Technical Appendix to Accompany

"Motivating Cost Reduction in Regulated Industries with Rolling Incentive Schemes"

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This Technical Appendix provides detailed proofs of the formal conclusions in the text.

Lemma 1. *When the firm operates under IRIS, it implements immediately any cost reduction it achieves.*

Proof of Lemma 1. Under IRIS, if the firm first implements the achieved cost reduction in period $\widehat{t} \in \{1, ..., 5\}$, then $p_t = c_0$ for $t = 1, ..., \widehat{t}+1$ and $p_t = c_0 - \Delta$ for $t = \widehat{t}+2, ..., 6$.[1] Suppose the firm achieves the Δ cost reduction in period $t \in \{1, 2\}$. If the firm implements the cost reduction immediately, the discounted present value (PDV) of its profit is $\Delta\,[\,Q_t(c_0) + \delta\,Q_{t+1}(c_0)\,]$. If the firm delays the implementation to period $t+l$, the PDV of its profit is $\delta^l\,\Delta\,[\,Q_{t+l}(c_0) + \delta\,Q_{t+l+1}(c_0)\,]$. Therefore, the firm will implement the achieved cost reduction immediately if:

$$\Delta\,[\,Q_t(c_0) + \delta\,Q_{t+1}(c_0)\,] \;\geq\; \delta^l\,\Delta\,[\,Q_{t+l}(c_0) + \delta\,Q_{t+l+1}(c_0)\,]$$

$$\Leftrightarrow \quad Q_t(c_0) + \delta\,Q_{t+1}(c_0) \;\geq\; \delta^l\,[\,Q_{t+l}(c_0) + \delta\,Q_{t+l+1}(c_0)\,]\,. \qquad (1)$$

The inequality in (1) holds because Assumption D implies:

$$Q_t(c_0) \;>\; \delta\,Q_{t+1}(c_0) \;\geq\; ... \;\geq\; \delta^l\,Q_{t+l}(c_0) \;\text{ for all }\; l \in \{\,1, ..., 6-t\,\}\,, \;\text{ and}$$

$$\delta\,Q_{t+1}(c_0) \;>\; \delta^2\,Q_{t+2}(c_0) \;\geq\; ... \;\geq\; \delta^{l+1}\,Q_{t+l+1}(c_0) \;\text{ for all }\; l \in \{\,1, ..., 6-t-1\,\}\,. \;\blacksquare$$

Lemma 2. *Suppose the firm operates under SR. If the firm achieves the cost reduction in period 1, it implements the cost reduction immediately. If the firm achieves the cost reduction in period 2, it implements the cost reduction immediately if*

$$Q_2(c_0) \;\geq\; \delta\,[\,Q_3(c_0) + \delta\,Q_4(c_0)\,] \qquad (2)$$

and otherwise implements the cost reduction in period 3.

Proof of Lemma 2. The proof consists of three Conclusions (A, B, and C). Each Conclusion pertains to the setting where the firm operates under SR.

Conclusion A. The firm always implements immediately a cost reduction achieved in period 1.

[1] The firm will not delay the implementation of an achieved cost reduction to period 6. The discounted present value (PDV) of the firm's profit from such a delay is $\delta^5\,\Delta\,Q_6(c_0)$. The PDV of the firm's profit from implementing the cost reduction in period 5 is $\delta^4\,\Delta\,[\,Q_5(c_0) + \delta\,Q_6(c_0)\,] > \delta^5\,\Delta\,Q_6(c_0)$.

Proof. If the firm implements the cost reduction achieved in period 1 immediately, the PDV of its profit is $\pi_1 \equiv \Delta\,[\,Q_1(c_0) + \delta\,Q_2(c_0)\,]$ because $p_1 = p_2 = c_0$ and $p_3 = p_4 = p_5 = p_6 = c_0 - \Delta$.

If the firm first implements in period 2 the cost reduction achieved in period 1, the PDV of its profit is $\pi_2 \equiv \delta\,\Delta\,Q_2(c_0)$ because $p_1 = p_2 = c_0$ and $p_3 = p_4 = p_5 = p_6 = c_0 - \Delta$. It is apparent that $\pi_2 = \pi_1 - \Delta\,Q_1(c_0) < \pi_1$.

If the firm first implements in period 3 the cost reduction achieved in period 1, the PDV of its profit is $\pi_3 \equiv \delta^2\,\Delta\,[\,Q_3(c_0) + \delta\,Q_4(c_0)\,]$ because $p_1 = p_2 = p_3 = p_4 = c_0$ and $p_5 = p_6 = c_0 - \Delta$. Assumption D implies:

$$\begin{aligned}\pi_3 &= \delta\,\Delta\left[\,\delta\,Q_3(c_0) + \delta^2\,Q_4(c_0)\,\right] < \delta\,\Delta\,[\,Q_2(c_0) + \delta\,Q_3(c_0)\,]\\ &= \Delta\left[\,\delta\,Q_2(c_0) + \delta^2\,Q_3(c_0)\,\right] < \Delta\,[\,Q_1(c_0) + \delta\,Q_2(c_0)\,] = \pi_1\,.\end{aligned}$$

If the firm first implements in period 4 the cost reduction achieved in period 1, the PDV of its profit is $\pi_4 \equiv \delta^3\,\Delta\,Q_4(c_0)$ because $p_1 = p_2 = p_3 = p_4 = c_0$ and $p_5 = p_6 = c_0 - \Delta$. It is apparent that $\pi_4 = \pi_3 - \delta^2\,\Delta\,Q_3(c_0) < \pi_3\,(< \pi_1)$.

If the firm implements in period 5 the cost reduction achieved in period 1, the PDV of its profit is $\pi_5 \equiv \delta^4\,\Delta\,[\,Q_5(c_0) + \delta\,Q_6(c_0)\,]$ because $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = c_0$. Assumption D implies:

$$\begin{aligned}\pi_5 &= \delta^3\,\Delta\left[\,\delta\,Q_5(c_0) + \delta^2\,Q_6(c_0)\,\right] < \delta^3\,\Delta\,[\,Q_4(c_0) + \delta\,Q_5(c_0)\,]\\ &= \delta^2\,\Delta\left[\,\delta\,Q_4(c_0) + \delta^2\,Q_5(c_0)\,\right] < \delta^2\,\Delta\,[\,Q_3(c_0) + \delta\,Q_4(c_0)\,] = \pi_3\ (< \pi_1)\,.\end{aligned}$$

If the firm implements in period 6 the Δ cost reduction achieved in period 1, the PDV of its profit is $\pi_6 \equiv \delta^5\,\Delta\,Q_6(c_0)$ because $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = c_0$. It is apparent that $\pi_6 = \pi_5 - \delta^4\,\Delta\,Q_5(c_0) < \pi_5\,(< \pi_1)$. □

Conclusion B. The firm never delays beyond period 3 the implementation of a cost reduction achieved in period 2.

Proof. If the firm implements in period 3 the cost reduction it achieves in period 2, the PDV of its profit is $\pi_L \equiv \delta\,\Delta\,[\,Q_3(c_0) + \delta\,Q_4(c_0)\,]$ because $p_1 = p_2 = p_3 = p_4 = c_0$ and $p_5 = p_6 = c_0 - \Delta$. We will show that the maximum PDV of profit the firm can secure by delaying the implementation of the achieved cost reduction beyond period 3 is always less π_L.

If the firm delays to period 4 the implementation of the cost reduction achieved in period 2, the PDV of its profit is $\delta^2\,\Delta\,Q_4(c_0)$ because $p_1 = p_2 = p_3 = p_4 = c_0$ and $p_5 = p_6 = c_0 - \Delta$. It is apparent that $\delta^2\,\Delta\,Q_4(c_0) < \delta\,\Delta\,[\,Q_3(c_0) + \delta\,Q_4(c_0)\,] = \pi_L$.

If the firm delays to period 5 the implementation of the cost reduction achieved in period 2, the PDV of its profit is $\delta^3\,\Delta\,[\,Q_5(c_0) + \delta\,Q_6(c_0)]$ because $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = c_0$. Assumption D implies:

$$\delta^3\,\Delta\,[\,Q_5(c_0) + \delta\,Q_6(c_0)\,] = \delta^2\,\Delta\left[\,\delta\,Q_5(c_0) + \delta^2\,Q_6(c_0)\,\right] < \delta^2\,\Delta\,[\,Q_4(c_0) + \delta\,Q_5(c_0)\,]$$

$$= \delta\,\Delta\left[\,\delta\,Q_4(c_0)+\delta^2\,Q_5(c_0)\,\right] \;<\; \delta\,\Delta\,[\,Q_3(c_0)+\delta\,Q_4(c_0)\,] \;=\; \pi_L\,. \tag{3}$$

If the firm delays to period 6 the implementation of the Δ cost reduction achieved in period 2, the PDV of its profit is $\delta^4\,\Delta\,Q_6(c_0)$ because $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = c_0$. It is apparent that:

$$\delta^4\,\Delta\,Q_6(c_0) \;<\; \delta^3\,\Delta\,[\,Q_5(c_0)+\delta\,Q_6(c_0)\,] \;<\; \delta\,\Delta\,[\,Q_3(c_0)+\delta\,Q_4(c_0)\,] \;=\; \pi_L\,. \tag{4}$$

The last inequality in (4) reflects (3). □

Conclusion C. If the firm achieves the cost reduction in period 2, it implements the cost reduction immediately if (2) holds, and otherwise implements the cost reduction in period 3.

Proof. If the firm implements the achieved cost reduction in period 2, the PDV of its profit is $\Delta\,Q_2(c_0)$ because $p_1 = p_2 = c_0$ and $p_3 = p_4 = p_5 = p_6 = c_0 - \Delta$. If the firm delays the implementation the achieved cost reduction in period 2 to period 3, the PDV of its profit is $\delta\,\Delta\,[\,Q_3(c_0)+\delta\,Q_4(c_0)\,]$ because $p_1 = p_2 = p_3 = p_4 = c_0$ and $p_5 = p_6 = c_0 - \Delta$. Therefore, Conclusion B implies that the firm will implement the cost reduction immediately if the inequality in (2) holds, and otherwise delay the implementation to period 3. □ ■

Corollary to Lemma 2. *Suppose Assumption G holds in the setting of Lemma 2. Then if the firm achieves the cost reduction in period 2, it implements the cost reduction immediately if and only if* $\widetilde{\delta} \equiv \delta\,g \le \widetilde{\widetilde{\delta}} = \frac{1}{2}\left[\sqrt{5}-1\right] \approx 0.618$.

Proof of the Corollary to Lemma 2. Define $Q_0 \equiv Q_1(c_0)$. Then when Assumption G holds, the inequality in (2) holds if and only if:

$$\delta\,g^2\,Q_0 + \delta^2\,g^3\,Q_0 \;\le\; g\,Q_0 \;\;\Leftrightarrow\;\; \delta\,g + \delta^2\,g^2 \;\le\; 1 \;\;\Leftrightarrow\;\; \widetilde{\delta}^2 + \widetilde{\delta} - 1 \;\le\; 0$$

$$\Leftrightarrow\;\; \widetilde{\delta} \;\le\; \frac{1}{2}\left[-1+\sqrt{1+4}\,\right] \;=\; \frac{1}{2}\left[\sqrt{5}-1\right] \;\approx\; 0.618\,. \;\; ■$$

Proposition 1. $0 < \phi_2^S < \phi_2^I < 1$.

Proof of Proposition 1. First consider the firm's problem in period 2 of the NID setting after no cost reduction is achieved in period 1. Under SR in this setting, the firm retains the full benefit of a cost reduction that is achieved in period 2 only for that period. Therefore, the firm's problem is:

$$\underset{\phi_2}{\text{Maximize}}\;\; \phi_2\,\Delta\,Q_2(c_0) - K_2(\phi_2)$$

$$\Rightarrow\;\; K_2'(\phi_2^S) \;=\; \Delta\,Q_2(c_0)\;\;\text{at an interior optimum.} \tag{5}$$

Under IRIS, if no cost reduction is achieved in period 1, the firm retains the full benefit of a cost reduction achieved in period 2 during both period 2 and period 3. Therefore, the firm's problem in period 2 (in both the ID setting and the NID setting) is:

$$\underset{\phi_2}{\text{Maximize}} \;\; \phi_2\, \Delta \left[\, Q_2(c_0) + \delta\, Q_3(c_0) \,\right] - K_2(\phi_2)$$

$$\Rightarrow \;\; K_2'(\phi_2^I) \;=\; \Delta \left[\, Q_2(c_0) + \delta\, Q_3(c_0) \,\right] \text{ at an interior optimum.} \tag{6}$$

First suppose that $\phi_2^S = 0$. Then (5) implies that $\Delta\, Q_2(c_0) \leq K_2'(0)$, which violates the maintained assumption that $K_2'(0) = 0$. Therefore, $\phi_2^S > 0$.

Now suppose that $\phi_2^I = 0$. Then (6) implies that $\Delta \left[\, Q_2(c_0) + \delta\, Q_3(c_0) \,\right] \leq K_2'(0)$, which violates the maintained assumption that $K_2'(0) = 0$. Therefore, $\phi_2^I > 0$.

Next suppose that $\phi_2^S = 1$. Then (5) implies that $K_2'(1) \leq \Delta\, Q_2(c_0)$, which violates the maintained assumption that $K_2'(1) > \Delta \left[\, Q_2(c_0) + \delta\, Q_3(c_0) \,\right]$. Therefore, $\phi_2^S < 1$.

Finally suppose that $\phi_2^I = 1$. Then (6) implies that $K_2'(1) \leq \Delta \left[\, Q_2(c_0) + \delta\, Q_3(c_0) \,\right]$, which violates the maintained assumption that $K_2'(1) > \Delta \left[\, Q_2(c_0) + \delta\, Q_3(c_0) \,\right]$. Therefore, $\phi_2^I < 1$.

Because $\phi_2^S \in (0,1)$ and $\phi_2^I \in (0,1)$, (5) and (6) imply that $K_2'(\phi_2^I) > K_2'(\phi_2^S) \Rightarrow \phi_2^I > \phi_2^S$. The conclusion here reflects the convexity of $K_2(\cdot)$.

Now consider the firm's problem in period 2 of the ID setting after no cost reduction is achieved in period 1. Under SR in this setting, the firm delays to period 3 the implementation of a cost reduction achieved in period 2. Therefore, the firm's problem is:

$$\underset{\phi_2}{\text{Maximize}} \;\; \phi_2\, \delta\, \Delta \left[\, Q_3(c_0) + \delta\, Q_4(c_0) \,\right] - K_2(\phi_2)$$

$$\Rightarrow \;\; K_2'(\phi_2^S) \;=\; \delta\, \Delta \left[\, Q_3(c_0) + \delta\, Q_4(c_0) \,\right] \text{ at an interior optimum.} \tag{7}$$

First suppose that $\phi_2^S = 0$. Then (7) implies that $\delta\, \Delta \left[\, Q_3(c_0) + \delta\, Q_4(c_0) \,\right] \leq K_2'(0)$, which violates the maintained assumption that $K_2'(0) = 0$. Therefore, $\phi_2^S > 0$.

Next suppose that $\phi_2^S = 1$. Then (7) implies that $K_2'(1) \leq \delta\, \Delta \left[\, Q_3(c_0) + \delta\, Q_4(c_0) \,\right]$, which violates the maintained assumption that $K_2'(1) > \delta\, \Delta \left[\, Q_3(c_0) + \delta\, Q_4(c_0) \,\right]$. Therefore, $\phi_2^S < 1$.

Because $\phi_2^S \in (0,1)$ and $\phi_2^I \in (0,1)$, (6) and (7) imply that $K_2'(\phi_2^I) > K_2'(\phi_2^S) \Rightarrow \phi_2^I > \phi_2^S$. The conclusion here reflects the convexity of $K_2(\cdot)$. ■

Proposition 2. $0 < \phi_1^I < \phi_1^S < 1$.

<u>Proof of Proposition 2</u>. Under SR in the NID setting, the firm retains the full benefit of a cost reduction that is achieved in period 1 both in period 1 and in period 2. Therefore, (5) implies that the firm's problem in period 1 under SR is:

$$\underset{\phi_1}{\text{Maximize}} \;\; \phi_1\, \Delta \left[\, Q_1(c_0) + \delta\, Q_2(c_0) \,\right] + \left[\, 1 - \phi_1 \,\right] \delta \left[\, \phi_2^S\, \Delta\, Q_2(c_0) - K_2(\phi_2^S) \,\right] - K_1(\phi_1)\,. \tag{8}$$

(8) implies that at an interior solution to this problem:

$$K_1'(\phi_1^S) \;=\; \Delta \left[\, Q_1(c_0) + \delta\, Q_2(c_0) \,\right] - \delta \left[\, \phi_2^S\, \Delta\, Q_2(c_0) - K_2(\phi_2^S) \,\right]. \tag{9}$$

Under IRIS in both the NID setting and the ID setting, the firm retains for two periods the full benefit of an achieved cost reduction, whether the reduction is achieved in period 1 or period 2. Therefore, (6) implies that the firm's problem in period 1 under IRIS is:

$$\underset{\phi_1}{\text{Maximize}} \quad \phi_1 \Delta [\, Q_1(c_0) + \delta\, Q_2(c_0)\,] \\ + [\, 1 - \phi_1\,]\, \delta \left\{ \phi_2^I \Delta [\, Q_2(c_0) + \delta\, Q_3(c_0)\,] - K_2(\phi_2^I) \right\} - K_1(\phi_1)\,. \qquad (10)$$

(10) implies that at an interior solution to this problem:

$$K_1'(\phi_1^I) \;=\; \Delta [\, Q_1(c_0) + \delta\, Q_2(c_0)\,] - \delta \left\{ \phi_2^I \Delta [\, Q_2(c_0) + \delta\, Q_3(c_0)\,] - K_2(\phi_2^I) \right\}. \qquad (11)$$

Observe that:

$$\phi_2^I \Delta [\, Q_2(c_0) + \delta\, Q_3(c_0)\,] - K_2(\phi_2^I) \;=\; \max_{\phi_2} \{\, \phi_2 \Delta [\, Q_2(c_0) + \delta\, Q_3(c_0)\,] - K_2(\phi_2)\,\} \\ > \; \phi_2^S \Delta [\, Q_2(c_0) + \delta\, Q_3(c_0)\,] - K_2(\phi_2^S) \;>\; \phi_2^S \Delta\, Q_2(c_0) - K_2(\phi_2^S)\,. \qquad (12)$$

The first inequality in (12) holds because $\phi_2^S \neq \phi_2^I$, from Proposition 1.

(10) implies that $\phi_1^I > 0$ in the NID setting if:

$$\Delta [\, Q_1(c_0) + \delta\, Q_2(c_0)\,] - \delta \left[\phi_2^I \Delta (\, Q_2(c_0) + \delta\, Q_3(c_0)\,) - K_2(\phi_2^I) \right] \;>\; K_1'(0)\,. \qquad (13)$$

Because $K_1'(0) = 0$ by assumption, the inequality in (13) holds if:

$$\Delta [\, Q_1(c_0) + \delta\, Q_2(c_0)\,] \;>\; \delta \left[\phi_2^I \Delta (\, Q_2(c_0) + \delta\, Q_3(c_0)\,) - K_2(\phi_2^I) \right].$$

This inequality holds because Assumption D implies:

$$Q_1(c_0) + \delta\, Q_2(c_0) \;>\; \delta\, Q_2(c_0) + \delta^2\, Q_3(c_0)$$

$$\Rightarrow \quad Q_1(c_0) + \delta\, Q_2(c_0) \;>\; \phi_2^I \left[\, \delta\, Q_2(c_0) + \delta^2\, Q_3(c_0)\,\right]$$

$$\Rightarrow \quad \Delta [\, Q_1(c_0) + \delta\, Q_2(c_0)\,] \;>\; \delta\, \phi_2^I \Delta [\, Q_2(c_0) + \delta\, Q_3(c_0)\,]$$

$$\Rightarrow \quad \Delta [\, Q_1(c_0) + \delta\, Q_2(c_0)\,] \;>\; \delta \left[\phi_2^I \Delta (\, Q_2(c_0) + \delta\, Q_3(c_0)\,) - K_2(\phi_2^I) \right].$$

(10) implies that $\phi_1^I < 1$ in the NID setting if:

$$\Delta [\, Q_1(c_0) + \delta\, Q_2(c_0)\,] - \delta \left\{ \phi_2^I \Delta [\, Q_2(c_0) + \delta\, Q_3(c_0)\,] - K_2(\phi_2^I) \right\} \;<\; K_1'(1)\,. \qquad (14)$$

$\Delta [\, Q_1(c_0) + \delta\, Q_2(c_0)\,] < K_1'(1)$, by assumption. Furthermore, $\phi_2^I \Delta [\, Q_2(c_0) + \delta\, Q_3(c_0)\,] - K_2(\phi_2^I) \geq 0$ because $\phi_2^I = \arg\max_{\phi} \{\, \phi \Delta [\, Q_2(c_0) + \delta\, Q_3(c_0)\,] - K_2(\phi)\,\}$ and $K_2(0) = 0$. Therefore, the inequality in (14) holds.

(8) implies that $\phi_1^S > 0$ in the NID setting if:

$$\Delta [\, Q_1(c_0) + \delta\, Q_2(c_0)\,] - \delta \left[\phi_2^S \Delta\, Q_2(c_0) - K_2(\phi_2^S) \right] \;>\; K_1'(0)\,. \qquad (15)$$

$K_1'(0) = 0$ by assumption. Therefore, (15) holds if:

$$\Delta\,[\,Q_1(c_0) + \delta\,Q_2(c_0)\,] - \delta\,\big[\,\phi_2^S\,\Delta\,Q_2(c_0) - K_2(\phi_2^S)\,\big] \;>\; 0\,.$$

This inequality holds because:

$$\begin{aligned}
& Q_1(c_0) + \delta\,Q_2(c_0) \;>\; \delta\,Q_2(c_0) \quad\Rightarrow\quad Q_1(c_0) + \delta\,Q_2(c_0) \;>\; \phi_2^S\,\delta\,Q_2(c_0) \\
& \Rightarrow\quad \Delta\,[\,Q_1(c_0) + \delta\,Q_2(c_0)\,] - \delta\,\phi_2^S\,\Delta\,Q_2(c_0) \;>\; 0 \\
& \Rightarrow\quad \Delta\,[\,Q_1(c_0) + \delta\,Q_2(c_0)\,] - \delta\,\big[\,\phi_2^S\,\Delta\,Q_2(c_0) - K_2(\phi_2^S)\,\big] \;>\; 0\,.
\end{aligned}$$

(8) implies that $\phi_1^S < 1$ in the NID setting if:

$$\Delta\,[\,Q_1(c_0) + \delta\,Q_2(c_0)\,] - \delta\,\big[\,\phi_2^S\,\Delta\,Q_2(c_0) - K_2(\phi_2^S)\,\big] \;<\; K_1'(1)\,. \tag{16}$$

$\Delta\,[\,Q_1(c_0) + \delta\,Q_2(c_0)\,] \;<\; K_1'(1)$, by assumption. Furthermore, $\phi_2^S\,\Delta\,Q_2(c_0) - K_2(\phi_2^S) \geq 0$ because $\phi_2^S \;=\; \arg\max_\phi\,\{\,\phi\,\Delta\,Q_2(c_0) - K_2(\phi)\,\}$ and $K_2(0) = 0$. Therefore, the inequality in (16) holds.

To prove that $\phi_1^I \;<\; \phi_1^S$ in the NID setting, observe that:

$$\begin{aligned}
\phi_1^I \;&=\; \arg\max_\phi\,\{\,\phi\,\Delta\,[\,Q_1(c_0) + \delta\,Q_2(c_0)\,] \\
&\qquad + \;[\,1-\phi\,]\;\delta\,\big[\,\phi_2^I\,\Delta\,(\,Q_2(c_0) + \delta\,Q_3(c_0)\,) - K_2(\phi_2^I)\,\big] - K_1(\phi)\,\} \\
&<\; \arg\max_\phi\,\{\,\phi\,\Delta\,[\,Q_1(c_0) + \delta\,Q_2(c_0)\,] \\
&\qquad + \;[\,1-\phi\,]\;\delta\,\big[\,\phi_2^S\,\Delta\,Q_2(c_0) - K_2(\phi_2^S)\,\big] - K_1(\phi)\,\} \;=\; \phi_1^S\,.
\end{aligned} \tag{17}$$

The equalities in (17) reflect (9) and (11) since $\phi_1^S \in (0,1)$ and $\phi_1^I \in (0,1)$. The inequality in (17) reflects (12) and the fact that the firm's profit-maximizing choice of ϕ_1 increases as the firm's expected profit following first-period failure to achieve a cost reduction declines, holding constant the firm's expected profit following first period success in securing a cost reduction.[2]

(7) implies that the firm's problem in period 1 under SR in the ID setting is:

$$\begin{aligned}
\underset{\phi_1}{\text{Maximize}} \quad & \phi_1\,\Delta\,[\,Q_1(c_0) + \delta\,Q_2(c_0)\,] \\
& + \;[\,1-\phi_1\,]\,\big[\,\phi_2^S\,\Delta\,\big(\,\delta^2\,Q_3(c_0) + \delta^3\,Q_4(c_0)\,\big) - \delta\,K_2(\phi_2^S)\,\big] - K_1(\phi_1)\,.
\end{aligned} \tag{18}$$

(18) implies that at an interior solution to this problem:

$$K_1'(\phi_1^S) \;=\; \Delta\,[\,Q_1(c_0) + \delta\,Q_2(c_0)\,] - \delta\,\big[\,\phi_2^S\,\Delta\,\big(\,\delta\,Q_3(c_0) + \delta^2\,Q_4(c_0)\,\big) - K_2(\phi_2^S)\,\big]\,. \tag{19}$$

(18) implies that $\phi_1^S > 0$ in the ID setting if:

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[2]Formally, if $\phi_1 \in (0,1) \;= \arg\max_\phi\,\{\,\phi\,A + [\,1-\phi\,]\,B - K_1(\phi)\,\}$, then $A - B \;=\; K_1'(\phi_1) \Rightarrow \frac{d\phi_1}{dB} \;=\; -\,\frac{1}{K_1''(\phi_1)} < 0$.

$$\Delta\left[\,Q_1(c_0) + \delta\,Q_2(c_0)\,\right] - \left[\,\phi_2^S\,\Delta\left(\delta^2\,Q_3(c_0) + \delta^3\,Q_4(c_0)\right) - \delta\,K_2(\phi_2^S)\,\right] \;>\; K_1'(0)\,. \tag{20}$$

$K_1'(0) = 0$ by assumption. Therefore, (20) holds if:

$$\Delta\left[\,Q_1(c_0) + \delta\,Q_2(c_0)\,\right] - \left[\,\phi_2^S\,\Delta\left(\delta^2\,Q_3(c_0) + \delta^3\,Q_4(c_0)\right) - \delta\,K_2(\phi_2^S)\,\right] \;>\; 0\,.$$

This inequality holds because:

$$\begin{aligned}
& Q_1(c_0) + \delta\,Q_2(c_0) \;>\; \delta^2\,Q_3(c_0) + \delta^3\,Q_4(c_0) \\
\Rightarrow\;\; & \Delta\left[\,Q_1(c_0) + \delta\,Q_2(c_0)\,\right] \;>\; \Delta\left[\,\delta^2\,Q_3(c_0) + \delta^3\,Q_4(c_0)\,\right] \\
\Rightarrow\;\; & \Delta\left[\,Q_1(c_0) + \delta\,Q_2(c_0)\,\right] - \phi_2^S\,\Delta\left[\,\delta^2\,Q_3(c_0) + \delta^3\,Q_4(c_0)\,\right] \;>\; 0 \\
\Rightarrow\;\; & \Delta\left[\,Q_1(c_0) + \delta\,Q_2(c_0)\,\right] - \left[\,\phi_2^S\,\Delta\left(\delta^2\,Q_3(c_0) + \delta^3\,Q_4(c_0)\right) - \delta\,K_2(\phi_2^S)\,\right] \;>\; 0.
\end{aligned} \tag{21}$$

The first inequality in (21) holds because Assumption D implies that $Q_1(c_0) > \delta\,Q_2(c_0) > \delta^2\,Q_3(c_0)$ and $Q_2(c_0) > \delta\,Q_3(c_0) > \delta^2\,Q_4(c_0)$.

(18) implies that $\phi_1^S < 1$ in the ID setting if:

$$\Delta\left[\,Q_1(c_0) + \delta\,Q_2(c_0)\,\right] - \left[\,\phi_2^S\,\Delta\left(\delta^2\,Q_3(c_0) + \delta^3\,Q_4(c_0)\right) - \delta\,K_2(\phi_2^S)\,\right] \;<\; K_1'(1)\,. \tag{22}$$

$\Delta\left[\,Q_1(c_0) + \delta\,Q_2(c_0)\,\right] < K_1'(1)$, by assumption. Furthermore, $\phi_2^S\,\Delta\left[\,\delta^2\,Q_3(c_0) + \delta^3\,Q_4(c_0)\,\right] - \delta\,K_2(\phi_2^S) \geq 0$ because $\phi_2^S = \arg\max_\phi \left\{\,\phi\,\Delta\left[\,\delta\,Q_3(c_0) + \delta^2\,Q_4(c_0)\,\right] - K_2(\phi)\,\right\}$ in the ID setting (from (7)) and because $K_2(0) = 0$. Therefore, the inequality in (22) holds.

To prove that $\phi_1^I < \phi_1^S$ in the ID setting, first observe that:

$$\begin{aligned}
\phi_2^I\,\Delta\left[\,Q_2(c_0) + \delta\,Q_3(c_0)\,\right] - K_2(\phi_2^I) \;&=\; \max_{\phi_2}\,\left\{\,\phi_2\,\Delta\left[\,Q_2(c_0) + \delta\,Q_3(c_0)\,\right] - K_2(\phi_2)\,\right\} \\
>\; \phi_2^S\,\Delta\left[\,Q_2(c_0) + \delta\,Q_3(c_0)\,\right] - K_2(\phi_2^S) \;&>\; \phi_2^S\,\Delta\,\delta\left[\,Q_3(c_0) + \delta\,Q_4(c_0)\,\right] - K_2(\phi_2^S)\,.
\end{aligned} \tag{23}$$

The equality in (23) reflects (7). The first inequality in (23) holds because $\phi_2^I \neq \phi_2^S$, from Proposition 1. The last inequality in (23) reflects Assumption D.

Now observe that:

$$\begin{aligned}
\phi_1^I \;&=\; \arg\max_\phi\,\{\,\phi\,\Delta\left[\,Q_1(c_0) + \delta\,Q_2(c_0)\,\right] \\
&\qquad\qquad + \;[\,1-\phi\,]\;\delta\left[\,\phi_2^I\,\Delta\,(\,Q_2(c_0) + \delta\,Q_3(c_0)\,) - K_2(\phi_2^I)\,\right] - K_1(\phi)\,\} \\
&<\; \arg\max_\phi\,\{\,\phi\,\Delta\left[\,Q_1(c_0) + \delta\,Q_2(c_0)\,\right] \\
&\qquad\qquad + \;[\,1-\phi\,]\;\delta\left[\,\phi_2^S\,\Delta\,\delta\,(\,Q_3(c_0) + \delta\,Q_4(c_0)\,) - K_2(\phi_2^S)\,\right] - K_1(\phi)\,\} \\
&=\; \phi_1^S\,.
\end{aligned} \tag{24}$$

The equalities in (24) reflect (10) and (18). The inequality in (24) reflects (23) and the fact that the firm's profit-maximizing choice of ϕ_1 increases as the firm's expected profit following first-period "failure" declines, holding constant the firm's expected profit following first period "success." ■

Definition. $\Phi^j \equiv \phi_1^j + [\,1 - \phi_1^j\,]\,\phi_2^j$ for $j \in \{S, R\}$. (25)

Lemma 3. *Suppose Assumption G holds and Assumption K with $\gamma = 2$ holds. Then:*

$$\Phi^S \gtreqless \Phi^I \;\Leftrightarrow\; G^{NID} \gtreqless 0 \textit{ in the NID setting; and}$$

$$\Phi^S \gtreqless \Phi^I \;\Leftrightarrow\; G^{ID} \gtreqless 0 \textit{ in the ID setting, where} \tag{26}$$

$$G^{NID} \equiv \Delta Q_0 \left\{ [\,2 + 4\,\widetilde{\delta} + (\,\widetilde{\delta}\,)^2\,]\,k_2 - g\,[\,(\,1+\widetilde{\delta}\,)^3 - 1\,]\,\Delta Q_0 \right\} - 2\,k_1\,k_2 \textit{ and}$$

$$G^{ID} \equiv \Delta Q_0\,[\,1+\widetilde{\delta}\,] \left\{ [\,2 - \widetilde{\delta} - (\,\widetilde{\delta}\,)^3\,]\,\widetilde{k}_2 - \widetilde{\delta}\,[\,1+\widetilde{\delta}\,]\,[\,1-(\,\widetilde{\delta}\,)^3\,]\,\Delta Q_0 \right\} - 2\,[\,1-\widetilde{\delta}\,]\,k_1\,\widetilde{k}_2\,.$$

Proof of Lemma 3. Define $Q_t \equiv Q_t(c_0)$ for $t = 1, ..., 6$. (5) and (6) imply that under the maintained assumptions in the NID setting:

$$\phi_2^S = \frac{\Delta}{k_2}\,Q_2 = \frac{\Delta\,g}{k_2}\,Q_0 \text{ and}$$

$$\phi_2^I = \frac{\Delta}{k_2}\,[\,Q_2 + \delta\,Q_3\,] = \frac{\Delta}{k_2}\left[\,g\,Q_0 + \delta\,g^2\,Q_0\,\right] = \frac{\Delta\,g\,[\,1+\delta\,g\,]}{k_2}\,Q_0\,. \tag{27}$$

(9) implies that in the NID setting:

$$\begin{aligned}
\phi_1^S &= \frac{1}{k_1}\left\{\,\Delta\,[\,Q_1 + \delta\,Q_2\,] - \delta\left[\,\phi_2^S\,\Delta\,Q_2 - K_2(\phi_2^S)\,\right]\right\}\\
&= \frac{1}{k_1}\left\{\,\Delta\,[\,Q_0 + \delta\,g\,Q_0\,] - \delta\left[\,\phi_2^S\,\Delta\,g\,Q_0 - K_2(\phi_2^S)\,\right]\right\}\\
&= \frac{1}{k_1}\left\{\,\Delta\,[\,1 + \delta\,g\,]\,Q_0 - \delta\left[\,\phi_2^S\,\Delta\,g\,Q_0 - K_2(\phi_2^S)\,\right]\right\}.
\end{aligned} \tag{28}$$

(11) implies:

$$\begin{aligned}
\phi_1^I &= \frac{1}{k_1}\left\{\,\Delta\,[\,Q_1 + \delta\,Q_2\,] - \delta\left[\,\phi_2^I\,\Delta\,(Q_2 + \delta\,Q_3) - K_2(\phi_2^I)\,\right]\right\}\\
&= \frac{1}{k_1}\left\{\,\Delta\,[\,Q_0 + \delta\,g\,Q_0\,] - \delta\left[\,\phi_2^I\,\Delta\left(g\,Q_0 + \delta\,g^2\,Q_0\right) - K_2(\phi_2^I)\,\right]\right\}\\
&= \frac{1}{k_1}\left\{\,\Delta\,[\,1 + \delta\,g\,]\,Q_0 - \delta\left[\,\phi_2^I\,\Delta\,g\,(1+\delta\,g)\,Q_0 - K_2(\phi_2^I)\,\right]\right\}.
\end{aligned} \tag{29}$$

(27) implies:

$$K_2(\phi_2^S) = \frac{k_2}{2}\left[\frac{\Delta\,g}{k_2}\,Q_0\right]^2 = \frac{\Delta^2\,g^2}{2\,k_2}\,[\,Q_0\,]^2\,;$$

$$K_2(\phi_2^I) \;=\; \frac{k_2}{2}\left[\frac{\Delta\, g\,(1+\delta\, g)}{k_2}\,Q_0\right]^2 \;=\; \frac{\Delta^2\, g^2}{2\,k_2}\,[\,1+\delta\, g\,]^2\,[\,Q_0\,]^2\,. \tag{30}$$

(27) and (30) imply:

$$\phi_2^S\,\Delta\,Q_2(c_0) - K_2(\phi_2^S) \;=\; \frac{\Delta\, g}{k_2}\,Q_0\,\Delta\, g\,Q_0 - \frac{\Delta^2\, g^2}{2\,k_2}\,[\,Q_0\,]^2 \;=\; \frac{\Delta^2\, g^2}{2\,k_2}\,[\,Q_0\,]^2\,. \tag{31}$$

(27) and (30) also imply:

$$\begin{aligned}
&\phi_2^I\,\Delta\,[\,Q_2+\delta\,Q_3\,] - K_2(\phi_2^I)\\
&\qquad=\; \frac{\Delta\, g\,[\,1+\delta\, g\,]}{k_2}\,Q_0\,\Delta\left[\,g\,Q_0+\delta\, g^2\,Q_0\,\right] - \frac{\Delta^2\, g^2}{2\,k_2}\,[\,1+\delta\, g\,]^2\,[\,Q_0\,]^2\\
&\qquad=\; \frac{\Delta\, g\,[\,1+\delta\, g\,]}{k_2}\,Q_0\,\Delta\, g\,[\,1+\delta\, g\,]\,Q_0 - \frac{\Delta^2\, g^2}{2\,k_2}\,[\,1+\delta\, g\,]^2\,[\,Q_0\,]^2\\
&\qquad=\; \frac{\Delta^2\, g^2}{2\,k_2}\,[\,1+\delta\, g\,]^2\,[\,Q_0\,]^2\,.
\end{aligned} \tag{32}$$

(28) and (31) imply:

$$\phi_1^S \;=\; \frac{\Delta}{k_1}\,[\,1+\delta\, g\,]\,Q_0 - \frac{\delta}{k_1}\left[\frac{\Delta^2\, g^2}{2\,k_2}\right][\,Q_0\,]^2 \;=\; \frac{\Delta\,Q_0}{k_1}\left[1+\delta\, g - \frac{\delta\,\Delta\, g^2}{2\,k_2}\,Q_0\right]. \tag{33}$$

(27) and (33) imply:

$$\phi_1^S\,\phi_2^S \;=\; \frac{\Delta^2\, g}{k_1\,k_2}\left[1+\delta\, g - \frac{\delta\,\Delta\, g^2}{2\,k_2}\,Q_0\right][\,Q_0\,]^2\,. \tag{34}$$

(29) and (32) imply:

$$\begin{aligned}
\phi_1^I \;&=\; \frac{\Delta}{k_1}\,[\,1+\delta\, g\,]\,Q_0 \;-\; \frac{\delta}{k_1}\left[\frac{\Delta^2\, g^2}{2\,k_2}\right][\,1+\delta\, g\,]^2\,[\,Q_0\,]^2\\
&=\; \frac{\Delta\,Q_0}{k_1}\left[1+\delta\, g - \frac{\delta\,\Delta\, g^2}{2\,k_2}\,(\,1+\delta\, g\,)^2\,Q_0\right].
\end{aligned} \tag{35}$$

(27) and (35) imply:

$$\phi_1^I\,\phi_2^I \;=\; \frac{\Delta^2\, g\,[\,1+\delta\, g\,]}{k_1\,k_2}\left[1+\delta\, g - \frac{\delta\,\Delta\, g^2}{2\,k_2}\,(\,1+\delta\, g\,)^2\,Q_0\right][\,Q_0\,]^2\,. \tag{36}$$

(25), (27), (33), and (34) imply:

$$\begin{aligned}
\Phi^S \;=\; \phi_1^S+\phi_2^S-\phi_1^S\,\phi_2^S \;&=\; \frac{\Delta\,Q_0}{k_1}\left[1+\delta\, g - \frac{\delta\,\Delta\, g^2}{2\,k_2}\,Q_0\right] + \frac{\Delta\, g}{k_2}\,Q_0\\
&\quad -\; \frac{\Delta^2\, g}{k_1\,k_2}\left[1+\delta\, g - \frac{\delta\,\Delta\, g^2}{2\,k_2}\,Q_0\right][\,Q_0\,]^2\,.
\end{aligned} \tag{37}$$

(25), (27), (35), and (36) imply:

$$\Phi^I \;=\; \phi_1^I + \phi_2^I - \phi_1^I\,\phi_2^I \;=\; \frac{\Delta\,Q_0}{k_1}\left[1+\delta\,g - \frac{\delta\,\Delta\,g^2}{2\,k_2}\,(1+\delta\,g)^2\;Q_0\right] + \frac{\Delta\,g\,[\,1+\delta\,g\,]}{k_2}\;Q_0$$

$$- \;\frac{\Delta^2\,g\,[\,1+\delta\,g\,]}{k_1\,k_2}\left[1+\delta\,g - \frac{\delta\,\Delta\,g^2}{2\,k_2}\,(1+\delta\,g)^2\;Q_0\right][\,Q_0\,]^2\;. \quad (38)$$

(37) and (38) imply that because $\delta > 0$:

$$\begin{aligned}
\Phi^S - \Phi^I \;=\;& \frac{\Delta\,Q_0}{k_1}\left[\frac{\delta\,\Delta\,g^2}{2\,k_2}\right]Q_0\left[\,(1+\delta\,g)^2 - 1\,\right] - \frac{\Delta\,g}{k_2}\;Q_0\,[\,1+\delta\,g-1\,] \\
&+\;\frac{\Delta^2\,g\,[\,1+\delta\,g\,]}{k_1\,k_2}\left[1+\delta\,g - \frac{\delta\,\Delta\,g^2}{2\,k_2}\,(1+\delta\,g)^2\;Q_0\right][\,Q_0\,]^2 \\
&-\;\frac{\Delta^2\,g}{k_1\,k_2}\left[1+\delta\,g - \frac{\delta\,\Delta\,g^2}{2\,k_2}\;Q_0\right][\,Q_0\,]^2 \\
=\;& \frac{\delta\,\Delta^2\,g^2}{2\,k_1\,k_2}\,[\,Q_0\,]^2\left[\,2\,\delta\,g + \delta^2\,g^2\,\right] - \frac{\Delta\,\delta\,g^2}{k_2}\;Q_0 \\
&+\;\frac{\Delta^2\,g}{k_1\,k_2}\,[\,Q_0\,]^2\left\{[\,1+\delta\,g\,]\left[1+\delta\,g - \frac{\delta\,\Delta\,g^2}{2\,k_2}\,(1+\delta\,g)^2\;Q_0\right]\right. \\
&\qquad\left. -\;\left[1+\delta\,g - \frac{\delta\,\Delta\,g^2}{2\,k_2}\;Q_0\right]\right\} \\
=\;& \frac{\delta^2\,\Delta^2\,g^3}{2\,k_1\,k_2}\,[\,Q_0\,]^2\,[\,2+\delta\,g\,] - \frac{\Delta\,\delta\,g^2}{k_2}\;Q_0 \\
&+\;\frac{\Delta^2\,g}{k_1\,k_2}\,[\,Q_0\,]^2\left\{[\,1+\delta\,g\,]\,[\,1+\delta\,g-1\,] + \frac{\delta\,\Delta\,g^2}{2\,k_2}\left[\,1-(1+\delta\,g)^3\,\right]Q_0\right\} \\
\stackrel{s}{=}\;& \frac{\delta^2\,\Delta\,g^2\,Q_0}{2\,k_1}\,[\,2+\delta\,g\,] - \delta\,g \\
&+\;\frac{\Delta\,Q_0}{k_1}\left\{\delta\,g\,[\,1+\delta\,g\,] - \frac{\delta\,\Delta\,g^2}{2\,k_2}\left[\,(1+\delta\,g)^3 - 1\,\right]Q_0\right\} \\
\stackrel{s}{=}\;& \Delta\,\delta^2\,g^2\,Q_0\,[\,2+\delta\,g\,]\,k_2 - 2\,\delta\,g\,k_1\,k_2 + 2\,k_2\,\Delta\,Q_0\,\delta\,g\,[\,1+\delta\,g\,] \\
&-\;\delta\,\Delta^2\,g^2\left[\,(1+\delta\,g)^3 - 1\,\right][\,Q_0\,]^2 \\
\stackrel{s}{=}\;& \Delta\,\delta\,g\,Q_0\,[\,2+\delta\,g\,]\,k_2 - 2\,k_1\,k_2 + 2\,k_2\,\Delta\,Q_0\,[\,1+\delta\,g\,] \\
&-\;\Delta^2\,g\left[\,(1+\delta\,g)^3 - 1\,\right][\,Q_0\,]^2
\end{aligned}$$

$$= \Delta Q_0 \left\{ \delta g [2 + \delta g] k_2 + 2 k_2 [1 + \delta g] - g \left[(1 + \delta g)^3 - 1 \right] \Delta Q_0 \right\} - 2 k_1 k_2$$

$$= \Delta Q_0 \left\{ [\delta g (2 + \delta g) + 2 (1 + \delta g)] k_2 - g \left[(1 + \delta g)^3 - 1 \right] \Delta Q_0 \right\} - 2 k_1 k_2$$

$$= \Delta Q_0 \left\{ \left[2 + 4 \delta g + (\delta g)^2 \right] k_2 - g \left[(1 + \delta g)^3 - 1 \right] \Delta Q_0 \right\} - 2 k_1 k_2 . \tag{39}$$

(7) implies that under the maintained conditions in the ID setting:

$$\phi_2^S = \frac{\Delta \delta [Q_3(c_0) + \delta Q_4(c_0)]}{k_2} = \frac{\Delta \delta Q_0 [g^2 + \delta g^3]}{k_2}$$

$$= \frac{\Delta Q_0 \delta g^2 [1 + \delta g]}{k_2} = \frac{\Delta Q_0 \delta g [1 + \delta g]}{k_2 / g} = \frac{\Delta Q_0 \widetilde{\delta} [1 + \widetilde{\delta}]}{\widetilde{k}_2} . \tag{40}$$

When Assumption G holds and the ID setting prevails, the PDV of the firm's profit in period 2 when it achieves the Δ cost reduction in that period is:

$$\pi_2^S = \Delta \delta [Q_3(c_0) + \delta Q_4(c_0)] = \Delta \delta \left[g^2 Q_0 + \delta g^3 Q_0 \right]$$

$$= \Delta g^2 \delta Q_0 [1 + \delta g] = \Delta Q_0 g \widetilde{\delta} [1 + \widetilde{\delta}]. \tag{41}$$

(40) and (41) imply:

$$\phi_2^S \pi_2^S - K(\phi_2^S) = \frac{\Delta Q_0 \widetilde{\delta} [1 + \widetilde{\delta}]}{\widetilde{k}_2} \Delta Q_0 g \widetilde{\delta} [1 + \widetilde{\delta}] - \frac{k_2}{2} \left[\frac{\Delta Q_0 \widetilde{\delta} (1 + \widetilde{\delta})}{\widetilde{k}_2} \right]^2$$

$$= \frac{1}{\widetilde{k}_2} \left[\Delta Q_0 \widetilde{\delta} (1 + \widetilde{\delta}) \right]^2 \left[g - \frac{k_2}{2 \widetilde{k}_2} \right] = \frac{1}{\widetilde{k}_2} \left[\Delta Q_0 \widetilde{\delta} (1 + \widetilde{\delta}) \right]^2 \left[g - \frac{g}{2} \right]$$

$$= \frac{g \left[\Delta Q_0 \widetilde{\delta} (1 + \widetilde{\delta}) \right]^2}{2 \widetilde{k}_2} . \tag{42}$$

(19) and (42) imply:

$$\phi_1^S = \frac{1}{k_1} \left\{ \Delta [Q_1(c_0) + \delta Q_2(c_0)] - \delta \left[\phi_2^S \pi_2^S - K(\phi_2^S) \right] \right\}$$

$$= \frac{1}{k_1} \left\{ \Delta Q_0 [1 + \delta g] - \delta \frac{g \left[\Delta Q_0 \widetilde{\delta} (1 + \widetilde{\delta}) \right]^2}{2 \widetilde{k}_2} \right\}$$

$$= \frac{1}{k_1} \left\{ \Delta Q_0 [1 + \widetilde{\delta}] - \delta g \frac{\left[\Delta Q_0 \widetilde{\delta} (1 + \widetilde{\delta}) \right]^2}{2 \widetilde{k}_2} \right\}$$

$$= \frac{1}{k_1} \left\{ \Delta Q_0 [1 + \widetilde{\delta}] - \frac{\widetilde{\delta}}{2\widetilde{k}_2} \left[\Delta Q_0 \widetilde{\delta} (1 + \widetilde{\delta}) \right]^2 \right\}$$

$$= \frac{\Delta Q_0 [1 + \widetilde{\delta}]}{k_1} \left[1 - \frac{(\widetilde{\delta})^3}{2\widetilde{k}_2} \Delta Q_0 (1 + \widetilde{\delta}) \right]. \tag{43}$$

(6) implies that under the specified conditions:

$$\phi_2^I = \frac{\Delta [Q_2(c_0) + \delta Q_3(c_0)]}{k_2} = \frac{\Delta Q_0 [g + \delta g^2]}{k_2}$$

$$= \frac{\Delta Q_0 g [1 + \delta g]}{k_2} = \frac{\Delta Q_0 [1 + \delta g]}{k_2 / g} = \frac{\Delta Q_0 [1 + \widetilde{\delta}]}{\widetilde{k}_2}. \tag{44}$$

(44) implies:

$$\phi_2^I \Delta Q_0 g [1 + \delta g] - K(\phi_2^I) = \frac{\Delta Q_0 [1 + \widetilde{\delta}]}{\widetilde{k}_2} \Delta Q_0 g [1 + \widetilde{\delta}] - \frac{k_2}{2} \left[\frac{\Delta Q_0 (1 + \widetilde{\delta})}{\widetilde{k}_2} \right]^2$$

$$= g \widetilde{k}_2 \left[\frac{\Delta Q_0 (1 + \widetilde{\delta})}{\widetilde{k}_2} \right]^2 - \frac{k_2}{2} \left[\frac{\Delta Q_0 (1 + \widetilde{\delta})}{\widetilde{k}_2} \right]^2$$

$$= g \widetilde{k}_2 \left[\frac{\Delta Q_0 (1 + \widetilde{\delta})}{\widetilde{k}_2} \right]^2 - \frac{g \widetilde{k}_2}{2} \left[\frac{\Delta Q_0 (1 + \widetilde{\delta})}{\widetilde{k}_2} \right]^2$$

$$= \frac{g \widetilde{k}_2}{2} \left[\frac{\Delta Q_0 (1 + \widetilde{\delta})}{\widetilde{k}_2} \right]^2 = \frac{g}{2\widetilde{k}_2} \left[\Delta Q_0 (1 + \widetilde{\delta}) \right]^2. \tag{45}$$

(11) and (45) imply:

$$\phi_1^I = \frac{1}{k_1} \left\{ \Delta [Q_1(c_0) + \delta Q_2(c_0)] - \delta \left[\phi_2^I \Delta g Q_0 (1 + g\delta) - K(\phi_2^I) \right] \right\}$$

$$= \frac{1}{k_1} \left\{ \Delta Q_0 [1 + g\delta] - \delta \frac{g}{2\widetilde{k}_2} \left[\Delta Q_0 (1 + \widetilde{\delta}) \right]^2 \right\}$$

$$= \frac{1}{k_1} \left\{ \Delta Q_0 [1 + \widetilde{\delta}] - \frac{\widetilde{\delta}}{2\widetilde{k}_2} \left[\Delta Q_0 (1 + \widetilde{\delta}) \right]^2 \right\}$$

$$= \frac{\Delta Q_0 [1 + \widetilde{\delta}]}{k_1} \left[1 - \frac{\widetilde{\delta}}{2\widetilde{k}_2} \Delta Q_0 (1 + \widetilde{\delta}) \right]. \tag{46}$$

(40) and (43) imply:

$$\phi_1^S\,\phi_2^S \;=\; \frac{\widetilde{\delta}\,[\,\Delta\,Q_0\,(\,1+\widetilde{\delta}\,)\,]^2}{k_1\,\widetilde{k}_2}\left[1-\frac{(\widetilde{\delta})^3}{2\,\widetilde{k}_2}\;\Delta\,Q_0\,(\,1+\widetilde{\delta}\,)\right]. \tag{47}$$

(44) and (46) imply:

$$\phi_1^I\,\phi_2^I \;=\; \frac{[\,\Delta\,Q_0\,(\,1+\widetilde{\delta}\,)\,]^2}{k_1\,\widetilde{k}_2}\left[1-\frac{\widetilde{\delta}}{2\,\widetilde{k}_2}\;\Delta\,Q_0\,(\,1+\widetilde{\delta}\,)\right]. \tag{48}$$

(25), (40), (43), and (47) imply:

$$\begin{aligned}
\Phi^S \;&=\; \phi_1^S+\phi_2^S-\phi_1^S\,\phi_2^S \\
&=\; \frac{\Delta\,Q_0\,[\,1+\widetilde{\delta}\,]}{k_1}\left[1-\frac{(\widetilde{\delta})^3}{2\,\widetilde{k}_2}\,\Delta\,Q_0\,(\,1+\widetilde{\delta}\,)\right]+\frac{\Delta\,Q_0\,\widetilde{\delta}\,[\,1+\widetilde{\delta}\,]}{\widetilde{k}_2} \\
&\quad -\;\frac{\widetilde{\delta}\,[\,\Delta\,Q_0\,(\,1+\widetilde{\delta}\,)\,]^2}{k_1\,\widetilde{k}_2}\left[1-\frac{(\widetilde{\delta})^3}{2\,\widetilde{k}_2}\;\Delta\,Q_0\,(\,1+\widetilde{\delta}\,)\right].
\end{aligned} \tag{49}$$

(25), (44), (46), and (48) imply:

$$\begin{aligned}
\Phi^I \;&=\; \phi_1^I+\phi_2^I-\phi_1^I\,\phi_2^I \\
&=\; \frac{\Delta\,Q_0\,[\,1+\widetilde{\delta}\,]}{k_1}\left[1-\frac{\widetilde{\delta}}{2\,\widetilde{k}_2}\;\Delta\,Q_0\,(\,1+\widetilde{\delta}\,)\right]+\frac{\Delta\,Q_0\,[\,1+\widetilde{\delta}\,]}{\widetilde{k}_2} \\
&\quad -\;\frac{[\,\Delta\,Q_0\,(\,1+\widetilde{\delta}\,)\,]^2}{k_1\,\widetilde{k}_2}\left[1-\frac{\widetilde{\delta}}{2\,\widetilde{k}_2}\;\Delta\,Q_0\,(\,1+\widetilde{\delta}\,)\right].
\end{aligned} \tag{50}$$

(49) and (50) imply:

$$\begin{aligned}
\Phi^S-\Phi^I \;&=\; \frac{\widetilde{\delta}}{2\,k_1\,\widetilde{k}_2}\;\Big[\Delta\,Q_0\,(\,1+\widetilde{\delta}\,)\Big]^2\,[\,1-(\,\widetilde{\delta}\,)^2\,]\;-\;\frac{\Delta\,Q_0\,[\,1+\widetilde{\delta}\,]}{\widetilde{k}_2}\;[\,1-\widetilde{\delta}\,] \\
&\quad +\;\frac{\Big[\Delta\,Q_0\,(\,1+\widetilde{\delta}\,)\Big]^2}{k_1\,\widetilde{k}_2}\;[\,1-\widetilde{\delta}\,]-\frac{\Big[\Delta\,Q_0\,(\,1+\widetilde{\delta}\,)\Big]^3}{k_1\,\widetilde{k}_2}\;\frac{\widetilde{\delta}}{2\,\widetilde{k}_2}\,[\,1-(\,\widetilde{\delta}\,)^3\,] \\
&\stackrel{s}{=}\; \frac{\widetilde{\delta}}{2\,k_1\,\widetilde{k}_2}\;\Delta\,Q_0\,[\,1+\widetilde{\delta}\,]\,[\,1-(\,\widetilde{\delta}\,)^2\,]-\frac{1-\widetilde{\delta}}{\widetilde{k}_2} \\
&\quad +\;\frac{\Delta\,Q_0\,[\,1+\widetilde{\delta}\,]\,[\,1-\widetilde{\delta}\,]}{k_1\,\widetilde{k}_2}-\frac{\Big[\Delta\,Q_0\,(\,1+\widetilde{\delta}\,)\Big]^2}{2\,k_1\,(\,\widetilde{k}_2\,)^2}\;\widetilde{\delta}\;[\,1-(\,\widetilde{\delta}\,)^3\,]
\end{aligned}$$

$$= \Delta Q_0 [1+\widetilde{\delta}] \left\{ \frac{\widetilde{\delta}[1-(\widetilde{\delta})^2]}{2k_1 \widetilde{k}_2} + \frac{1-\widetilde{\delta}}{k_1 \widetilde{k}_2} - \frac{\widetilde{\delta}[1-(\widetilde{\delta})^3]}{2k_1(\widetilde{k}_2)^2} \Delta Q_0 [1+\widetilde{\delta}] \right\} - \frac{1-\widetilde{\delta}}{\widetilde{k}_2}$$

$$\stackrel{s}{=} \Delta Q_0 [1+\widetilde{\delta}] \left\{ \widetilde{\delta}[1-(\widetilde{\delta})^2]\widetilde{k}_2 + 2[1-\widetilde{\delta}]\widetilde{k}_2 - \widetilde{\delta}[1-(\widetilde{\delta})^3]\Delta Q_0 [1+\widetilde{\delta}] \right\}$$
$$- 2k_1 \widetilde{k}_2 [1-\widetilde{\delta}]$$

$$\stackrel{s}{=} \Delta Q_0 [1+\widetilde{\delta}] \left\{ [\widetilde{\delta} - (\widetilde{\delta})^3 + 2 - 2\widetilde{\delta}]\widetilde{k}_2 - \widetilde{\delta}[1+\widetilde{\delta}][1-(\widetilde{\delta})^3]\Delta Q_0 \right\}$$
$$- 2k_1 \widetilde{k}_2 [1-\widetilde{\delta}]$$

$$\stackrel{s}{=} \Delta Q_0 [1+\widetilde{\delta}] \left\{ [2 - \widetilde{\delta} - (\widetilde{\delta})^3]\widetilde{k}_2 - \widetilde{\delta}[1+\widetilde{\delta}][1-(\widetilde{\delta})^3]\Delta Q_0 \right\}$$
$$- 2[1-\widetilde{\delta}]k_1 \widetilde{k}_2 . \quad \blacksquare \tag{51}$$

Proposition 3. *Suppose Assumption G with $g = 1$ and Assumption K with $\gamma = 2$ holds. Then $\Phi^S > \Phi^I$ in the NID setting if $\Delta Q_0 > \sqrt{k_1 k_2}$.*

Proof of Proposition 3. When $g = 1$, the first term in G^{NID} (as defined below (26)) is:

$$\Delta Q_0 \left[\left(2+4\delta+\delta^2\right) k_2 - \left(3+3\delta+\delta^2\right)\delta \Delta Q_0 \right]$$
$$> \Delta Q_0 \left[\left(2+4\delta+\delta^2\right) Q(1+\delta)\Delta - \left(3+3\delta+\delta^2\right)\delta \Delta Q_0 \right] \tag{52}$$
$$= [\Delta Q_0]^2 \left[\left(2+4\delta+\delta^2\right)(1+\delta) - \left(3+3\delta+\delta^2\right)\delta \right]$$
$$= [\Delta Q_0]^2 \left[2+4\delta+\delta^2 + \left(2+4\delta+\delta^2\right)\delta - \left(3+3\delta+\delta^2\right)\delta \right]$$
$$= [\Delta Q_0]^2 \left[2+4\delta+\delta^2 + (-1+\delta)\delta \right]$$
$$= [\Delta Q_0]^2 \left[2+4\delta+\delta^2 - \delta + \delta^2 \right] = \Delta^2 (Q_0)^2 \left[2+3\delta+2\delta^2 \right]. \tag{53}$$

The inequality in (52) reflects the maintained assumption that for $t \in \{2,3\}$, $K_2'(1) = k_2 > \Delta[Q_t(c_0) + \delta Q_{t+1}(c_0)] = \Delta[1+\delta]Q_0 \Rightarrow Q_0 < \frac{k_2}{[1+\delta]\Delta}$. (53) and Lemma 3 imply that $\Phi^S > \Phi^I$ if:

$$\Delta^2 (Q_0)^2 \left[2+3\delta+2\delta^2 \right] > 2k_1 k_2 \quad \Leftrightarrow \quad \Leftrightarrow \quad \Delta Q_0 > \sqrt{\frac{2k_1 k_2}{2+3\delta+2\delta^2}} . \tag{54}$$

The inequality in (54) holds if $\Delta Q_0 > \sqrt{k_1 k_2}$ because $2+3\delta+2\delta^2 > 2$.

Let Q denote Q_0 in the ensuing analysis. Then Lemma 3 implies that when $g = 1$, $\Phi^S > \Phi^I$ in the ID setting if:

$$\Lambda \equiv [1+\delta]\,\Delta\,Q\,\big\{\,\big[\,2-\delta-\delta^3\,\big]\,k_2 - \Delta\,\delta\,[1+\delta]\,\big[\,1-\delta^3\,\big]\,Q\,\big\}$$

$$- 2\,[1-\delta]\,k_1\,k_2 \;>\; 0\,. \tag{55}$$

The maintained assumption that $K_2'(1) > \max\{\,\Delta\,[\,Q_2(c_0)+\delta\,Q_3(c_0)\,]\,,\,\Delta\,[\,Q_3(c_0)+\delta\,Q_4(c_0)\,]\,\}$ implies that $Q \leq \frac{k_2}{[1+\delta]\,\Delta}$ in the present setting, which, in turn, implies:

$$\Lambda \geq [1+\delta]\,\Delta\,Q\,\big\{\,\big[\,2-\delta-\delta^3\,\big]\,[1+\delta]\,\Delta\,Q - \Delta\,\delta\,[1+\delta]\,\big[\,1-\delta^3\,\big]\,Q\,\big\} - 2\,[1-\delta]\,k_1\,k_2$$

$$= [1+\delta]^2\,[\,\Delta\,Q\,]^2\,\big[\,2-\delta-\delta^3-\delta\left(1-\delta^3\right)\big] - 2\,[1-\delta]\,k_1\,k_2$$

$$= [1+\delta]^2\,[\,\Delta\,Q\,]^2\,\big[\,2-\delta-\delta^3-\delta+\delta^4\,\big] - 2\,[1-\delta]\,k_1\,k_2$$

$$= [1+\delta]^2\,[\,\Delta\,Q\,]^2\,\big[\,2\,(1-\delta)-\delta^3\,(1-\delta)\big] - 2\,[1-\delta]\,k_1\,k_2$$

$$= [1+\delta]^2\,[1-\delta]\,\big[\,2-\delta^3\,\big]\,[\,\Delta\,Q\,]^2 - 2\,[1-\delta]\,k_1\,k_2$$

$$> 0 \;\text{ if }\; [1+\delta]^2\,[1-\delta]\,\big[\,2-\delta^3\,\big]\,[\,\Delta\,Q\,]^2 \;>\; 2\,[1-\delta]\,k_1\,k_2$$

$$\Leftrightarrow\quad [1+\delta]^2\,\big[\,2-\delta^3\,\big]\,[\,\Delta\,Q\,]^2 \;>\; 2\,k_1\,k_2$$

$$\Leftrightarrow\quad Q^2 \;>\; \frac{2\,k_1\,k_2}{\Delta^2\,[1+\delta]^2\,\big[\,2-\delta^3\,\big]} \quad\Leftrightarrow\quad \Delta\,Q \;>\; \sqrt{\frac{2\,k_1\,k_2}{[1+\delta]^2\,\big[\,2-\delta^3\,\big]}}\,. \tag{56}$$

Define $g(\delta) \equiv [1+\delta]^2\,\big[\,2-\delta^3\,\big]$. The conclusion in the Proposition follows from (55) and (56) if $g(\delta) \geq 2$ for all $\delta \in (0,1)$. Observe that:

$$g(0) = 2\,;\;\; g(1) = 4\,;\;\text{ and}$$

$$g'(\delta) = -3\,\delta^2\,[1+\delta]^2 + 2\,\big[\,2-\delta^3\,\big]\,[1+\delta]$$

$$= [1+\delta]\,\big\{\,2\,\big[\,2-\delta^3\,\big] - 3\,\delta^2\,[1+\delta]\,\big\} \;=\; [1+\delta]\,\big[\,4-3\,\delta^2-5\,\delta^3\,\big]$$

$$\Rightarrow\quad g'(0) = 4 \;\text{ and }\; g'(1) = -8\,. \tag{57}$$

(57) implies:

$$g''(\delta) = -\,[1+\delta]\,\big[\,6\,\delta+15\,\delta^2\,\big] + 4 - 3\,\delta^2 - 5\,\delta^3$$

$$= 4-3\,\delta^2-5\,\delta^3-6\,\delta-15\,\delta^2-6\,\delta^2-15\,\delta^3 \;=\; 4-6\,\delta-24\,\delta^2-20\,\delta^3\,. \tag{58}$$

(57) and (58) imply that: (i) $g(0) = 2 < 4 = g(1)$; (ii) $g(\delta)$ is increasing for small δ; and (ii) $g(\delta)$ is decreasing for large δ. Consequently, $g(\delta) \geq 2$ for all $\delta \in (0,1)$. ■

Proposition 4. *Suppose Assumption G with $g = 1$ and Assumption K with $\gamma = 2$ holds. Then $\Phi^S > \Phi^I$ in the ID setting if $g\,\big[\,1+g\,\delta^2\,\big] < 1$ and $\Delta\,Q_0\,[\,1+g\,\delta\,]$ is sufficiently close to $k_1 = k_2$.*

Proof of Proposition 4. Lemma 3 implies that $\Phi^S > \Phi^I$ in the ID setting when $k_1 = k_2 = \Delta\, Q_0\,[\,1 + g\,\delta\,]$ if:

$$[\,1+\delta\,g\,]\,\Delta\,Q_0 \left\{ \left[\,2 - \delta\,g - (\delta\,g)^3\,\right] \frac{k_2}{g} - \Delta\;g\,\delta\,[\,1+g\,\delta\,]\left[\,1-(g\,\delta)^3\,\right] Q_0 \right\}$$

$$> \; 2\,[\,1-g\,\delta\,]\,k_1\,\frac{k_2}{g} \tag{59}$$

$$\Leftrightarrow \;\; k_1 \left\{ \left[\,2-\delta\,g-(\delta\,g)^3\,\right] \frac{k_1}{g} - k_1\,g\,\delta\,\left[\,1-(g\,\delta)^3\,\right] \right\} \; > \; 2\,\frac{1}{g}\,[\,1-g\,\delta\,]\,[\,k_1\,]^2 \tag{60}$$

$$\Leftrightarrow \;\; \left[\,2-\delta\,g-(\delta\,g)^3\,\right] \frac{1}{g}\,[\,k_1\,]^2 - [\,k_1\,]^2\;g\,\delta\left[\,1-(g\,\delta)^3\,\right] \; > \; 2\,\frac{1}{g}\,[\,1-g\,\delta\,]\,[\,k_1\,]^2$$

$$\Leftrightarrow \;\; \frac{1}{g}\left[\,2-\delta\,g-(\delta\,g)^3 - 2\,(1-g\,\delta)\,\right][\,k_1\,]^2 - [\,k_1\,]^2\;g\,\delta\left[\,1-(g\,\delta)^3\,\right] \; > \; 0$$

$$\Leftrightarrow \;\; \frac{1}{g}\left[\,2-\delta\,g-(\delta\,g)^3 - 2\,(1-g\,\delta)\,\right] - g\,\delta\left[\,1-(g\,\delta)^3\,\right] \; > \; 0$$

$$\Leftrightarrow \;\; \frac{1}{g}\;\left[\,2-\delta\,g-(\delta\,g)^3 - 2 + 2\,g\,\delta\,\right] - g\,\delta\left[\,1-(g\,\delta)^3\,\right] \; > \; 0$$

$$\Leftrightarrow \;\; \frac{1}{g}\;\left[\,\delta\,g-(\delta\,g)^3\,\right] - \;g\,\delta\left[\,1-(g\,\delta)^3\,\right] \; > \; 0$$

$$\Leftrightarrow \;\; \delta - \delta^3\,g^2 - g\,\delta + (g\,\delta)^4 \;\; > \;0 \quad \Leftrightarrow \quad 1 - \delta^2\,g^2 - g + g^4\,\delta^3 \;\; > \; 0\,. \tag{61}$$

(60) reflects the assumption that $k_1 = k_2 = \Delta\,Q_0\,[\,1+g\,\delta\,]$. The last inequality in (61) holds because $1-\delta^2\,g^2 - g = 1 - g\left[\,1+g\,\delta^2\,\right] > 0$, by assumption.

The inequality in (59) will continue to hold when $k_1 = k_2$ is increased marginally to ensure that $\Delta\,Q_0\,[\,1+g\,\delta\,] < \min\{\,k_1,\,k_2\,\}$. ■

Proposition 5. *$E_d\{W^S\} > E_d\{W^I\}$ in the NID setting if $\Phi^S > \Phi^I$.*

Proof of Proposition 5. Under the specified conditions, when the firm operates under SR in the NID setting: (i) $p_1 = p_2 = c_0$; (ii) $p_3 = p_4 = p_5 = p_6 = c_0$ if the firm never achieves a cost reduction; and (iii) $p_3 = p_4 = p_5 = p_6 = c_0 - \Delta$ if the firm ever achieves a cost reduction. Therefore, the PDV of expected consumer surplus under SR in this setting is:

$$\begin{aligned} E_d\{W^S\} \;&=\; W_1(c_0) + \delta\,W_2(c_0) \\ &\quad + \;\Phi^S\left[\,\delta^2\,W_3(c_0-\Delta) + \delta^3\,W_4(c_0-\Delta) + \delta^4\,W_5(c_0-\Delta) + \delta^5\,W_6(c_0-\Delta)\,\right] \\ &\quad + \;\left[\,1-\Phi^S\,\right]\left[\,\delta^2\,W_3(c_0) + \delta^3\,W_4(c_0) + \delta^4\,W_5(c_0) + \delta^5\,W_6(c_0)\,\right] \\ &=\; W_1(c_0) + \delta\,W_2(c_0) + \delta^2\,W_3(c_0) + \delta^3\,W_4(c_0) + \delta^4\,W_5(c_0) + \delta^5\,W_6(c_0) \end{aligned}$$

$$+ \;\Phi^S \,\delta^2 \,[\, W_3(c_0 - \Delta) - W_3(c_0)\,] + \Phi^S \,\delta^3 \,[\, W_4(c_0 - \Delta) - W_4(c_0)\,]$$

$$+ \;\Phi^S \,\delta^4 \,[\, W_5(c_0 - \Delta) - W_5(c_0)\,] + \Phi^S \,\delta^5 \,[\, W_6(c_0 - \Delta) - W_6(c_0)\,]\,. \qquad (62)$$

Under the specified conditions, when the firm operates under IRIS: (i) $p_1 = p_2 = c_0$; (ii) $p_5 = p_6 = c_0$ if the firm never achieves success; (iii) $p_5 = p_6 = c_0 - \Delta$ if the firm ever achieves success; (iv) $p_3 = c_0 - \Delta$ if the firm achieves success in period 1; (v) $p_3 = c_0$ if the firm does not achieve success in period 1; (vi) $p_4 = c_0 - \Delta$ if the firm achieves success (in period 1 or period 2); and (vii) $p_4 = c_0$ if the firm does not achieve success. Therefore, the PDV of expected consumer surplus under IRIS in this setting is:

$$\begin{aligned} E_d\{W^I\} \;&=\; W_1(c_0) + \delta\, W_2(c_0) + \phi_1^I\, \delta^2\, W_3(c_0 - \Delta) + \left[\, 1 - \phi_1^I \,\right] \delta^2\, W_3(c_0) \\ &+\; \Phi^I\, \delta^3\, W_4(c_0 - \Delta) + \delta^3 \left[\, 1 - \Phi^I \,\right] W_4(c_0) + \;\Phi^I \delta^4\, W_5(c_0 - \Delta) \\ &+\; \delta^4 \left[\, 1 - \Phi^I \,\right]\; W_5(c_0) + \Phi^I \delta^5\, W_6(c_0 - \Delta) + \delta^5 \left[\, 1 - \Phi^I \,\right]\; W_6(c_0) \\ &=\; W_1(c_0) + \delta\, W_2(c_0) + \delta^2\, W_3(c_0) + \delta^3\, W_4(c_0) + \delta^4\, W_5(c_0) + \delta^5\, W_6(c_0) \\ &+\; \phi_1^I\, \delta^2\, [\, W_3(c_0 - \Delta) - W_3(c_0)\,] + \delta^3\, \Phi^I\, [\, W_4(c_0 - \Delta) - W_4(c_0)\,] \\ &+\; \delta^4\, \Phi^I\, [\, W_5(c_0 - \Delta) - W_5(c_0)\,] + \delta^5\, \Phi^I\, [\, W_6(c_0 - \Delta) - W_6(c_0)\,]\,. \qquad (63) \end{aligned}$$

(62) and (63) imply:

$$\begin{aligned} E_d\{W^S\} - E_d\{W^I\} \;&=\; \left[\, \Phi^S - \phi_1^I \,\right] \delta^2\, [\, W_3(c_0 - \Delta) - W_3(c_0)\,] \\ &+\; \left[\, \Phi^S - \Phi^I \,\right]\; \{\, \delta^3\, [\, W_4(c_0 - \Delta) - W_4(c_0)\,] + \delta^4\, [\, W_5(c_0 - \Delta) - W_5(c_0)\,] \\ &+\; \delta^5\, [\, W_6(c_0 - \Delta) - W_6(c_0)\,]\,\}. \qquad (64) \end{aligned}$$

If $\Phi^S > \Phi^I$, then $\Phi^S > \phi_1^I$. Consequently, (64) implies that $E_d\{W^S\} > E_d\{W^I\}$ when $\Phi^S > \Phi^I$ (because $\delta > 0$, by assumption). ■

Proposition 6. *Suppose Assumptions G and K hold. Then* $E_d\{W^S\} > E_d\{W^I\}$ *in the NID setting if* $\widetilde{\delta}\,[1 + \widetilde{\delta}\,]^{\frac{1}{\gamma-1}} < 1$.[3]

<u>Proof of Proposition 6</u>. (6) and (7) imply that under the specified conditions:

$$k_2 \left(\phi_2^S\right)^{\gamma-1} \;=\; \Delta\; g\, Q_0 \;\;\Rightarrow\;\; \phi_2^S \;=\; \left[\frac{\Delta\, g\, Q_0}{k_2}\right]^{\frac{1}{\gamma - 1}} \;=\; \left[\frac{\Delta\, Q_0}{\widetilde{k}_2}\right]^{\frac{1}{\gamma - 1}} \quad \text{and}$$

$$k_2 \left(\phi_2^I\right)^{\gamma-1} \;=\; \Delta \left[\, g\, Q_0 + g^2\, \delta\, Q_0 \,\right]$$

[3]Recall that $\widetilde{\delta} \equiv g\,\delta$.

$$\Rightarrow \quad \phi_2^I \; = \; \left[\frac{\Delta\, Q_0\, g \left(1 + \widetilde{\delta}\right)}{k_2} \right]^{\frac{1}{\gamma - 1}} \; = \; \left[\frac{\Delta\, Q_0 \left(1 + \widetilde{\delta}\right)}{\widetilde{k}_2} \right]^{\frac{1}{\gamma - 1}}$$

$$\Rightarrow \quad \phi_2^I \; = \; [\, 1 + \widetilde{\delta}\,]^{\frac{1}{\gamma - 1}} \left[\frac{\Delta\, Q_0}{\widetilde{k}_2} \right]^{\frac{1}{\gamma - 1}} \; = \; \phi_2^S\, [\, 1 + \widetilde{\delta}\,]^{\frac{1}{\gamma - 1}}. \tag{65}$$

When Assumption G holds, $W_t(p) \; = \; g\, W_t(p)$. Therefore, (64) implies:

$$E_d\{W^S\} \; > \; E_d\{W^I\} \;\text{ if }\; \Phi^S - \phi_1^I + \left[\, \Phi^S - \Phi^I \,\right] [\, \widetilde{\delta} + \widetilde{\delta}^2 + \widetilde{\delta}^3 \,] \; > \; 0\,. \tag{66}$$

First suppose that $\Phi^S \geq \Phi^I$. Proposition 2 implies that $\Phi^S > \phi_1^I$. Therefore, (66) implies that $E_d\{W^S\} > \; E_d\{W^I\}$ when $\Phi^S \geq \; \Phi^I$.

Now suppose that $\Phi^S < \; \Phi^I$. (66) holds in this case if:

$$\Phi^S - \phi_1^I + \left[\, \Phi^S - \Phi^I \,\right] [\, \widetilde{\delta} + \widetilde{\delta}^2 + \widetilde{\delta}^3 + \widetilde{\delta}^4 + \ldots \,] \; > \; 0$$

$$\Leftrightarrow \quad \Phi^S - \phi_1^I + \left[\, \Phi^S - \Phi^I \,\right] \frac{\widetilde{\delta}}{1 - \widetilde{\delta}} \; > \; 0$$

$$\Leftrightarrow \quad \left[\, \Phi^S - \phi_1^I \,\right] \left[\, 1 - \widetilde{\delta}\,\right] + \widetilde{\delta} \left[\, \Phi^S - \Phi^I \,\right] \; > \; 0$$

$$\Leftrightarrow \quad \left[\, 1 - \widetilde{\delta}\,\right] \left[\, \phi_1^S + \phi_2^S \left(1 - \phi_1^S\right) - \phi_1^I \right]$$

$$+ \; \widetilde{\delta} \left[\, \phi_1^S + \phi_2^S \left(1 - \phi_1^S\right) - \left(\phi_1^I + \phi_2^I \left[\, 1 - \phi_1^I \,\right]\right) \right] \; > \; 0$$

$$\Leftrightarrow \quad \phi_1^S + \phi_2^S \left[\, 1 - \phi_1^S \,\right] - \phi_1^I - \widetilde{\delta} \left[\, \phi_1^S + \phi_2^S \left(1 - \phi_1^S\right) - \phi_1^I \,\right]$$

$$+ \; \widetilde{\delta} \left[\, \phi_1^S + \phi_2^S \left(1 - \phi_1^S\right) - \left(\phi_1^I + \phi_2^I \left[\, 1 - \phi_1^I \,\right]\right) \right] \; > \; 0$$

$$\Leftrightarrow \quad \phi_1^S + \phi_2^S \left[\, 1 - \phi_1^S \,\right] - \phi_1^I - \widetilde{\delta}\, \phi_2^I \left[\, 1 - \phi_1^I \,\right] \; > \; 0$$

$$\Leftrightarrow \quad - \left[\, 1 - \phi_1^S \,\right] + \phi_2^S \left[\, 1 - \phi_1^S \,\right] + 1 - \phi_1^I - \widetilde{\delta}\, \phi_2^I \left[\, 1 - \phi_1^I \,\right] \; > \; 0$$

$$\Leftrightarrow \quad \left[\, 1 - \phi_1^I \,\right] \left[\, 1 - \widetilde{\delta}\, \phi_2^I \,\right] - \left[\, 1 - \phi_1^S \,\right] \left[\, 1 - \phi_2^S \,\right] \; > \; 0\,. \tag{67}$$

Proposition 2 implies:

$$1 - \phi_1^I \; > \; 1 - \phi_1^S\,. \tag{68}$$

(65) implies that when $\widetilde{\delta}\, [\, 1 + \widetilde{\delta}\,]^{\frac{1}{\gamma - 1}} < 1$:

$$\phi_2^S - \widetilde{\delta}\, \phi_2^I \; = \; \phi_2^S - \widetilde{\delta}\, \phi_2^S\, [\, 1 + \widetilde{\delta}\,]^{\frac{1}{\gamma - 1}} \; = \; \phi_2^S \left[1 - \widetilde{\delta} \left(1 + \widetilde{\delta}\right)^{\frac{1}{\gamma - 1}} \right] \; > \; 0$$

$$\Rightarrow \quad \phi_2^S \; > \; \widetilde{\delta}\, \phi_2^I \quad \Rightarrow \quad 1 - \widetilde{\delta}\, \phi_2^I \; > \; 1 - \phi_2^S\,. \tag{69}$$

(68) and (69) imply that the inequality in (67) holds. ■

Corollary to Proposition 6. *Suppose Assumptions G and K hold. Then* $E_d\{W^S\} > E_d\{W^I\}$ *in the NID setting if* $\gamma \geq 2$.

Proof of the Corollary to Proposition 6.

The Corollary follows directly from Proposition 6 because $\widetilde{\delta}\,[\,1+\widetilde{\delta}\,]^{\frac{1}{\gamma-1}} < 1$ under the specified conditions. This is the case because:

$$\widetilde{\delta}\,[\,1+\widetilde{\delta}\,]^{\frac{1}{\gamma-1}} \;\leq\; \widetilde{\delta}\,[\,1+\widetilde{\delta}\,] \;<\; 1\,.$$

The first inequality here holds because $\gamma \geq 2$, by assumption. The last inequality here holds because $\widetilde{\delta} = g\,\delta < \widehat{\widehat{\delta}}$ in the NID setting and because $\widehat{\widehat{\delta}}\,[\,1+\widehat{\widehat{\delta}}\,] = 1$, by definition. (Recall the proof of the Corollary to Lemma 2.) ■

Proposition 7. *Suppose Assumption K holds, Assumption G with* $g = 1$ *holds, and* $\delta > \widehat{\widehat{\delta}}$. *Then* $E_d\{W^I\} > E_d\{W^S\}$ *when* δ *is sufficiently large (in the ID setting).*

Proof of Proposition 7. Lemma 2 implies that when the firm operates under SR in the ID setting: (i) $p_1 = p_2 = c_0$; (ii) $p_5 = p_6 = c_0$ if the firm never achieves success;[4] (iii) $p_5 = p_6 = c_0 - \Delta$ if the firm ever achieves success; (iv) $p_3 = p_4 = c_0 - \Delta$ if the firm achieves success in period 1; and (v) $p_3 = p_4 = c_0$ if the firm does not achieve success in period 1. Therefore, expected consumer surplus under SR in this setting is:

$$\begin{aligned}
E_d\{W^S\} \;=\;& W_1(c_0) + \delta\,W_2(c_0) + \phi_1^S\left[\,\delta^2\,W_3(c_0-\Delta) + \delta^3\,W_4(c_0-\Delta)\,\right] \\
&+ \left[\,1-\phi_1^S\,\right]\left[\,\delta^2\,W_3(c_0) + \delta^3\,W_4(c_0)\,\right] + \left[\,1-\Phi^S\right]\left[\,\delta^4\,W_5(c_0) + \delta^5\,W_6(c_0)\,\right] \\
&+ \Phi^S\left[\,\delta^4\,W_5(c_0-\Delta) + \delta^5\,W_6(c_0-\Delta)\,\right] \\
=\;& W_1(c_0) + \delta\,W_2(c_0) + \delta^2\,W_3(c_0) + \delta^3\,W_4(c_0) + \delta^4\,W_5(c_0) + \delta^5\,W_6(c_0) \\
&+ \delta^2\,\phi_1^S\,[\,W_3(c_0-\Delta) - W_3(c_0)\,] + \delta^3\,\phi_1^S\,[\,W_4(c_0-\Delta) - W_4(c_0)\,] \\
&+ \delta^4\,\Phi^S\,[\,W_5(c_0-\Delta) - W_5(c_0)\,] + \delta^5\,\Phi^S\,[\,W_6(c_0-\Delta) - W_6(c_0)\,]\,. \qquad (70)
\end{aligned}$$

(63) and (70) imply that in the ID setting:

$$\begin{aligned}
E_d\{W^S\} - E_d\{W^I\} \;=\;& \delta^2\left[\,\phi_1^S - \phi_1^I\,\right][\,W_3(c_0-\Delta) - W_3(c_0)\,] \\
&+ \delta^3\left[\,\phi_1^S - \Phi^I\,\right][\,W_4(c_0-\Delta) - W_4(c_0)\,]
\end{aligned}$$

[4] The firm achieves "success" when it achieves the Δ cost reduction.

$$+ \ \delta^4 \left[\Phi^S - \Phi^I \right] \left[W_5(c_0 - \Delta) - W_5(c_0) \right]$$

$$+ \ \delta^5 \left[\Phi^S - \Phi^I \right] \left[W_6(c_0 - \Delta) - W_6(c_0) \right]$$

$$\overset{s}{=} \ \left[\phi_1^S - \phi_1^I \right] \left[W_3(c_0 - \Delta) - W_3(c_0) \right] + \delta \left[\phi_1^S - \Phi^I \right] \left[W_4(c_0 - \Delta) - W_4(c_0) \right]$$

$$+ \ \delta^2 \left[\Phi^S - \Phi^I \right] \left[W_5(c_0 - \Delta) - W_5(c_0) \right] + \delta^3 \left[\Phi^S - \Phi^I \right] \left[W_6(c_0 - \Delta) - W_6(c_0) \right]. \quad (71)$$

Define $Q_0 \equiv Q(c_0)$. (6) implies that under the specified conditions:

$$k_2 \left(\phi_2^I \right)^{\gamma - 1} = \Delta \left[g \, Q_0 + g^2 \, \delta \, Q_0 \right] \ \Rightarrow \ \phi_2^I = \left[\frac{\Delta \, Q_0 \, g \, (1 + g \, \delta)}{k_2} \right]^{\frac{1}{\gamma - 1}} . \quad (72)$$

(7) implies that under the maintained conditions:

$$k_2 \left(\phi_2^S \right)^{\gamma - 1} = \Delta \, \delta \left[g^2 \, Q_0 + g^3 \, \delta \, Q_0 \right] \ \Rightarrow \ \phi_2^S = \left[\frac{\Delta \, Q_0 \, \delta \, g^2 \, (1 + g \, \delta)}{k_2} \right]^{\frac{1}{\gamma - 1}}$$

$$\Rightarrow \ \phi_2^S = (\delta \, g)^{\frac{1}{\gamma - 1}} \left[\frac{\Delta \, Q_0 \, g \, (1 + g \, \delta)}{k_2} \right]^{\frac{1}{\gamma - 1}} = \phi_2^I \, (\delta \, g)^{\frac{1}{\gamma - 1}} . \quad (73)$$

Define $\phi_2^{Lim} \equiv \left(\frac{2 \, \Delta \, Q_0}{k_2} \right)^{\frac{1}{\gamma - 1}}$. Then (72) and (73) imply that under the specified conditions:

$$\phi_2^I = \left(\frac{\Delta \, Q_0 \, \left[1 + \delta \right]}{k_2} \right)^{\frac{1}{\gamma - 1}} \ \rightarrow \ \phi_2^{Lim} \ \text{ as } \ \delta \ \rightarrow \ 1 \quad \text{and}$$

$$\phi_2^S = \left(\frac{\Delta \, Q_0 \, \delta \left[1 + \delta \right]}{k_2} \right)^{\frac{1}{\gamma - 1}} \ \rightarrow \ \phi_2^{Lim} \ \text{ as } \ \delta \ \rightarrow \ 1$$

$$\Rightarrow \ \lim_{\delta \rightarrow 1} \left(\phi_2^I - \phi_2^S \right) = 0 . \quad (74)$$

Define $\phi_1^{Lim} \equiv \left(\frac{2 \, \Delta \, Q_0 - \left[2 \, \phi_2^{Lim} \, \Delta \, Q_0 \, - K_2(\phi_2^{Lim}) \right]}{k_1} \right)^{\frac{1}{\gamma - 1}}$. (9), (11), and (74) imply that under the specified conditions:

$$\phi_1^S = \left(\frac{\Delta \, Q_0 \left[1 + \delta \right] - \delta \left[\phi_2^S \, \Delta \, Q_0 \, \delta \ (1 + \delta \,) - K_2(\phi_2^S) \right]}{k_1} \right)^{\frac{1}{\gamma - 1}}$$

$$\rightarrow \left(\frac{2 \, \Delta \, Q_0 - \left[2 \, \phi_2^{Lim} \, \Delta \, Q_0 \ - K_2(\phi_2^{Lim}) \right]}{k_1} \right)^{\frac{1}{\gamma - 1}} = \phi_1^{Lim} \ \text{ as } \ \delta \ \rightarrow \ 1 \, ;$$

$$\phi_1^I = \left(\frac{\Delta \, Q_0 \left[1 + \delta \right] - \delta \left[\phi_2^I \, \Delta \, Q_0 \ (1 + \delta \,) - K_2(\phi_2^I) \right]}{k_1} \right)^{\frac{1}{\gamma - 1}}$$

$$\rightarrow \left(\frac{2\,\Delta\,Q_0 - \left[\,2\,\phi_2^{Lim}\,\Delta\,Q_0 - K_2(\phi_2^{Lim})\,\right]}{k_1} \right)^{\frac{1}{\gamma-1}} \;=\; \phi_1^{Lim} \;\text{ as }\; \delta \,\rightarrow\, 1$$

$$\Rightarrow \quad \lim_{\delta \to 1} \left(\phi_1^I - \phi_1^S \right) \;=\; 0\,. \tag{75}$$

(74) and (75) imply:

$$\lim_{\delta \to 1} \left(\Phi^I - \Phi^S \right) \;=\; 0\,. \tag{76}$$

(71), (75), and (76) imply:

$$\begin{aligned}
\lim_{\delta \to 1} \left(E_d\{W^S\} - E_d\{W^I\} \right) \;&=\; \lim_{\delta \to 1} \delta^2 \left[\,\phi_1^S - \phi_1^I\,\right] [\,W_3(c_0 - \Delta) - W_3(c_0)\,] \\
&\quad +\; \lim_{\delta \to 1} \delta^3 \left[\,\phi_1^S - \Phi^I\,\right] [\,W_4(c_0 - \Delta) - W_4(c_0)\,] \\
&\quad +\; \lim_{\delta \to 1} \delta^4 \left[\,\Phi^S - \Phi^I\,\right] [\,W_5(c_0 - \Delta) - W_5(c_0)\,] \\
&\quad +\; \lim_{\delta \to 1} \delta^5 \left[\,\Phi^S - \Phi^I\,\right] [\,W_6(c_0 - \Delta) - W_6(c_0)\,] \\
&=\; \lim_{\delta \to 1} \delta^3 \left[\,\phi_1^S - \Phi^I\,\right] [\,W_4(c_0 - \Delta) - W_4(c_0)\,] \\
&<\; \lim_{\delta \to 1} \delta^3 \left[\,\Phi^S - \Phi^I\,\right] [\,W_4(c_0 - \Delta) - W_4(c_0)\,] \;=\; 0\,.
\end{aligned}$$

The inequality here holds because, from (25), $\Phi^S \;=\; \phi_1^S + \phi_2^S \left[\,1 - \phi_1^I\,\right] > \phi_1^S$. ■

Proposition 8. *Suppose Assumption K holds, Assumption G holds, and* $\widetilde{\delta} \geq \widehat{\widehat{\delta}}$. *Then* $E_d\{W^I\} > E_d\{W^S\}$ *for sufficiently large* k_1 *(in the ID setting).*

Proof of Proposition 8. (75) implies that under the specified conditions:

$$\lim_{k_1 \to \infty} \phi_1^S \;=\; 0 \;\text{ and }\; \lim_{k_1 \to \infty} \phi_1^I \;=\; 0\,. \tag{77}$$

(71) implies that under the specified conditions:

$$\begin{aligned}
E_d\{W^S\} - E_d\{W^I\} \;&=\; \delta^2 \left[\,\phi_1^S - \phi_1^I\,\right] D_{W3}(c_0, \Delta) + \delta^3 \left[\,\phi_1^S - \Phi^I\,\right] D_{W4}(c_0, \Delta) \\
&\quad +\; \delta^4 \left[\,\Phi^S - \Phi^I\,\right] D_{W5}(c_0, \Delta) + \delta^5 \left[\,\Phi^S - \Phi^I\,\right] D_{W6}(c_0, \Delta)
\end{aligned} \tag{78}$$

where $D_{Wt}(c_0, \Delta) \;=\; W_t(c_0 - \Delta) - W_t(c_0) \;=\; g^{t-1}\, D_{W1}(c_0, \Delta) \;>\; 0$ for $t \in \{1, ..., 6\}$. (78) implies:

$$E_d\{W^S\} - E_d\{W^I\} \;=\; A(k_1)\, D_{W1}(c_0, \Delta)$$

$$\text{where }\; A(k_1) \;\equiv\; \widetilde{\delta}^2 \left[\,\phi_1^S - \phi_1^I\,\right] + \widetilde{\delta}^3 \left[\,\phi_1^S - \Phi^I\,\right] + \widetilde{\delta}^4 \left[\,\Phi^S - \Phi^I\,\right] + \widetilde{\delta}^5 \left[\,\Phi^S - \Phi^I\,\right]. \tag{79}$$

$\frac{\partial D_{W1}(c_0,\Delta)}{\partial k_1} = 0$. Therefore, (79) implies:

$$\lim_{k_1 \to \infty} \left(E_d\{W^I\} - E_d\{W^S\} \right) \; > \; 0 \;\; \text{if} \;\; \lim_{k_1 \to \infty} A(k_1) \; < \; 0\,.$$

(79) implies that $\lim_{k_1 \to \infty} A(k_1) < 0$ if: (i) $\lim_{k_1 \to \infty} \left(\phi_1^S - \phi_1^I \right) = 0$; (ii) $\lim_{k_1 \to \infty} \left(\phi_1^S - \Phi^I \right) < 0$; and (iii) $\lim_{k_1 \to \infty} \left(\Phi^S - \Phi^I \right) < 0$. We complete the proof by showing that (i), (ii), and (iii) hold.

(77) implies that $\lim_{k_1 \to \infty} \left(\phi_1^S - \phi_1^I \right) = 0$.

(25) and (77) imply:

$$\lim_{k_1 \to \infty} \left(\phi_1^S - \Phi^I \right) \; = \; \lim_{k_1 \to \infty} \left(\phi_1^S - \phi_1^I - \left[1 - \phi_1^I \right] \phi_2^I \right) \; = \; - \lim_{k_1 \to \infty} \phi_2^I \; = \; -\phi_2^I \; < \; 0\,.$$

(25) and (77) also imply:

$$\lim_{k_1 \to \infty} \left(\Phi^S - \Phi^I \right) \; = \; \lim_{k_1 \to \infty} \left(\phi_1^S + \left[1 - \phi_1^S \right] \phi_2^S - \phi_1^I - \left[1 - \phi_1^I \right] \phi_2^I \right)$$

$$= \; \lim_{k_1 \to \infty} \left(\phi_2^S - \phi_2^I \right) \; = \; \phi_2^S - \phi_2^I \; < \; 0\,.$$

The inequality here reflects Proposition 1. ■

Proposition 9. *Suppose Assumption K with $\gamma = 2$ holds and Assumption G holds. Then $E_d\{W^I\} > E_d\{W^S\}$ when $\Delta\, Q_0$ is sufficiently small or $k_1 = k_2 \equiv k$ is sufficiently large in the ID setting.*

Proof of Proposition 9. Define $\widetilde{k}_2 \equiv \frac{k_2}{g}$, $x \equiv \frac{\Delta\, Q_0}{\widetilde{k}_2} = \frac{\Delta\, g\, Q_0}{k_2}$, and $\widetilde{\delta} \equiv g\, \delta$. (40) and (44) imply that under the maintained assumptions:

$$\phi_2^S \; = \; \frac{\Delta\, Q_0\, g^2\, \delta\, [\, 1 + g\, \delta\,]}{k_2} \; = \; x\, g\, \delta\, [\, 1 + g\, \delta\,] \; = \; x\, \widetilde{\delta}\, [\, 1 + \widetilde{\delta}\,] \;\; \text{and} \tag{80}$$

$$\phi_2^I \; = \; \frac{\Delta\, g\, Q_0\, [\, 1 + g\, \delta\,]}{k_2} \; = \; x\, [\, 1 + \delta\, g\,] \; = \; x\, [\, 1 + \widetilde{\delta}\,]. \tag{81}$$

(46) and (81) imply:

$$\phi_1^I \; = \; \frac{\Delta\, [\, 1 + g\, \delta\,]\, Q_0 - \delta \left[\, g\, (1 + g\, \delta)\, \Delta\, Q_0\, \phi_2^I - K_2(\phi_2^I)\, \right]}{k_1}$$

$$= \; \frac{\Delta\, [\, 1 + g\, \delta\,]\, Q_0 - \delta \left[\, k_2 \left(\phi_2^I \right)^2 - \frac{k_2}{2} \left(\phi_2^I \right)^2 \right]}{k_1} \; = \; \frac{\Delta\, [\, 1 + g\, \delta\,]\, Q_0 - \delta \left[\frac{k_2 \left(\phi_2^I \right)^2}{2} \right]}{k_1}$$

$$= \frac{\Delta\,[\,1 + g\,\delta\,]\,Q_0 - \delta\,g\left[\frac{\widetilde{k}_2\left(\phi_2^I\right)^2}{2}\right]}{k_1} \;=\; \frac{\Delta\,[\,1 + \widetilde{\delta}\,]\,Q_0}{k_1} - \widetilde{\delta}\,\frac{\widetilde{k}_2}{k_1}\left[\frac{\left(\phi_2^I\right)^2}{2}\right]$$

$$= \frac{\Delta\,[\,1 + \widetilde{\delta}\,]\,Q_0}{\widetilde{k}_2}\,\frac{\widetilde{k}_2}{k_1} - \widetilde{\delta}\,\frac{\widetilde{k}_2}{k_1}\left[\frac{\left(\phi_2^I\right)^2}{2}\right]. \tag{82}$$

(81) and (82) imply:

$$\phi_1^I \;=\; \phi_2^I\,\frac{\widetilde{k}_2}{k_1} - \widetilde{\delta}\,\frac{\widetilde{k}_2}{k_1}\,\frac{\left(\phi_2^I\right)^2}{2} \;=\; \frac{\widetilde{k}_2}{k_1}\,\phi_2^I\left[1 - \frac{\widetilde{\delta}}{2}\,\phi_2^I\right]$$

$$= x\,\frac{\widetilde{k}_2}{k_1}\,[\,1 + \widetilde{\delta}\,]\left[1 - \frac{\widetilde{\delta}}{2}\,x\left(1 + \widetilde{\delta}\right)\right]. \tag{83}$$

(41) – (43) imply:

$$\phi_1^S \;=\; \frac{\Delta\,[\,1 + g\,\delta\,]\,Q_0 - \delta\left[\,g^2\,\delta\,(1 + g\,\delta)\,\Delta\,Q_0\,\phi_2^S - K_2(\phi_2^S)\,\right]}{k_1}$$

$$= \frac{\Delta\,[\,1 + \widetilde{\delta}\,]\,Q_0 - \delta\left[k_2\left(\phi_2^S\right)^2 - \frac{k_2}{2}\left(\phi_2^S\right)^2\right]}{k_1}$$

$$= \frac{\Delta\,[\,1 + \widetilde{\delta}\,]\,Q_0 - \delta\left[\,\frac{k_2}{2}\,(\phi_2^S)^2\,\right]}{k_1} \;=\; \frac{\Delta\,[\,1 + \widetilde{\delta}\,]\,Q_0 - \delta\,g\left[\frac{\widetilde{k}_2}{2}\left(\phi_2^S\right)^2\right]}{k_1}$$

$$= \frac{\Delta\,[\,1 + \widetilde{\delta}\,]\,Q_0 - \widetilde{\delta}\left[\frac{\widetilde{k}_2}{2}(\phi_2^S)^2\right]}{k_1} \;=\; \frac{\Delta\,[\,1 + \widetilde{\delta}\,]\,Q_0}{\widetilde{k}_2}\,\frac{\widetilde{k}_2}{k_1} - \widetilde{\delta}\,\frac{\widetilde{k}_2}{k_1}\left[\frac{1}{2}\left(\phi_2^S\right)^2\right]$$

$$= [\,1 + \widetilde{\delta}\,]\,x\,\frac{\widetilde{k}_2}{k_1} - \widetilde{\delta}\,\frac{\widetilde{k}_2}{k_1}\left[\frac{1}{2}\left(\phi_2^S\right)^2\right]$$

$$= x\,[\,1 + \widetilde{\delta}\,]\,\frac{\widetilde{k}_2}{k_1} - \frac{\widetilde{\delta}}{2}\,\frac{\widetilde{k}_2}{k_1}\left[\left(1 + \widetilde{\delta}\right)\widetilde{\delta}\,x\right]^2 \;=\; \frac{\widetilde{k}_2}{k_1}\,x\,[\,1 + \widetilde{\delta}\,]\left[1 - \frac{\widetilde{\delta}^3\left(1 + \widetilde{\delta}\right)}{2}\,x\right]. \tag{84}$$

(25), (80), (81), (83), and (84) imply:

$$\Phi^I \;=\; \phi_1^I + \left[\,1 - \phi_1^I\,\right]\phi_2^I \;=\; \frac{\widetilde{k}_2}{k_1}\,x\,[\,1 + \widetilde{\delta}\,]\left[1 - \frac{\widetilde{\delta}}{2}\,x\,(\,1 + \widetilde{\delta}\,)\right]$$

$$+ \left[1 - \frac{\widetilde{k}_2}{k_1}\,x\left(1 + \widetilde{\delta}\right)\left(1 - \frac{\widetilde{\delta}}{2}\,x\,[\,1 + \widetilde{\delta}\,]\right)\right]x\,[\,1 + \widetilde{\delta}\,]\,; \tag{85}$$

$$\Phi^S \;=\; \phi_1^S + \left[\,1 - \phi_1^S\,\right]\phi_2^S \;=\; \frac{\widetilde{k}_2}{k_1}\,x\,[\,1 + \widetilde{\delta}\,]\left[1 - \frac{\widetilde{\delta}^3\,(\,1 + \widetilde{\delta}\,)}{2}\,x\right]$$

$$+ \left[1 - \frac{\widetilde{k}_2}{k_1}\, x \left(1 + \widetilde{\delta}\right) \left(1 - \frac{\widetilde{\delta}^3 \left[\, 1 + \widetilde{\delta}\,\right]}{2}\, x\right) \right] \left[\, 1 + \widetilde{\delta}\,\right] \widetilde{\delta}\, x\,. \tag{86}$$

(71) implies that in the ID setting:

$$E_d\{W^I\} - E_d\{W^S\} \overset{s}{=} \left[\, \phi_1^I - \phi_1^S \,\right] D_{W3} + \delta \left[\, \Phi^I - \phi_1^S \,\right] D_{W4}$$
$$+ \; \delta^2 \left[\, \Phi^I - \Phi^S \,\right] D_{W5} + \delta^3 \left[\, \Phi^I - \Phi^S \,\right] D_{W6} \tag{87}$$

where $D_{Wt} \equiv W_t(c_0 - \Delta) - W_t(c_0) > 0$. Assumption G implies that $D_{Wt} = g\, D_{W(t-1)}$. Therefore, (87) implies:

$$E_d\{W^I\} - E_d\{W^S\} \overset{s}{=} \left[\, \phi_1^I - \phi_1^S \,\right] D_{W3} + g\,\delta \left[\, \Phi^I - \phi_1^S \,\right] D_{W3}$$
$$+ \; [\, g\,\delta\,]^2 \left[\, \Phi^I - \Phi^S \,\right] D_{W3} + [\, g\,\delta\,]^3 \left[\, \Phi^I - \Phi^S \,\right] D_{W3}$$
$$\overset{s}{=} \phi_1^I - \phi_1^S + \widetilde{\delta} \left[\, \Phi^I - \phi_1^S \,\right] + \widetilde{\delta}^{\,2} \left[\, \Phi^I - \Phi^S \,\right] + \widetilde{\delta}^{\,3} \left[\, \Phi^I - \Phi^S \,\right] \tag{88}$$

where $\widetilde{\delta} \equiv \delta\, g$. (88) implies:

$$E_d\{W^I\} > E_d\{W^S\} \;\text{ if}$$
$$\phi_1^I - \phi_1^S + \widetilde{\delta} \left[\, \Phi^I - \phi_1^S \,\right] + \widetilde{\delta}^2 \left[\, \Phi^I - \Phi^S \,\right] + \widetilde{\delta}^3 \left[\, \Phi^I - \Phi^S \,\right] > 0\,. \tag{89}$$

(83) – (86) and (89) imply that $E_d\{W^I\} > E_d\{W^S\}$ if:

$$\frac{\widetilde{k}_2}{k_1}\, x \left[\, 1 + \widetilde{\delta}\,\right] \left[1 - \frac{\widetilde{\delta}}{2}\, x \left(1 + \widetilde{\delta}\right) \right] - \frac{\widetilde{k}_2}{k_1}\, x \left[\, 1 + \widetilde{\delta}\,\right] \left[1 - \frac{\widetilde{\delta}^3 \left(1 + \widetilde{\delta}\right)}{2}\, x \right]$$
$$+ \, \widetilde{\delta} \left\{ \frac{\widetilde{k}_2}{k_1}\, x \left[\, 1 + \widetilde{\delta}\,\right] \left[1 - \frac{\widetilde{\delta}}{2}\, x \left(1 + \widetilde{\delta}\right) \right] \right.$$
$$+ \left[1 - \frac{\widetilde{k}_2}{k_1}\, x \left(1 + \widetilde{\delta}\right) \left(1 - \frac{\widetilde{\delta}}{2}\, x \left[\, 1 + \widetilde{\delta}\,\right] \right) \right] x \left[\, 1 + \widetilde{\delta}\,\right]$$
$$\left. - \frac{\widetilde{k}_2}{k_1}\, x \left[\, 1 + \widetilde{\delta}\,\right] \left[1 - \frac{\widetilde{\delta}^3 \left(1 + \widetilde{\delta}\right)}{2}\, x \right] \right\}$$
$$+ \left[\widetilde{\delta}^2 + \widetilde{\delta}^3 \right] \left\{ \frac{\widetilde{k}_2}{k_1}\, x \left[\, 1 + \widetilde{\delta}\,\right] \left[1 - \frac{\widetilde{\delta}}{2}\, x \left(1 + \widetilde{\delta}\right) \right] \right.$$
$$+ \left[1 - \frac{\widetilde{k}_2}{k_1}\, x \left(1 + \widetilde{\delta}\right) \left(1 - \frac{\widetilde{\delta}}{2}\, x \left[\, 1 + \widetilde{\delta}\,\right] \right) \right] x \left[\, 1 + \widetilde{\delta}\,\right]$$

$$- \frac{\widetilde{k}_2}{k_1}\, x\,[\,1+\widetilde{\delta}\,]\left[1-\frac{\widetilde{\delta}^3\left(1+\widetilde{\delta}\right)}{2}\,x\right]$$

$$- \left[1-\frac{\widetilde{k}_2}{k_1}\, x\left(1+\widetilde{\delta}\right)\left(1-\frac{\widetilde{\delta}^3\left[1+\widetilde{\delta}\right]}{2}\,x\right)\right][\,1+\widetilde{\delta}\,]\,\widetilde{\delta}\,x\Bigg\} > 0 \qquad (90)$$

$$\Leftrightarrow \quad \frac{\widetilde{k}_2}{k_1}\left[1-\frac{\widetilde{\delta}}{2}\,x\left(1+\widetilde{\delta}\right)\right]-\frac{\widetilde{k}_2}{k_1}\left[1-\frac{\widetilde{\delta}^3\,[\,1+\widetilde{\delta}\,]}{2}\,x\right]$$

$$+\,\widetilde{\delta}\left[\frac{\widetilde{k}_2}{k_1}\left(1-\frac{\widetilde{\delta}}{2}\,x\left[1+\widetilde{\delta}\right]\right)+1-\frac{\widetilde{k}_2}{k_1}\,x\left(1+\widetilde{\delta}\right)\left(1-\frac{\widetilde{\delta}}{2}\,x\left[1+\widetilde{\delta}\right]\right)\right.$$

$$\left.-\,\frac{\widetilde{k}_2}{k_1}\left(1-\frac{\widetilde{\delta}^3\left[1+\widetilde{\delta}\right]}{2}\,x\right)\right]$$

$$+\left[\widetilde{\delta}^2+\widetilde{\delta}^3\right]\Bigg\{\frac{\widetilde{k}_2}{k_1}\left[1-\frac{\widetilde{\delta}}{2}\,x\left(1+\widetilde{\delta}\right)\right]+1-\frac{\widetilde{k}_2}{k_1}\,x\,[\,1+\widetilde{\delta}\,]\left[1-\frac{\widetilde{\delta}}{2}\,x\left(1+\widetilde{\delta}\right)\right]$$

$$-\,\frac{\widetilde{k}_2}{k_1}\left[1-\frac{\widetilde{\delta}^3\left(1+\widetilde{\delta}\right)}{2}\,x\right]-\widetilde{\delta}\left[1-\frac{\widetilde{k}_2}{k_1}\,x\left(1+\widetilde{\delta}\right)\left(1-\frac{\widetilde{\delta}^3\left[1+\widetilde{\delta}\right]}{2}\,x\right)\right]\Bigg\} > 0 \qquad (91)$$

$$\Leftrightarrow \quad \frac{\widetilde{k}_2}{k_1}\,\frac{\widetilde{\delta}^3\,[\,1+\widetilde{\delta}\,]}{2}\,x-\frac{\widetilde{k}_2}{k_1}\,\frac{\widetilde{\delta}}{2}\,x\,[\,1+\widetilde{\delta}\,]$$

$$+\,\widetilde{\delta}\left[-\frac{\widetilde{\delta}}{2}\,\frac{\widetilde{k}_2}{k_1}\,x\left(1+\widetilde{\delta}\right)+1-\frac{\widetilde{k}_2}{k_1}\,x\left(1+\widetilde{\delta}\right)\left(1-\frac{\widetilde{\delta}}{2}\,x\left[1+\widetilde{\delta}\right]\right)+\frac{\widetilde{k}_2}{k_1}\,\frac{\widetilde{\delta}^3\left(1+\widetilde{\delta}\right)}{2}\,x\right]$$

$$+\left[\widetilde{\delta}^2+\widetilde{\delta}^3\right]\Bigg\{-\frac{\widetilde{k}_2}{k_1}\,\frac{\widetilde{\delta}}{2}\,x\,[\,1+\widetilde{\delta}\,]+1-\frac{\widetilde{k}_2}{k_1}\,x\,[\,1+\widetilde{\delta}\,]\left[1-\frac{\widetilde{\delta}}{2}\,x\left(1+\widetilde{\delta}\right)\right]$$

$$+\,\frac{\widetilde{k}_2}{k_1}\,\frac{\widetilde{\delta}^3\,[\,1+\widetilde{\delta}\,]}{2}\,x-\widetilde{\delta}\left[1-\frac{\widetilde{k}_2}{k_1}\,x\left(1+\widetilde{\delta}\right)\left(1-\frac{\widetilde{\delta}^3\left[1+\widetilde{\delta}\right]}{2}\,x\right)\right]\Bigg\} > 0\,. \qquad (92)$$

(91) reflects the fact that $x\,[\,1+\widetilde{\delta}\,] > 0$.

As $x \equiv \frac{\Delta Q_0}{\widetilde{k}_2} \to 0$, the inequality in (92) becomes:

$$\widetilde{\delta}+[\,\widetilde{\delta}^2+\widetilde{\delta}^3\,]\,[\,1-\widetilde{\delta}\,] > 0\,. \qquad (93)$$

The inequality in (93) holds because Assumption G implies that $\widetilde{\delta} < 1$. Therefore, $E_d\{W^I\} >$

$E_d\{W^S\}$ when $\Delta\, Q_0$ is sufficiently small.

Finally, suppose $k_1 = k_2 \equiv k$, so $\frac{\widetilde{k}_2}{k_1} = \frac{1}{g}$. As $k \to \infty$, $x \equiv \frac{\Delta\, g\, Q_0}{k} \to 0$ and the inequality in (92) becomes the inequality in (93). Because this inequality holds, $E_d\{W^I\} > E_d\{W^S\}$ when k is sufficiently large. ■

Proposition 10. *Suppose Assumption K with $\gamma = 2$ holds, Assumption G holds, $k_2 \leq k_1\, g$, and $\widetilde{\delta} > \widehat{\widetilde{\delta}}$. Then $E_d\{W^S\} > E_d\{W^I\}$ when $\widetilde{\delta}$ is sufficiently close to $\widehat{\widetilde{\delta}}$ and $\frac{\Delta\, Q_0\, g\, [\, 1 + \widehat{\widetilde{\delta}}\,]}{k_2}$ is sufficiently close to 1 (in the ID setting).*

Proof of Proposition 10. Recall from the proof of the Corollary to Lemma 2 that $\widehat{\widetilde{\delta}}$ is the value of $g\,\delta$ for which:

$$g\,\delta\,[\,1 + g\,\delta\,] \;=\; (g\,\delta)^2 + \delta\, g \;=\; 1\,. \tag{94}$$

Initially suppose that $\delta\, g = \widehat{\widetilde{\delta}}$ and $\frac{\Delta\, Q_0\, [\, 1 + \widehat{\widetilde{\delta}}\,]}{k_2} = 1$. Then (81) implies that under the specified conditions:

$$\phi_2^I \;=\; \frac{\Delta\, Q_0\, g\, [\, 1 + \widehat{\widetilde{\delta}}\,]}{k_2} \;=\; 1\,. \tag{95}$$

(71) implies that under the specified conditions:

$$\begin{aligned}
E_d\{W^S\} - E_d\{W^I\} \;&=\; A_D\; D_W \\
\text{where}\;\; A_D \;&\equiv\; [\,g\,\delta\,]^2 \left[\,\phi_1^S - \phi_1^I\,\right] + [\,g\,\delta\,]^3 \left[\,\phi_1^S - \Phi^I\,\right] + [\,g\,\delta\,]^4 \left[\,\Phi^S - \Phi^I\,\right] \\
&\quad + [\,g\,\delta\,]^5 \left[\,\Phi^S - \Phi^I\,\right] \;\; \text{and} \\
D_W \;&\equiv\; W(c_0 - \Delta) - W(c_0) \;>\; 0\,.
\end{aligned} \tag{96}$$

We will show that $E_d\{W^S\} > E_d\{W^I\}$ by showing that $A_D > 0$ when $\frac{\Delta\, Q_0\, g\, [\, 1 + \widehat{\widetilde{\delta}}\,]}{k_2} = 1$ and $\delta\, g = \widehat{\widetilde{\delta}}$. The continuity of $E_d\{W^S\} - E_d\{W^I\}$ then ensures that $E_d\{W^S\} > E_d\{W^I\}$ when δ is sufficiently close to $\widehat{\widetilde{\delta}}$ and $\frac{\Delta\, Q_0\, g\, [\, 1 + \widehat{\widetilde{\delta}}\,]}{k}$ is less than, but sufficiently close to, 1.

(82) and (95) imply that under the specified conditions:

$$\begin{aligned}
\phi_1^I \;&=\; \frac{\Delta\, Q_0\, [\, 1 + g\,\delta\,] - \delta \left[\, \phi_2^I\, \Delta\, Q_0\, g\, (1 + \delta\, g) - K_2(\phi_2^I)\,\right]}{k_1} \\
&=\; \frac{\Delta\, Q_0\, [\, 1 + g\,\delta\,] - \delta\, [\, \Delta\, Q_0\, g\, (1 + \delta\, g) - K_2(1)\,]}{k_1} \\
&=\; \frac{\Delta\, Q_0\, [\, 1 + g\,\delta\,] - \delta \left[\, \Delta\, Q_0\, g\, (1 + g\,\delta) - \frac{k_2}{2}\,\right]}{k_1} \;=\; \frac{\Delta\, Q_0\, [\, 1 + g\,\delta\,]\, [\, 1 - g\,\delta\,] + \delta \left[\, \frac{k_2}{2}\,\right]}{k_1} \\
&=\; \left[\, \frac{k_2}{k_1}\,\right] \frac{\Delta\, Q_0\, [\, 1 + g\,\delta\,]\, [\, 1 - g\,\delta\,] + \delta \left[\, \frac{k_2}{2}\,\right]}{k_2}
\end{aligned}$$

$$= \left[\frac{k_2}{k_1}\right] \frac{\Delta Q_0 \,[\,1 + g\,\delta\,]\,[\,1 - g\,\delta\,]}{k_2} + \left[\frac{k_2}{k_1}\right] \frac{\delta}{2}$$

$$= \frac{k_2}{k_1}\left[\frac{1}{g} - \delta + \frac{\delta}{2}\right] \;=\; \frac{k_2}{k_1}\left[\frac{1}{g} - \frac{\delta}{2}\right]. \qquad (97)$$

The penultimate equality in (97) follows from (95) because $\delta\, g = \widehat{\widetilde{\delta}}$, by assumption.

(80) and (95) imply that when $\delta\, g = \widetilde{\delta}$ under the specified conditions:

$$\phi_2^S \;=\; \frac{\Delta Q_0\, \delta\, g^2\,[\,1 + g\,\delta\,]}{k_2} \;=\; \delta\, g \;=\; \widetilde{\delta}\,. \qquad (98)$$

(84), (95), and (98) imply that when $\delta\, g = \widehat{\widetilde{\delta}}$ under the specified conditions:

$$\phi_1^S \;=\; \frac{\Delta Q_0\,[\,1 + \delta\, g\,] - \delta\left[\,\phi_2^S\, \Delta Q_0\, g^2\, \delta\;(1 + g\,\delta) - K_2(\phi_2^S)\,\right]}{k_1}$$

$$= \left[\frac{k_2}{k_1}\right] \frac{\Delta Q_0\,[\,1 + g\,\delta\,] - \delta\left[\,\phi_2^S\, \Delta Q_0\, \delta\, g^2\,(1 + g\,\delta) - K_2(\phi_2^S)\,\right]}{k_2}$$

$$= \frac{k_2}{k_1}\left[\frac{1}{g} - \frac{\delta\left[\,\phi_2^S\, \Delta Q_0\, \delta\, g^2\;(1 + g\,\delta) - K_2(\phi_2^S)\,\right]}{k_2}\right]$$

$$= \frac{k_2}{k_1}\left[\frac{1}{g} - \frac{\delta\left[\,\delta\, \Delta Q_0\, \delta\, g^2\,(1 + g\,\delta) - \frac{k_2}{2}\,\delta^2 g^2\,\right]}{k_2}\right]$$

$$= \frac{k_2}{k_1}\left[\frac{1}{g} - \frac{\delta\,[\,\delta\, \Delta Q_0\, \delta\, g^2\,(1 + g\,\delta)\,]}{k_2} + \frac{\frac{k_2}{2}\,\delta^3\, g^2}{k_2}\right] \;=\; \frac{k_2}{k_1}\left[\frac{1}{g} - \delta^3\, g + \frac{\frac{k_2}{2}\,\delta^3\, g^2}{k_2}\right]$$

$$= \frac{k_2}{k_1}\left[\frac{1}{g} - \delta^3\, g + \frac{1}{2}\,\delta^3\, g^2\right] \;=\; \frac{k_2}{k_1}\left[\frac{1}{g} - \frac{1}{2}\,\delta^3\, g^2\right]$$

$$= \frac{k_2}{g\,k_1}\left[1 - \frac{1}{2}\,(\,\widetilde{\delta}\,)^3\right] \;<\; 1\,. \qquad (99)$$

The inequality in (99) holds because $\widetilde{\delta} < 1$ and $\frac{k_2}{g\,k_1} \leq 1$, by assumption.

(96) implies that when $\delta\, g = \widehat{\widetilde{\delta}}$ and $\widetilde{\delta} = g\,\delta$:

$$A_D \;\stackrel{s}{=}\; \phi_1^S - \phi_1^I + \widetilde{\delta}\left[\,\phi_1^S - \Phi^I\,\right] + \widetilde{\delta}^{\,2}\left[\,\Phi^S - \Phi^I\,\right] + \widetilde{\delta}^{\,3}\left[\,\Phi^S - \Phi^I\,\right] \;>\; 0$$

$$\text{if}\;\; \phi_1^S - \phi_1^I + \widetilde{\delta}\left[\,\phi_1^S - \Phi^I\,\right] + \widetilde{\delta}^{\,2}\left[\,\phi_1^S - \Phi^I\,\right] + \widetilde{\delta}^{\,3}\left[\,\phi_1^S - \Phi^I\,\right] \;>\; 0$$

$$\Leftrightarrow\;\; \phi_1^S - \phi_1^I + \widetilde{\delta}\left[\,\phi_1^S - 1\,\right] + \widetilde{\delta}^{\,2}\left[\,\phi_1^S - 1\,\right] + \widetilde{\delta}^{\,3}\left[\,\phi_1^S - 1\,\right] \;>\; 0\,. \qquad (100)$$

The last equivalence here holds because $\Phi^I = 1$ when $\phi_2^I = 1$ (from (95)).

Observe that when $\delta\, g = \widehat{\widetilde{\delta}}$:

$$\phi_1^S - \phi_1^I + \widetilde{\delta}\,[\,\phi_1^S - 1\,] + \widetilde{\delta}^2\,[\,\phi_1^S - 1\,] + \widetilde{\delta}^3\,[\,\phi_1^S - 1\,]$$

$$= \phi_1^S - \phi_1^I - \widetilde{\delta}\,[\,1 - \phi_1^S\,] \sum_{t=0}^{2} \widetilde{\delta}^t \;=\; \phi_1^S - \phi_1^I - \widetilde{\delta}\,[\,1 - \phi_1^S\,] \left[\frac{1 - \widetilde{\delta}^3}{1 - \widetilde{\delta}} \right]$$

$$= \phi_1^S - \phi_1^I - \widetilde{\delta}\,[\,1 - \phi_1^S\,] \; \frac{[\,1 - \widetilde{\delta}\,]\,[\,\widetilde{\delta}^2 + \widetilde{\delta} + 1\,]}{1 - \widetilde{\delta}}$$

$$= \phi_1^S - \phi_1^I - \widetilde{\delta}\,[\,1 - \phi_1^S\,] \left[\widetilde{\delta}^2 + \widetilde{\delta} + 1 \right] \;=\; \phi_1^S - \phi_1^I - 2\,\widetilde{\delta}\,[\,1 - \phi_1^S\,]\,. \tag{101}$$

The last equality in (101) reflects (94). (100) and (101) imply:

$$A_D \;>\; 0 \;\text{ if }\; \phi_1^S - \phi_1^I \;>\; 2\,\widetilde{\delta}\;[\,1 - \phi_1^S\,]$$

$$\Leftrightarrow\; \frac{k_2}{k_1} \left[\frac{1}{g} - \frac{1}{2}\,\delta^3\,g^2 - \left(\frac{1}{g} - \frac{\delta}{2} \right) \right] \;>\; 2\,\widetilde{\delta} \left[1 - \frac{k_2}{k_1} \left(\frac{1}{g} - \frac{1}{2}\,\delta^3\,g^2 \right) \right]$$

$$\Leftrightarrow\; \frac{\delta}{2} - \frac{1}{2}\,\delta^3 g^2 \;>\; 2\,\widetilde{\delta} \left[\frac{k_1}{k_2} - \left(\frac{1}{g} - \frac{1}{2}\,\delta^3\,g^2 \right) \right]$$

$$\Leftrightarrow\; \frac{\delta\,g}{2} - \frac{1}{2}\,\delta^3 g^3 \;>\; 2\,\widetilde{\delta} \left[\frac{k_1\,g}{k_2} - \left(1 - \frac{1}{2}\,\delta^3\,g^3 \right) \right]$$

$$\Leftrightarrow\; \frac{\widetilde{\delta}}{2} - \frac{1}{2}\,\widetilde{\delta}^3 \;>\; 2\,\widetilde{\delta} \left[\frac{k_1\,g}{k_2} - \left(1 - \frac{1}{2}\,\widetilde{\delta}^3 \right) \right]. \tag{102}$$

Observe that:

$$\frac{\widetilde{\delta}}{2} - \frac{1}{2}\,\widetilde{\delta}^3 \;>\; 2\,\widetilde{\delta} \left[1 - \left(1 - \frac{1}{2}\,\widetilde{\delta}^3 \right) \right] \;\Leftrightarrow\; \frac{\widetilde{\delta}}{2} - \frac{1}{2}\,\widetilde{\delta}^3 \;>\; 2\,\widetilde{\delta} \left[\frac{1}{2}\,\widetilde{\delta}^3 \right]$$

$$\Leftrightarrow\; \frac{\widetilde{\delta}}{2} - \frac{1}{2}\,\widetilde{\delta}^3 \;>\; \widetilde{\delta}^4 \;\Leftrightarrow\; 1 - \widetilde{\delta}^2 \;>\; 2\,\widetilde{\delta}^3 \;\Leftrightarrow\; 1 \;>\; \widetilde{\delta}^2 + 2\,\widetilde{\delta}^3\,. \tag{103}$$

The last inequality in (103) holds because:

$$\widetilde{\delta}^2 + 2\,\widetilde{\delta}^3 \;=\; \widetilde{\delta}^2\,[\,1 + \widetilde{\delta}\,] + \widetilde{\delta}^3 \;=\; \widetilde{\delta} + \widetilde{\delta}^3 \;=\; \widetilde{\delta}\,[\,1 + \widetilde{\delta}^2\,] \;<\; \widetilde{\delta}\,[\,1 + \widetilde{\delta}\,] \;=\; 1\,.$$

The second and last equalities here reflect (94).

Because $k_1\,g \le k_2$ by assumption, (102) and (103) imply that $A_D > 0$ when $\phi_2^I = 1$ and $\delta\,g = \widehat{\widetilde{\delta}}$. ■

Proposition 11. *Suppose Assumption K with $\gamma = 2$ holds, Assumption G holds, and $\widetilde{\delta} \geq \widehat{\widetilde{\delta}}$. Then $E_d\{W^S\} > E_d\{W^I\}$ when k_2 is sufficiently large and $\frac{\Delta Q_0}{k_1}[1+\widetilde{\delta}]$ is sufficiently close to 1 (in the ID setting).*

<u>Proof of Proposition 11</u>. Define $\widetilde{k}_2 \equiv \frac{k_2}{g}$, $x \equiv \frac{\Delta g Q_0}{k_2} = \frac{\Delta Q_0}{\widetilde{k}_2}$, and $\widetilde{\delta} \equiv g\,\delta$. Recall that $E_d\{W^S\} > E_d\{W^I\}$ if the inequality in (92) is reversed. Because $x\,\widetilde{k}_2 = \Delta Q_0$, the inequality in (92) is reversed if:

$$\begin{aligned}
&\frac{\Delta Q_0}{k_1}\frac{\widetilde{\delta}^3[1+\widetilde{\delta}]}{2} - \frac{\Delta Q_0}{k_1}\frac{\widetilde{\delta}}{2}[1+\widetilde{\delta}] \\
&+ \widetilde{\delta}\left[-\frac{\widetilde{\delta}}{2}\frac{\Delta Q_0}{k_1}\left(1+\widetilde{\delta}\right) + 1 - \frac{\Delta Q_0}{k_1}\left(1+\widetilde{\delta}\right)\left(1-\frac{\widetilde{\delta}}{2}x[1+\widetilde{\delta}]\right) + \frac{\Delta Q_0}{k_1}\frac{\widetilde{\delta}^3\left(1+\widetilde{\delta}\right)}{2}\right] \\
&+ \left[\widetilde{\delta}^2 + \widetilde{\delta}^3\right]\left\{-\frac{\Delta Q_0}{k_1}\frac{\widetilde{\delta}}{2}[1+\widetilde{\delta}] + 1 - \frac{\Delta Q_0}{k_1}[1+\widetilde{\delta}]\left[1-\frac{\widetilde{\delta}}{2}x\left(1+\widetilde{\delta}\right)\right]\right. \\
&\left.\quad + \frac{\Delta Q_0}{k_1}\frac{\widetilde{\delta}^3[1+\widetilde{\delta}]}{2} - \widetilde{\delta}\left[1-\frac{\Delta Q_0}{k_1}\left(1+\widetilde{\delta}\right)\left(1-\frac{\widetilde{\delta}^3[1+\widetilde{\delta}]}{2}x\right)\right]\right\} < 0\,. \qquad (104)
\end{aligned}$$

$x \equiv \frac{\Delta g Q_0}{k_2} \to 0$ as $k_2 \to \infty$. Therefore, as $k_2 \to \infty$, the inequality in (104) holds if:

$$\begin{aligned}
&\frac{\Delta Q_0}{k_1}\frac{\widetilde{\delta}^3[1+\widetilde{\delta}]}{2} - \frac{\Delta Q_0}{k_1}\frac{\widetilde{\delta}}{2}[1+\widetilde{\delta}] \\
&+ \widetilde{\delta}\left[-\frac{\widetilde{\delta}}{2}\frac{\Delta Q_0}{k_1}\left(1+\widetilde{\delta}\right) + 1 - \frac{\Delta Q_0}{k_1}\left(1+\widetilde{\delta}\right) + \frac{\Delta Q_0}{k_1}\frac{\widetilde{\delta}^3\left(1+\widetilde{\delta}\right)}{2}\right] \\
&+ \left[\widetilde{\delta}^2 + \widetilde{\delta}^3\right]\left\{-\frac{\Delta Q_0}{k_1}\frac{\widetilde{\delta}}{2}[1+\widetilde{\delta}] + 1 - \frac{\Delta Q_0}{k_1}[1+\widetilde{\delta}]\right. \\
&\left.\quad + \frac{\Delta Q_0}{k_1}\frac{\widetilde{\delta}^3[1+\widetilde{\delta}]}{2} - \widetilde{\delta}\left[1-\frac{\Delta Q_0}{k_1}\left(1+\widetilde{\delta}\right)\right]\right\} < 0\,. \qquad (105)
\end{aligned}$$

Define $y \equiv \frac{\Delta Q_0}{k_1}[1+\widetilde{\delta}]$. Then the inequality in (105) holds if:

$$\begin{aligned}
&\frac{\widetilde{\delta}}{2}y[\widetilde{\delta}^2-1] + \widetilde{\delta}\left[1+\frac{\widetilde{\delta}}{2}y(\widetilde{\delta}^2-1) - y\right] \\
&+ \left[\widetilde{\delta}^2+\widetilde{\delta}^3\right]\left[1-\widetilde{\delta}+y\frac{\widetilde{\delta}}{2}(\widetilde{\delta}^2-1) - y(1-\widetilde{\delta})\right] < 0
\end{aligned}$$

$$\Leftrightarrow \quad \frac{\widetilde{\delta}}{2}\, y\, [\, \widetilde{\delta}^2 - 1\,] + \widetilde{\delta} \left[1 - y + \frac{\widetilde{\delta}}{2}\, y\, (\, \widetilde{\delta}^2 - 1\,) \right]$$

$$+ \left[\widetilde{\delta}^2 + \widetilde{\delta}^3 \right] \left\{ [\, 1 - \widetilde{\delta}\,]\, [\, 1 - y\,] + y\, \frac{\widetilde{\delta}}{2}\, [\, \widetilde{\delta}^2 - 1\,] \right\} < 0$$

$$\Leftrightarrow \quad \frac{\widetilde{\delta}}{2}\, y\, [\, 1 - \widetilde{\delta}^2\,] \left[1 + \widetilde{\delta} + \widetilde{\delta}^2 + \widetilde{\delta}^3 \right]$$

$$> \left\{ \widetilde{\delta} + [\, \widetilde{\delta}^2 + \widetilde{\delta}^3\,]\, [\, 1 - \widetilde{\delta}\,] \right\} [\, 1 - y\,]\,. \tag{106}$$

Assumption G requires $g < \frac{1}{\delta} \Rightarrow \widetilde{\delta} = \delta\, g < 1$. Therefore, the inequality in (106) holds as $y \equiv \frac{\Delta Q_0}{k_1} [\, 1 + \widetilde{\delta}\,] \to 1$. ■

Proposition 12. *$\phi_1^I < \phi_1^S$ and $\phi_2^I > \phi_2^S$ in the setting with innovation persistence.*

Proof of Proposition 12. Initially suppose the NID setting prevails, so the firm always implements an achieved cost reduction immediately.

Under standard rebasing (SR) in this setting, the firm retains the full benefit of a cost reduction that is achieved in period 2 only for that period. Therefore, the firm's problem in period 2, given that it implemented first-period success probability ϕ_1^S but did not achieve a cost reduction in period 1, is:

$$\underset{\phi_2}{\text{Maximize}} \quad \left[\phi_2 + \alpha\, \phi_1^S \right] \Delta\, Q_2(c_0) - K_2(\phi_2)$$

$$\Rightarrow \quad K_2'(\phi_2^S) \;=\; \Delta\, Q_2(c_0) \;\; \text{at an interior optimum.} \tag{107}$$

Under IRIS in this setting, the firm retains the full benefit of a cost reduction achieved in period 2 during both period 2 and period 3. Therefore, the firm's problem in period 2, given that it implemented first-period success probability ϕ_1^I but did not achieve a cost reduction in period 1, is:

$$\underset{\phi_2}{\text{Maximize}} \quad \left[\phi_2 + \alpha\, \phi_1^I \right] \Delta\, [\, Q_2(c_0) + \delta\, Q_3(c_0)\,] - K_2(\phi_2)$$

$$\Rightarrow \quad K_2'(\phi_2^I) \;=\; \Delta\, [\, Q_2(c_0) + \delta\, Q_3(c_0)\,] \;\; \text{at an interior optimum.} \tag{108}$$

Under SR, the firm retains the full benefit of a cost reduction that is achieved in period 1 both in period 1 and in period 2. Therefore, the firm's problem in period 1 under SR in the NID setting is:

$$\underset{\phi_1}{\text{Maximize}} \quad \phi_1\, \Delta\, [\, Q_1(c_0) + \delta\, Q_2(c_0)\,]$$

$$+ \; [\, 1 - \phi_1\,]\; \delta \left[\left(\phi_2^S + \alpha\, \phi_1 \right) \Delta\, Q_2(c_0) - K_2(\phi_2^S) \right] - K_1(\phi_1)\,. \tag{109}$$

(109) implies that at an interior solution to this problem:

$$K_1'(\phi_1^S) \;=\; \Delta\left[\,Q_1(c_0)+\delta\,Q_2(c_0)\,\right]-\delta\left[\,\left(\phi_2^S+\alpha\,\phi_1^S\right)\Delta\,Q_2(c_0)-K_2(\phi_2^S)\,\right]$$
$$+\;\delta\,\alpha\left[\,1-\phi_1^S\,\right]\Delta\,Q_2(c_0)\,. \tag{110}$$

Under IRIS, the firm retains for two periods the full benefit of an achieved cost reduction, whether the reduction is achieved in period 1 or period 2. Therefore, the firm's problem in period 1 under IRIS is:

$$\underset{\phi_1}{\text{Maximize}} \quad \phi_1\,\Delta\left[\,Q_1(c_0)+\delta\,Q_2(c_0)\,\right]$$
$$+\left[\,1-\phi_1\,\right]\delta\left\{\left[\,\phi_2^I+\alpha\,\phi_1\,\right]\Delta\left[\,Q_2(c_0)+\delta\,Q_3(c_0)\,\right]-K_2(\phi_2^I)\right\}-K_1(\phi_1)\,. \tag{111}$$

(111) implies that at an interior solution to this problem:

$$K_1'(\phi_1^I) \;=\; \Delta\left[\,Q_1(c_0)+\delta\,Q_2(c_0)\,\right]-\delta\left\{\left[\,\phi_2^I+\alpha\,\phi_1^I\,\right]\Delta\left[\,Q_2(c_0)+\delta\,Q_3(c_0)\,\right]-K_2(\phi_2^I)\right\}$$
$$+\;\delta\,\alpha\left[\,1-\phi_1^I\,\right]\,\Delta\left[\,Q_2(c_0)+\delta\,Q_3(c_0)\,\right]. \tag{112}$$

(107) and (108) imply:

$$K_2'(\phi_2^I) \;=\; \Delta\left[\,Q_2(c_0)+\delta\,Q_3(c_0)\,\right] \;>\; \Delta\,Q_2(c_0) \;=\; K_2'(\phi_2^S) \quad\Rightarrow\quad \phi_2^I \;>\; \phi_2^S\,. \tag{113}$$

The implication ($\Rightarrow$) in (113) reflects the convexity of $K_2(\cdot)$.

To prove that $\phi_1^S > \phi_1^I$, suppose that $\phi_1^I \geq \phi_1^S$. Then:

$$\left[\,\phi_2^I+\alpha\,\phi_1^I\,\right]\Delta\left[\,Q_2(c_0)+\delta\,Q_3(c_0)\,\right]-K_2(\phi_2^I)$$
$$=\; \max_{\phi_2}\;\left\{\left[\,\phi_2+\alpha\,\phi_1^I\,\right]\Delta\left[\,Q_2(c_0)+\delta\,Q_3(c_0)\,\right]-K_2(\phi_2)\right\}$$
$$>\; \left[\,\phi_2^S+\alpha\,\phi_1^I\,\right]\Delta\left[\,Q_2(c_0)+\delta\,Q_3(c_0)\,\right]-K_2(\phi_2^S)$$
$$\geq\; \left[\,\phi_2^S+\alpha\,\phi_1^S\,\right]\Delta\left[\,Q_2(c_0)+\delta\,Q_3(c_0)\,\right]-K_2(\phi_2^S)$$
$$>\; \left[\,\phi_2^S+\alpha\,\phi_1^S\,\right]\Delta\,Q_2(c_0)-K_2(\phi_2^S)\,. \tag{114}$$

The equality in (114) reflects (108). The first inequality in (114) reflects (113). The second inequality in (114) reflects the maintained assumption that $\phi_1^I \geq \phi_1^S$.

Observe that:

$$\phi_1^I \;=\; \arg\max_{\phi_1}\;\left\{\,\phi_1\,\Delta\left[\,Q_1(c_0)+\delta\,Q_2(c_0)\,\right]\right.$$
$$\left.+\;\left[\,1-\phi_1\,\right]\delta\left\{\left[\,\phi_2^I+\alpha\,\phi_1\,\right]\Delta\left[\,Q_2(c_0)+\delta\,Q_3(c_0)\,\right]-K_2(\phi_2^I)\right\}-K_1(\phi_1)\right\}$$
$$<\; \arg\max_{\phi_1}\;\left\{\,\phi_1\,\Delta\left[\,Q_1(c_0)+\delta\,Q_2(c_0)\,\right]\right.$$
$$\left.+\;\left[\,1-\phi_1\,\right]\delta\left\{\left[\,\phi_2^S+\alpha\,\phi_1\,\right]\Delta\,Q_2(c_0)-K_2(\phi_2^S)\right\}-K_1(\phi_1)\right\} \;=\; \phi_1^S\,. \tag{115}$$

The first equality in (115) reflects (111). The inequality in (115) follows from (114) because the value of ϕ_1 that maximizes the PDV of the firm's expected profit increases as the firm's expected profit following first-period failure declines.[5] The final equality in (115) reflects (109).

The conclusion in (115) that $\phi_1^I < \phi_1^S$ contradicts the maintained assumption that $\phi_1^I \geq \phi_1^S$. Therefore, by contradiction:

$$\phi_1^S \; > \; \phi_1^I \, . \tag{116}$$

Now suppose the ID setting prevails. Then when the firm operates under SR in this setting, the firm delays to period 3 the implementation of a cost reduction achieved in period 2. Therefore, the firm's problem in period 2, given that it implemented first-period success probability ϕ_1^S but did not achieve a cost reduction in period 1, is:

$$\underset{\phi_2}{\text{Maximize}} \quad \left[\phi_2 + \alpha \, \phi_1^S \right] \Delta \, \delta \left[Q_3(c_0) + \delta \, Q_4(c_0) \right] - K_2(\phi_2)$$

$$\Rightarrow \quad K_2'(\phi_2^S) \; = \; \Delta \, \delta \left[Q_3(c_0) + \delta \, Q_4(c_0) \right] \quad \text{at an interior optimum.} \tag{117}$$

The firm's choice of second-period success probability under IRIS in the ID setting is as specified in (108).

Under SR, the firm retains the full benefit of a cost reduction that is achieved in period 1 both in period 1 and in period 2. Therefore, (117) implies that the firm's problem in period 1 under SR in the ID setting is:

$$\underset{\phi_1}{\text{Maximize}} \quad \phi_1 \, \Delta \left[Q_1(c_0) + \delta \, Q_2(c_0) \right]$$
$$+ \; \left[1 - \phi_1 \right] \, \delta \left[\left(\phi_2^S + \alpha \, \phi_1 \right) \, \delta \, \Delta \left(\, Q_3(c_0) + \delta \, Q_4(c_0) \, \right) - K_2(\phi_2^S) \right] - K_1(\phi_1) \, . \tag{118}$$

(118) implies that at an interior solution to this problem:

$$K_1'(\phi_1^S) \; = \; \Delta \left[Q_1(c_0) + \delta \, Q_2(c_0) \right] - \delta \left[\left(\phi_2^S + \alpha \, \phi_1^S \right) \, \delta \, \Delta \left(\, Q_3(c_0) + \delta \, Q_4(c_0) \, \right) - K_2(\phi_2^S) \right]$$
$$+ \; \delta \, \alpha \left[1 - \phi_1^S \right] \, \delta \, \Delta \left[Q_3(c_0) + \delta \, Q_4(c_0) \right] . \tag{119}$$

The firm's choice of first-period success probability under IRIS in the ID setting is as specified in (108).

(108) and (117) imply:

$$K_2'(\phi_2^I) \; = \; \Delta \left[Q_2(c_0) + \delta \, Q_3(c_0) \right]$$
$$> \; \Delta \, \delta \left[Q_3(c_0) + \delta \, Q_4(c_0) \right] \; = \; K_2'(\phi_2^S) \quad \Rightarrow \quad \phi_2^I \; > \; \phi_2^S \, . \tag{120}$$

The inequality in (120) reflects Assumption D. The implication in (120) reflects the convexity of $K_2(\cdot)$.

[5]Formally, if $\phi_1^I \in (0,1) = \underset{\phi_1}{\arg\max} \, \{ \, \phi_1 \, A + [\, 1 - \phi_1 \,] \, B - K_1(\phi_1) \, \}$, then $A - B = K_1'(\phi_1^I) \Rightarrow \frac{d\phi_1^I}{dB} = - \frac{1}{K_1''(\phi_1^I)} < 0$.

To prove that $\phi_1^S > \phi_1^I$, suppose that $\phi_1^I \geq \phi_1^S$. Then:

$$
\begin{aligned}
&\left[\, \phi_2^I + \alpha\, \phi_1^I \,\right] \Delta \left[\, Q_2(c_0) + \delta\, Q_3(c_0) \,\right] - K_2(\phi_2^I) \\
&\quad = \max_{\phi_2} \left\{ \left[\, \phi_2 + \alpha\, \phi_1^I \,\right] \Delta \left[\, Q_2(c_0) + \delta\, Q_3(c_0) \,\right] - K_2(\phi_2) \right\} \\
&\quad > \left[\, \phi_2^S + \alpha\, \phi_1^I \,\right] \Delta \left[\, Q_2(c_0) + \delta\, Q_3(c_0) \,\right] - K_2(\phi_2^S) \\
&\quad \geq \left[\, \phi_2^S + \alpha\, \phi_1^S \,\right] \Delta \left[\, Q_2(c_0) + \delta\, Q_3(c_0) \,\right] - K_2(\phi_2^S) \\
&\quad > \left[\, \phi_2^S + \alpha\, \phi_1^S \,\right] \Delta\, \delta \left[\, Q_3(c_0) + \delta\, Q_4(c_0) \,\right] - K_2(\phi_2^S)\,. \qquad (121)
\end{aligned}
$$

The equality in (121) reflects (108). The first inequality in (121) reflects (120). The second inequality in (121) reflects the maintained assumption that $\phi_1^I \geq \phi_1^S$. The last inequality in (121) reflects Assumption D.

Observe that:

$$
\begin{aligned}
\phi_1^I \;&=\; \arg\max_{\phi_1} \left\{ \phi_1\, \Delta \left[\, Q_1(c_0) + \delta\, Q_2(c_0) \,\right] \right. \\
&\qquad\qquad \left. + \left[\, 1 - \phi_1 \,\right] \delta \left\{ \left[\, \phi_2^I + \alpha\, \phi_1 \,\right] \Delta \left[\, Q_2(c_0) + \delta\, Q_3(c_0) \,\right] - K_2(\phi_2^I) \right\} - K_1(\phi_1) \right\} \\
&<\; \arg\max_{\phi_1} \left\{ \phi_1\, \Delta \left[\, Q_1(c_0) + \delta\, Q_2(c_0) \,\right] \right. \\
&\qquad\qquad \left. + \left[\, 1 - \phi_1 \,\right] \delta \left\{ \left[\, \phi_2^S + \alpha\, \phi_1 \,\right] \Delta\, \delta \left[\, Q_3(c_0) + \delta\, Q_4(c_0) \,\right] - K_2(\phi_2^S) \right\} - K_1(\phi_1) \right\} \\
&=\; \phi_1^S\,. \qquad (122)
\end{aligned}
$$

The first equality in (122) reflects (111). The inequality in (122) follows from (121) because the value of ϕ_1 that maximizes the PDV of the firm's expected profit increases as the firm's expected profit following first-period failure declines. The final equality in (122) reflects (118).

The conclusion in (122) that $\phi_1^I < \phi_1^S$ contradicts the maintained assumption that $\phi_1^I \geq \phi_1^S$. Therefore, by contradiction:

$$
\phi_1^S \;>\; \phi_1^I\,. \quad \blacksquare \qquad (123)
$$

Proposition 13. *Suppose Assumptions G and K hold. Then when the NID setting prevails, $E_d\{W^S\} > E_d\{W^I\}$ if $\widetilde{\delta}\,[\,1+\widetilde{\delta}\,]^{\frac{1}{\gamma-1}} < 1$ in the setting with innovation persistence.*

Proof of Proposition 13. (107) and (108) imply that in the NID setting:

$$
k_2 \left(\phi_2^S\right)^{\gamma-1} \;=\; \Delta\, g\, Q_0 \;\;\Rightarrow\;\; \phi_2^S \;=\; \left[\frac{\Delta\, g\, Q_0}{k_2}\right]^{\frac{1}{\gamma-1}} \;=\; \left[\frac{\Delta\, Q_0}{\widetilde{k}_2}\right]^{\frac{1}{\gamma-1}} \quad \text{and}
$$

$$
k_2 \left(\phi_2^I\right)^{\gamma-1} \;=\; \Delta \left[\, g\, Q_0 + g^2\, \delta\, Q_0 \,\right]
$$

$$\Rightarrow \quad \phi_2^I \;=\; \left[\frac{\Delta\, Q_0\, g \left(1+\widetilde{\delta}\right)}{k_2} \right]^{\frac{1}{\gamma-1}} \;=\; \left[\frac{\Delta\, Q_0 \,\left(1+\widetilde{\delta}\right)}{\widetilde{k}_2} \right]^{\frac{1}{\gamma-1}}$$

$$= \; [\, 1+\widetilde{\delta}\,]^{\frac{1}{\gamma-1}} \left[\frac{\Delta\, Q_0}{\widetilde{k}_2} \right]^{\frac{1}{\gamma-1}} \;=\; \phi_2^S \, [\, 1+\widetilde{\delta}\,]^{\frac{1}{\gamma-1}}. \tag{124}$$

(25) implies that in the setting with innovation persistence:

$$\Phi^j \;=\; \phi_1^S + \left[1-\phi_1^S \right] \left[\phi_2^S + \alpha\, \phi_1^S \right] \quad \text{for } j \in \{ S, I \}\,. \tag{125}$$

(64) implies that under the specified conditions:

$E_d \left\{ W^S \right\} \; > \; E_d \left\{ W^I \right\}$ if

$$\widetilde{\delta}^2 \left[\Phi^S - \phi_1^I \right] [\, S_0(c_0-\Delta) - S_0(c_0)\,]$$

$$+ \left[\Phi^S - \Phi^I \right] [\, S_0(c_0-\Delta) - S_0(c_0)\,] [\, \widetilde{\delta}^3 + \widetilde{\delta}^4 + \widetilde{\delta}^5 \,] \; > \; 0$$

$$\Leftrightarrow \quad \Phi^S - \phi_1^I + \left[\Phi^S - \Phi^I \right] [\, \widetilde{\delta} + \widetilde{\delta}^2 + \widetilde{\delta}^3 \,] \; > \; 0\,. \tag{126}$$

First suppose that $\Phi^S \geq \Phi^I$. (123) implies that $\Phi^S > \phi_1^I$. Therefore, (126) implies that $E_d \left\{ W^S \right\} \; > \; E_d \left\{ W^I \right\}$ when $\Phi^S \geq \Phi^I$.

Now suppose that $\Phi^S < \Phi^I$. (125) implies that (126) holds in this case if:

$$\Phi^S - \phi_1^I + \left[\Phi^S - \Phi^I \right] \, [\, \widetilde{\delta} + \widetilde{\delta}^2 + \widetilde{\delta}^3 + \widetilde{\delta}^4 + \ldots \,] \; > \; 0$$

$$\Leftrightarrow \quad \Phi^S - \phi_1^I + \left[\Phi^S - \Phi^I \right] \, \frac{\widetilde{\delta}}{1-\widetilde{\delta}} \; > \; 0$$

$$\Leftrightarrow \quad \left[\Phi^S - \phi_1^I \right] \left[1 - \widetilde{\delta} \right] + \widetilde{\delta} \left[\Phi^S - \Phi^I \right] \; > \; 0$$

$$\Leftrightarrow \quad \left[1 - \widetilde{\delta} \right] \left[\phi_1^S + \phi_2^S \left(1-\phi_1^S\right) + \alpha\, \phi_1^S \left(1-\phi_1^S\right) - \phi_1^I \right]$$

$$+ \, \widetilde{\delta} \, [\, \phi_1^S + \phi_2^S \left(1-\phi_1^S\right) + \alpha\, \phi_1^S \left(1-\phi_1^S\right)$$

$$- \left(\phi_1^I + \phi_2^I \left[1-\phi_1^I \right] + \alpha\, \phi_1^I \left[1-\phi_1^I \right] \right)] \; > \; 0$$

$$\Leftrightarrow \quad \phi_1^S + \phi_2^S \left[1-\phi_1^S \right] + \alpha\, \phi_1^S \left[1-\phi_1^S \right] - \phi_1^I$$

$$- \, \widetilde{\delta} \, \left[\phi_1^S + \phi_2^S \left(1-\phi_1^S\right) + \alpha\, \phi_1^S \left(1-\phi_1^S\right) - \phi_1^I \right]$$

$$+ \, \widetilde{\delta} \, [\, \phi_1^S + \phi_2^S \left(1-\phi_1^S\right) + \alpha\, \phi_1^S \left(1-\phi_1^S\right)$$

$$- \, \left(\phi_1^I + \phi_2^I \left[1-\phi_1^I \right] + \alpha\, \phi_1^I \left[1-\phi_1^I \right] \right)] \; > \; 0$$

$$\Leftrightarrow \quad \phi_1^S + \phi_2^S \left[1 - \phi_1^S \right] + \alpha \, \phi_1^S \left[1 - \phi_1^S \right] - \phi_1^I - \widetilde{\delta} \, \phi_2^I \left[1 - \phi_1^I \right]$$

$$- \, \alpha \, \widetilde{\delta} \, \phi_1^I \left[1 - \phi_1^I \right] \; > \; 0$$

$$\Leftrightarrow \quad - \left[1 - \phi_1^S \right] + \phi_2^S \left[1 - \phi_1^S \right] + \alpha \, \phi_1^S \left[1 - \phi_1^S \right] + 1 - \phi_1^I - \widetilde{\delta} \, \phi_2^I \left[1 - \phi_1^I \right]$$

$$- \, \alpha \, \widetilde{\delta} \, \phi_1^I \left[1 - \phi_1^I \right] \; > \; 0$$

$$\Leftrightarrow \quad \left[1 - \phi_1^I \right] \left[1 - \widetilde{\delta} \, \phi_2^I - \alpha \, \widetilde{\delta} \, \phi_1^I \right] - \left[1 - \phi_1^S \right] \left[1 - \phi_2^S - \alpha \, \phi_1^S \right] \; > \; 0 \, . \tag{127}$$

(123) implies:

$$1 - \phi_1^I \; > \; 1 - \phi_1^S \, . \tag{128}$$

Furthermore, (124) implies that when $\widetilde{\delta} \, [\, 1 + \widetilde{\delta} \,]^{\frac{1}{\gamma - 1}} < 1$:

$$\phi_2^S - \widetilde{\delta} \, \phi_2^I \; = \; \phi_2^S - \widetilde{\delta} \, \phi_2^S \, [\, 1 + \widetilde{\delta} \,]^{\frac{1}{\gamma - 1}} \; = \; \phi_2^S \left[1 - \widetilde{\delta} \left(1 + \widetilde{\delta} \right)^{\frac{1}{\gamma - 1}} \right] \; > \; 0 \, . \tag{129}$$

$\widetilde{\delta} \leq 1$ because $\widetilde{\delta} \, [\, 1 + \widetilde{\delta} \,]^{\frac{1}{\gamma - 1}} < 1$. Therefore, (123) implies:

$$\phi_1^I \; < \; \phi_1^S \quad \Rightarrow \quad \alpha \, \phi_1^I \; < \; \alpha \, \phi_1^S \quad \Rightarrow \quad \alpha \, \widetilde{\delta} \, \phi_1^I \; < \; \alpha \, \phi_1^S \, . \tag{130}$$

Because $\phi_2^S + \alpha \, \phi_1^S < 1$, by assumption, (129) and(130) imply:

$$1 - \widetilde{\delta} \, \phi_2^I - \alpha \, \widetilde{\delta} \, \phi_1^I \; > \; 1 - \phi_2^S - \alpha \, \phi_1^S \; > \; 0 \, . \tag{131}$$

(128) and (131) imply that the inequality in (127) holds. ■

Corollary to Proposition 13. Suppose Assumptions G and K hold. Then when the NID setting prevails, $E_d\{W^S\} > E_d\{W^I\}$ if $\gamma \geq 2$ in the setting with innovation persistence.

Proof of the Corollary to Proposition 13. The Corollary follows directly from Proposition 13 because $\widetilde{\delta} \, [\, 1 + \widetilde{\delta} \,]^{\frac{1}{\gamma - 1}} < 1$ under the specified conditions. This is the case because:

$$\widetilde{\delta} \, [\, 1 + \widetilde{\delta} \,]^{\frac{1}{\gamma - 1}} \; \leq \; \widetilde{\delta} \, [\, 1 + \widetilde{\delta} \,] \; < \; 1 \, .$$

The weak inequality here holds because $\gamma \geq 2$. The strict inequality here holds because $\widetilde{\delta} \; = \; g \, \delta \; < \; \widetilde{\delta}$ in the NID setting and because $\widetilde{\delta} \, [\, 1 + \widetilde{\delta} \,] \; = 1$, by definition. ■