Technical Appendix to Accompany

"Pareto Gains from Limiting Compensation Options"

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Conclusion A1. $E\,\Pi_M(\phi^*_{HM}) \ > \ E\,\Pi(\phi^*_H) \ \ if \ \ \frac{\theta_H}{\theta_L} \ \geq \ \max\left\{ 2.8, \ \frac{10.212\ K'(0.5)}{\theta_L[\,\overline{x}-\underline{x}\,]^2} \right\}$.

The proof of Conclusion A1 relies upon the following finding.

Finding A1. $g\,(0) \ = \ 0$ and $g(\phi_H) \ > \ 0$ for all $\phi_H \ \in \ (\,0, \widetilde{\phi}_H\,]$, where:

$$g\,(\phi_H) \ \ \equiv \ \ \widetilde{E}'_M\,(\phi_H) - \widetilde{E}'\,(\phi_H)\,. \tag{1}$$

Proof. The proofs of Findings 8 and 9 in the paper establish that:

$$\widetilde{E}'_M(\phi_H) \ = \ \frac{\Gamma_M}{2\,[\,2\,\phi_H\,(\theta_H-\theta_L)+\theta_L\,]^3} \tag{2}$$

where

$$\Gamma_M \ \equiv \ (\theta_L)^3\,[\,\theta_H-\theta_L\,]\,[\,\overline{x}-\underline{x}\,]^2 + 2\,\phi_H\,(\theta_L)^2\,[\,\theta_H-\theta_L\,]^2\,[\,\overline{x}-\underline{x}\,]^2$$
$$+ \ 3\,(\phi_H)^2\,\theta_L\,[\,\theta_H-\theta_L\,]^3\,[\,\overline{x}-\underline{x}\,]^2 + \ 2\,(\phi_H)^3\,[\,\theta_H-\theta_L\,]^4\,[\,\overline{x}-\underline{x}\,]^2\,; \tag{3}$$

and $\quad \widetilde{E}'(\phi_H) \ = \ \dfrac{\Gamma}{2\,[\,\phi_H\,(\theta_H-2\,\theta_L)+\theta_L\,]^3}$ (4)

where

$$\Gamma \ \equiv \ (\theta_L)^3\,[\,\theta_H-\theta_L\,]\,[\,\overline{x}-\underline{x}\,]^2 + (\phi_H)^3\,[\,\theta_H-2\,\theta_L\,]\,[\,\theta_H-\theta_L\,]\,(\theta_L)^2\,[\,\overline{x}-\underline{x}\,]^2$$
$$+ \ 3\,(\phi_H)^2\,[\,\theta_H-\theta_L\,]\,(\theta_L)^3\,[\,\overline{x}-\underline{x}\,]^2 - \phi_H\,(\theta_L)^2\,[\,\theta_H-\theta_L\,]\,[\,\theta_H+2\,\theta_L\,]\,[\,\overline{x}-\underline{x}\,]^2\,.$$

(2) and (4) imply:

$$g\,(\phi_H) \ = \ \frac{\phi_H^2\,\theta_H\,[\,\theta_H-\theta_L\,]^2\,[\,\overline{x}-\underline{x}\,]^2\ \Gamma}{2\,[\,\phi_H\,(\,\theta_H-2\,\theta_L\,)+\theta_L\,]^3\,[\,2\,\phi_H\,(\,\theta_H-\theta_L\,)+\theta_L\,]^3} \tag{5}$$

where $\Gamma \ \equiv \ 2\,\phi_H^4\,(\theta_H)^4 + [\,9-16\,\phi_H\,]\,\phi_H^3\,(\theta_H)^3\,\theta_L + \phi_H^2\,(\theta_H)^2\,(\theta_L)^2\,\left[\,25-57\,\phi_H+42\,\phi_H^2\,\right]$

$$+ \ \phi_H\,\theta_H\,(\theta_L)^3\,\left[\,22+\phi_H\left(-81+96\,\phi_H-44\,\phi_H^2\,\right)\right]$$
$$+ \ 2\,[\,1-2\,\phi_H\,]^2\,(\theta_L)^4\,[\,3+2\,(\,-2+\phi_H\,)\,\phi_H\,]\,.$$

Finding 9 in the paper establishes that: (i) $\widetilde{E}'(\widetilde{\phi}_H) = 0$; and (ii) $\widetilde{E}\,(\phi_H)$ is concave for $\phi_H \leq \frac{2\,\theta_L}{\theta_H+\theta_L}$ and convex for $\phi_H > \frac{2\,\theta_L}{\theta_H+\theta_L}$. Therefore, $\widetilde{\phi}_H \ \leq \ \frac{2\,\theta_L}{\theta_H+\theta_L}$.

We now establish that:

$$f(\phi_H) \;\equiv\; \phi_H\,[\,\theta_H - 2\,\theta_L\,] + \theta_L \;\geq\; 0 \text{ for all } \phi_H \in [\,0,\, \frac{2\,\theta_L}{\theta_H + \theta_L}\,]. \tag{6}$$

To do so, observe that $f(\phi_H) > 0$ if $y \equiv \frac{\theta_H}{\theta_L} \geq 2$. If $y < 2$, then $f(\phi_H)$ is a decreasing function of ϕ_H. Furthermore:

$$f\left(\frac{2\,\theta_L}{\theta_H + \theta_L}\right) = \left[\frac{2\,\theta_L}{\theta_H + \theta_L}\right][\,\theta_H - 2\,\theta_L\,] + \theta_L$$

$$= \frac{\theta_L\,[\,2\,(\,\theta_H - 2\,\theta_L\,) + \theta_H + \theta_L\,]}{\theta_H + \theta_L} = \frac{3\,\theta_L\,[\,\theta_H - \theta_L\,]}{\theta_H + \theta_L} > 0\,.$$

(5) and (6) imply that $g(\phi_H)$ and Γ have the same sign for all $\phi_H \in (\,0, \widetilde{\phi}_H\,]$. We now show that $\Gamma > 0$ for all $\phi_H \in [\,0, \widetilde{\phi}_H\,]$. To do so, it can be verified that:

$$\begin{aligned}
\Gamma(\phi_H) \;=\;& 6\,(\theta_L)^4 + 2\,\phi_H\,[\,11\,\theta_H - 16\,\theta_L\,]\,(\theta_L)^3 + \phi_H^2\,(\theta_L)^2\left[\,25\,(\theta_H)^2 - 81\,\theta_H\,\theta_L + 60\,(\theta_L)^2\,\right] \\
&+ 3\,\phi_H^3\,\theta_L\,[\,\theta_H - 4\,\theta_L\,]\,[\,3\,\theta_H - 4\,\theta_L\,]\,[\,\theta_H - \theta_L\,] \\
&+ 2\,\phi_H^4\,[\,\theta_H - \theta_L\,]^2\,[\,\theta_H - 4\,\theta_L\,]\,[\,\theta_H - 2\,\theta_L\,]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad \frac{\Gamma(\phi_H)}{(\theta_L)^4} \;=\;& 6 + 2\,\phi_H\left[11\left(\frac{\theta_H}{\theta_L}\right) - 16\right] + \phi_H^2\left[25\left(\frac{\theta_H}{\theta_L}\right)^2 - 81\left(\frac{\theta_H}{\theta_L}\right) + 60\right] \\
&+ 3\,\phi_H^3\left[\frac{\theta_H}{\theta_L} - 4\right]\left[3\left(\frac{\theta_H}{\theta_L}\right) - 4\right]\left[\frac{\theta_H}{\theta_L} - 1\right] \\
&+ 2\,\phi_H^4\left[\frac{\theta_H}{\theta_L} - 1\right]^2\left[\frac{\theta_H}{\theta_L} - 4\right]\left[\frac{\theta_H}{\theta_L} - 2\right]
\end{aligned}$$

$$\begin{aligned}
=\;& 6 + 2\,\phi_H\,[\,11\,y - 16\,] + \phi_H^2\left[\,25\,y^2 - 81\,y + 60\,\right] \\
&+ 3\,\phi_H^3\,[\,y - 4\,]\,[\,3\,y - 4\,]\,[\,y - 1\,] + 2\,\phi_H^4\,[\,y - 1\,]^2\,[\,y - 4\,]\,[\,y - 2\,]\,.
\end{aligned} \tag{7}$$

$$\text{Define:} \qquad \widetilde{\Gamma}(\phi_H) \;\equiv\; \frac{\Gamma(\phi_H)}{(\theta_L)^4}\,. \tag{8}$$

It can be verified that:

$$25\,(y)^2 - 81\,y + 60 \;=\; 25\;[\,y - r_1\,]\,[\,y - r_2\,] \tag{9}$$

where $r_1 \approx 1.14629$ and $r_2 \approx 2.09371$.

(7), (8), and (9) imply:

$$\widetilde{\Gamma}(\phi_H) = 6 + 22\,\phi_H \left[y - \frac{16}{11} \right] + 25\;\phi_H^2\,[\,y - r_1\,]\,[\,y - r_2\,]$$
$$+\; 9\,\phi_H^3\,[\,y-4\,]\left[y - \frac{4}{3} \right][\,y-1\,] + 2\,\phi_H^4\;\;[\,y-1\,]^2\,[\,y-4\,]\,[\,y-2\,]\,. \tag{10}$$

We now demonstrate that $\widetilde{\Gamma}(\phi_H) > 0$ for all relevant values of y.

Case 1. $y \geq 4$.

(10) implies that $\widetilde{\Gamma}(\phi_H) > 0$ in this case. ▲

Case 2. $y \in [\,r_2,\, 4\,)$.

(10) implies that $\widetilde{\Gamma}(0) > 0$ in this case. Because $\widetilde{E}'(\widetilde{\phi}_H) = 0$, (1) and Findings 9 and 10 in the paper imply that $g(\widetilde{\phi}_H) > 0$. Therefore, (5) and (8) imply that $\widetilde{\Gamma}(\widetilde{\phi}_H) > 0$.

Differentiating (10) provides:

$$\widetilde{\Gamma}'(\phi_H) = 22 \left[y - \frac{16}{11} \right] + 50\,\phi_H\,[\,y - r_1\,]\,[\,y - r_2\,]$$
$$+\; 27\,\phi_H^2\,[\,y-4\,]\left[y - \frac{4}{3} \right][\,y-1\,] + 8\,\phi_H^3\;\;[\,y-1\,]^2\,[\,y-4\,]\,[\,y-2\,]\,. \tag{11}$$

(11) implies that $\widetilde{\Gamma}'(0) > 0$ for all $y \in [\,r_2,\, 4\,)$. Differentiating (11) provides:

$$\widetilde{\Gamma}''(\phi_H) = 50\,[\,y - r_1\,]\,[\,y - r_2\,] + 54\,\phi_H\,[\,y-4\,]\left[y - \frac{4}{3} \right][\,y-1\,]$$
$$+\; 24\,\phi_H^2\;\;[\,y-1\,]^2\,[\,y-4\,]\,[\,y-2\,]\,. \tag{12}$$

(12) implies that $\widetilde{\Gamma}''(0) \geq 0$ for all $y \in [\,r_2,\, 4\,)$. Differentiating (12) provides:

$$\widetilde{\Gamma}'''(\phi_H) = 54\,[\,y-4\,]\left[y - \frac{4}{3} \right][\,y-1\,] + 48\,\phi_H\;\;[\,y-1\,]^2\,[\,y-4\,]\,[\,y-2\,]\,. \tag{13}$$

(13) implies that $\widetilde{\Gamma}'''(\phi_H) < 0$ for all $y \in [\,r_2,\, 4\,)$. Therefore, $\widetilde{\Gamma}''(\phi_H)$ is a decreasing function of ϕ_H, so $\widetilde{\Gamma}(\phi_H)$ is either convex for all $\phi_H \in [\,0,\, \widetilde{\phi}_H\,]$ or initially convex and then concave on this interval. Consequently, $\widetilde{\Gamma}(\phi_H) > 0$ for all $\phi_H \in [\,0, \widetilde{\phi}_H\,]$ because $\widetilde{\Gamma}(0) > 0$, $\widetilde{\Gamma}'(0) > 0$, and $\widetilde{\Gamma}(\widetilde{\phi}_H) > 0$. ▲

Case 3. $y \in [2, r_2)$.

(10) implies that $\widetilde{\Gamma}(0) > 0$ in this case. (11) implies that $\widetilde{\Gamma}'(0) > 0$ in this case. (12) implies that $\widetilde{\Gamma}''(\phi_H) < 0$ for all $\phi_H \in [0, \widetilde{\phi}_H]$ in this case, so $\widetilde{\Gamma}(\phi_H)$ is a concave function of ϕ_H. Consequently, $\widetilde{\Gamma}(\phi_H) > 0$ for all $\phi_H \in [0, \widetilde{\phi}_H]$ in this case because $\widetilde{\Gamma}(\widetilde{\phi}_H) > 0$. ▲

Case 4. $y \in [1.20635, 2)$.

(12) implies that $\widetilde{\Gamma}''(0) = 50[y - r_1][y - r_2] < 0$. Therefore, $h(0) < 0$, where $h(\phi_H) \equiv \widetilde{\Gamma}''(\phi_H)$.

Differentiating (13) provides:

$$h''(\phi_H) = \widetilde{\Gamma}''''(\phi_H) = 48[y-1]^2[y-4][y-2] > 0. \tag{14}$$

(14) implies that $h(\phi_H)$ is convex in ϕ_H. Therefore, because $h(0) < 0$, if $h(\widetilde{\phi}_H) \leq 0$, then $h(\phi_H) \leq 0$ (and so $\widetilde{\Gamma}''(\phi_H) \leq 0$) for all $\phi_H \in [0, \widetilde{\phi}_H]$. Consequently, if $h(\widetilde{\phi}_H) \leq 0$, then $\widetilde{\Gamma}(\phi_H)$ is concave in ϕ_H for all $\phi_H \in [0, \widetilde{\phi}_H]$ and so $\widetilde{\Gamma}(\phi_H) > 0$ for all $\phi_H \in [0, \widetilde{\phi}_H]$ (because $\widetilde{\Gamma}(0) > 0$ and $\widetilde{\Gamma}(\widetilde{\phi}_H) > 0$). We now proceed to complete the proof by demonstrating that $h(\widetilde{\phi}_H) \leq 0$ in this case.

(12) implies:

$$\begin{aligned} h(\widetilde{\phi}_H) &= 50[y-r_1][y-r_2] + 54\,\widetilde{\phi}_H[y-4]\left[y - \frac{4}{3}\right][y-1] \\ &\quad + 24(\widetilde{\phi}_H)^2[y-1]^2[y-4][y-2]. \end{aligned} \tag{15}$$

Solving $\widetilde{E}'(\phi_H) = 0$ provides:

$$\widetilde{\phi}_H = \frac{-(\theta_H + \theta_L) + \sqrt{(\theta_H - \theta_L)(\theta_H + 7\,\theta_L)}}{2[\theta_H - 2\,\theta_L]} = \frac{-(y+1) + \sqrt{(y-1)(y+7)}}{2[y-2]}. \tag{16}$$

Define $\widetilde{\psi}(y) \equiv h(\widetilde{\phi}_H)$. Then (15) and (16) imply:

$$\begin{aligned} \widetilde{\psi}(y) &= 50[y-r_1][y-r_2] \\ &\quad + 54\left[\frac{-(y+1) + \sqrt{(y-1)(y+7)}}{2(y-2)}\right][y-4]\left[y-\frac{4}{3}\right][y-1] \\ &\quad + 24\left[\frac{-(y+1) + \sqrt{(y-1)(y+7)}}{2(y-2)}\right]^2[y-1]^2[y-4][y-2]. \end{aligned} \tag{17}$$

Figure A1 demonstrates that $\widetilde{\psi}(y) > 0$ for $y \in [r_1, 1.20635)$ and $\widetilde{\psi}(y) < 0$ for $y \in (1.20635, 2)$. Therefore, $\widetilde{\psi}(y) \equiv h(\widetilde{\phi}_H) \leq 0$ for $y \in [1.20635, 2)$. ▲

Case 5. $y \in (r_1,\, 1.20635)$.

(12) implies that $\widetilde{\Gamma}''(0) < 0$. (14) implies that $\widetilde{\Gamma}''''(\phi_H) > 0$, so $\widetilde{\Gamma}''(\phi_H)$ is a convex function of ϕ_H. Hence, either $\widetilde{\Gamma}(\phi_H)$ is concave for all $\phi_H \in [0,\, \widetilde{\phi}_H]$ or $\widetilde{\Gamma}(\phi_H)$ is first concave and then convex in this interval. We also know that $\widetilde{\Gamma}(0) > 0$ and $\widetilde{\Gamma}(\widetilde{\phi}_H) > 0$. Therefore, $\widetilde{\Gamma}(\phi_H) > 0$ for all $\phi_H \in [0,\, \widetilde{\phi}_H]$ if $\widetilde{\Gamma}'(\widetilde{\phi}_H) \leq 0$. We now proceed to complete the proof by demonstrating that $\widetilde{\Gamma}'(\widetilde{\phi}_H) \leq 0$ in this case.

Define $\widetilde{\varphi}(y) \equiv \widetilde{\Gamma}'(\widetilde{\phi}_H)$. Then (11) implies:

$$\begin{aligned}\widetilde{\varphi}(y) \;\approx\; & 22\left[y - \frac{16}{11}\right] + 50\left[\frac{-(y+1) + \sqrt{(y-1)(y+7)}}{2(y-2)}\right][y - r_1][y - r_2] \\ & + 27\left[\frac{-(y+1) + \sqrt{(y-1)(y+7)}}{2(y-2)}\right]^2 [y-4]\left[y - \frac{4}{3}\right][y-1] \\ & + 8\left[\frac{-(y+1) + \sqrt{(y-1)(y+7)}}{2(y-2)}\right]^3 [y-1]^2[y-4][y-2]. \qquad (18)\end{aligned}$$

Figure A2 demonstrates that $\widetilde{\varphi}(y) < 0$ for all $y \in (r_1,\, 1.20635)$. Therefore, $\widetilde{\Gamma}'(\widetilde{\phi}_H) \leq 0$ if $y \in [r_1,\, 1.20635]$. ▲

Case 6. $y \in (1,\, r_1]$.

(12) implies that $\widetilde{\Gamma}''(\phi_H) > 0$, so $\widetilde{\Gamma}(\phi_H)$ is convex in ϕ_H. Also, $\widetilde{\Gamma}(0) > 0$ and $\widetilde{\Gamma}(\widetilde{\phi}_H) > 0$. Therefore, $\widetilde{\Gamma}(\phi_H) > 0$ for all $\phi_H \in [0, \widetilde{\phi}_H]$ if $\widetilde{\Gamma}'(\widetilde{\phi}_H) \leq 0$. Define $\widetilde{k}(y) \equiv \widetilde{\Gamma}'(\widetilde{\phi}_H)$. Then (11) implies:

$$\begin{aligned}\widetilde{k}(y) \;=\; & 22\left[y - \frac{16}{11}\right] + 50\left[\frac{-(y+1) + \sqrt{(y-1)(y+7)}}{2(y-2)}\right][y - r_1][y - r_2] \\ & + 27\left[\frac{-(y+1) + \sqrt{(y-1)(y+7)}}{2(y-2)}\right]^2 [y-4]\left[y - \frac{4}{3}\right][y-1] \\ & + 8\left[\frac{-(y+1) + \sqrt{(y-1)(y+7)}}{2(y-2)}\right]^3 [y-1]^2[y-4][y-2]. \qquad (19)\end{aligned}$$

Figure A3 reveals that $\widetilde{k}(y) < 0$ for all $y \in (1,\, r_1]$. Therefore, $\widetilde{\Gamma}(\phi_H) > 0$ for all $\phi_H \in [0,\, \widetilde{\phi}_H]$ in the present case. ▲

Cases 1 – 6 imply that $\widetilde{\Gamma}(\phi_H)$ (and thus $\Gamma(\phi_H)$) is strictly positive for all $\phi_H \in [0,\, \widetilde{\phi}_H]$. Therefore, $g(\phi_H) > 0$ for all $\phi_H \in [0,\, \widetilde{\phi}_H]$. Consequently, (1) and (5) imply that $g(0) = 0$ and $g(\phi_H) > 0$ for all $\phi_H \in (0,\, \widetilde{\phi}_H]$. ■

Proof of Conclusion A1. We first establish that $\phi^*_{HM} > \frac{1}{2}$ when $\theta_L \geq \frac{K'(\frac{1}{2})}{0.2742\,[\,\overline{x} - \underline{x}\,]^2}$. ϕ^*_{HM} is determined by:

$$\widetilde{E}'_M\left(\phi^*_{HM}\right) - K'\left(\phi^*_{HM}\right) \;=\; 0 \tag{20}$$

where $\widetilde{E}'_M\left(\phi_H\right)$ is defined in (2). The proof of Finding 8 in the paper establishes that $\widetilde{E}_M\left(\phi_H\right)$ is concave in ϕ_H. Furthermore, $K\left(\phi_H\right)$ is convex in ϕ_H, by assumption. Therefore, if $\widetilde{E}'_M\left(\frac{1}{2}\right) > K'(\frac{1}{2})$, then $\phi^*_{HM} > \frac{1}{2}$.

From (2):

$$\widetilde{E}'_M(\,\frac{1}{2}\,) \;=\; \frac{\Gamma_M}{2\left(\theta_H\right)^3} \tag{21}$$

where Γ_M is as specified in (3). (3) and (21) imply:

$$\widetilde{E}'_M(\,\frac{1}{2}\,) \;=\; \frac{[\,\theta_H - \theta_L\,]\,[\,\overline{x} - \underline{x}\,]^2\;\widetilde{\Gamma}_M}{2\left(\theta_H\right)^3} \tag{22}$$

$$\text{where}\;\;\widetilde{\Gamma}_M \;=\; \theta_H\left(\theta_L\right)^2 + \frac{1}{4}\,[\,\theta_H - \theta_L\,]^2\;[\,\theta_H + 2\,\theta_L\,]\,. \tag{23}$$

(22) and (23) imply:

$$\widetilde{E}'_M(\,\frac{1}{2}\,) \;=\; \frac{[\,\theta_H - \theta_L\,]\,[\,\overline{x} - \underline{x}\,]^2\left[\,\theta_H\left(\theta_L\right)^2 + \frac{1}{4}\left(\,\theta_H - \theta_L\,\right)^2\;\left(\theta_H + 2\,\theta_L\right)\right]}{2\;\left(\theta_H\right)^3}\,. \tag{24}$$

Differentiating (24) with respect to θ_H and simplifying provides:

$$\frac{\partial\widetilde{E}'_M\left(\frac{1}{2}\right)}{\partial\theta_H} \;=\; \frac{[\,\overline{x} - \underline{x}\,]^2\left[\,\left(\theta_H\right)^4 - \;\left(\theta_H\right)^2\left(\theta_L\right)^2 - 2\,\theta_H\,\theta_L^3 + 6\left(\theta_L\right)^4\,\right]}{8\left(\theta_H\right)^4}\,. \tag{25}$$

It can be verified that the expression to the right of the equality in (25) is positive for all $\theta_H > \theta_L > 0$. Therefore, $\widetilde{E}'_M(\frac{1}{2})$ is increasing in θ_H. Consequently, if $\widetilde{E}'_M\left(\frac{1}{2}\right) > K'(\frac{1}{2})$ at $\theta_H = 2.8\;\theta_L$, then $\widetilde{E}'_M\left(\frac{1}{2}\right) > K'(\frac{1}{2})$ for all $\theta_H > 2.8\;\theta_L$.

(24) implies:

$$\begin{aligned}
\widetilde{E}'_M\left(\,\frac{1}{2}\,\right)\Big|_{\theta_H\,=\,2.8\,\theta_L} \;&=\; \frac{[\,\overline{x} - \underline{x}\,]^2\,[\,2.8\,\theta_L - \theta_L\,]\left[\,(2.8\,\theta_L)\left(\theta_L\right)^2 + \frac{1}{4}\left(\,2.8\,\theta_L - \theta_L\,\right)^2\;\left(2.8\,\theta_L + 2\,\theta_L\right)\right]}{2\left(2.8\;\theta_L\right)^3} \\
&=\; \frac{[\,\overline{x} - \underline{x}\,]^2\;\theta_L\,[\,1.8\,]\left[\,2.8 + \frac{1}{4}\left(\,1.8\right)^2\;(4.8)\,\right]}{2\left(2.8\right)^3} \\
&\approx\; \theta_L\;0.2742\,[\,\overline{x} - \underline{x}\,]^2 \;=\; \frac{0.2742}{2.8}\;[\,\overline{x} - \underline{x}\,]^2\;\theta_H\,.
\end{aligned} \tag{26}$$

Therefore, $\phi^*_{HM} > \frac{1}{2}$ for all $\theta_H \geq 2.8\,\theta_L$ when

$$\frac{0.2742}{2.8}\,[\,\overline{x}-\underline{x}\,]^2\,\theta_H \;\geq\; K'(\frac{1}{2}) \;\Leftrightarrow\; \theta_H \;\geq\; \left[\frac{2.8}{0.2742}\right]\frac{K'(\frac{1}{2})}{[\,\overline{x}-\underline{x}\,]^2} \;=\; \frac{10.212\;K'(\frac{1}{2})}{[\,\overline{x}-\underline{x}\,]^2}$$

$$\Leftrightarrow\; \frac{\theta_H}{\theta_L} \;\geq\; \frac{10.212\;K'(\frac{1}{2})}{\theta_L\,[\,\overline{x}-\underline{x}\,]^2}\,.$$

We now establish that $E\,\Pi_M(\phi^*_{HM}) \;>\; E\,\Pi(\phi^*_H)$. The proof of Lemma 6 in the paper establishes that:

$$E\,\Pi(\phi^*_H) - E\,\Pi_M(\phi^*_{HM}) \;=\; \frac{[\,\theta_H-\theta_L\,]^2\,[\,\overline{x}-\underline{x}\,]^2\,\Gamma}{2\,[\,\phi^*_H\,(\theta_H-2\,\theta_L)+\theta_L\,]\,[\,2\,\phi^*_{HM}\,(\theta_H-\theta_L)+\theta_L\,]} \tag{27}$$

where, for $y \;\equiv\; \frac{\theta_H}{\theta_L}$:

$$\Gamma \;\equiv\; \phi^*_H\,[\,2\,\phi^*_H-\phi^*_{HM}\,]\,\phi^*_{HM}\,\theta_H + [\,-\,\phi^*_H+\phi^*_{HM}\,]\,[\,-\,\phi^*_{HM}+\phi^*_H\,(-\,1+2\,\phi^*_{HM})\,]\,\theta_L$$

$$=\; \theta_L\,\{\,\phi^*_H\,[\,2\,\phi^*_H-\phi^*_{HM}\,]\,\phi^*_{HM}\,y + [\,\phi^*_{HM}-\phi^*_H\,]\,[\,\phi^*_H\,(\,2\,\phi^*_{HM}-1\,)-\phi^*_{HM}\,]\,\}\,. \tag{28}$$

(27) and (28) imply:

$$E\,\Pi_M(\phi^*_{HM}) \;\gtreqless\; E\,\Pi(\phi^*_H) \quad\Leftrightarrow\quad \Gamma \;\lesseqgtr\; 0\,. \tag{29}$$

Because $\phi^*_{HM} \geq \frac{1}{2}$, (28) implies:

$$\Gamma \;\leq\; 0 \quad\text{if}\quad \phi^*_H \;\leq\; \frac{1}{4}\,. \tag{30}$$

(30) holds because when $\phi^*_H \;\leq\; \frac{1}{4}$ and $\phi^*_{HM} \;\geq\; \frac{1}{2}$:

$$\phi^*_H\,[\,2\,\phi^*_{HM}-1\,]-\phi^*_{HM} \;\leq\; \frac{1}{4}\,[\,2\,\phi^*_{HM}-1\,]-\phi^*_{HM} \;=\; -\,\frac{1}{2}\,\phi^*_{HM}-\frac{1}{4} \;<\; 0$$

$$\Rightarrow\quad [\,\phi^*_{HM}-\phi^*_H\,]\,[\,\phi^*_H\,(\,2\,\phi^*_{HM}-1\,)-\phi^*_{HM}\,] \;<\; 0\,. \tag{31}$$

Furthermore:

$$2\,\phi^*_H \;\leq\; \frac{1}{2} \;\leq\; \phi^*_{HM} \quad\Rightarrow\quad 2\,\phi^*_H-\phi^*_{HM} \;\leq\; 0$$

$$\Rightarrow\quad \phi^*_H\,[\,2\,\phi^*_H-\phi^*_{HM}\,]\,\phi^*_{HM}\,y \;\leq\; 0\,. \tag{32}$$

(28), (31), and (32) imply that (30) holds.

(16) implies that when $K\,(\phi_H) \;=\; 0$ for all $\phi_H \in [\,0,1\,]$:

$$\phi^*_H \;=\; \frac{-\,(\theta_H+\theta_L)+\sqrt{[\,\theta_H-\theta_L\,]\,[\,\theta_H+7\,\theta_L\,]}}{2\,[\,\theta_H-2\,\theta_L\,]}\,. \tag{33}$$

(33) implies that under the maintained assumptions that $K(0) \geq 0$ and $K'(\phi_H) > 0$:

$$\phi_H^* \; < \; \frac{-\,(\theta_H + \theta_L) + \sqrt{[\,\theta_H - \theta_L\,]\,[\,\theta_H + 7\,\theta_L\,]}}{2\,[\,\theta_H - 2\,\theta_L\,]}. \tag{34}$$

(34) implies that $\phi_H^* \leq \frac{1}{4}$ if:

$$\frac{-\,(\theta_H + \theta_L) + \sqrt{[\,\theta_H - \theta_L\,]\,[\,\theta_H + 7\,\theta_L\,]}}{2\,[\,\theta_H - 2\,\theta_L]} \;\leq\; \frac{1}{4}$$

$$\Leftrightarrow \quad \frac{-\,(y+1) + \sqrt{[\,y-1\,]\,[\,y+7\,]}}{2\,[\,y-2\,]} \;\leq\; \frac{1}{4}\,. \tag{35}$$

When $y > 2$, (35) implies that $\phi_H^* \leq \frac{1}{4}$ if:

$$-\,2\,[\,y+1\,] + 2\sqrt{[\,y-1\,]\,[\,y+7\,]} \;\leq\; y-2$$

$$\Leftrightarrow \quad 2\sqrt{[\,y-1\,]\,[\,y+7\,]} \;\leq\; 3\,y \quad \Leftrightarrow \quad 4\,[\,y-1\,]\,[\,y+7\,] \;\leq\; [\,3\,y\,]^2$$

$$\Leftrightarrow \quad 4\left[\,y^2 + 6\,y - 7\,\right] \;\leq\; 9\,y^2 \quad \Leftrightarrow \quad 5\,y^2 - 24\,y + 28 \;\geq\; 0\,. \tag{36}$$

It can be verified that $5\,y^2 - 24\,y + 28 \geq 0$ if $y \geq 2.8$. Therefore, (36) implies that $\phi_H^* \leq \frac{1}{4}$ if $y \equiv \frac{\theta_H}{\theta_L} \geq 2.8 \Leftrightarrow \theta_H \geq 2.8\,\theta_L$. Consequently, $\Gamma < 0$ (from (28)) and so $E\,\Pi_M(\phi_{HM}^*) > E\,\Pi(\phi_H^*)$ (from (29)) under the maintained conditions. ■