tag{170}$$

(144), (158), (161), and (163) imply:

$$\frac{\partial q_i \, (\cdot)}{\partial w_1} \; = \; - \, \frac{1}{\Xi} \, \{ - \gamma \, [\, 1 + \gamma \, (\, n - 2 \,) \,]^2 + \; \gamma \, [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, 1 + \gamma \, (\, n - 1 \,) \,] \, [\, v_1 f_{11} + v_2 f_{21} \,]$$

$$- \; \gamma^2 \, [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, v_1 f_{11} + v_2 f_{21} \,] \, \} \, ;$$

$$\frac{\partial q_1 \, (\cdot)}{\partial w_1} \; = \; \frac{1}{\Xi} \, \{ \, \gamma \, [\, 1 + \gamma \, (\, n - 2 \,) \,]^2 - [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, 1 + \gamma \, (\, n - 1 \,) \,] \, [\, 2 + \gamma \, (\, n - 3 \,) \,]$$

$$+ \; [\, 1 + \gamma \, (\, n - 1 \,) \,] \left\{ \, [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, 2 + \gamma \, (\, n - 2 \,) \,] - \gamma^2 \, [\, n - 1 \,] \, \right\} [\, v_1 f_{11} + v_2 f_{21} \,]$$

$$+ \; \gamma \left[\, \gamma^2 \, (\, n - 1 \,) - (\, 1 + \gamma \, [\, n - 2 \,] \,) \, (\, 2 + \gamma \, [\, n - 2 \,] \,) \, \right] [\, v_1 f_{11} + v_2 f_{21} \,] \, \} \, ;$$

$$\frac{\partial P_1 \, (\cdot)}{\partial w_1} \; = \; \frac{1}{[\, 2 \, (\, 1 - \gamma \,) + \gamma \, (\, n - 1 \,) \,] \, [\, 2 - \gamma + 2 \, \gamma \, (\, n - 1 \,) \,]}$$

$$\cdot \, \{ \, \gamma \, [\, 1 + \gamma \, (\, n - 2 \,) \,] + \; [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, 2 \, (\, 1 - \gamma \,) + \gamma \, (\, n - 1 \,) \,]$$

$$- \;\; [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, 2 + \gamma \, (\, n - 2 \,) \,] \, [\, v_1 f_{11} + v_2 f_{21} \,] \, \} \, . \tag{171}$$

(140), (158), and (170) imply:

$$0 \; = \; \sum_{i=2}^{n} [\, w_i - c_1^u \, f_{1i} - c_2^u \, f_{2i} \,] \, \frac{\partial q_i \, (\cdot)}{\partial w_1} - \left[\, c_1^d + c_1^u \, f_{11} + c_2^u \, f_{21} \, \right] \frac{\partial q_1 \, (\cdot)}{\partial w_1}$$

$$+ \; \frac{\partial q_1 \, (\cdot)}{\partial w_1} \; \frac{1}{[\, 2 \, (\, 1 - \gamma \,) + \gamma \, (\, n - 1 \,) \,] \, [\, 2 - \gamma + 2 \, \gamma \, (\, n - 1 \,) \,]}$$

$$\cdot \left\{ \gamma \, [\, 1 + \gamma \, (\, n - 2 \,) \,] \, \sum_{i=1}^{n} \left(\, \alpha_i + w_i + c_i^d \, \right) \right.$$

$$+ \; [\, 1 + \gamma \, (\, n - 2 \,) \,] \, [\, 2 \, (\, 1 - \gamma \,) + \gamma \, (\, n - 1 \,) \,] \left[\, \alpha_1 + w_1 + c_1^d \, \right]$$

$$
\begin{aligned}
& - \gamma^2 [\, n-1\,] \sum_{i=1}^{n} \alpha_i - \gamma\,[\,2\,(\,1-\gamma\,) + \gamma\,(\,n-1\,)\,] \sum_{j=2}^{n} \alpha_j \\
& - [\,1+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]\,[\,v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,] \\
& + \gamma\,[\,2+\gamma\,(\,n-2\,)\,] \sum_{i=1}^{n} [\,v_1\,(\,w_i - c_1^u\,)\,f_{1i} + v_2\,(\,w_i - c_2^u\,)\,f_{2i}\,] \Bigg\}
\end{aligned}
$$

$$
+ \frac{\partial P_1(\cdot)}{\partial w_1}\; \frac{\alpha_1\,[\,1+\gamma\,(\,n-2\,)\,] - \gamma \sum_{j=2}^{n} \alpha_j}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}
$$

$$
\begin{aligned}
+ \frac{\partial P_1(\cdot)}{\partial w_1}\, \frac{1}{\Xi} \Bigg\{ & \gamma\,[\,1+\gamma\,(\,n-2\,)\,]^2 \sum_{j=1}^{n} \left(\,\alpha_j + w_j + c_j^d\,\right) \\
& - [\,1+\gamma\,(\,n-2\,)\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,] \left[\,\alpha_1 + w_1 + c_1^d\,\right] \\
& + \gamma\,\{\,[\,2+\gamma\,(\,n-3\,)\,]\,[\,1+\gamma\,(\,n-1\,)\,] - \gamma\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\,\} \sum_{j=1}^{n} \alpha_j \\
& - \gamma\,[\,2+\gamma\,(\,n-3\,)\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,\alpha_1 \\
& + [\,1+\gamma\,(\,n-1\,)\,] \left\{\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,] - \gamma^2\,[\,n-1\,]\,\right\} \\
& \quad \cdot [\,v_1\,(\,w_1 - c_1^u\,)\,f_{11} + v_2\,(\,w_1 - c_2^u\,)\,f_{21}\,] \\
& + \gamma \left[\,\gamma^2\,(\,n-1\,) - (\,1+\gamma\,[\,n-2\,]\,)\,(\,2+\gamma\,[\,n-2\,]\,)\,\right] \\
& \quad \cdot \sum_{j=1}^{n} [\,v_1\,(\,w_j - c_1^u\,)\,f_{1j} + v_2\,(\,w_j - c_2^u\,)\,f_{2j}\,] \Bigg\}. \qquad (172)
\end{aligned}
$$

(172) can be written as:

$$
\begin{aligned}
0 \;=\; & \sum_{\substack{j=2 \\ j \neq i}}^{n} [\,w_j - c_1^u\,f_{1j} - c_2^u\,f_{2j}\,]\; \frac{\partial q_j(\cdot)}{\partial w_1} + [\,w_i - c_1^u\,f_{1i} - c_2^u\,f_{2i}\,]\; \frac{\partial q_i(\cdot)}{\partial w_1} \\
& - \left[\,c_1^d + c_1^u\,f_{11} + c_2^u\,f_{21}\,\right]\; \frac{\partial q_1(\cdot)}{\partial w_1} \\
& + \frac{\partial q_1(\cdot)}{\partial w_1}\; \frac{1}{[\,2\,(\,1-\gamma\,) + \gamma\,(\,n-1\,)\,]\,[\,2-\gamma+2\,\gamma\,(\,n-1\,)\,]}
\end{aligned}
$$

$$
\cdot \left\{ \gamma \left[1 + \gamma \left(n - 2 \right) \right] \sum_{\substack{j=2 \\ j \neq i}}^{n} \left(\alpha_j + w_j + c_j^d \right) \right.
$$

$$
+ \gamma \left[1 + \gamma \left(n - 2 \right) \right] \left[\alpha_1 + w_1 + c_1^d \right] + \gamma \left[1 + \gamma \left(n - 2 \right) \right] \left[\alpha_i + w_i + c_i^d \right]
$$

$$
+ \left[1 + \gamma \left(n - 2 \right) \right] \left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right] \left[\alpha_1 + w_1 + c_1^d \right]
$$

$$
- \gamma^2 \left[n - 1 \right] \sum_{i=1}^{n} \alpha_i - \gamma \left[2 \left(1 - \gamma \right) + \gamma \left(n - 1 \right) \right] \sum_{j=2}^{n} \alpha_j
$$

$$
- \left[1 + \gamma \left(n - 1 \right) \right] \left[2 + \gamma \left(n - 2 \right) \right] \left[v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right]
$$

$$
+ \gamma \left[2 + \gamma \left(n - 2 \right) \right] \left[v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right]
$$

$$
+ \gamma \left[2 + \gamma \left(n - 2 \right) \right] \left[v_1 \left(w_i - c_1^u \right) f_{1i} + v_2 \left(w_i - c_2^u \right) f_{2i} \right]
$$

$$
\left. + \gamma \left[2 + \gamma \left(n - 2 \right) \right] \sum_{\substack{j=2 \\ j \neq i}}^{n} \left[v_1 \left(w_i - c_1^u \right) f_{1i} + v_2 \left(w_i - c_2^u \right) f_{2i} \right] \right\}
$$

$$
+ \frac{\partial P_1 (\cdot)}{\partial w_1} \frac{\alpha_1 \left[1 + \gamma \left(n - 2 \right) \right] - \gamma \sum_{j=2}^{n} \alpha_j}{\left[1 - \gamma \right] \left[1 + \gamma \left(n - 1 \right) \right]}
$$

$$
+ \frac{\partial P_1 (\cdot)}{\partial w_1} \frac{1}{\Xi} \left\{ \gamma \left[1 + \gamma \left(n - 2 \right) \right]^2 \sum_{\substack{j=2 \\ j \neq i}}^{n} \left(\alpha_j + w_j + c_j^d \right) \right.
$$

$$
+ \gamma \left[1 + \gamma \left(n - 2 \right) \right]^2 \left[\alpha_1 + w_1 + c_1^d \right] + \gamma \left[1 + \gamma \left(n - 2 \right) \right]^2 \left[\alpha_i + w_i + c_i^d \right]
$$

$$
- \left[1 + \gamma \left(n - 2 \right) \right] \left[1 + \gamma \left(n - 1 \right) \right] \left[2 + \gamma \left(n - 3 \right) \right] \left[\alpha_1 + w_1 + c_1^d \right]
$$

$$
+ \gamma \left\{ \left[2 + \gamma \left(n - 3 \right) \right] \left[1 + \gamma \left(n - 1 \right) \right] - \gamma \left[n - 1 \right] \left[1 + \gamma \left(n - 2 \right) \right] \right\} \sum_{j=1}^{n} \alpha_j
$$

$$
- \gamma \left[2 + \gamma \left(n - 3 \right) \right] \left[1 + \gamma \left(n - 1 \right) \right] \alpha_1
$$

$$
+ \left[1 + \gamma \left(n - 1 \right) \right] \left\{ \left[1 + \gamma \left(n - 2 \right) \right] \left[2 + \gamma \left(n - 2 \right) \right] - \gamma^2 \left[n - 1 \right] \right\}
$$

$$
\cdot \left[v_1 \left(w_1 - c_1^u \right) f_{11} + v_2 \left(w_1 - c_2^u \right) f_{21} \right]
$$

$$
+ \gamma \left[\gamma^2 \left(n - 1 \right) - \left(1 + \gamma \left[n - 2 \right] \right) \left(2 + \gamma \left[n - 2 \right] \right) \right]
$$

$$\cdot \left[v_1 (w_1 - c_1^u) f_{11} + v_2 (w_1 - c_2^u) f_{21} \right]$$

$$+ \gamma \left[\gamma^2 (n-1) - (1 + \gamma [n-2]) (2 + \gamma [n-2]) \right]$$

$$\cdot \left[v_1 (w_i - c_1^u) f_{1i} + v_2 (w_i - c_2^u) f_{2i} \right]$$

$$+ \gamma \left[\gamma^2 (n-1) - (1 + \gamma [n-2]) (2 + \gamma [n-2]) \right]$$

$$\left. \cdot \sum_{\substack{j=2 \\ j \neq i}}^{n} \left[v_1 (w_j - c_1^u) f_{1j} + v_2 (w_j - c_2^u) f_{2j} \right] \right\}, \qquad (173)$$

where $\frac{\partial q_i(\cdot)}{\partial w_1}$, $\frac{\partial q_1(\cdot)}{\partial w_1}$, and $\frac{\partial P_1(\cdot)}{\partial w_1}$ are as specified in (171).

(173) can be written as:

$$\left[\tau_1 + \tau_2 (v_1 f_{11} + v_2 f_{21}) \right] w_1 + \left[\tau_3 + \tau_4 (v_1 f_{1i} + v_2 f_{2i}) \right] w_i$$

$$+ \sum_{\substack{j=2 \\ j \neq i}}^{n} \left[\tau_5 + \tau_6 (v_1 f_{1j} + v_2 f_{2j}) \right] w_j + \xi_2 = 0 \qquad (174)$$

where

$$\tau_1 = \frac{\partial q_1(\cdot)}{\partial w_1} \frac{\gamma [1 + \gamma (n-2)] + [1 + \gamma (n-2)] [2 (1-\gamma) + \gamma (n-1)]}{[2 (1-\gamma) + \gamma (n-1)] [2 - \gamma + 2 \gamma (n-1)]}$$

$$+ \frac{\partial P_1(\cdot)}{\partial w_1} \frac{\gamma [1 + \gamma (n-2)]^2 - [1 + \gamma (n-2)] [1 + \gamma (n-1)] [2 + \gamma (n-3)]}{\Xi};$$

$$\tau_2 = \frac{\partial q_1(\cdot)}{\partial w_1} \frac{- [1 + \gamma (n-1)] [2 + \gamma (n-2)] + \gamma [2 + \gamma (n-2)]}{[2 (1-\gamma) + \gamma (n-1)] [2 - \gamma + 2 \gamma (n-1)]}$$

$$+ \frac{\partial P_1(\cdot)}{\partial w_1} \frac{1}{\Xi} \{ [1 + \gamma (n-1)] \big([1 + \gamma (n-2)] [2 + \gamma (n-2)] - \gamma^2 [n-1] \big)$$

$$+ \gamma \left[\gamma^2 (n-1) - (1 + \gamma [n-2]) (2 + \gamma [n-2]) \right] \};$$

$$\tau_3 = \frac{\partial q_i(\cdot)}{\partial w_1} + \frac{\partial q_1(\cdot)}{\partial w_1} \frac{\gamma [1 + \gamma (n-2)]}{[2 (1-\gamma) + \gamma (n-1)] [2 - \gamma + 2 \gamma (n-1)]}$$

$$+ \frac{\partial P_1(\cdot)}{\partial w_1} \frac{\gamma [1 + \gamma (n-2)]^2}{\Xi};$$

$$\tau_4 = \frac{\partial q_1(\cdot)}{\partial w_1} \frac{\gamma [2 + \gamma (n-2)]}{[2 (1-\gamma) + \gamma (n-1)] [2 - \gamma + 2 \gamma (n-1)]}$$

$$+ \frac{\partial P_1(\cdot)}{\partial w_1} \frac{\gamma[\gamma^2(n-1) - (1+\gamma[n-2])(2+\gamma[n-2])]}{\Xi};$$

$$\tau_5 = \frac{\partial q_j(\cdot)}{\partial w_1} + \frac{\partial q_1(\cdot)}{\partial w_1} \frac{\gamma[1+\gamma(n-2)]}{[2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]}$$

$$+ \frac{\partial P_1(\cdot)}{\partial w_1} \frac{\gamma[1+\gamma(n-2)]^2}{\Xi};$$

$$\tau_6 = \frac{\partial q_1(\cdot)}{\partial w_1} \frac{\gamma[2+\gamma(n-2)]}{[2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]}$$

$$+ \frac{\partial P_1(\cdot)}{\partial w_1} \frac{\gamma[\gamma^2(n-1) - (1+\gamma[n-2])(2+\gamma[n-2])]}{\Xi}$$

$$\xi_2 = \sum_{\substack{j=2 \\ j \neq i}}^{n} [-c_1^u f_{1j} - c_2^u f_{2j}] \frac{\partial q_j(\cdot)}{\partial w_1} + [-c_1^u f_{1i} - c_2^u f_{2i}] \frac{\partial q_i(\cdot)}{\partial w_1} - [c_1^d + c_1^u f_{11} + c_2^u f_{21}] \frac{\partial q_1(\cdot)}{\partial w_1}$$

$$+ \frac{\partial q_1(\cdot)}{\partial w_1} \frac{1}{[2(1-\gamma)+\gamma(n-1)][2-\gamma+2\gamma(n-1)]}$$

$$\cdot \left\{ \gamma[1+\gamma(n-2)] \sum_{\substack{j=2 \\ j \neq i}}^{n} (\alpha_j + c_j^d) \right.$$

$$+ \gamma[1+\gamma(n-2)][\alpha_1 + c_1^d] + \gamma[1+\gamma(n-2)][\alpha_i + c_i^d]$$

$$+ [1+\gamma(n-2)][2(1-\gamma)+\gamma(n-1)][\alpha_1 + c_1^d]$$

$$- \gamma^2[n-1] \sum_{i=1}^{n} \alpha_i - \gamma[2(1-\gamma)+\gamma(n-1)] \sum_{j=2}^{n} \alpha_j$$

$$- [1+\gamma(n-1)][2+\gamma(n-2)][v_1(-c_1^u) f_{11} + v_2(-c_2^u) f_{21}]$$

$$+ \gamma[2+\gamma(n-2)][v_1(-c_1^u) f_{11} + v_2(-c_2^u) f_{21}]$$

$$+ \gamma[2+\gamma(n-2)][v_1(-c_1^u) f_{1i} + v_2(-c_2^u) f_{2i}]$$

$$\left. + \gamma[2+\gamma(n-2)] \sum_{\substack{j=2 \\ j \neq i}}^{n} [v_1(-c_1^u) f_{1i} + v_2(-c_2^u) f_{2i}] \right\}$$

$$+ \frac{\partial P_1(\cdot)}{\partial w_1} \frac{\alpha_1 [1 + \gamma(n-2)] - \gamma \sum_{j=2}^{n} \alpha_j}{[1-\gamma][1+\gamma(n-1)]}$$

$$+ \frac{\partial P_1(\cdot)}{\partial w_1} \frac{1}{\Xi} \Bigg\{ \gamma [1 + \gamma(n-2)]^2 \sum_{\substack{j=2 \\ j \neq i}}^{n} \left(\alpha_j + c_j^d \right)$$

$$+ \gamma [1 + \gamma(n-2)]^2 \left[\alpha_1 + c_1^d \right] + \gamma [1 + \gamma(n-2)]^2 \left[\alpha_i + c_i^d \right]$$

$$- [1 + \gamma(n-2)][1 + \gamma(n-1)][2 + \gamma(n-3)] \left[\alpha_1 + c_1^d \right]$$

$$+ \gamma \{ [2 + \gamma(n-3)][1 + \gamma(n-1)] - \gamma [n-1][1 + \gamma(n-2)] \} \sum_{j=1}^{n} \alpha_j$$

$$- \gamma [2 + \gamma(n-3)][1 + \gamma(n-1)] \alpha_1$$

$$+ [1 + \gamma(n-1)] \left\{ [1 + \gamma(n-2)][2 + \gamma(n-2)] - \gamma^2 [n-1] \right\}$$

$$\cdot [v_1(-c_1^u) f_{11} + v_2(-c_2^u) f_{21}]$$

$$+ \gamma \left[\gamma^2 (n-1) - (1 + \gamma[n-2])(2 + \gamma[n-2]) \right]$$

$$\cdot [v_1(-c_1^u) f_{11} + v_2(-c_2^u) f_{21}]$$

$$+ \gamma \left[\gamma^2 (n-1) - (1 + \gamma[n-2])(2 + \gamma[n-2]) \right]$$

$$\cdot [v_1(-c_1^u) f_{1i} + v_2(-c_2^u) f_{2i}]$$

$$+ \gamma \left[\gamma^2 (n-1) - (1 + \gamma[n-2])(2 + \gamma[n-2]) \right]$$

$$\cdot \sum_{\substack{j=2 \\ j \neq i}}^{n} [v_1(-c_1^u) f_{1j} + v_2(-c_2^u) f_{2j}] \Bigg\}. \tag{175}$$

Define:

$$\varsigma_1 = \kappa_1 + \kappa_2 [v_1 f_{11} + v_2 f_{21}]; \quad \varsigma_2 = \kappa_3 + \kappa_4 [v_1 f_{1i} + v_2 f_{2i}];$$

$$\varsigma_3 = \kappa_5 + \kappa_6 [v_1 f_{1j} + v_2 f_{2j}]; \quad \varsigma_4 = \tau_1 + \tau_2 [v_1 f_{11} + v_2 f_{21}];$$

$$\varsigma_5 = \tau_3 + \tau_4 [v_1 f_{1i} + v_2 f_{2i}]; \quad \varsigma_6 = \tau_5 + \tau_6 [v_1 f_{1j} + v_2 f_{2j}]. \tag{176}$$

(168), (174), and (176) imply that for $i \in \{2, ..., n\}$:

$$\varsigma_1 w_1 + \varsigma_2 w_i + \sum_{\substack{j=2 \\ j \neq i}}^{n} \varsigma_3 w_j + \xi_1 = 0 \quad \text{and} \tag{177}$$

$$\varsigma_4 w_1 + \varsigma_5 w_i + \sum_{\substack{j=2 \\ j \neq i}}^{n} \varsigma_6 w_j + \xi_2 = 0$$

$$\Leftrightarrow \qquad \varsigma_4 w_1 + \sum_{j=2}^{n} \varsigma_6 w_j + \xi_2 = 0\,. \tag{178}$$

The equivalence in (178) follows from (176) because $\tau_3 = \tau_5$ and $\tau_4 = \tau_6$, from (175).[3]

[3] Observe from (171) that $\frac{\partial q_i(\cdot)}{\partial w_1} = \frac{\partial q_j(\cdot)}{\partial w_1}$ for all $i, j \in \{2, ..., n\}$.