Appendix to Accompany

"Designing Optimal Gain Sharing Plans to Promote Energy Conservation"

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The two problems considered in the text are the following:

Problem [P]

$$\underset{s_i,\, F_i \geq -\underline{F}}{\text{Maximize}} \quad \sum_{i=1}^{2} \phi_i \left\{ [1 - s_i]\, G_i - F_i + \alpha\, \pi_i(F_i, s_i) \right\} \tag{1}$$

subject to, for $j \neq i$, $i, j \in \{1, 2\}$:

$$\pi_i(F_i, s_i) \equiv F_i + s_i G_i - K(G_i, k_i) \geq 0\,; \text{ and} \tag{2}$$

$$\pi_i(F_i, s_i) \geq F_j + s_j G_{ji} - K(G_{ji}, k_i)\,, \tag{3}$$

where

$$G_{ji} = \underset{G}{\arg\max} \left\{ F_j + s_j G - K(G, k_i) \right\} \quad \text{and} \quad G_i = G_{ii}\,. \tag{4}$$

Problem [P-1]

$$\underset{s,\, F \geq -\underline{F}}{\text{Maximize}} \quad \sum_{i=1}^{2} \phi_i \left\{ [1 - s]\, G_i - F + \alpha\, [F + s\, G_i - K(G_i, k_i)] \right\} \tag{5}$$

subject to, for $i = 1, 2$:

$$F + s\, G_i - K(G_i, k_i) \geq 0\,, \tag{6}$$

where

$$G_i = \underset{G}{\arg\max} \left\{ F + s\, G - K(G, k_i) \right\}\,. \tag{7}$$

Observation 1. *Suppose $\underline{F} \geq G_i^* - K(G_i^*, k_i)$ and the regulator knows $k = k_i$. Then she can secure the same expected payoff she achieves in the full information setting by awarding the firm the entire realized gain (so $s = 1$) and setting the fixed payment to ensure exactly zero expected profit for the firm (i.e., $F = -\{G_i^* - K(G_i^*, k_i)\}$).*

Proof. It is apparent from (4) that the firm will implement expected gain G_i^* when $s = 1$. The firm's expected profit will be $F + G_i^* - K(G_i^*, k_i) = 0$ when $F = -\{G_i^* - K(G_i^*, k_i)\}$. This gain sharing plan is feasible under the maintained assumptions. Because the plan maximizes the total expected surplus $(G - K(G, k_i))$ and eliminates the firm's rent, the plan secures for the regulator the same expected payoff she achieves in the full information setting. ■

Observation 2. *Suppose* $\underline{F} < G_i^* - K(G_i^*, k_i)$ *and the regulator knows the prevailing cost environment* (k_i). *Then the regulator optimally sets* $F = \underline{F}$ *and* $s < 1$. *The share of the realized gain delivered to the firm* (s) *declines as the maximum loss the firm can be compelled to bear* ($\underline{F}$) *declines.*

Proof. Let [P-k] denote the regulator's problem when she knows the prevailing cost parameter is k. This problem is:

$$\underset{s,\, F \geq -\underline{F}}{\text{Maximize}} \quad -F + [1-s]\,G + \alpha\,\{\,F + s\,G - K(G,k)\,\}$$

$$\text{subject to:} \qquad F + s\,G - K(G,k) \;\geq\; 0\,, \tag{8}$$

$$\text{where} \qquad K_G(G,k) \;=\; s\,. \tag{9}$$

Let λ denote the Lagrange multiplier associated with constraint (8), and let $\underline{\lambda}$ denote the Lagrange multiplier associated with the $F \geq -\underline{F}$ constraint. Then the necessary conditions for a solution to [P-k] are:

$$s: \qquad G\,[-1+\alpha+\lambda\,] + [1-s]\,\frac{dG}{ds} \;=\; 0\,; \quad \text{and} \tag{10}$$

$$F: \qquad -1+\alpha+\lambda+\underline{\lambda} \;=\; 0\,. \tag{11}$$

From (9):

$$\frac{dG}{ds} \;=\; \frac{1}{K_{GG}(G,k)} \;>\; 0. \tag{12}$$

It is readily verified that the $F \geq -\underline{F}$ constraint binds at the solution to [P-k] when $\underline{F} < G_i^* - K(G_i^*, k_i)$. Consequently, $\underline{\lambda} > 0$. Therefore, from (10), (11), and (12):

$$[1-s]\,\frac{dG}{ds} \;=\; G\,\underline{\lambda} \;>\; 0 \quad \Rightarrow \quad s \;<\; 1\,.$$

If $\underline{F}$ is sufficiently small that constraint (8) does not bind at the solution to [P-k], then $\lambda = 0$, and so $\underline{\lambda} = 1-\alpha > 0$, from (11). Consequently, $F = \underline{F}$. Furthermore, from (10) and (12):

$$\frac{1-s}{K_{GG}(G,k)} \;=\; [1-\alpha]\,G \quad \Rightarrow \quad 1-s \;=\; [1-\alpha]\,G\,K_{GG}(G,k)$$

$$\Rightarrow \quad s \;=\; 1-[1-\alpha]\,G\,K_{GG}(G,k) \;\equiv\; \widetilde{s}\,.$$

Let $\underline{F}^o$ denote the largest value of $\underline{F}$ for which constraint (8) does not bind at the solution to [P-k]. Then as $\underline{F}$ increases from $\underline{F}^o$ to $\underline{F}^* \equiv G_i^* - K(G_i^*, k_i)$, s increases monotonically from $\widetilde{s}$ to 1. This is the case because the $F \geq -\underline{F}$ constraint and constraint (8) both bind at the solution to [P-k] for all $\underline{F} \in (\underline{F}^o, \underline{F}^*)$. Therefore:

$$-\underline{F} + s\,G - K(G,k) \;=\; 0\,. \tag{13}$$

Differentiating (13) and using (9) provides:

$$-d\,\underline{F} + \left\{ G + [\, s - K_G(G,k)\,] \frac{dG}{ds} \right\} ds \;=\; 0$$

$$\Rightarrow \quad -\,d\,\underline{F} + G\;ds \;=\; 0 \quad \Rightarrow \quad \frac{ds}{d\,\underline{F}} \;=\; \frac{1}{G} \;>\; 0\,. \quad \blacksquare$$

Lemma 1. $\Delta\pi(F,s)$ *is strictly increasing in* s*.*

Proof.

$$\Delta\pi(F,s) \;=\; \max_G \,\{\, F + s\,G - K(G,k_1)\,\} \;-\; \max_G \,\{\, F + s\,G - K(G,k_2)\,\}\,. \tag{14}$$

(14) and the envelope theorem imply:

$$\frac{d\Delta\pi(F,s)}{ds} \;=\; G_1(s) - G_2(s) \;>\; 0\,, \quad \text{where} \;\; G_i(s) \;=\; \max_G \,\{\, s\,G - K(G,k_i)\,\}\,. \tag{15}$$

The inequality in (15) holds because $K_G(G_1(s),k_1) \;=\; s \;=\; K_G(G_2(s),k_2)$, $K_{GG}(G,k_i) > 0$ for $i = 1,2$, and $K_G(G,k_2) > K_G(G,k_1)$ for all $G > 0$. $\blacksquare$

Conclusion 1. *There exist two distinct values of* $\underline{F}$*, namely* $\underline{F}_L < \underline{F}_H$*, such that at the solution to [P-1], the optimal single gain sharing plan has the following features:*

(i) *If* $\underline{F} \geq \underline{F}_H$ *, then* $s \;=\; \overline{s} \;<\; 1$, $\frac{d\overline{s}}{d\underline{F}} = 0$*, and* $\pi_2 = 0$*.*

(ii) *If* $\underline{F} \in (\underline{F}_L, \underline{F}_H)$*, then* $s \in (\underline{s}, \overline{s})$, $\frac{ds}{d\underline{F}} > 0$, $F \;=\; -\,\underline{F}$ *, and* $\pi_2 = 0$*.*

(iii) *If* $\underline{F} \;\leq\; \underline{F}_L$*, then* $s \;=\; \underline{s} \;<\; \overline{s}$, $\frac{ds}{d\underline{F}} \;=\; 0$, $F \;=\; -\,\underline{F}$*, and* $\pi_2 \;\geq\; 0$*, with strict inequality if and only if* $\underline{F} \;<\; \underline{F}_L$*.*

Proof. Let λ_i denote the Lagrange multiplier associated with constraint (6), and let $\underline{\lambda}$ denote the Lagrange multiplier associated with the $F \;\geq\; -\underline{F}$ constraint. Then the necessary conditions for a solution to [P-1] include:

$$s: \qquad \sum_{i=1}^{2} G_i\,[-\phi_i\,(1-\alpha) + \lambda_i] + \sum_{i=1}^{2} \phi_i\,[1-s]\,\frac{dG_i}{ds} \;=\; 0\,; \quad \text{and} \tag{16}$$

$$F: \qquad -\,1 + \alpha + \lambda_1 + \lambda_2 + \underline{\lambda} \;=\; 0\,. \tag{17}$$

From (7):

$$s \;=\; K_G(G_i,k_i) \quad \Rightarrow \quad \frac{dG_i}{ds} \;=\; \frac{1}{K_{GG}(G_i,k_i)} \;>\; 0\,. \tag{18}$$

Since $K(G,k_2) \;>\; K(G,k_1)$ for all $G \;>\; 0$, constraint (6) does not bind for $i \;=\; 1$. Therefore, $\lambda_1 \;=\; 0$ at the solution to [P-1]. Consequently, (17) provides:

$$\lambda_2 \;=\; 1 - \alpha - \underline{\lambda}\,. \tag{19}$$

Define problem [P-1]$'$ to be problem [P-1] without the participation constraints (6) imposed. (19) implies that $\underline{\lambda} = 1 - \alpha > 0$ at the solution to [P-1]$'$, and so $F = -\underline{F}$. Furthermore, from (16):

$$[1-s]\sum_{i=1}^{2} \phi_i \frac{dG_i}{ds} \;=\; \phi_1\,[1-\alpha]\,G_1 + \phi_2\,[1-\alpha]\,G_2 \tag{20}$$

Let $\underline{s}$ denote the value of s that solves (20). Then $(-\underline{F}, \underline{s})$ is the solution to [P-1]$'$.

Define $\underline{F}_L$ to be the largest value of $\underline{F}$ for which no participation constraint binds at the solution to [P-1] (so $\underline{F}_L = \max_G \{\, \underline{s}\,G - K(G,k_2)\}\,$). Observe that if $\underline{F} \le \underline{F}_L$, then $(-\underline{F}, \underline{s})$, the solution to [P-1]$'$, is a feasible solution to [P-1], and so is the solution to [P-1]. Note from (20) that $\frac{d\,s}{d\underline{F}} = 0$ when $\underline{F} < \underline{F}_L$.

Now define problem [P-1]$''$ to be problem [P-1] without the $F \ge -\underline{F}$ constraint imposed. (19) implies that $\lambda_2 = 1 - \alpha > 0$ at the solution to [P-1]$''$, and so $\pi_2 = 0$. Furthermore, from (16):

$$-\,\phi_1\,[1-\alpha]\,G_1 + [1-\alpha]\,G_2\,[1-\phi_2] + [1-s]\sum_{i=1}^{2}\phi_i\frac{dG_i}{ds} \;=\; 0$$

$$\Leftrightarrow \quad [1-s]\sum_{i=1}^{2}\phi_i\frac{dG_i}{ds} \;=\; \phi_1\,[1-\alpha]\,[G_1 - G_2] \;>\; 0\,. \tag{21}$$

The inequality in (21) holds because $G_1 > G_2$ from (7), since $K_G(G,k_2) > K_G(G,k_1)$. Since $\frac{dG_i}{ds} > 0$ for $i = 1, 2$ from (18), (21) implies that $s < 1$. Let $\overline{s}$ denote the value of s that solves the equality in (21).

Define $\underline{F}_H$ to be the smallest value of $\underline{F}$ for which the solution to [P-1]$''$ is a feasible solution (and thus the solution) to [P-1].

It remains to show that $\underline{s} < \overline{s}$, and so $\underline{F}_L < \underline{F}_H$, since:

$$-\,\underline{F}_L + \max_G\,\{\underline{s}\,G - K(G,k_2)\} \;=\; 0 \;=\; -\,\underline{F}_H + \max_G\,\{\,\overline{s}\,G - K(G,k_2)\}$$

$$\Rightarrow \;\; \underline{F}_L \;=\; \underline{F}_H + \max_G\,\{\underline{s}\,G - K(G,k_2)\} - \max_G\,\{\overline{s}\,G - K(G,k_2)\} < \underline{F}_H \;\text{ when }\; \underline{s} < \overline{s}\,.$$

First observe from (20) and (21) that $\overline{s} \ne \underline{s}$. Now suppose that $\underline{s} > \overline{s}$, and so $\underline{F}_L > \underline{F}_H$. Consider two values of $\underline{F}$, namely $\underline{F}_1$ and $\underline{F}_2$, such that $\underline{F}_1 \ne \underline{F}_2$ and $\underline{F}_1, \underline{F}_2 \in (\underline{F}_H, \underline{F}_L)$. If $\underline{F} = \underline{F}_i$ for $i = 1$ or $i = 2$, then $(-\underline{F}_H, \overline{s})$, the solution to [P-1]$'$, remains a feasible solution to [P-1] since $\underline{F}_i > \underline{F}_H$. Hence, $(-\underline{F}_H, \overline{s})$ is a solution to [P-1].

Furthermore, $(-\underline{F}_i, \underline{s})$, the solution to [P-1]$''$ when $\underline{F} = \underline{F}_i$, remains a feasible solution to [P-1] since $\underline{F}_i < \underline{F}_L$. Hence, $(-\underline{F}_i, \underline{s})$ is a solution to [P-1]. Therefore, the regulator is indifferent between the $(-\underline{F}_H, \overline{s})$ and the $(-\underline{F}_i, \underline{s})$ plans for $i = 1$ and $i = 2$. Consequently, the regulator must be indifferent between the $(-\underline{F}_1, \underline{s})$ plan and the $(-\underline{F}_2, \underline{s})$ plan. However, the regulator strictly prefers the $(-\underline{F}_2, \underline{s})$ plan to the $(-\underline{F}_1, \underline{s})$ plan because the former provides systematically less compensation for the firm and the two plans generate the same total expected surplus. Therefore, by contradiction, it must be the case that $\underline{s} < \overline{s}$, and so $\underline{F}_L < \underline{F}_H$.

Three possibilities arise at the solution to [P-1]: (i) the participation constraint (6) when $k = k_2$ is the unique binding constraint; (ii) the $F \geq -\underline{F}$ constraint is the unique binding constraint; or (iii) both constraints bind. We have shown that possibility (i) arises if and only if $\underline{F} \geq \underline{F}_H$. We have also shown that possibility (ii) arises if and only if $\underline{F} \leq \underline{F}_L$. Therefore, possibility (iii) arises if and only if $\underline{F} \in (\underline{F}_L, \underline{F}_H)$. In this case, $F = -\underline{F}$ and:

$$-\underline{F} + s\,G_2 - K(G_2, k_2) = 0 \;\Rightarrow\; -d\underline{F} + G_2\,ds = 0 \;\Rightarrow\; \frac{ds}{d\underline{F}} = \frac{1}{G_2} > 0\,. \quad \blacksquare$$

Conclusion 2. *There exist two values of $\underline{F}$, namely $\underline{F}_L < \widehat{\underline{F}}_H$, such that, at the solution to [P], the optimal pair of gain sharing plans $\{(F_1, s_1), (F_2, s_2)\}$ has the following properties:*

(i) If $\underline{F} \geq \widehat{\underline{F}}_H$, then $s_1 = 1$, $s_2 = \overline{s}_2 < 1$, $F_1 < F_2$, $\frac{d\,\overline{s}_2}{d\underline{F}} = 0$, and $\widehat{\pi}_2 = 0$.

(ii) If $\underline{F} \in [\underline{F}_L, \widehat{\underline{F}}_H)$, then $s_2 \leq s_1 < 1$, $F_2 \geq F_1 = -\underline{F}$, and $\widehat{\pi}_2 = 0$. In addition, if $K_{GGG}(G, k_i) \geq 0$ and $K_{GG}(G, k_2) \geq K_{GG}(G, k_1)$ for all G and for $k_i \in \{k_1, k_2\}$, then there exists an $\widehat{\underline{F}}_L \in [\underline{F}_L, \widehat{\underline{F}}_H)$, such that $s_1 = s_2$ for $\underline{F} \in [\underline{F}_L, \widehat{\underline{F}}_L]$, whereas $s_2 < s_1$ for $\underline{F} \in (\widehat{\underline{F}}_L, \widehat{\underline{F}}_H)$. Furthermore, $\frac{ds_1}{d\underline{F}} = \frac{ds_2}{d\underline{F}} > 0$ for $\underline{F} \in (\underline{F}_L, \widehat{\underline{F}}_L)$, whereas $\frac{ds_1}{d\underline{F}} > 0$, $\frac{ds_2}{d\underline{F}} < 0$, $\frac{dF_1}{d\underline{F}} < 0$, and $\frac{dF_2}{d\underline{F}} > 0$ for $\underline{F} \in (\widehat{\underline{F}}_L, \widehat{\underline{F}}_H)$.

(iii) If $\underline{F} < \underline{F}_L$, then $s_1 = s_2 = \underline{s}$, $F_1 = F_2 = -\underline{F}$, $\frac{d\,s}{d\underline{F}} = 0$, and $\widehat{\pi}_2 > 0$.

Proof. Let λ_i and λ_{ij} denote the Lagrange multipliers associated with constraints (2) and (3), respectively. Also let $\underline{\lambda}_i$ denote the Lagrange multiplier associated with the $F_i \geq -\underline{F}$ constraint. Then the necessary conditions for a solution to [P] include:

$$s_i: \quad G_i\,[-\phi_i\,(1-\alpha) + \lambda_i + \lambda_{ij}] - \lambda_{ji}\,G_{ij} + \phi_i\,[1 - s_i]\,\frac{dG_i}{ds_i} = 0; \text{ and} \tag{22}$$

$$F_i: \quad -\phi_i\,[1-\alpha] + \lambda_i + \lambda_{ij} - \lambda_{ji} + \underline{\lambda}_i = 0\,. \tag{23}$$

(22) and (23) provide:

$$\phi_i\,[1 - s_i]\,\frac{dG_i}{ds_i} = \lambda_{ji}\,[G_{ij} - G_i] + \underline{\lambda}_i\,G_i \quad \text{for } j \neq i, \quad i, j \in \{1, 2\}\,. \tag{24}$$

From (4):

$$K_G(G_i, k_i) = s_i \text{ and } K_G(G_{ij}, k_j) = s_i \;\Rightarrow\; G_{21} \geq G_2 \text{ and } G_1 \geq G_{12}\,. \tag{25}$$

The inequalities in (25) hold because $K_G(G, k_1) < K_G(G, k_2)$ and $K(\cdot)$ is an increasing, convex function of G. The inequalities in (25) hold as strict inequalities if a positive expected gain is induced when $k = k_1$.

The following lemmas constitute the remainder of the proof of the Conclusion.

Lemma A1. The participation constraint (2) when $k = k_1$ does not bind at the solution to [P].

Proof. The conclusion holds because the firm's expected profit is strictly higher when $k = k_1$ than when $k = k_2$ under any non-trivial gain sharing plan.[1] $\square$

Lemma A2. $G_1 > G_2$, $F_2 \geq F_1$, and $s_2 \leq s_1$ under any feasible solution to [P] that entails a non-trivial gain sharing plan.

Proof. To show that $G_1 > G_2$, observe that the incentive compatibility constraints (3) ensure:

$$\pi_1(s_1, F_1) - \pi_1(s_2, F_2) \ \geq \ 0 \ \geq \ \pi_2(s_1, F_1) - \pi_2(s_2, F_2)$$

$$\Rightarrow \quad \pi_1(s_1, F_1) + \pi_2(s_2, F_2) \ \geq \ \pi_2(s_1, F_1) + \pi_1(s_2, F_2)\,. \tag{26}$$

Further observe that:

$$\pi_1(s_1, F_1) + \pi_2(s_2, F_2) \ = \ F_1 + s_1\, G_1 - K(G_1, k_1) \ + \ F_2 + s_2\, G_2 - K(G_2, k_2)\,; \text{ and} \tag{27}$$

$$\pi_2(s_1, F_1) + \pi_1(s_2, F_2) \ \geq \ F_1 + s_1\, G_1 - K(G_1, k_2) \ + \ F_2 + s_2\, G_2 - K(G_2, k_1)\,. \tag{28}$$

The inequality in (28) holds because G_i is not necessarily the profit-maximizing expected gain under the (s_i, F_i) gain sharing plan when $k = k_j$ for $j \neq i$. (26), (27), and (28) provide:

$$\begin{aligned} 0 \ &\leq \ \pi_1(s_1, F_1) + \pi_2(s_2, F_2) - [\pi_2(s_1, F_1) + \pi_1(s_2, F_2)] \\ &\leq \ K(G_1, k_2) - K(G_2, k_2) - [K(G_1, k_1) - K(G_2, k_1)] \\ &= \ \int_{G_2}^{G_1} \left[\frac{\partial}{\partial G} K(G, k_2) - \frac{\partial}{\partial G} K(G, k_1) \right] dG \quad \Rightarrow \quad G_1 \ > \ G_2\,. \end{aligned} \tag{29}$$

To show that $s_1 \geq s_2$, observe that:

$$\pi_2(s_1, F_1) + \pi_1(s_2, F_2) \ \geq \ F_1 + s_1\, G_2 - K(G_2, k_2) + F_2 + s_2\, G_1 - K(G_1, k_2)\,. \tag{30}$$

The inequality in (30) holds because G_j is not necessarily the profit-maximizing expected gain under the (s_i, F_i) gain sharing plan when $k = k_j$ for $j \neq i$. (26), (27), and (30) provide:

$$0 \ \leq \ \pi_1(s_1, F_1) + \pi_2(s_2, F_2) - [\pi_2(s_1, F_1) + \pi_1(s_2, F_2)] \ \leq \ [G_1 - G_2][s_1 - s_2]. \tag{31}$$

(31) implies that $s_1 \geq s_2$, since $G_1 > G_2$. Therefore, because incentive compatibility ensures it cannot be the case that $F_1 > F_2$ and $s_1 > s_2$, it must be the case that $F_2 \geq F_1$. $\square$

[1] A non-trivial gain sharing plan (F, s) is one: (i) that the firm selects either when $k = k_1$ or when $k = k_2$; and (ii) in which the firm implements a strictly positive expected gain ($G > 0$) when it operates under the plan.

Lemma A3. The $F_2 \geq -\underline{F}$ limited liability constraint does not bind at the solution to [P].

Proof. From Lemma A2, $F_2 \geq F_1$ under any feasible nontrivial gain sharing plan. Consequently, the $F_2 \geq -\underline{F}$ limited liability constraint will be satisfied at the solution to [P] as long as the $F_1 \geq -\underline{F}$ constraint is imposed. Therefore, the $F_2 \geq -\underline{F}$ limited liability constraint does not bind at the solution to [P]. □

Lemmas A1 and A3 imply that $\lambda_1 = 0$ and $\underline{\lambda}_2 = 0$ at the solution to [P].

Lemma A4. When the regulator offers two distinct, non-trivial gain sharing plans to the firm, the firm cannot be indifferent between the two plans both when $k = k_1$ and when $k = k_2$.

Proof.

$$\frac{\partial}{\partial s}\left\{ \max_{G_i} \; [\, s\, G_i - K(G_i, k_1)] \;\; - \;\; \max_{G_i} \; [\, s\, G_i - K(G_i, k_2)] \right\} \;\; = \;\; G_{i1} - G_{i2} \; \geq \; 0\, . \qquad (32)$$

The inequality in (32), which follows from (25), implies that:

$$\max_{G} \; \{\, s_1\, G - K(G, k_1)\} \;\; - \;\; \max_{G} \; \{\, s_1\, G - K(G, k_2)\}$$

$$\geq \; \max_{G} \; \{\, s_2\, G - K(G, k_1)\} \;\; - \;\; \max_{G} \; \{\, s_2\, G - K(G, k_2)\}\, . \qquad (33)$$

When the firm is indifferent between the two plans both when $k = k_1$ and when $k = k_2$, the weak inequality in (33) will hold as an equality. Consequently, it must be the case that a zero expected gain ($G = 0$) is induced under both plans. But then the plans are not distinct, non-trivial plans. Therefore, when the regulator offers two distinct, non-trivial gain sharing plans to the firm, only one of the incentive compatibility constraints will bind. □

Lemma A5. If neither participation constraint (2) binds at the solution to [P], then the regulator optimally offers only a single gain sharing plan.

Proof. If neither participation constraint binds at the solution to [P], then $\lambda_1 = \lambda_2 = 0$. Consequently, from (23):

$$\underline{\lambda}_1 \;\; = \;\; [1 - \alpha]\, \phi_1 + \lambda_{21} - \lambda_{12} \quad \text{and} \quad \underline{\lambda}_2 \;\; = \;\; [1 - \alpha]\, \phi_2 + \lambda_{12} - \lambda_{21}\, . \qquad (34)$$

Since $\underline{\lambda}_2 = 0$ from Lemma A3, (34) implies that $\lambda_{21} > 0$. (34) also implies that $\underline{\lambda}_1 = \underline{\lambda}_1 + \underline{\lambda}_2 = 1 - \alpha > 0$. Therefore, $F_1 = -\underline{F}$.

(24) and (25) imply:

$$\phi_2\, [1 - s_2]\, \frac{dG_2}{ds_2} \;\; = \;\; \lambda_{12}\, [G_{21} - G_2] + \underline{\lambda}_2\, G_2$$

$$\Rightarrow \quad s_2 \; < \; 1 \text{ if and only if } \underline{\lambda}_2 > 0 \text{ or } \lambda_{12} > 0\, . \qquad (35)$$

Lemma A4 implies that $\lambda_{12} = 0$, since $\lambda_{21} > 0$. Consequently, $s_2 = 1$, from (35). But then it cannot be optimal for the regulator to offer two distinct gain sharing plans because the single (F_2, s_2) plan would deliver no more rent to the firm and would generate a higher level of expected total surplus. $\square$

Lemma A6. Suppose $\underline{F}$ is sufficiently large that the $F_i \geq -\underline{F}$ constraints do not bind at the solution to [P]. Then $s_2 < s_1 = 1$, $\lambda_2 > 0$, and $\lambda_{12} > 0$ at the solution to [P].

Proof. Since $\underline{\lambda}_1 = \underline{\lambda}_2 = 0$ in this case, (23) and Lemma A1 imply that $\lambda_2 = \lambda_1 + \lambda_2 = 1 - \alpha > 0$. (23) also implies that $\lambda_{12} = \lambda_{21} + [1-\alpha]\,\phi_1 > 0$. Therefore, $\lambda_{21} = 0$, from Lemma A4. Consequently, from (24):

$$\phi_1\,[1 - s_1]\,\frac{dG_1}{ds_1} \;=\; 0 \quad\Rightarrow\quad s_1 \;=\; 1\,.$$

(24) implies that when $\underline{\lambda}_2 = 0$:

$$\phi_2\,[1 - s_2]\,\frac{dG_2}{ds_2} \;=\; \lambda_{12}\,[G_{21} - G_2] \;>\; 0 \quad\Rightarrow\quad s_2 \;<\; 1\,. \tag{36}$$

The first inequality in (36) reflects (25). $\square$

Lemma A7. Suppose the participation constraint (2) when $k = k_2$ and the $F_1 \geq -\underline{F}$ limited liability constraint both bind at the solution to [P]. Then $s_2 \leq s_1 < 1$.

Proof. Since $\underline{\lambda}_1 > 0$ in this case, (24) implies:

$$\phi_1[1 - s_1]\,\frac{dG_1}{ds_1} \;>\; 0 \quad\Rightarrow\quad s_1 \;<\; 1\,.$$

Furthermore, $s_2 \leq s_1$ from Lemma A2. Therefore, since $\underline{\lambda}_2 > 0$ from Lemma A3, (24) implies that $\lambda_{12} > 0$. $\square$

Define $\widehat{\underline{F}}_H$ to be the smallest value of $\underline{F}$ for which the $F_1 \geq -\underline{F}$ constraint does not bind at the solution to [P]. Then Lemma A6 implies that when $\underline{F} \geq \widehat{\underline{F}}_H$, $s_2 < s_1 = 1$, $\widehat{\pi}_2 = 0$, $F_2 > F_1 = -\widehat{\underline{F}}_H$, and the firm secures the same expected profit under the two gain sharing plans in the low cost environment at the solution to [P].

Recall that $\underline{F}_L = \max\limits_{G} \{\underline{s}\,G - K(G, k_2)\}$ is the largest value of $\underline{F}$ for which no participation constraint binds at the solution to [P-1]. Lemma A5 implies that the solution to [P] is the solution to [P-1] when $\underline{F} \leq \underline{F}_L$. Therefore, from the proof of Conclusion 1, $s_1 = s_2 = \underline{s}$, $F_1 = F_2 = -\underline{F}$, $\frac{d\,\underline{s}}{d\underline{F}} = 0$, and $\widehat{\pi}_2 > 0$ at the solution to [P] when $\underline{F} \leq \underline{F}_L$.

The definition of $\underline{F}_L$ and Lemma A1 imply that $\widehat{\pi}_2 = 0$ at the solution to [P] when $\underline{F} > \underline{F}_L$. Furthermore, if the $F_1 \geq -\underline{F}$ constraint binds and $s_1 = s_2 = \widehat{\underline{s}}$ at the solution to [P], it must be the case that $\frac{d\widehat{\underline{s}}}{d\underline{F}} > 0$ (to ensure $\widehat{\pi}_2 = 0$) when $\underline{F} > \underline{F}_L$.

Lemma A8. $\underline{\widehat{F}}_L < \underline{\widehat{F}}_H$.

Proof. We first show that $\underline{\widehat{F}}_L \neq \underline{\widehat{F}}_H$. To do so, suppose $\underline{\widehat{F}}_L = \underline{\widehat{F}}_H$. Lemma A5 and (34) imply that $(-\underline{\widehat{F}}_L, \underline{s})$ is the optimal plan when $\underline{F} = \underline{\widehat{F}}_L = \underline{\widehat{F}}_H$. Furthermore, $\widehat{\underline{s}} < 1$ and $\widehat{\pi}_2 = 0$ under this plan. Lemma A6 implies that the $\{(F_2, s_2), (-\underline{\widehat{F}}_H, 1)\}$ gain sharing program is also optimal and $\widehat{\pi}_2 = 0$ under this program. Notice that the firm strictly prefers the $(-\underline{\widehat{F}}_H, 1)$ plan to the $(-\underline{\widehat{F}}_L, \underline{s})$ plan because $\underline{\widehat{F}}_L = \underline{\widehat{F}}_H$ and $\widehat{\underline{s}} < 1$. Therefore, it cannot be the case that $\widehat{\pi}_2 = 0$ under both plans. Hence, by contradiction, $\underline{\widehat{F}}_L \neq \underline{\widehat{F}}_H$.

Now suppose $\underline{\widehat{F}}_L > \underline{\widehat{F}}_H$, and consider a value of $\underline{F} \in (\underline{\widehat{F}}_H, \underline{\widehat{F}}_L)$. Since $\underline{F} > \underline{\widehat{F}}_H$, the $\{(F_2, s_2), (-\underline{\widehat{F}}_H, 1)\}$ gain sharing program identified in Lemma A6 is a solution to [P]. Since $\underline{F} < \underline{\widehat{F}}_L$, the $(-\underline{F}, \widehat{\underline{s}})$ gain sharing plan identified in Lemma A5 is also a solution to [P]. As $\underline{F}$ increases in this range, the regulator's expected payoff increases under the $(-\underline{F}, \widehat{\underline{s}})$ plan because the payment to the firm $(-\underline{F})$ declines. In contrast, the regulator's expected payoff does not change under the $\{(F_2, s_2), (-\underline{\widehat{F}}_H, 1)\}$ program because this program does not change as $\underline{F}$ increases. Therefore, both of the identified solutions cannot be optimal and so, by contradiction, $\underline{\widehat{F}}_L \leq \underline{\widehat{F}}_H$.

Since $\underline{\widehat{F}}_L \leq \underline{\widehat{F}}_H$ and $\underline{\widehat{F}}_L \neq \underline{\widehat{F}}_H$, it must be the case that $\underline{\widehat{F}}_L < \underline{\widehat{F}}_H$. □

Lemma A9. Suppose $\underline{F} \in [\underline{F}_L, \underline{\widehat{F}}_H)$. Then $s_2 \leq s_1 < 1$, $F_2 \geq F_1 = -\underline{F}$, and $\widehat{\pi}_2 = 0$. In addition, if $K_{GGG}(G, k_i) \geq 0$ and $K_{GG}(G, k_2) \geq K_{GG}(G, k_1)$ for all G and for $k_i \in \{k_1, k_2\}$, then there exists an $\underline{\widehat{F}}_L \in [\underline{F}_L, \underline{\widehat{F}}_H)$, such that $s_1 = s_2$ for $\underline{F} \in [\underline{F}_L, \underline{\widehat{F}}_L]$, whereas $s_2 < s_1$ for $\underline{F} \in (\underline{\widehat{F}}_L, \underline{\widehat{F}}_H)$. Furthermore, $\frac{ds_1}{d\underline{F}} = \frac{ds_2}{d\underline{F}} > 0$ for $\underline{F} \in (\underline{F}_L, \underline{\widehat{F}}_L)$, whereas $\frac{ds_1}{d\underline{F}} > 0$, $\frac{ds_2}{d\underline{F}} < 0$, $\frac{dF_1}{d\underline{F}} < 0$, and $\frac{dF_2}{d\underline{F}} > 0$ for $\underline{F} \in (\underline{\widehat{F}}_L, \underline{\widehat{F}}_H)$.

Proof. If $\underline{F} \in [\underline{F}_L, \underline{\widehat{F}}_H)$, then the participation constraint (2) when $k = k_2$ and the $F_1 \geq -\underline{F}$ constraint both bind at the solution to [P]. Consequently, $\widehat{\pi}_2 = 0$ and $F_1 = -\underline{F}$. Furthermore: (i) $F_2 \geq F_1$ from Lemma A2; (ii) $s_2 \leq s_1 < 1$ from Lemma A7; and (iii) $\lambda_{12} > 0$ from the proof of Lemma A7.

From (1), the regulator maximizes:

$$\sum_{i=1}^{2} \phi_i \{ [1 - s_i] \, G_i - F_i + \alpha \, \pi_i (F_i, s_i) \}$$
$$= \sum_{i=1}^{2} \phi_i \, \{ G_i - K(G_i, k_i) - [1 - \alpha] \, \pi_i (F_i, s_i) \} \, . \tag{37}$$

When $s_2 < s_1$, the regulator can be viewed as choosing the optimal value of s_2. The corresponding optimal values of F_2 and s_1 are then readily determined because $\widehat{\pi}_2 = 0$ and $\lambda_{12} > 0$. Differentiating (37), recognizing that $\frac{d\pi_2(\cdot)}{ds_2} = 0$, provides:

$$\sum_{i=1}^{2} \phi_i \left\{ [1 - K_G(G_i, k_i)] \, \frac{dG_i}{ds_i} \right\} ds_i - \phi_1 \, [1 - \alpha] \, G_1 \, ds_1$$

$$= \sum_{i=1}^{2} \phi_i \left[\frac{1 - K_G(G_i, k_i)}{K_{GG}(G_i, k_i)} \right] ds_i - \phi_1 [1 - \alpha] G_1 \, ds_1 \;=\; 0 \, . \tag{38}$$

The first equality in (38) holds because $\frac{dG_i}{ds_i} = \frac{1}{K_{GG}(G_i, k_i)}$, since $K_G(G_i, k_i) = s_i$ from (25).

Since $\widehat{\pi}_2 = 0$:

$$F_2 + s_2 G_2 - K(G_2, k_2) \;=\; 0 \quad \Rightarrow \quad dF_2 + G_2 \, ds_2 \;=\; 0 \, . \tag{39}$$

Since $\lambda_{12} > 0$:

$$-\underline{F} + s_1 G_1 - K(G_1, k_1) \;=\; F_2 + s_2 G_{21} - K(G_{21}, k_1) \, . \tag{40}$$

Differentiating (40), using (39), provides:

$$G_1 \, ds_1 \;=\; dF_2 + G_{21} \, ds_2 \;=\; [G_{21} - G_2] \, ds_2 \, . \tag{41}$$

(38) and (41) imply that when $s_2 < s_1$ at the solution to [P]:

$$\phi_1 \left[\frac{1 - K_G(G_1, k_1)}{K_{GG}(G_1, k_1)} \right] G_1 \, ds_1 + \phi_2 \left[\frac{1 - K_G(G_2, k_2)}{K_{GG}(G_2, k_2)} \right] \left[\frac{G_1}{G_{21} - G_2} \right] ds_1$$

$$- \; \phi_1 [1 - \alpha] G_1 \, ds_1 \;=\; 0$$

$$\Rightarrow \quad \phi_1 \left[\frac{1 - K_G(G_1, k_1)}{K_{GG}(G_1, k_1)} \right] \frac{1}{G_1} + \phi_2 \left[\frac{1 - K_G(G_2, k_2)}{K_{GG}(G_2, k_2)} \right] \left[\frac{1}{G_{21} - G_2} \right] - \phi_1 [1 - \alpha] \;=\; 0 \, . \tag{42}$$

G_2 and G_{21} are readily calculated for any given s_2. Given G_2 and G_{21}, G_1 can be derived from (42). We now show that G_1 (and therefore s_1) is uniquely determined by s_2 and that s_1 is a monotone decreasing function of s_2.

Differentiating (42) provides:

$$\phi_1 \left\{ \left[\frac{1 - K_G(G_1, k_1)}{K_{GG}(G_1, k_1)} \right] \left[- \frac{1}{G_1^2} \right] \right.$$

$$\left. + \; \frac{-K_{GG}^2(G_1, k_1) - [1 - K_G(G_1, k_1)] \, K_{GGG}(G_1, k_1)}{K_{GG}^2(G_1, k_1)} \left[\frac{1}{G_1} \right] \right\} \left[\frac{dG_1}{ds_1} \right] ds_1$$

$$+ \; \phi_2 \left\{ \frac{1 - K_G(G_2, k_2)}{K_{GG}(G_2, k_2)} \left[\frac{1}{(G_{21} - G_2)^2} \right] \right.$$

$$\left. + \; \frac{- K_{GG}^2(G_2, k_2) - [1 - K_G(G_2, k_2)] \, K_{GGG}(G_2, k_2)}{K_{GG}^2(G_2, k_2)} \left[\frac{1}{G_{21} - G_2} \right] \right\} \frac{dG_2}{ds_2} \, ds_2$$

$$+ \; \phi_2 \left[\frac{1 - K_G(G_2, k_2)}{K_{GG}(G_2, k_2)} \right] \left[- \frac{1}{(G_{21} - G_2)^2} \right] \left[\frac{dG_{21}}{ds_2} \right] ds_2 \;=\; 0 \, . \tag{43}$$

Since $\frac{dG_i}{ds_i} = \frac{1}{K_{GG}(G_i,k_i)}$, the terms that multiply ds_1 in (43) can be written as:

$$\frac{\phi_1}{K_{GG}^2(G_1,k_1)\,G_1^2}\,\{\,-\,[1-K_G(G_1,k_1)]\,K_{GG}(G_1,k_1) - G_1\,K_{GG}^2(G_1,k_1)$$
$$-\,G_1\,[1-K_G(G_1,k_1)]\,K_{GGG}(G_1,k_1)\,\}\,\frac{1}{K_{GG}(G_1,k_1)}\,ds_1 \;<\; 0\,. \tag{44}$$

The inequality in (44) holds when $K_{GGG}(\cdot) \geq 0$ because $K_G(G_1,k_1) = s_1 < 1$.

Similarly, the terms that multiply ds_2 in (43) can be written as:

$$\frac{\phi_2}{K_{GG}^2(G_2,k_2)\,[G_{21}-G_2]^2}\,\{\,[1-K_G(G_2,k_2)]\,K_{GG}(G_2,k_2) - [G_{21}-G_2]\,K_{GG}^2(G_2,k_2)$$
$$-\;[G_{21}-G_2]\,[1-K_G(G_2,k_2)]\,K_{GGG}(G_2,k_2)\,\}\,\frac{1}{K_{GG}(G_2,k_2)}$$
$$+\,\phi_2\left[\frac{1-K_G(G_2,k_2)}{K_{GG}(G_2,k_2)}\right]\left[-\,\frac{1}{(G_{21}-G_2)^2}\right]\frac{1}{K_{GG}(G_{21},k_1)}$$
$$<\;\frac{\phi_2\,[1-K_G(G_2,k_2)]}{K_{GG}(G_2,k_2)\,[G_{21}-G_2]^2}\left[\frac{1}{K_{GG}(G_2,k_2)} - \frac{1}{K_{GG}(G_{21},k_1)}\right] \;\leq\; 0\,. \tag{45}$$

The first inequality in (45) holds when $K_{GGG}(G,k) \geq 0$ since $K_G(G_2,k_2) = s_2 < 1$ and $G_{21} > G_2$. The last inequality in (45) holds because $K_{GG}(G_{21},k_1) \leq K_{GG}(G_2,k_2)$ when $K_{GGG}(G,k_i) \geq 0$ and $K_{GG}(G,k_2) \geq K_{GG}(G,k_1)$ for all G and for $k_i \in \{k_1,k_2\}$.

(43), (44), and (45) imply that for each s_2, there is a unique s_1 that decreases as s_2 increases (so $\frac{ds_1}{ds_2} < 0$) at the solution to [P]. Lemma 1 implies that the firm's profit in the low cost environment at the solution to [P] increases as s_2 increases and s_1 decreases. Therefore, since $\lambda_{12} > 0$, there is a unique F_1 that increases as s_2 increases.

Let $\overline{s}_2$ denote the value of s_2 at the solution to [P] when $\underline{F} = \widehat{\underline{F}}_H$. Also let $\widehat{s}$ denote the largest share of the realized gain awarded the supplier when $s_1 = s_2$ at the solution to [P]. In addition, let $\widehat{\underline{F}}_L \geq \underline{F}_L$ denote the value $\underline{F}_L$ at which $s_1 = s_2 = \widehat{s}$ at the solution to [P]. Since $F_1 = -\underline{F}$ when $\underline{F} \in [\underline{F}_L, \widehat{\underline{F}}_H)$, it follows that s_2 increases from $\overline{s}_2$ to $\widehat{s}$ as $\underline{F}$ declines from $\widehat{\underline{F}}_H$ to $\underline{F}_L$. Therefore, $s_2 < s_1$ and $\frac{ds_1}{d\underline{F}} > 0$, $\frac{ds_2}{d\underline{F}} < 0$, $\frac{dF_1}{d\underline{F}} < 0$, and $\frac{dF_2}{d\underline{F}} > 0$ when $\underline{F} \in (\widehat{\underline{F}}_L, \widehat{\underline{F}}_H)$. □ ■

Condition 1. $K_{GGk}(G,k) \geq K_{GGG}(G,k)\left[\frac{K_{Gk}(G,k)}{K_{GG}(G,k)}\right]$ for all G and k.

Condition 2. $K_{GGk}(G,k) \leq K_{GGG}(G,k)\left[\frac{K_{Gk}(G,k)}{K_{GG}(G,k)}\right]$ for all G and k.

Conclusion 3. *Suppose the regulator's objective function is a concave function of s_2. Then at the solution to [P]:*

(i) s_2 increases as ϕ_2 increases or as α increases;

(ii) s_2 decreases as k_2 increases if Condition 1 holds; and

(iii) s_2 increases as k_1 increases if $\underline{F} > \widehat{\underline{F}}_H$ or if $\underline{F} \leq \widehat{\underline{F}}_L$ and Condition 2 holds.

Proof. Let (F_i, s_i) denote the gain sharing plan the firm chooses when $k = k_i$. Then consumer surplus when $k = k_i$ is:

$$CS_i \equiv -F_i + [1-s_i]\,G_i\,. \tag{46}$$

Total surplus when $k = k_i$ is:

$$T_i \equiv G_i - K(G_i, k_i). \tag{47}$$

The firm's rent when $k = k_i$ is:

$$R_i \equiv F_i + s_i\,G_i - K(G_i, k_i)\,. \tag{48}$$

The regulator's objective is to maximize:

$$W \equiv \sum_{i=1}^{2} \phi_i\,[CS_i + \alpha\,R_i] = \sum_{i=1}^{2} \phi_i\,[T_i - (1-\alpha)R_i]\,. \tag{49}$$

Case I. $\underline{F} \geq \widehat{\underline{F}}_H$.

The regulator can be viewed as determining the optimal s_2. Conclusion 2 implies that once s_2 is determined, F_2 is set to ensure the firm earns no rent when $k = k_2$. Furthermore, $s_1 = 1$ and F_1 is chosen so that the firm is indifferent between the (F_1, s_1) plan and the (F_2, s_2) plan when $k = k_1$. This indifference implies:

$$R_1 = F_2 + s_2\,G_{21} - K(G_{21}, k_1)\,, \tag{50}$$

where G_{21} is the success probability the firm would implement under the (F_2, s_2) plan in the low cost environment.

Because $R_2 = 0$:

$$0 = \frac{dR_2}{ds_2} = \frac{\partial R_2}{\partial s_2} + \frac{\partial R_2}{\partial G_2}\left[\frac{dG_2}{ds_2}\right] + \frac{\partial R_2}{\partial F_2}\left[\frac{dF_2}{ds_2}\right] = G_2 + \frac{dF_2}{ds_2} \;\Rightarrow\; \frac{dF_2}{ds_2} = -\,G_2\,. \tag{51}$$

The third equality in (51) reflects the envelope theorem and the fact that $\frac{\partial R_2}{\partial s_2} = G_2$ and $\frac{\partial R_2}{\partial F_2} = 1$, from (48).

$\frac{dW}{ds_2} = 0$ at the solution to [P]. We will determine how changes in parameter values affect $\frac{dW}{ds_2}$. If $\frac{dW}{ds_2}$ becomes positive (negative) as a parameter increases, then the optimal s_2 will increase (decrease), given the presumed concavity of W.

From (48):

$$K_G(G_i, k_i) \ = \ s_i \quad \Rightarrow \quad \frac{dG_i}{ds_i} \ = \ \frac{1}{K_{GG}(G_i, k_i)} \quad \text{for } i = 1, 2\,. \tag{52}$$

Because $s_1 = 1$, T_1 is not affected by changes in s_2, i.e., $\frac{dT_1}{ds_2} = 0$.

From (47), using (52):

$$\begin{aligned} \frac{dT_2}{ds_2} &= \frac{\partial T_2}{\partial s_2} + \frac{\partial T_2}{\partial G_2}\left[\frac{dG_2}{ds_2}\right] + \frac{\partial T_2}{\partial F_2}\left[\frac{dF_2}{ds_2}\right] \ = \ \frac{\partial T_2}{\partial G_2}\left[\frac{dG_2}{ds_2}\right] \\ &= [1 - K_G(G_2, k_2)]\,\frac{dG_2}{ds_2} \ = \ [1 - s_2]\,\frac{dG_2}{ds_2} \ = \ \frac{1 - s_2}{K_{GG}(G_2, k_2)}\,. \end{aligned} \tag{53}$$

The second equality in (53) holds because $\frac{\partial T_2}{\partial s_2} = \frac{\partial T_2}{\partial F_2} = 0$, from (47). The last two equalities in (53) reflect (52). From (48):

$$\begin{aligned} \frac{dR_1}{ds_2} &= \frac{\partial R_1}{\partial s_2} + \frac{\partial R_1}{\partial G_{21}}\left[\frac{dG_{21}}{ds_2}\right] + \frac{\partial R_1}{\partial F_2}\left[\frac{dF_2}{ds_2}\right] \\ &= \frac{\partial R_1}{\partial s_2} + \frac{\partial R_1}{\partial F_2}\left[\frac{dF_2}{ds_2}\right] \ = \ G_{21} + \frac{dF_2}{ds_2} \ = \ G_{21} - G_2\,. \end{aligned} \tag{54}$$

The second equality in (54) reflects the envelope theorem. The third equality in (54) follows from (50). The last equality in (54) reflects (51).

(49), (53), and (54) imply:

$$\frac{dW}{ds_2} \ = \ \phi_2\left[\frac{1 - s_2}{K_{GG}(G_2, k_2)}\right] - \phi_1\,[1 - \alpha]\,[G_{21} - G_2]\ . \tag{55}$$

Differentiating (55) with respect to α provides:

$$\frac{d}{d\alpha}\left(\frac{dW}{ds_2}\right) \ = \ \phi_1\,[G_{21} - G_2] \ > \ 0. \tag{56}$$

The inequality in (56) implies that the optimal s_2 increases as α increases.

Differentiating (55) with respect to ϕ_1 provides:

$$\frac{d}{d\phi_1}\left(\frac{dW}{ds_2}\right) \ = \ -\,[1 - s_2]\,\frac{1}{K_{GG}(G_2, k_2)} - [1 - \alpha]\,[G_{21} - G_2] \ < \ 0.$$

This inequality implies that the optimal s_2 decreases as ϕ_1 increases.

Differentiating (55) with respect to k_2 provides:

$$\frac{d}{dk_2}\left(\frac{dW}{ds_2}\right) \;=\; \phi_1\,[1-\alpha]\,\frac{dG_2}{dk_2} - \phi_2\,[1-s_2]\,\frac{K_{GGG}(G_2,k_2)\frac{dG_2}{dk_2} + K_{GGk}(G_2,k_2)}{[K_{GG}(G_2,k_2)]^2} \;<\; 0. \quad (57)$$

The inequality in (57) holds when Condition 1 holds because $\frac{dG_2}{dk_2} = -\frac{K_{Gk}(G_2,k_2)}{K_{GG}(G_2,k_2)} < 0$, since $s_2 = K_G(G_2,k_2)$. The inequality in (57) implies that the optimal s_2 decreases as k_2 increases.

Differentiating (55) with respect to k_1 provides:

$$\frac{d}{dk_1}\left(\frac{dW}{ds_2}\right) \;=\; -\,\phi_1\,[1-\alpha]\,\frac{dG_{21}}{dk_1} \;>\; 0. \quad (58)$$

The inequality in (58) holds because $\frac{dG_{21}}{dk_1} = -\frac{K_{Gk}(G_{21},k_1)}{K_{GG}(G_{21},k_1)} < 0$, since $K_G(G_{21},k_1) = s_2$. The inequality in (58) implies that the optimal s_2 increases as k_1 increases.

Case II. $\underline{F} \in (\underline{\widehat{F}}_L, \underline{\widehat{F}}_H)$.

The regulator can again be viewed as determining the optimal s_2. Conclusion 2 implies that once s_2 is determined, F_2 is set to ensure the firm earns no rent when $k = k_2$. Furthermore, $F_1 = -\underline{F}$ and s_1 is chosen so that the firm is indifferent between the (F_1, s_1) and (F_2, s_2) plans when $k = k_1$.

$\frac{dT_2}{ds_2}$ in this case is as specified in (53). Furthermore, from (47), using (52):

$$\begin{aligned}\frac{dT_1}{ds_2} &= \frac{\partial T_1}{\partial G_1}\left[\frac{dG_1}{ds_2}\right] = [1-K_G(G_1,k_1)]\,\frac{dG_1}{ds_2}\\ &= [1-s_1]\,\frac{dG_1}{ds_1}\left[\frac{ds_1}{ds_2}\right] = \left[\frac{1-s_1}{K_{GG}(G_1,k_1)}\right]\frac{ds_1}{ds_2}. \quad (59)\end{aligned}$$

From (50), (51), and the envelope theorem:

$$-\,\underline{F} + s_1\,G_1 - K(G_1,k_1) \;=\; F_2 + s_2\,G_{21} - K(G_{21},k_1)$$

$$\Rightarrow \quad G_1\,\frac{ds_1}{ds_2} = \frac{dF_2}{ds_2} + G_{21} \quad \Rightarrow \quad \frac{ds_1}{ds_2} = \frac{G_{21}-G_2}{G_1} \;>\; 0\,. \quad (60)$$

In addition, from (50):

$$\begin{aligned}\frac{dR_1}{ds_2} &= \frac{\partial R_1}{\partial s_2} + \frac{\partial R_1}{\partial G_{21}}\left[\frac{dG_{21}}{ds_2}\right] + \frac{\partial R_1}{\partial F_2}\left[\frac{dF_2}{ds_2}\right]\\ &= \frac{\partial R_1}{\partial s_2} + \frac{\partial R_1}{\partial F_2}\left[\frac{dF_2}{ds_2}\right] = G_{21} + \frac{dF_2}{ds_2} = G_{21} - G_2\,. \quad (61)\end{aligned}$$

The second equality in (61) reflects the envelope theorem. The third equality in (61) holds because $\frac{\partial R_1}{\partial s_2} = G_{21}$ and $\frac{\partial R_1}{\partial F_2} = 1$, from (50). The last equality in (61) reflects (51).

(49), (53), (59), and (61) imply:

$$\frac{dW}{ds_2} \;=\; \phi_1 \left[\frac{1 - s_1}{K_{GG}(G_1, k_1)} \right] \frac{ds_1}{ds_2} + \phi_2 \left[\frac{1 - s_2}{K_{GG}(G_2, k_2)} \right] - \phi_1 \left[1 - \alpha \right] \left[G_{21} - G_2 \right] . \tag{62}$$

Differentiating (62) with respect to α provides:

$$\frac{d}{d\alpha} \left(\frac{dW}{ds_2} \right) \;=\; \phi_1 \left[G_{21} - G_2 \right] \;>\; 0 \, .$$

This inequality implies that the optimal s_2 increases as α increases.

Differentiating (62) with respect to ϕ_1 provides:

$$\begin{aligned} \frac{d}{d\phi_1} \left(\frac{dW}{ds_2} \right) \;&=\; - \left[1 - \alpha \right] \left[G_{21} - G_2 \right] - \frac{1 - s_2}{K_{GG}(G_2, k_2)} + \frac{1 - s_1}{K_{GG}(G_1, k_1)} \left[\frac{ds_1}{ds_2} \right] \\ &=\; - \; \frac{1 - s_2}{K_{GG}(G_2, k_2)} - \frac{\phi_2}{\phi_1} \left[\frac{1 - s_2}{K_{GG}(G_2, k_2)} \right] \;<\; 0 \, . \end{aligned} \tag{63}$$

The last equality in (63) follows from (62), since $\frac{dW}{ds_2} = 0$ at the optimal value of s_2. The inequality in (63) implies that the optimal s_2 decreases as ϕ_1 increases.

Differentiating (62) with respect to k_2 provides:

$$\frac{d}{dk_2} \left(\frac{dW}{ds_2} \right) \;=\; \phi_1 \left[1 - \alpha \right] \frac{dG_2}{dk_2} - \phi_2 \left[1 - s_2 \right] \frac{K_{GGG}(G_2, k_2) \frac{dG_2}{dk_2} + K_{GGk}(G_2, k_2)}{\left[K_{GG}(G_2, k_2) \right]^2} \;<\; 0 \, .$$

This inequality holds when Condition 1 holds because $\frac{dG_2}{dk_2} = - \frac{K_{Gk}(G_2, k_2)}{K_{GG}(G_2, k_2)} < 0$, since $s_2 = K_G(G_2, k_2)$. The inequality implies that the optimal s_2 decreases as k_2 increases.

The proofs for the settings in which $\underline{F} \leq \widehat{\underline{F}}_L$ are analogous, and so are omitted. ■