Technical Appendix to Accompany

"Employing Gain-Sharing Regulation to Promote Forward Contracting in the Electricity Sector"

by David P. Brown and David E. M. Sappington

This Technical Appendix provides detailed proofs of Lemma 1, Lemma 2, and Corollary 1 from the text.

Equations From the Text

$$Q(\cdot) \;=\; a^I - b^I\, w + \overline{Q} + \eta\,. \tag{1}$$

$$w(\cdot) \;=\; a + \varepsilon - b\,Q \quad \text{where} \;\; a \;=\; \frac{a^I + \overline{Q}}{b^I},\;\; \varepsilon \;=\; \frac{\eta}{b^I},\; \text{and}\;\; b \;=\; \frac{1}{b^I}\,. \tag{2}$$

$$\pi_i^G \;=\; w\,q_i + \left[\,p^f - w\,\right] F_i - c_0\,q_i - \frac{1}{2}\,c\,q_i^2\,. \tag{3}$$

$$\pi^B \;=\; R(\varepsilon) - w\left[\overline{Q} + b^I\,\varepsilon - \sum_{i=1}^{n} F_i\right] - p^f \sum_{i=1}^{n} F_i - K - \Phi(\cdot)\,. \tag{4}$$

Lemma 1. *In equilibrium under forward contracting, given ε and $F_1, ..., F_n$:*

$$w(\varepsilon) \;=\; \frac{b+c}{D}\left[(\,b+c\,)\,(\,a+\varepsilon\,) + n\,b\,c_0 - b^2 \sum_{i=1}^{n} F_i\right],$$

$$p^f \;=\; E\left\{w(\varepsilon)\right\} \;=\; \frac{b+c}{D}\left[a\,(\,b+c\,) + n\,b\,c_0 - b^2 \sum_{i=1}^{n} F_i\right], \;\; \textit{and}$$

$$q_i(\varepsilon) \;=\; \frac{[\,b+c\,]\,[\,a+\varepsilon-c_0\,] + b\,[\,b\,n + c\,]\,F_i - b^2\,F_{-i}}{D} \quad \textit{for}\;\; i \;=\; 1, ..., n\,,$$

$$\textit{where} \quad D \;\equiv\; b^2\,[\,n+1\,] + c\,[\,b\,(n+2) + c\,] \;>\; 0\,. \tag{5}$$

Proof. (3) implies that when ε is realized, Gi's problem is:

$$\underset{q_i \,\geq\, 0}{Maximize} \;\; \pi_i^G(\varepsilon) \;=\; w(\varepsilon)\,[\,q_i - F_i\,] + p^f\,F_i - c_0\,q_i - \frac{1}{2}\,c\,(q_i)^2\,. \tag{6}$$

(2) and (6) imply that the necessary conditions for an interior maximum include:

$$\frac{\partial \pi_i^G(\varepsilon)}{\partial q_i} \;=\; w(\varepsilon) + [\,q_i - F_i\,]\,\frac{\partial w(\cdot)}{\partial Q} - c_0 - c\,q_i \;=\; 0\,. \tag{7}$$

Define $Q_{-i} \equiv \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j$. Then (2) and (7) imply that Gi's profit-maximizing choice of $q_i > 0$ is determined by:

$$a + \varepsilon - b\,[\,q_i + Q_{-i}\,] - b\,[\,q_i - F_i\,] - c_0 - c\,q_i \;=\; 0$$

$$\Rightarrow \quad [\,2\,b + c\,]\,q_i \;=\; a + \varepsilon - b\,Q_{-i} + b\,F_i - c_0$$

$$\Rightarrow \quad q_i \;=\; \frac{1}{2\,b + c}\,[\,a + \varepsilon - c_0 + b\,F_i\,] - \frac{b}{2\,b + c}\,Q_{-i}\,. \tag{8}$$

(8) implies that in equilibrium:

$$\begin{aligned}
Q_{-i} \;&=\; \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{1}{2\,b + c}\,[\,a + \varepsilon - c_0 + b\,F_j\,] - \frac{b}{2\,b + c}\,Q_{-j} \right) \\
&=\; \frac{n-1}{2\,b + c}\,[\,a + \varepsilon - c_0\,] + \frac{b}{2\,b + c} \sum_{\substack{j=1 \\ j \neq i}}^{n} F_j \;-\; \frac{b}{2\,b + c} \sum_{\substack{j=1 \\ j \neq i}}^{n} Q_{-j} \\
&=\; \frac{n-1}{2\,b + c}\,[\,a + \varepsilon - c_0\,] + \frac{b}{2\,b + c}\,F_{-i} \\
&\qquad - \frac{b}{2\,b + c}\,[\,Q_{-1} + \ldots + Q_{-(i-1)} + Q_{-(i+1)} + \ldots + Q_{-n}\,] \\
&=\; \frac{n-1}{2\,b + c}\,[\,a + \varepsilon - c_0\,] + \frac{b}{2\,b + c}\,F_{-i} - \frac{b}{2\,b + c}\,[\,(n-1)\,q_i + (n-2)\,Q_{-i}\,]\,.
\end{aligned} \tag{9}$$

(9) implies:

$$Q_{-i}\left[1 + \frac{b\,(n-2)}{2\,b + c}\right] \;=\; \frac{n-1}{2\,b + c}\,[\,a + \varepsilon - c_0\,] + \frac{b}{2\,b + c}\,F_{-i} - \left[\frac{b\,(n-1)}{2\,b + c}\right] q_i$$

$$\Rightarrow \quad Q_{-i}\left[\frac{b\,n + c}{2\,b + c}\right] \;=\; \frac{n-1}{2\,b + c}\,[\,a + \varepsilon - c_0\,] + \frac{b}{2\,b + c}\,F_{-i} - \left[\frac{b\,(n-1)}{2\,b + c}\right] q_i$$

$$\Rightarrow \quad Q_{-i} \;=\; \frac{n-1}{b\,n + c}\,[\,a + \varepsilon - c_0\,] + \frac{b}{b\,n + c}\,F_{-i} - \left[\frac{b\,(n-1)}{b\,n + c}\right] q_i\,. \tag{10}$$

(8) and (10) imply that in equilibrium:

$$\begin{aligned}
q_i \;&=\; \frac{1}{2\,b + c}\,[\,a + \varepsilon - c_0 + b\,F_i\,] \\
&\qquad - \frac{b}{[\,2\,b + c\,]\,[\,b\,n + c\,]}\,[\,(n-1)\,(a + \varepsilon - c_0) + b\,F_{-i} - b\,(n-1)\,q_i\,]
\end{aligned}$$

$$\Rightarrow \quad q_i \left[1 - \frac{b^2\,(n-1)}{(\,2\,b + c\,)\,(\,b\,n + c\,)} \right] \;=\; \frac{b\,n + c - b\,[\,n-1\,]}{[\,2\,b + c\,]\,[\,b\,n + c\,]}\,[\,a + \varepsilon - c_0\,]$$

$$+ \;\frac{b}{2\,b + c}\;F_i - \frac{b^2}{[\,2\,b + c\,]\,[\,b\,n + c\,]}\;F_{-i}$$

$$\Rightarrow \quad q_i \left[\frac{(\,2\,b + c\,)\,(\,b\,n + c\,) - b^2\,(\,n-1\,)}{(\,2\,b + c\,)\,(\,b\,n + c\,)} \right] \;=\; \frac{b + c}{[\,2\,b + c\,]\,[\,b\,n + c\,]}\,[\,a + \varepsilon - c_0\,]$$

$$+ \;\frac{b\,[\,b\,n + c\,]}{[\,2\,b + c\,]\,[\,b\,n + c\,]}\;F_i - \frac{b^2}{[\,2\,b + c\,]\,[\,b\,n + c\,]}\;F_{-i}\,. \tag{11}$$

Observe that:

$$[\,2\,b + c\,]\,[\,b\,n + c\,] - b^2\,[\,n - 1\,] \;=\; 2\,b^2\,n + 2\,b\,c + b\,c\,n + c^2 - b^2 n + b^2$$

$$=\; b^2\,n + 2\,b\,c + b\,c\,n + b^2 + c^2 \;=\; b^2\,[\,n + 1\,] + c\,[\,b\,(n+2) + c\,]\,. \tag{12}$$

(11) and (12) imply:

$$q_i(\varepsilon) \;=\; \frac{[\,b + c\,]\,[\,a + \varepsilon - c_0\,] + b\,[\,b\,n + c\,]\,F_i - b^2\,F_{-i}}{b^2\,[\,n + 1\,] + c\,[\,b\,(n+2) + c\,]}\,. \tag{13}$$

Because $\sum_{i=1}^{n} F_{-i} \;=\; [\,n - 1\,] \sum_{i=1}^{n} F_i$:

$$\sum_{i=1}^{n} (\,[\,b\,n + c\,]\,F_i - b\,F_{-i}\,) \;=\; [\,b\,n + c\,] \sum_{i=1}^{n} F_i - b\,[\,n - 1\,] \sum_{i=1}^{n} F_i \;=\; [\,b + c\,] \sum_{i=1}^{n} F_i\,. \tag{14}$$

(13) and (14) imply that in equilibrium:

$$Q(\varepsilon) \;=\; \sum_{i=1}^{n} q_i \;=\; \frac{n\,[\,b + c\,]\,[\,a + \varepsilon - c_0\,] + b\,[\,b + c\,] \sum_{i=1}^{n} F_i}{b^2\,[\,n + 1\,] + c\,[\,b\,(n+2) + c\,]}\,. \tag{15}$$

(2) and (15) imply:

$$w(\varepsilon) \;=\; a + \varepsilon - \frac{n\,b\,[\,b + c\,]\,[\,a + \varepsilon - c_0\,] + b^2\,[\,b + c\,] \sum_{i=1}^{n} F_i}{b^2\,[\,n + 1\,] + c\,[\,b\,(n+2) + c\,]}\,. \tag{16}$$

Observe that:

$$b^2\,[\,n + 1\,] + c\,[\,b\,(n+2) + c\,] - n\,b\,[\,b + c\,]$$

$$=\; b^2\,n + b^2 + b\,c\,n + 2\,b\,c + c^2 - b^2\,n - b\,c\,n$$

$$=\; b^2 + 2\,b\,c + c^2 \;=\; [\,b + c\,]^2\,. \tag{17}$$

(16) and (17) imply:

$$w(\varepsilon) \;=\; \frac{[\,b+c\,]^2\,[\,a+\varepsilon\,] + n\,b\,[\,b+c\,]\,c_0 - b^2\,[\,b+c\,]\sum\limits_{i=1}^{n} F_i}{b^2\,[\,n+1\,] + c\,[\,b\,(n+2) + c\,]}$$

$$=\; \frac{[\,b+c\,]\left[\,(\,b+c\,)\,(\,a+\varepsilon\,) + n\,b\,c_0 - b^2\sum\limits_{i=1}^{n} F_i\right]}{b^2\,[\,n+1\,] + c\,[\,b\,(n+2) + c\,]}$$

$$\Rightarrow\; p^f \;=\; E\{w(\varepsilon)\} \;=\; \frac{[\,b+c\,]\left[\,(\,b+c\,)\,(\,a+E\{\varepsilon\}\,) + n\,b\,c_0 - b^2\sum\limits_{i=1}^{n} F_i\right]}{b^2\,[\,n+1\,] + c\,[\,b\,(n+2) + c\,]}. \;\blacksquare \qquad (18)$$

Lemma 2. *At a symmetric equilibrium under forward contracting, for* $i = 1, ..., n$ *:*

$$F_i \;=\; \frac{b^2\,[\,b+c\,]\,[\,n-1\,]\,\left[\,a^I + \overline{Q} - b^I\,c_0\,\right]}{[\,b\,n+c\,]\,[\,2\,b\,(\,b+c\,) + c\,(\,b\,n+c\,)\,] + b^3\,[\,n-1\,]^2}$$

$$\Rightarrow\; \frac{\partial F_i}{\partial a^I} \;>\; 0,\;\; \frac{\partial F_i}{\partial \overline{Q}} \;>\; 0,\;\; \frac{\partial F_i}{\partial b^I} \;<\; 0,\;\; \frac{\partial F_i}{\partial c_0} \;<\; 0,\; \textit{and}\;\; \frac{\partial F_i}{\partial c} \;<\; 0\,. \qquad (19)$$

Proof. (3) implies that because $p^f = E\{w(\varepsilon)\}$:

$$E\left\{\pi_i^G(\varepsilon)\right\} \;=\; E\left\{\,w(\varepsilon)\,q_i(\varepsilon) - c_0\,q_i(\varepsilon) - \frac{c}{2}\,[\,q_i(\varepsilon)\,]^2 + \left[\,p^f - w(\varepsilon)\,\right]F_i\,\right\}$$

$$=\; E\left\{\,w(\varepsilon)\,q_i(\varepsilon) - c_0\,q_i(\varepsilon) - \frac{c}{2}\,[\,q_i(\varepsilon)\,]^2\,\right\}. \qquad (20)$$

Lemma 1 implies:

$$w(\varepsilon)\,q_i(\varepsilon) \;=\; \frac{b+c}{D^2}\left\{\,[\,b+c\,]\,[\,a+\varepsilon-c_0\,] + b\,[\,b\,n+c\,]\,F_i - b^2\,F_{-i}\,\right\}$$

$$\cdot\left[\,(\,b+c\,)\,(\,a+\varepsilon\,) + n\,b\,c_0 - b^2\sum_{i=1}^{n} F_i\,\right]$$

$$=\; \frac{b+c}{D^2}\Bigg\{\,[\,b+c\,]^2\,[\,a+\varepsilon\,]\,[\,a+\varepsilon-c_0\,] + n\,b\,c_0\,[\,b+c\,]\,[\,a+\varepsilon-c_0\,]$$

$$-\; b^2\,[\,b+c\,]\,[\,a+\varepsilon-c_0\,]\sum_{i=1}^{n} F_i + b\,[\,b+c\,]\,[\,b\,n+c\,]\,[\,a+\varepsilon\,]\,F_i$$

$$+\; b^2\,n\,[\,b\,n+c\,]\,c_0\,F_i - b^3\,[\,b\,n+c\,]\,F_i\sum_{i=1}^{n} F_i$$

$$- \; b^2\,[\,b+c\,]\,[\,a+\varepsilon\,]\,F_{-i} - b^3\,n\,c_0\,F_{-i} + b^4\,F_{-i}\sum_{i=1}^{n} F_i \Bigg\}. \tag{21}$$

(21) implies that when $E\{\,\varepsilon\,\} = 0$:

$$E\{\,w(\varepsilon)\,q_i(\varepsilon)\,\} \;=\; \frac{b+c}{D^2}\Bigg\{\;[\,b+c\,]^2\;a\,[\,a-c_0\,] + [\,b+c\,]^2\;E\left\{\,\varepsilon^2\right\} + n\,b\,c_0\,[\,b+c\,]\,[\,a-c_0\,]$$

$$- \; b^2\,[\,b+c\,]\,[\,a-c_0\,]\sum_{i=1}^{n} F_i + a\,b\,[\,b+c\,]\,[\,b\,n+c\,]\,F_i$$

$$+ \; b^2\,n\,[\,b\,n+c\,]\,c_0\,F_i - b^3\,[\,b\,n+c\,]\,F_i\;\sum_{i=1}^{n} F_i$$

$$- \; a\,b^2\,[\,b+c\,]\,F_{-i} - b^3\,n\,c_0\,F_{-i} + b^4\,F_{-i}\;\sum_{i=1}^{n} F_i\Bigg\}$$

$$\Rightarrow \quad \frac{\partial E\{\,w(\varepsilon)\,q_i(\varepsilon)\,\}}{\partial F_i} \;=\; \frac{b+c}{D^2}\Bigg\{ -\; b^2\,[\,b+c\,]\,[\,a-c_0\,] + a\,b\,[\,b+c\,]\,[\,b\,n+c\,]$$

$$+ \; b^2\,n\,[\,b\,n+c\,]\,c_0 - b^3\,[\,b\,n+c\,]\left[\,F_i + \sum_{i=1}^{n} F_i\,\right] + b^4\,F_{-i}\Bigg\}. \tag{22}$$

Lemma 1 implies:

$$-\,c_0\,E\{\,q_i(\varepsilon)\,\} \;=\; -\,\frac{c_0}{D}\,E\left\{\,[\,b+c\,]\,[\,a+\varepsilon-c_0\,] + b\,[\,b\,n+c\,]\,F_i - b^2\,F_{-i}\,\right\}$$

$$\Rightarrow \quad \frac{-\,c_0\,\partial E\{\,q_i(\varepsilon)\,\}}{\partial F_i} \;=\; -\,\frac{c_0}{D}\;b\,[\,b\,n+c\,]\,. \tag{23}$$

Lemma 1 also implies:

$$-\,\frac{c}{2}\,E\left\{[\,q_i(\varepsilon)\,]^2\right\} \;=\; -\,\frac{c}{2\,D^2}\,E\Bigg\{\;\big([\,b+c\,]\,[\,a+\varepsilon-c_0\,] + b\,[\,b\,n+c\,]\,F_i - b^2\,F_{-i}\big)$$

$$\cdot\;\big([\,b+c\,]\,[\,a+\varepsilon-c_0\,] + b\,[\,b\,n+c\,]\,F_i - b^2\,F_{-i}\big)\;\Bigg\}$$

$$= \; -\,\frac{c}{2\,D^2}\,E\Bigg\{\;[\,b+c\,]^2\,[\,a+\varepsilon-c_0\,]^2 + 2\,b\,[\,b\,n+c\,]\,[\,b+c\,]\,[\,a+\varepsilon-c_0\,]\,F_i$$

$$- \;\; 2\,b^2\,[\,b+c\,]\,[\,a+\varepsilon-c_0\,]\,F_{-i} \; - 2\,b^3\,[\,b\,n+c\,]\,F_i\;F_{-i}$$

$$+ \; b^2\,[\,b\,n+c\,]^2\,(F_i)^2 + b^4\,(F_{-i})^2\;\Bigg\}$$

$$= \; - \, \frac{c}{2\,D^2} \Big\{ \, [\, b + c\,]^2 \, [\, a - c_0\,]^2 + [\, b + c\,]^2 \; E\left\{ \varepsilon^2 \right\}$$

$$+ \; 2\,b\,[\,b\,n + c\,]\,[\,b + c\,]\,[\,a - c_0\,]\,F_i - 2\,b^2\,[\,b + c\,]\,[\,a - c_0\,]\,F_{-i}$$

$$- \; 2\,b^3\,[\,b\,n + c\,]\,F_i\;F_{-i} \; + b^2\,[\,b\,n + c\,]^2\,(F_i)^2 + b^4\,(F_{-i})^2 \; \Big\}$$

$$\Rightarrow \quad - \, \frac{c}{2} \, \frac{\partial E\left\{ [\, q_i(\varepsilon)\,]^2 \right\}}{\partial F_i} \; = \; - \, \frac{c}{2\,D^2} \Big\{ \, 2\,b\,[\,b\,n + c\,]\,[\,b + c\,]\,[\,a - c_0\,]$$

$$- \; 2\,b^3\,[\,b\,n + c\,]\,F_{-i} \; + 2\,b^2\,[\,b\,n + c\,]^2\,F_i \Big\}. \qquad (24)$$

(20), (22), (23), and (24) imply:

$$\frac{\partial E\left\{ \pi_i^G(\varepsilon) \right\}}{\partial F_i} \; = \; \frac{b + c}{D^2} \Big\{ - \, b^2\,[\,b + c\,]\,[\,a - c_0\,] + a\,b\,[\,b + c\,]\,[\,b\,n + c\,]$$

$$+ \; b^2\,n\,[\,b\,n + c\,]\,c_0 - b^3\,[\,b\,n + c\,] \; [\,F_i + \sum_{i=1}^{n} F_i \;] \; + b^4\,F_{-i} \Big\} - \frac{c_0}{D}\,b\,[\,b\,n + c\,]$$

$$- \, \frac{c}{2\,D^2} \Big\{ \, 2\,b\,[\,b\,n + c\,]\,[\,b + c\,]\,[\,a - c_0\,] - 2\,b^3\,[\,b\,n + c\,]\,F_{-i} \; + 2\,b^2\,[\,b\,n + c\,]^2\,F_i \Big\}. \qquad (25)$$

(25) implies that Gi's profit-maximizing choice of F_i is determined by:

$$[\,b + c\,] \Big\{ - \, b^2\,[\,b + c\,]\,[\,a - c_0\,] + a\,b\,[\,b + c\,]\,[\,b\,n + c\,]$$

$$+ \; b^2\,n\,[\,b\,n + c\,]\,c_0 - b^3\,[\,b\,n + c\,] \; [\,F_i + \sum_{i=1}^{n} F_i \;] \; + b^4\,F_{-i} \Big\}$$

$$- \, c_0\,b\,[\,b\,n + c\,] \left[b^2\,(\,n + 1\,) + c\,(\,b\,[\,n + 2\,] + c\,) \right]$$

$$- \; \frac{c}{2} \Big\{ \, 2\,b\,[\,b\,n + c\,]\,[\,b + c\,]\,[\,a - c_0\,] - 2\,b^3\,[\,b\,n + c\,]\,F_{-i} \; + 2\,b^2\,[\,b\,n + c\,]^2\,F_i \Big\} \; = \; 0$$

$$\Leftrightarrow \quad [\,b + c\,] \Big\{ - \, b^2\,[\,b + c\,]\,[\,a - c_0\,] + a\,b\,[\,b + c\,]\,[\,b\,n + c\,]$$

$$+ \; b^2\,n\,[\,b\,n + c\,]\,c_0 - b^3\,[\,b\,n + c\,] \; F_{-i} \; + b^4\,F_{-i} \Big\}$$

$$- \, c_0\,b\,[\,b\,n + c\,] \left[b^2\,(\,n + 1\,) + c\,(\,b\,[\,n + 2\,] + c\,) \right]$$

$$- \; \frac{c}{2} \Big\{ \, 2\,b\,[\,b\,n + c\,]\,[\,b + c\,]\,[\,a - c_0\,] - 2\,b^3\,[\,b\,n + c\,]\,F_{-i} \Big\}$$

$$= \; [\,b\,n + c\,]\,b^2\,[\,2\,b\,(\,b + c\,) + c\,(\,b\,n + c\,)\,]\,F_i$$

$$\Leftrightarrow \quad X\,F_i \;=\; Y - \,b^5\,[\,n-1\,]\,F_{-i}\,, \tag{26}$$

where $X \;\equiv\; b^2\,[\,b\,n+c\,]\,[\,2\,b\,(\,b+c\,)+c\,(\,b\,n+c\,)\,]$ and

$$\begin{aligned} Y \;\equiv\;\; & [\,b+c\,]\,\big\{-\,b^2\,[\,b+c\,]\,[\,a-c_0\,]+a\,b\,[\,b+c\,]\,[\,b\,n+c\,]+b^2\,n\,[\,b\,n+c\,]\,c_0\,\big\} \\ & -\,c_0\,b\,[\,b\,n+c\,]\,\big[\,b^2\,(\,n+1\,)+c\,(\,b\,[\,n+2\,]+c\,)\,\big] \\ & -\,b\,c\,[\,b\,n+c\,]\,[\,b+c\,]\,[\,a-c_0\,]\,. \end{aligned} \tag{27}$$

The coefficient on F_{-i} in (26) reflects the fact that:

$$\begin{aligned} & [\,b+c\,]\,\big[\,b^4-b^3\,(\,b\,n+c\,)\,\big]+b^3\,c\,[\,b\,n+c\,] \\ & \quad = \; -\,b^3\,[\,b\,(n-1)+c\,]\,[\,b+c\,]+b^3\,c\,[\,b\,n+c\,] \\ & \quad = \; b^3\,\{\,c\,[\,b\,n+c\,]-[\,b+c\,]\,[\,b\,(n-1)+c\,]\,\} \\ & \quad = \; b^3\,\big\{\,b\,c\,n+c^2-b^2\,[\,n-1\,]-b\,c-b\,c\,[\,n-1\,]-c^2\,\big\} \\ & \quad = \; b^3\,\big\{\,b\,c\,n-b^2\,n+b^2-b\,c-b\,c\,n+b\,c\,\big\} \;=\; -\,b^5\,[\,n-1\,]\,. \end{aligned}$$

(27) implies that $Y \;=\; y_a\,a+y_0\,c_0$ where:

$$\begin{aligned} y_a \;&=\; b\,[\,b+c\,]\;\{\,[\,b+c\,]\,[\,b\,n+c\,]-b\,[\,b+c\,]-c\,[\,b\,n+c\,]\,\} \\ &=\; b\,[\,b+c\,]\;\{\;b\,[\,b\,n+c\,]-b\,[\,b+c\,]\,\} \\ &=\; b^2\,[\,b+c\,]\,[\,b\,n+c-(\,b+c\,)\,] \;=\; b^3\,[\,b+c\,]\,[\,n-1\,]\,; \end{aligned}$$

$$\begin{aligned} y_0 \;=\; & b^2\,[\,b+c\,]^2+b^2\,n\,[\,b+c\,]\,[\,b\,n+c\,]+b\,c\,[\,b+c\,]\,[\,b\,n+c\,] \\ & -\,b^3\,[\,b\,n+c\,]\,[\,n+1\,]-b^2\,c\,[\,b\,n+c\,]\,[\,n+2\,]-b\,c^2\,[\,b\,n+c\,] \\ =\; & b^2\,[\,b+c\,]^2+b\,[\,b\,n+c\,]\,\big[\,b\,n\,(b+c)+c\,(b+c)-b^2\,(n+1)-b\,c\,(n+2)-c^2\,\big] \\ =\; & b^2\,[\,b+c\,]^2+b\,[\,b\,n+c\,]\,\big[\,b^2\,n+b\,c\,n+b\,c+c^2-b^2\,n-b^2-b\,c\,n-2\,b\,c-c^2\,\big] \\ =\; & b^2\,[\,b+c\,]^2+b\,[\,b\,n+c\,]\,\big[-\,b\,c-b^2\,\big] \;=\; b^2\,[\,b+c\,]^2-b^2\,[\,b\,n+c\,]\,[\,b+c\,] \\ =\; & b^2\,[\,b+c\,]\,[\,b+c-\,b\,n-c\,] \;=\; -\,b^3\,[\,b+c\,]\,[\,n-1\,]\,. \end{aligned} \tag{28}$$

(28) implies:

$$Y \;=\; b^3\,[\,b+c\,]\,[\,n-1\,]\,[\,a-c_0\,]\,. \tag{29}$$

(2), (26), (27), and (29) imply that at a symmetric equilibrium:

$$X\,F_i \;=\; Y - \,b^5\,[\,n-1\,]^2\,F_i \quad\Leftrightarrow\quad \left[\,X + b^5\,(n-1)^2\,\right]F_i \;=\; Y$$

$$\Leftrightarrow \quad F_i \;=\; \frac{Y}{X + b^5\,[\,n-1\,]^2} \;=\; \frac{b^3\,[\,b+c\,]\,[\,n-1\,]\,[\,a-c_0\,]}{b^2\,[\,b\,n+c\,]\,[\,2\,b\,(\,b+c\,)+c\,(\,b\,n+c\,)\,]+b^5\,[\,n-1\,]^2}$$

$$=\; \frac{b\,[\,b+c\,]\,[\,n-1\,]\left[\,\frac{a^I+\overline{Q}}{b^I}-c_0\,\right]}{[\,b\,n+c\,]\,[\,2\,b\,(\,b+c\,)+c\,(\,b\,n+c\,)\,]+b^3\,[\,n-1\,]^2}$$

$$=\; \frac{b^2\,[\,b+c\,]\,[\,n-1\,]\,[\,a^I+\,\overline{Q}-b^I\,c_0\,]}{[\,b\,n+c\,]\,[\,2\,b\,(\,b+c\,)+c\,(\,b\,n+c\,)\,]+b^3\,[\,n-1\,]^2}\,. \tag{30}$$

It is apparent from (30) that $\frac{\partial F_i}{\partial a^I} > 0$, $\frac{\partial F_i}{\partial \overline{Q}} > 0$, and $\frac{\partial F_i}{\partial c_0} < 0$. Furthermore:

$$\frac{\partial F_i}{\partial c} \stackrel{s}{=} [\,b\,n+c\,]\,[\,2\,b\,(\,b+c\,)+c\,(\,b\,n+c\,)\,]+b^3\,[\,n-1\,]^2$$

$$-\,[\,b+c\,]\,\{\,[\,b\,n+c\,]\,[\,2\,b+b\,n+2\,c\,]+2\,b\,[\,b+c\,]+c\,[\,b\,n+c\,]\,\}$$

$$=\; 2\,b\,[\,b+c\,]\,[\,b\,n+c\,]+c\,[\,b\,n+c\,]^2+b^3\,[\,n-1\,]^2-2\,b\,[\,b+c\,]\,[\,b\,n+c\,]$$

$$-\;\,b\,n\,[\,b+c\,]\,[\,b\,n+c\,]-2\,c\,[\,b+c\,]\,[\,b\,n+c\,]-2\,b\,[\,b+c\,]^2-c\,[\,b+c\,]\,[\,b\,n+c\,]$$

$$=\; c\,[\,b\,n+c\,]^2+b^3\,[\,n-1\,]^2-b\,n\,[\,b+c\,]\,[\,b\,n+c\,]-3\,c\,[\,b+c\,]\,[\,b\,n+c\,]-2\,b\,[\,b+c\,]^2$$

$$=\; b^3\,[\,n-1\,]^2-2\,b\,[\,b+c\,]^2+[\,b\,n+c\,]\,\{\,c\,[\,b\,n+c\,]-b\,n\,[\,b+c\,]-3\,c\,[\,b+c\,]\,\}$$

$$=\; b^3\,[\,n-1\,]^2-2\,b\,[\,b+c\,]^2+[\,b\,n+c\,]\left[\,b\,c\,n+c^2-b^2\,n-b\,c\,n-3\,b\,c-3\,c^2\,\right]$$

$$=\; b^3\,[\,n-1\,]^2-2\,b\,[\,b+c\,]^2-[\,b\,n+c\,]\left[\,b^2\,n+3\,b\,c+2\,c^2\,\right]$$

$$=\; b^3\left[\,n^2-2\,n+1\,\right]-2\,b\left[\,b^2+2\,b\,c+c^2\,\right]-b\,n\left[\,b^2\,n+3\,b\,c+2\,c^2\,\right]$$

$$-\;c\left[\,b^2\,n+3\,b\,c+2\,c^2\,\right]$$

$$=\; b^3\,n^2-2\,b^3\,n+b^3-2\,b^3-4\,b^2\,c-2\,b\,c^2-b^3\,n^2-3\,b^2\,c\,n-2\,b\,c^2\,n$$

$$-\;b^2\,c\,n-3\,b\,c^2-2\,c^3$$

$$=\; -\,2\,b^3\,n-b^3-4\,b^2\,c-2\,b\,c^2-3\,b^2\,c\,n-2\,b\,c^2\,n-b^2\,c\,n-3\,b\,c^2-2\,c^3 \;<\; 0\,.$$

Because $b^I = \frac{1}{b}$, (30) implies:

$$\frac{\partial F_i}{\partial b} \stackrel{s}{=} \left\{\,[\,b\,n+c\,]\,[\,2\,b\,(\,b+c\,)+c\,(\,b\,n+c\,)\,]+b^3\,[\,n-1\,]^2\,\right\}$$

$$
\begin{aligned}
&\cdot \Big\{ 2\,b\,[\,b+c\,]\,[\,n-1\,]\,\big[\,a^I + \overline{Q} - b^I\,c_0\,\big] + b^2\,[\,n-1\,]\,\big[\,a^I + \overline{Q} - b^I\,c_0\,\big] \\
&\qquad\qquad + \left[\,\frac{c_0}{b^2}\,\right]\,b^2\,[\,b+c\,]\,[\,n-1\,]\,\Big\} \\
&- \Big\{\,b^2\,[\,b+c\,]\,[\,n-1\,]\,\big[\,a^I + \overline{Q} - b^I\,c_0\,\big]\,\Big\} \\
&\quad \cdot \Big\{\,n\,\left[\,2\,b\,(\,b+c\,) + c\,(\,b\,n+c\,)\,\right] + [\,b\,n+c\,]\,[\,4\,b+2\,c+c\,n\,] + 3\,b^2\,[\,n-1\,]^2\,\Big\} \\
= &\ \Big\{\,2\,b\,[\,b+c\,]\,[\,b\,n+c\,] + c\,[\,b\,n+c\,]^2 + b^3\,[\,n-1\,]^2\,\Big\} \\
&\cdot \Big\{\,2\,b\,[\,b+c\,]\,[\,n-1\,]\,\big[\,a^I + \overline{Q} - b^I\,c_0\,\big] + b^2\,[\,n-1\,]\,\big[\,a^I + \overline{Q} - b^I\,c_0\,\big] \\
&\qquad\qquad + c_0\,[\,b+c\,]\,[\,n-1\,]\,\Big\} \\
&- \Big\{\,b^2\,[\,b+c\,]\,[\,n-1\,]\,\big[\,a^I + \overline{Q} - b^I\,c_0\,\big]\,\Big\} \\
&\quad \cdot \Big\{\,2\,b\,n\,[\,b+c\,] + [\,b\,n+c\,]\,[\,4\,b+2\,c+2\,c\,n\,] + 3\,b^2\,[\,n-1\,]^2\,\Big\} \\
= &\ 4\,b^2\,[\,b\,n+c\,]\,[\,b+c\,]^2\,[\,n-1\,]\,\big[\,a^I + \overline{Q} - b^I\,c_0\,\big] \\
&+ 2\,b^3\,[\,b\,n+c\,]\,[\,b+c\,]\,[\,n-1\,]\,\big[\,a^I + \overline{Q} - b^I\,c_0\,\big] \\
&+ 2\,b\,c_0\,[\,b\,n+c\,]\,[\,b+c\,]^2\,[\,n-1\,] \\
&+ 2\,b\,c\,[\,b\,n+c\,]^2\,[\,b+c\,]\,[\,n-1\,]\,\big[\,a^I + \overline{Q} - b^I\,c_0\,\big] \\
&+ b^2\,c\,[\,b\,n+c\,]^2\,[\,n-1\,]\,\big[\,a^I + \overline{Q} - b^I\,c_0\,\big] \\
&+ c\,c_0\,[\,b\,n+c\,]^2\,[\,b+c\,]\,[\,n-1\,] \\
&+ 2\,b^4\,[\,b+c\,]\,[\,n-1\,]^3\,\big[\,a^I + \overline{Q} - b^I\,c_0\,\big] \\
&+ b^5\,[\,n-1\,]^3\,\big[\,a^I + \overline{Q} - b^I\,c_0\,\big] + b^3\,c_0\,[\,b+c\,]\,[\,n-1\,]^3 \\
&- 2\,b^3\,n\,[\,b+c\,]^2\,[\,n-1\,]\,\big[\,a^I + \overline{Q} - b^I\,c_0\,\big] \\
&- b^2\,[\,4\,b+2\,c+2\,c\,n\,]\,[\,b\,n+c\,]\,[\,b+c\,]\,[\,n-1\,]\,\big[\,a^I + \overline{Q} - b^I\,c_0\,\big] \\
&- 3\,b^4\,[\,b+c\,]\,[\,n-1\,]^3\,\big[\,a^I + \overline{Q} - b^I\,c_0\,\big] \\
= &\ \big[\,a^I + \overline{Q} - b^I\,c_0\,\big]\,[\,n-1\,]
\end{aligned}
$$

$$
\cdot \Big\{ 4\,b^2\,[\,b\,n + c\,]\,[\,b + c\,]^2 + 2\,b^3\,[\,b\,n + c\,]\,[\,b + c\,] + 2\,b\,c\,[\,b\,n + c\,]^2\,[\,b + c\,]
+ b^2\,c\,[\,b\,n + c\,]^2 + 2\,b^4\,[\,b + c\,]\,[\,n - 1\,]^2 + b^5\,[\,n - 1\,]^2 - 2\,b^3\,n\,[\,b + c\,]^2
- b^2\,[\,4\,b + 2\,c + 2\,c\,n\,]\,[\,b\,n + c\,]\,[\,b + c\,] - 3\,b^4\,[\,b + c\,]\,[\,n - 1\,]^2 \Big\}
+ 2\,b\,c_0\,[\,b\,n + c\,]\,[\,b + c\,]^2\,[\,n - 1\,] + c\,c_0\,[\,b\,n + c\,]^2\,[\,b + c\,]\,[\,n - 1\,]
+ b^3\,c_0\,[\,b + c\,]\,[\,n - 1\,]^3 \;>\; 0\,. \tag{31}
$$

The inequality in (31) implies that $\frac{\partial F_i}{\partial b^I} < 0$ because $b^I = \frac{1}{b}$. The inequality holds because $a^I + \overline{Q} - b^I\,c_0 > 0$, $n > 1$, and:

$$
4\,b^2\,[\,b\,n + c\,]\,[\,b + c\,]^2 + 2\,b^3\,[\,b\,n + c\,]\,[\,b + c\,]
+ 2\,b\,c\,[\,b\,n + c\,]^2\,[\,b + c\,] + b^2\,c\,[\,b\,n + c\,]^2
+ 2\,b^4\,[\,b + c\,]\,[\,n - 1\,]^2 + b^5\,[\,n - 1\,]^2 - 2\,b^3\,n\,[\,b + c\,]^2
- 4\,b^3\,[\,b\,n + c\,]\,[\,b + c\,] - 2\,c\,b^2\,[\,b\,n + c\,]\,[\,b + c\,]
- 2\,c\,n\,b^2\,[\,b\,n + c\,]\,[\,b + c\,] - 3\,b^4\,[\,b + c\,]\,[\,n - 1\,]^2
$$

$$
= [\,b + c\,] \Big\{ 2\,b^3\,[\,b\,n + c\,] + 2\,b\,c\,[\,b\,n + c\,]^2 + 2\,b^4\,[\,n - 1\,]^2
- 4\,b^3\,[\,b\,n + c\,] - 2\,c\,b^2\,[\,b\,n + c\,] - 2\,c\,n\,b^2\,[\,b\,n + c\,] - 3\,b^4\,[\,n - 1\,]^2 \Big\}
+ [\,b + c\,]^2 \Big[4\,b^2\,(\,b\,n + c\,) - 2\,b^3\,n \Big] + b^2\,c\,[\,b\,n + c\,]^2 + b^5\,[\,n - 1\,]^2
$$

$$
= [\,b + c\,] \Big\{ 2\,b\,c\,[\,b\,n + c\,]^2 - 2\,b^3\,[\,b\,n + c\,] - 2\,c\,b^2\,[\,b\,n + c\,]
- 2\,c\,n\,b^2\,[\,b\,n + c\,] - b^4\,[\,n - 1\,]^2 \Big\}
+ [\,b + c\,]^2 \Big[2\,b^3\,n + 4\,b^2\,c \Big] + b^2\,c\,[\,b\,n + c\,]^2 + b^5\,[\,n - 1\,]^2
$$

$$
= [\,b + c\,] \Big\{ 2\,b\,c\,\Big[\,b^2\,n^2 + 2\,b\,n\,c + c^2\,\Big] - 2\,b^4\,n - 2\,b^3\,c - 2\,c\,b^3\,n - 2\,c^2\,b^2
- 2\,c\,n^2\,b^3 - 2\,c^2\,n\,b^2 - b^4\,\Big[\,n^2 - 2\,n + 1\,\Big] \Big\}
+ [\,b + c\,]^2 \Big[2\,b^3\,n + 4\,b^2\,c \Big] + b^2\,c\,[\,b\,n + c\,]^2 + b^5\,[\,n - 1\,]^2
$$

$$
= [\,b + c\,] \Big\{ 2\,b^3\,c\,n^2 + 4\,b^2\,c^2\,n + 2\,b\,c^3 - 2\,b^4\,n - 2\,b^3\,c - 2\,c\,b^3\,n - 2\,c^2\,b^2
$$

$$- 2\,c\,n^2\,b^3 - 2\,c^2\,n\,b^2 - b^4\,n^2 + 2\,n\,b^4 - b^4 \Big\}$$

$$+ \,[\,b + c\,]^2 \left[\,2\,b^3\,n + 4\,b^2\,c\,\right] + \,b^2\,c\,[\,b\,n + c\,]^2 + b^5\,[\,n - 1\,]^2$$

$$= \,[\,b + c\,] \left[\,2\,b^2\,c^2\,n + 2\,b\,c^3 - 2\,b^3\,c - 2\,c\,b^3\,n - 2\,c^2\,b^2 - b^4\,n^2 - b^4\,\right]$$

$$+ \,\left[\,b^2 + 2\,b\,c + c^2\,\right] \left[\,2\,b^3\,n + 4\,b^2\,c\,\right] + \,b^2\,c\,[\,b\,n + c\,]^2 + b^5\,[\,n - 1\,]^2$$

$$= \,2\,b^3\,c^2\,n + 2\,b^2\,c^3 - 2\,b^4\,c - 2\,c\,b^4\,n - 2\,c^2\,b^3 - b^5\,n^2 - b^5$$

$$+ \,2\,b^2\,c^3\,n + 2\,b\,c^4 - 2\,b^3\,c^2 - 2\,c^2\,b^3\,n - 2\,c^3\,b^2 - b^4\,n^2\,c - b^4\,c$$

$$+ \,2\,b^5\,n + 4\,b^4\,c + 4\,b^4\,n\,c + 8\,b^3\,c^2 + \,2\,b^3\,n\,c^2 + 4\,b^2\,c^3$$

$$+ \,b^2\,c\,\left[\,b^2\,n^2 + 2\,b\,n\,c + c^2\,\right] + b^5\left[\,n^2 - 2\,n + 1\,\right]$$

$$= \,2\,b^3\,c^2\,n + 2\,b^2\,c^3 - 2\,b^4\,c - 2\,c\,b^4\,n - 2\,c^2\,b^3 - b^5\,n^2 - b^5$$

$$+ \,2\,b^2\,c^3\,n + 2\,b\,c^4 - 2\,b^3\,c^2 - 2\,c^2\,b^3\,n - 2\,c^3\,b^2 - b^4\,n^2\,c - b^4\,c$$

$$+ \,2\,b^5\,n + 4\,b^4\,c + 4\,b^4\,n\,c + 8\,b^3\,c^2 + \,2\,b^3\,n\,c^2 + 4\,b^2\,c^3$$

$$+ \,b^4\,n^2\,c + 2\,b^3\,n\,c^2 + b^2\,c^3 + b^5\,n^2 - 2\,b^5\,n + b^5$$

$$= \,b^3\,c^2\,[\,2\,n - 2 - 2 - 2\,n + 8 + \,2\,n + 2\,n\,] + b^2\,c^3\,[\,2 + \,2\,n - 2 + 4\,]$$

$$+ \,b^4\,c\,\left[-2 - 2\,n - n^2 - 1 + 4 + 4\,n + \,n^2\,\right]$$

$$+ \,b^5\left[\,- n^2 - 1 + 2\,n + n^2 - 2\,n + 1\,\right] + 2\,b\,c^4 + b^2\,c^3$$

$$= \,b^3\,c^2\,[\,4\,n + 4\,] + b^2\,c^3\,[\,2\,n + 4\,] + b^4\,c\,[\,2\,n + 1\,] + 2\,b\,c^4 + b^2\,c^3 \;\;>\;\; 0\,. \;\; \blacksquare$$

Corollary 1. *Suppose $c = 0$. Then at a symmetric equilibrium under forward contracting, for $i = 1, ..., n$, $F_i = \frac{n-1}{n^2+1}\left[\,a^I + \overline{Q} - b^I\,c_0\,\right] \;\Rightarrow\; F_i = \frac{1}{5}\left[\,a^I + \overline{Q} - b^I\,c_0\,\right]$ when $n = 2$ and when $n = 3$. Furthermore, $\frac{\partial F_i}{\partial n} < 0$ for all $n \geq 3$, and $\frac{\partial}{\partial n}\left(\sum_{j=1}^{n} F_j\right) > 0$ for all $n \geq 2$.*

Proof. (30) implies:

$$\lim_{c \to 0} F_i \;=\; \frac{b^3\,[\,n-1\,]\left[\,a^I + \,\overline{Q} - b^I c_0\,\right]}{b\,n\,[\,2\,b^2\,] + b^3\,[\,n-1\,]^2} \;=\; \frac{[\,n-1\,]\left[\,a^I + \,\overline{Q} - b^I c_0\,\right]}{2\,n + [\,n-1\,]^2}$$

$$= \;\frac{[\,n-1\,]\left[\,a^I + \,\overline{Q} - b^I c_0\,\right]}{2\,n + n^2 - 2\,n + 1} \;=\; \frac{n-1}{n^2+1}\left[\,a^I + \,\overline{Q} - b^I c_0\,\right]. \qquad (32)$$

Observe that: $\dfrac{2-1}{(2)^2+1} = \dfrac{1}{5}$ and $\dfrac{3-1}{(3)^2+1} = \dfrac{2}{10} = \dfrac{1}{5}$.

Therefore, (32) implies that if $c = 0$, then $F_i = \frac{1}{5}\left[a^I + \overline{Q} - b^I c_0 \right]$ when $n = 2$ and when $n = 3$.

(32) also implies that when $c = 0$:

$$\frac{\partial F_i}{\partial n} \stackrel{s}{=} n^2 + 1 - 2\,n\,[\,n-1\,] \;=\; -\,n^2 + 2\,n + 1$$

$$\Rightarrow \;\; \frac{\partial F_i}{\partial n} \gtreqless 0 \;\;\Leftrightarrow\;\; g(n) \;\equiv\; n^2 - 2\,n - 1 \lesseqgtr 0\,. \tag{33}$$

The roots of the equation $g(n) = 0$ are:

$$n \;=\; \frac{1}{2}\left[2 \pm \sqrt{4+4}\, \right] \;=\; 1 \pm \sqrt{2}\,. \tag{34}$$

(33) and (34) imply:

$$g(n) \begin{cases} < 0 \;\text{ for } n \in \left(1, 1+\sqrt{2}\,\right) \\ > 0 \;\text{ for } n > 1+\sqrt{2}\,. \end{cases} \tag{35}$$

(33) and (35) imply that when $c = 0$:

$$\frac{\partial F_i}{\partial n} \begin{cases} > 0 \;\text{ for } n \in \left(1, 1+\sqrt{2}\,\right) \\ = 0 \;\text{ for } n = 1+\sqrt{2} \\ < 0 \;\text{ for } n > 1+\sqrt{2}\,. \end{cases} \;\;\Rightarrow\;\; \frac{\partial F_i}{\partial n} < 0 \text{ for all } n \geq 3\,.$$

(32) further implies that when $c = 0$:

$$\frac{\partial}{\partial n}\left(\sum_{j=1}^{n} F_j \right) \stackrel{s}{=} \frac{\partial}{\partial n}\left(\frac{n\,[\,n-1\,]}{n^2+1} \right) \stackrel{s}{=} \left[n^2 + 1 \right] [\,2\,n-1\,] - 2\,n^2\,[\,n-1\,]$$

$$= \; 2\,n^3 + 2\,n - n^2 - 1 - 2\,n^3 + 2\,n^2 \;=\; n^2 + 2\,n - 1 \;>\; 0 \text{ for all } n \geq 2\,. \tag{36}$$

The strict inequality in (36) holds because $n^2 + 2\,n - 1$ is a strictly increasing function of n for all $n \geq 0$ that is strictly positive at $n = 2$. ■