Appendix to Accompany

"The Political Economy of Voluntary Public Service"

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Part I of this Appendix provides the proofs of the formal conclusions in the paper, after re-stating key equations from the paper. Part II of this Appendix has three sections. Section II.A presents additional numerical solutions to supplement those reported in Section 4 of the paper. Section II.B states and proves the additional analytic conclusions noted in Sections 3 and 4 of the paper. Section III.C states and proves the analytic conclusions noted in Section 5 of the paper.

I. **Proofs of Formal Conclusions in the Paper**

Key Equations in the Paper

$$N \int_{\underline{c}}^{\overline{c}} W_M(c)\, dG(c) \;=\; -\,T\, c^e, \;\text{ where }\; c^e \;\equiv\; \int_{\underline{c}}^{\overline{c}} c\, dG(c)\,. \tag{1}$$

$$W_i(c) \;>\; W_o(c) \quad \Leftrightarrow \quad \frac{T}{N_i}\,[\, w - c\,] \;>\; -\,F\,. \tag{2}$$

$$W_i(\widehat{c}\,) \;=\; W_o(\widehat{c}\,) \quad \Leftrightarrow \quad \widehat{c} \;=\; w + \frac{F\,N_i}{T}\,. \tag{3}$$

$$T\,w + A \;=\; [\,1 - G(\widehat{c}\,)\,]\,N\,F\,. \tag{4}$$

$$N\,G(\widehat{c}\,) \;\geq\; T\,. \tag{5}$$

$$N \int_{\underline{c}}^{\widehat{c}} \frac{T}{N_i}\,[\, w - c\,]\, dG(c) - N\,[\,1 - G(\widehat{c}\,)\,]\,F \,-\, A\,. \tag{6}$$

$$G(c_2(\widetilde{N})) - G(\widetilde{c}_1(\widetilde{N})) \;=\; \frac{1}{2} \tag{7}$$

$$\text{where }\; c_1(N) \;\equiv\; \widehat{c} - \frac{A_W}{N-T}, \quad c_2(N) \;\equiv\; \widehat{c} + \frac{A_W}{N}, \;\text{ and}$$

$$\widetilde{c}_1(N) \;\equiv\; c_1(2T) - [\,c_2(2T) - c_2(N)\,]. \tag{8}$$

Lemma 1. *When an optimal VJS replaces MJS: (i) welfare increases for individuals with the lowest c's ($c \in [\,\underline{c}, c_1\,]$ where $c_1 \equiv \widehat{c} - \frac{A}{N-T}$) and the highest c's ($c \in [\,c_2, \overline{c}\,]$ where $c_2 \equiv \widehat{c} + \frac{A}{T}$); whereas (ii) welfare declines for individuals with intermediate c's ($c \in (c_1, c_2)$).*

Lemma 2. *Under an optimal VJS, the rate at which an individual's expected increase in welfare from VJS (relative to MJS) varies with c is:*

$$W'_{\Delta i}(c) \;\equiv\; W'_i(c) - W'_M(c) \;=\; -\,\frac{N-T}{N} \;<\; 0 \quad \textit{for } c \in [\,\underline{c}, \widehat{c}\,)\,; \tag{9}$$

$$W'_{\Delta o}(c) \;\equiv\; W'_o(c) - W'_M(c) \;=\; \frac{T}{N} \;>\; 0 \quad \textit{for } c \in (\,\widehat{c}, \overline{c}\,]\,. \tag{10}$$

Corollary 1. $$|\,W'_{\Delta i}(c)\,| \;\gtreqless\; |\,W'_{\Delta o}(c)\,| \quad\Leftrightarrow\quad \frac{N-T}{N} \;\gtreqless\; \frac{T}{N} \quad\Leftrightarrow\quad N \;\gtreqless\; 2T\,. \tag{11}$$

Proof. (2) and (3) imply that individuals with $c \in [\,\underline{c}, \widehat{c}\,]$ opt in whereas individuals with $c \in (\,\widehat{c}, \overline{c}\,]$ opt out under VJS. Also, $W_i(c) = w - c$ (because $G(\widehat{c}\,) = \frac{T}{N}$) and $W_o(c) = -\,F$. Therefore:

$$W_{\Delta i}(c) \;=\; W_i(c) - W_M(c) \;=\; w - c - \left(-\,\frac{T}{N}\,c\right) \;=\; w - \left[\frac{N-T}{N}\right] c$$

$$\Rightarrow\quad W'_{\Delta i}(c) \;=\; -\,\frac{N-T}{N} \;<\; 0 \quad \text{for } c \in [\,\underline{c}, \widehat{c}\,)\,; \text{ and}$$

$$W_{\Delta o}(c) \;=\; W_o(c) - W_M(c) \;=\; -\,F - \left(-\,\frac{T}{N}\,c\right) \;=\; -\,F + \frac{T}{N}\,c$$

$$\Rightarrow\quad W'_{\Delta o}(c) \;=\; \frac{T}{N} \;>\; 0 \quad \text{for } c \in (\,\widehat{c}, \overline{c}\,]\,, \tag{12}$$

so Lemma 2 holds.

Because $G(\widehat{c}\,) = \frac{T}{N}$, the definition of $\widehat{c}$ implies:

$$w - \widehat{c} \;=\; -\,F \quad\Leftrightarrow\quad F \;=\; \widehat{c} - w\,. \tag{13}$$

c_1 is the largest realization of $c \in (\,\underline{c}, \widehat{c}\,)$ for which $W_i(c) \geq W_M(c)$. Therefore, (12) implies:

$$w - c_1 \;=\; -\,\frac{T}{N}\,c_1 \quad\Rightarrow\quad c_1\left[\frac{N-T}{N}\right] \;=\; w_1 \quad\Rightarrow\quad c_1 \;=\; \left[\frac{N}{N-T}\right] w\,. \tag{14}$$

Because $G(\widehat{c}\,) = \frac{T}{N}$ and the financing constraint holds:

$$T\,w + A \;=\; [\,N-T\,]\,F \quad\Rightarrow\quad T\,w + A \;=\; [\,N-T\,]\,[\,\widehat{c} - w\,]$$

$$\Rightarrow\quad T\,w + A \;=\; [\,N-T\,]\,\widehat{c} - N\,w + T\,w \quad\Rightarrow\quad w \;=\; \frac{1}{N}\,[\,(N-T)\,\widehat{c} - A\,]\,. \tag{15}$$

The second equality in the first line of (15) reflects (13). (14) and (15) imply:

$$c_1 \;=\; \left[\frac{N}{N-T}\right]\frac{1}{N}\,[\,(N-T)\,\widehat{c} - A\,] \;=\; \widehat{c} - \frac{A}{N-T}\,. \tag{16}$$

c_2 is the smallest realization of $c \in (\,\widehat{c}, \overline{c}\,)$ for which $W_0(c) \geq W_M(c)$. Therefore, (12) implies:

$$- F \;=\; - \frac{T}{N}\, c_2 \;\;\Rightarrow\;\; c_2 \;=\; \frac{N\,F}{T}\,. \tag{17}$$

(13) and (15) imply:

$$F \;=\; \widehat{c} - \frac{1}{N}\,[\,(N-T)\,\widehat{c} - A\,] \;=\; \frac{T}{N}\,\widehat{c} + \frac{A}{N}\,. \tag{18}$$

(17) and (18) imply:

$$c_2 \;=\; \frac{N}{T}\left[\frac{T}{N}\,\widehat{c} + \frac{A}{N}\right] \;=\; \widehat{c} + \frac{A}{T}\,. \tag{19}$$

Lemma 1 follows from (12), (16), and (19). ■

Lemma 3. *If $A = 0$, then VJS can be designed to ensure that every individual secures at least the level of expected welfare he secures under MJS, and that nearly all individuals secure strictly higher levels of expected welfare.*

Proof. As demonstrated in the text, the financing and adequate jury pool constraints are satisfied as equalities when $F = \frac{T}{N}\,\widehat{c}$ and $w = \left[\,\frac{N-T}{N}\,\right]\widehat{c}$. Furthermore, (3) implies $W_i(\,\widehat{c}\,) = W_o(\,\widehat{c}\,)$ because:

$$w + F\,\frac{N_i}{T} \;=\; \left[\frac{N-T}{N}\right]\widehat{c} + \frac{T}{N}\,\widehat{c}\left[\frac{T}{T}\right] \;=\; \widehat{c}\,.$$

Therefore, because $c_1 = c_2 = \widehat{c}$ when $A = 0$, Lemma 1 implies the proof is complete if $W_V(\,\widehat{c}\,) = W_M(\,\widehat{c}\,)$. This equality holds because:

$$W_V(\,\widehat{c}\,) \;=\; W_M(\,\widehat{c}\,) \;\;\Leftrightarrow\;\; w - \widehat{c} \;=\; -\frac{T}{N}\,\widehat{c}$$

$$\Leftrightarrow\;\; \left[\frac{N-T}{N}\right]\widehat{c} - \widehat{c} \;=\; -\frac{T}{N}\,\widehat{c} \;\;\Leftrightarrow\;\; -\frac{T}{N}\,\widehat{c} \;=\; -\frac{T}{N}\,\widehat{c}\,. \;\;■$$

Lemma 4. *Suppose $A > 0$. Then a VJS policy that secures a strict increase in expected welfare for some individuals (relative to MJS) necessarily reduces the expected welfare of some other individuals.*

Proof. Under an optimal VJS that ensures $W_V(c) > W_M(c)$ for some $c \in [\,\underline{c}, \overline{c}\,]$, there exists a $\widehat{c} \in (\,\underline{c}, \overline{c}\,)$ defined by:

$$W_i(\,\widehat{c}\,) \;=\; W_o(\,\widehat{c}\,) \;\;\Leftrightarrow\;\; w - \widehat{c} \;=\; -F \;\;\Leftrightarrow\;\; w \;=\; \widehat{c} - F\,. \tag{20}$$

Suppose the VJS policy can be designed to ensure $W_V(c) \geq W_M(c)$ for all $c \in [\,\underline{c}, \overline{c}\,]$. Then it must be the case that:

$$W_V(\,\widehat{c}\,) \;\geq\; W_M(\,\widehat{c}\,) \;\;\Leftrightarrow\;\; w - \widehat{c} \;\geq\; -\frac{T}{N}\,\widehat{c} \;\;\Leftrightarrow\;\; w \;\geq\; \left[\frac{N-T}{N}\right]\widehat{c} \tag{21}$$

$$\Leftrightarrow \;\; \widehat{c} - F \;\geq\; \left[\frac{N-T}{N}\right]\widehat{c} \;\;\Leftrightarrow\;\; \frac{T}{N}\,\widehat{c} \geq F \;\;\Leftrightarrow\;\; N\,F \;\leq\; T\,\widehat{c}\,. \tag{22}$$

The first equivalence in (22) reflects (20). Because $G(\widehat{c}) = \frac{T}{N}$, the last inequality in (21) implies:

$$\begin{aligned}[1 - G(\widehat{c})]\,N\,F - w\,T \;&\leq\; [1 - G(\widehat{c})]\,N\,F - \left[\frac{N-T}{N}\right]\widehat{c}\,T \\ = \left[\frac{N-T}{N}\right]N\,F - \left[\frac{N-T}{N}\right]\widehat{c}\,T \;&=\; \left[\frac{N-T}{N}\right][N\,F - T\,\widehat{c}\,] \;\leq\; 0\,.\end{aligned} \tag{23}$$

The inequality in (23) reflects (22). (23) implies that (4) cannot hold for any $A > 0$. Therefore, it cannot be the case that the VJS policy ensures $W_V(c) \geq W_M(c)$ for all $c \in [\,\underline{c}, \overline{c}\,]$. ■

Proposition 1. *Suppose $A = 0$. Then all individuals (weakly) prefer an optimal VJS policy to MJS, and majority rule always implements the optimal VJS policy.*

Proof. The proof follows immediately from the associated discussion in the text. ■

Proposition 2. *In the limit as $T/N \to 0$, majority rule favors MJS when $c^e > c^d$, favors VJS when $c^e < c^d$, and favors neither MJS nor VJS when $c^e = c^d$ (i.e., $A_M \gtreqless A_W \;\Leftrightarrow\; c^e \lesseqgtr c^d$).*

Proof. The average expected cost that individuals incur under VJS is $\frac{T}{N}\frac{\int_{\underline{c}}^{\widehat{c}} c\,dG(c)}{G(\widehat{c})} + \frac{A}{N}$. The corresponding average expected cost under MJS is $\frac{T}{N}\,c^e$. Therefore, the average expected net gain from VJS is:

$$\frac{T}{N}\,c^e - \frac{T}{N}\,\frac{\int_{\underline{c}}^{\widehat{c}} c\,dG(c)}{G(\widehat{c})} - \frac{A}{N}\,. \tag{24}$$

(24) and the definition of A_W imply:

$$\frac{T}{N}\,c^e - \frac{T}{N}\,\frac{\int_{\underline{c}}^{\widehat{c}} c\,dG(c)}{G(\widehat{c})} - \frac{A_W}{N} \;=\; 0 \;\;\Rightarrow\;\; \frac{A_W}{T} \;=\; c^e - \frac{\int_{\underline{c}}^{\widehat{c}} c\,dG(c)}{G(\widehat{c})}\,. \tag{25}$$

$$\text{Define:}\quad c_1(A) \;=\; \widehat{c} - \frac{A}{N-T} \quad\text{and}\quad c_2(A) \;=\; \widehat{c} + \frac{A}{T}\,. \tag{26}$$

Lemma 1 and the definitions of A_M and A_W imply:

$$A_M \;\gtreqless\; A_W \;\;\Leftrightarrow\;\; G\left(c_2(A_W)\right) - G\left(c_1(A_W)\right) \;\lesseqgtr\; \frac{1}{2}, \;\text{ and} \tag{27}$$

$$A_M \lesseqgtr A_W \quad \Leftrightarrow \quad \int_{c_1(A_W)}^{c_2(A_W)} dG(c) \gtreqless \frac{1}{2}\,. \tag{28}$$

From (25):

$$\widehat{c} + \frac{A_W}{T} = \widehat{c} + c^e - \frac{\int_{\underline{c}}^{\widehat{c}} c\, dG(c)}{G(\widehat{c}\,)}\,. \tag{29}$$

L'Hôpital's Rule implies:

$$\lim_{\widehat{c} \to \underline{c}} \left[\frac{\int_{\underline{c}}^{\widehat{c}} c\, dG(c)}{G(\widehat{c}\,)} \right] = \lim_{\widehat{c} \to \underline{c}} \left[\frac{\widehat{c}\, g(\widehat{c}\,)}{g(\widehat{c}\,)} \right] = \lim_{\widehat{c} \to \underline{c}} [\,\widehat{c}\,] = \underline{c}\,. \tag{30}$$

Because $G(\widehat{c}\,) = \frac{T}{N}$, (29) and (30) imply that as $T/N \to 0$:

$$\begin{aligned} \widehat{c} + \frac{A_W}{T} \;\to\; \lim_{\widehat{c} \to \underline{c}} \left[\widehat{c} + \frac{A_W}{T} \right] &= \lim_{\widehat{c} \to \underline{c}} \left[\widehat{c} + c^e - \frac{\int_{\underline{c}}^{\widehat{c}} c\, dG(c)}{G(\widehat{c}\,)} \right] \\ &= \underline{c} + c^e - \underline{c} = c^e\,. \end{aligned} \tag{31}$$

From (25):

$$\widehat{c} - \frac{A_W}{N-T} = \widehat{c} - \left[\frac{T}{N-T} \right] \frac{A_W}{T} = \widehat{c} - \frac{T}{N-T} \left[c^e - \frac{\int_{\underline{c}}^{\widehat{c}} c\, dG(c)}{G(\widehat{c}\,)} \right]. \tag{32}$$

Because $G(\widehat{c}\,) = \frac{T}{N}$, (30) and (32) imply that as $T/N \to 0$:

$$\begin{aligned} \widehat{c} - \frac{A_W}{N-T} \;&\to\; \lim_{T/N \to 0} \left[\widehat{c} - \frac{A_W}{N-T} \right] \\ &= \lim_{T/N \to 0} \left[\widehat{c} - \left(\frac{T}{N-T} \right) \left(c^e - \frac{\int_{\underline{c}}^{\widehat{c}} c\, dG(c)}{G(\widehat{c}\,)} \right) \right] = \underline{c} - [\,0\,]\,[\,c^e - \underline{c}\,] = \underline{c}\,. \end{aligned} \tag{33}$$

(27), (31), and (33) imply that when T/N is sufficiently small:

$$A_M \gtreqless A_W \;\Leftrightarrow\; G(c^e) - G(\underline{c}) \lesseqgtr \frac{1}{2} \;\Leftrightarrow\; G(c^e) \lesseqgtr G\left(c^d\right) \;\Leftrightarrow\; c^e \lesseqgtr c^d\,. \quad \blacksquare$$

Proposition 3. *In the limit as $T/N \to 1$, majority rule favors VJS when $c^e > c^d$, favors MJS when $c^e < c^d$, and favors neither MJS nor VJS when $c^e = c^d$ (i.e., $A_M \lesseqgtr A_W \Leftrightarrow c^e \lesseqgtr c^d$).*

<u>Proof</u>. As $N \to T$, nearly all individuals must opt in under VJS to ensure that every trial has a juror. Consequently:

$$\widehat{c} \to \bar{c} \;\text{ as }\; N \to T\,. \tag{34}$$

(25) and (34) imply that as $N \to T$:

$$\frac{A_W}{T} = c^e - \frac{\int_{\underline{c}}^{\widehat{c}} c\, dG(c)}{G(\widehat{c})} \to c^e - \frac{\int_{\underline{c}}^{\overline{c}} c\, dG(c)}{G(\overline{c})} = c^e - \frac{c^e}{1} = 0$$

$$\Rightarrow A_W \to 0 \text{ as } N \to T. \tag{35}$$

(26) and the definition of A_M imply:

$$G\left(\widehat{c} + \frac{A_M}{T}\right) - G\left(\widehat{c} - \frac{A_M}{N-T}\right) = \frac{1}{2}. \tag{36}$$

(34) and (35) imply:

$$G\left(\widehat{c} + \frac{A_M}{T}\right) \to 1 \text{ as } N \to T. \tag{37}$$

(26), (34), and (35) imply:

$$c_2(A_W) = \widehat{c} + \frac{A_W}{T} \to \overline{c} \text{ as } N \to T. \tag{38}$$

(26), (36), and (37) imply:

$$G\left(\widehat{c} - \frac{A_M}{N-T}\right) = G(c_1(A_M)) \to \frac{1}{2} \Rightarrow c_1(A_M) \to c^d \text{ as } N \to T. \tag{39}$$

(26) and (34) imply:

$$\lim_{N \to T} c_1(A_W) = \overline{c} - \lim_{N \to T} \left(\frac{A_W}{N-T}\right). \tag{40}$$

From Finding 4 in the proof of Proposition 5 below:

$$\frac{\partial A_W}{\partial N} = \int_{\underline{c}}^{\widehat{c}} G(c)\, dc. \tag{41}$$

(34), (35), (40), (41), and L'Hôpital's Rule imply:

$$\lim_{N \to T} \left(\frac{A_W}{N-T}\right) = \lim_{N \to T} \int_{\underline{c}}^{\widehat{c}} G(c)\, dc = \lim_{N \to T} \left\{ G(c)\, c \,\Big|_{\underline{c}}^{\widehat{c}} - \int_{\underline{c}}^{\widehat{c}} c\, g(c)\, dc \right\}$$

$$= G(c)\, c \,\Big|_{\underline{c}}^{\overline{c}} - \int_{\underline{c}}^{\overline{c}} c\, g(c)\, dc = \overline{c} - c^e. \tag{42}$$

(40) and (42) imply:

$$\lim_{N \to T} c_1(A_W) = c^e. \tag{43}$$

From (27):

$$A_W \gtreqless A_M \quad \Leftrightarrow \quad G\left(c_2\left(A_W\right)\right) - G\left(c_1\left(A_W\right)\right) \gtreqless \frac{1}{2}. \tag{44}$$

(38) and (43) imply that as $N \to T$:

$$G\left(c_2\left(A_W\right)\right) - G\left(c_1\left(A_W\right)\right) \;\to\; 1 - G\left(c^e\right). \tag{45}$$

(44) and (45) imply that as $N \to T$:

$$A_W \gtreqless A_M \Leftrightarrow 1 - G\left(c^e\right) \gtreqless \frac{1}{2} \Leftrightarrow G\left(c^e\right) \lesseqgtr \frac{1}{2} = G\left(c^d\right) \Leftrightarrow c^e \lesseqgtr c^d. \quad \blacksquare$$

Proposition 4. *Majority rule favors neither MJS nor VJS (so $A_M = A_W$) when $g(c)$ is the uniform density.*

Proof. For expositional ease, suppose $\underline{c} = 0$, so $g(c) = \frac{1}{\overline{c}}$. (We prove below that this normalization is without loss of generality.) Because $G(\widehat{c}) = \frac{T}{N}$:

$$\frac{\widehat{c}}{\overline{c}} = \frac{T}{N} \quad \Rightarrow \quad \widehat{c} = \frac{T}{N}\,\overline{c}. \tag{46}$$

From (25):

$$\frac{A_W}{T} = c^e - \frac{\int_0^{\widehat{c}} c\, dG(c)}{G(\widehat{c})} = \frac{\overline{c}}{2} - \frac{\frac{(\widehat{c})^2}{2}}{\widehat{c}} = \frac{1}{2}\left[\overline{c} - \widehat{c}\right]. \tag{47}$$

(26) and (47) imply:

$$c_2\left(A_W\right) = \widehat{c} + \frac{A_W}{T} = \widehat{c} + \frac{1}{2}\left[\overline{c} - \widehat{c}\right] = \frac{1}{2}\left[\overline{c} + \widehat{c}\right]. \tag{48}$$

(26) and (47) also imply:

$$\begin{aligned} c_1\left(A_W\right) &= \widehat{c} - \frac{A_W}{N-T} = \widehat{c} - \frac{A_W}{T}\left[\frac{T}{N-T}\right] = \widehat{c} - \frac{1}{2}\left[\overline{c} - \widehat{c}\right]\frac{T}{N-T} \\ &= \widehat{c}\left[1 + \frac{1}{2}\left(\frac{T}{N-T}\right)\right] - \frac{\overline{c}\,T}{2\left[N-T\right]} = \frac{\widehat{c}\left[2N-T\right] - \overline{c}\,T}{2\left[N-T\right]}. \end{aligned} \tag{49}$$

(48) and (49) imply:

$$\begin{aligned} \int_{c_1(A_W)}^{c_2(A_W)} dG(c) &= \frac{1}{\overline{c}}\left[c_2\left(A_W\right) - c_1\left(A_W\right)\right] = \frac{1}{\overline{c}}\left[\frac{\left[N-T\right]\left[\overline{c} + \widehat{c}\right] - \widehat{c}\left[2N-T\right] + \overline{c}\,T}{2\left[N-T\right]}\right] \\ &= \frac{\widehat{c}\left[N-T-2N+T\right] + \overline{c}\,N}{2\,\overline{c}\left[N-T\right]} = \frac{N\left[\overline{c} - \widehat{c}\right]}{2\,\overline{c}\left[N-T\right]}. \end{aligned} \tag{50}$$

(27) and (50) imply:

$$A_M \lesseqgtr A_W \;\Leftrightarrow\; \frac{N\,[\,\overline{c} - \widehat{c}\,]}{2\,\overline{c}\,[\,N - T\,]} \gtreqless \frac{1}{2} \;\Leftrightarrow\; N\,[\,\overline{c} - \widehat{c}\,] \gtreqless \overline{c}\,[\,N - T\,]$$

$$\Leftrightarrow\; N\,\widehat{c} \lesseqgtr T\,\overline{c} \;\Leftrightarrow\; \widehat{c} \lesseqgtr \frac{T}{N}\,\overline{c}. \tag{51}$$

(46) and (51) imply $A_M = A_W$. ■

Observation. *The expected welfare gain from VJS is proportional to* $A_W - A$ *when* $g(c)$ *is the uniform density.*

Proof. As in the proof of Proposition 4, assume $\underline{c} = 0$ without loss of generality. Then $G(\widehat{c}) = \frac{T}{N}$ and $\widehat{c} = \frac{T}{N}\,\overline{c}$ when $g(c)$ is the uniform density. The expected welfare gain from VJS (relative to MJS) given c is:

$$W_V(c) - W_M(c) \;=\; w - c - \frac{T}{N}\,[\,-c\,] \;=\; w - \left[\frac{N-T}{N}\right] c \;\text{ for }\; c \in [\,0, \widehat{c}\,]; \;\text{ and}$$

$$W_V(c) - W_M(c) \;=\; -F - \frac{T}{N}\,[\,-c\,] \;=\; -F + \left[\frac{T}{N}\right] c \;\text{ for }\; c \in [\,\widehat{c}, \overline{c}\,]. \tag{52}$$

(52) implies:

$$\int_0^{\frac{T}{N}\overline{c}} \left(w - \left[\frac{N-T}{N}\right] c \right) dc \;=\; \frac{T}{N}\,\overline{c} \left[w - \frac{\overline{c}}{2}\,\frac{T}{N} \left(\frac{N-T}{N}\right) \right], \;\text{ and}$$

$$\int_{\frac{T}{N}\overline{c}}^{\overline{c}} \left(-F + \left[\frac{T}{N}\right] c \right) dc \;=\; \left[\frac{N-T}{N}\right] \overline{c} \left[-F + \frac{\overline{c}}{2}\,\frac{T}{N} \left(\frac{N+T}{N}\right) \right]. \tag{53}$$

(53) implies:

$$\begin{aligned}
\int_0^{\overline{c}} (\,W_V(c) - W_M(c)\,)\,dc \;&=\; \overline{c} \left[\frac{T}{N}\,w - \left(\frac{N-T}{N}\right) F + \frac{\overline{c}}{2}\,\frac{T}{N} \left(\frac{N-T}{N}\right) \left(\frac{N+T}{N} - \frac{T}{N}\right) \right] \\
&=\; \overline{c} \left[\frac{T}{N}\,w - \left(\frac{N-T}{N}\right) F + \frac{\overline{c}}{2}\,\frac{T}{N} \left(\frac{N-T}{N}\right) \right] \\
&=\; \frac{\overline{c}}{N} \left[T\,w - \left(\frac{N-T}{N}\right) N\,F + \frac{\overline{c}}{2}\,T \left(\frac{N-T}{N}\right) \right] \\
&=\; \frac{\overline{c}}{N} \left[T\,w - (1 - G(\widehat{c}))\,N\,F + \frac{\overline{c}}{2}\,T\,(1 - G(\widehat{c})) \right] \\
&=\; \frac{\overline{c}}{N} \left[-A + \frac{\overline{c}}{2}\,T \left(\frac{\overline{c} - \widehat{c}}{\overline{c}}\right) \right] \;=\; \frac{\overline{c}}{N}\,[\,A_W - A\,].
\end{aligned} \tag{54}$$

(54) reflects (4) and (47). ■

Proposition 5. *Suppose $g(c)$ is symmetric about its mean, non-decreasing below its median, and strictly log concave. Then there exists a $\widetilde{N} > 2T$ such that majority rule favors MJS (i.e., $A_M < A_W$) for all $N \in [2T, \widetilde{N})$.*

Proof. Without loss of generality, assume $\underline{c} = 0$ and $\overline{c} = 1$, so $c^e = \frac{1}{2}$. $G(c)$ is strictly log concave when $g(c)$ is strictly log concave (Bagnoli and Bergstrom, 2005). Therefore:

$$\frac{\partial}{\partial c}\left(\frac{g(c)}{G(c)}\right) < 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial c}\left(\frac{G(c)}{g(c)}\right) > 0 \text{ for all } c \in [0,1]. \tag{55}$$

Suppose $A = A_W$. Then Lemma 1 implies:

$$c_1 = \widehat{c} - \frac{A_W}{N-T} \quad \text{and} \quad c_2 = \widehat{c} + \frac{A_W}{T}. \tag{56}$$

The definitions of A_W and A_M imply:

$$A_W > A_M \quad \Leftrightarrow \quad G(c_2) - G(c_1) > \frac{1}{2}. \tag{57}$$

Findings 1 – 12 below demonstrate that $G(c_2) - G(c_1) > \frac{1}{2}$ for all $N \in [2T, \widetilde{N})$ under the specified conditions, where $\widetilde{N}$ is defined in (7) above.

Finding 1. $\widehat{c} \leq \frac{1}{2}$.

Proof. $G(\widehat{c}) = \frac{T}{N} \leq \frac{1}{2}$ when $N \geq 2T$. Therefore, $\widehat{c} \leq \frac{1}{2}$ because $g(c)$ is symmetric and strictly increasing. □

Finding 2. $\frac{\partial \widehat{c}}{\partial N} = -\frac{T}{N^2 g(\widehat{c})} = -\frac{1}{T}\frac{[G(\widehat{c})]^2}{g(\widehat{c})} < 0$ *for all* $\widehat{c} > 0$.

Proof.

$$G(\widehat{c}) = \frac{T}{N} \quad \Rightarrow \quad g(\widehat{c})\left[\frac{\partial \widehat{c}}{\partial N}\right] = -\frac{T}{N^2}$$

$$\Rightarrow \quad \frac{\partial \widehat{c}}{\partial N} = -\frac{T}{N^2 g(\widehat{c})} = -\frac{1}{T}\left[\frac{T^2}{N^2 g(\widehat{c})}\right] = -\frac{1}{T}\frac{[G(\widehat{c})]^2}{g(\widehat{c})}. \quad \square$$

Finding 3. $\frac{A_W}{T} = \frac{1}{2} - \widehat{c} + \frac{N}{T}\int_0^{\widehat{c}} G(c)\, dc$.

Proof. From (25):

$$\frac{A_W}{T} = c^e - \frac{\int_0^{\widehat{c}} c\, g(c)\, dc}{G(\widehat{c})} = \frac{1}{2} - \frac{\int_0^{\widehat{c}} c\, g(c)\, dc}{G(\widehat{c})}. \tag{58}$$

Integration by parts provides:

$$\int_0^{\widehat{c}} c\, g(c)\, dc = [\, c\, G(c)\,]_0^{\widehat{c}} - \int_0^{\widehat{c}} G(c)\, dc = \widehat{c}\, G(\widehat{c}) - \int_0^{\widehat{c}} G(c)\, dc. \tag{59}$$

(58) and (59) imply:

$$\frac{A_W}{T} = \frac{1}{2} - \frac{\int_0^{\widehat{c}} c\, g(c)\, dc}{G(\widehat{c})} = \frac{1}{2} - \frac{\widehat{c}\, G(\widehat{c}) - \int_0^{\widehat{c}} G(c)\, dc}{G(\widehat{c})}$$

$$= \frac{1}{2} - \widehat{c} + \frac{\int_0^{\widehat{c}} G(c)\, dc}{G(\widehat{c})} = \frac{1}{2} - \widehat{c} + \frac{N}{T} \int_0^{\widehat{c}} G(c)\, dc\,. \quad \square$$

Finding 4. $\frac{\partial A_W}{\partial N} = \int_0^{\widehat{c}} G(c)\, dc\,.$

Proof. Finding 3 implies:

$$\frac{\partial A_W}{\partial N} = \frac{\partial}{\partial N}\left[T\left(\frac{1}{2} - \widehat{c}\right) + N \int_0^{\widehat{c}} G(c)\, dc \right]$$

$$= -\, T\, \frac{\partial \widehat{c}}{\partial N} + \int_0^{\widehat{c}} G(c)\, dc + N \left[G(\widehat{c})\, \frac{\partial \widehat{c}}{\partial N} \right]$$

$$= T\, \frac{\partial \widehat{c}}{\partial N} \left[-\, 1 + \frac{N}{T}\, G(\widehat{c}) \right] + \int_0^{\widehat{c}} G(c)\, dc = \int_0^{\widehat{c}} G(c)\, dc\,. \quad \square \tag{60}$$

Finding 5. $\frac{\partial}{\partial c}\left(\frac{G(c)}{g(c)}\right) > 0$ *for all* $c \;\Rightarrow\; g(\widetilde{c})\, G(c) - g(c)\, G(\widetilde{c}) < 0$ *for* $c < \widetilde{c}$.

Proof. For $c < \widetilde{c}$:

$$\frac{\partial}{\partial c}\left(\frac{G(c)}{g(c)}\right) > 0 \;\Rightarrow\; \frac{G(c)}{g(c)} < \frac{G(\widetilde{c})}{g(\widetilde{c})} \;\Rightarrow\; g(\widetilde{c})\, G(c) - g(c)\, G(\widetilde{c}) < 0\,. \quad \square$$

Finding 6. $\frac{\partial c_2}{\partial N} < 0\,.$

Proof. (56) and Findings 2 and 3 imply:

$$c_2 = \widehat{c} + \frac{A_W}{T} = \widehat{c} + \frac{1}{2} - \widehat{c} + \frac{N}{T} \int_0^{\widehat{c}} G(c)\, dc = \frac{1}{2} + \frac{N}{T} \int_0^{\widehat{c}} G(c)\, dc$$

$$\Rightarrow \frac{\partial c_2}{\partial N} = \frac{1}{T} \int_0^{\widehat{c}} G(c)\, dc + \frac{N}{T}\, G(\widehat{c}) \left[\frac{\partial \widehat{c}}{\partial N} \right] = \frac{1}{T} \int_0^{\widehat{c}} G(c)\, dc + \frac{\partial \widehat{c}}{\partial N}$$

$$= \frac{1}{T} \int_0^{\widehat{c}} G(c)\, dc - \frac{1}{T}\, \frac{[\, G(\widehat{c})\,]^2}{g(\widehat{c})} = \frac{1}{T\, g(\widehat{c})} \left[g(\widehat{c}) \int_0^{\widehat{c}} G(c)\, dc - (G(\widehat{c}))^2 \right]$$

$$= \frac{1}{T\, g(\widehat{c})} \left[\int_0^{\widehat{c}} [\, g(\widehat{c})\, G(c) - g(c)\, G(\widehat{c})\,]\, dc \right] < 0\,. \tag{61}$$

The inequality in (61) reflects Finding 5. □

Finding 7. $\frac{\partial c_1}{\partial N} < 0$.

<u>Proof</u>. Finding 3 implies:

$$A_W = T \left[\frac{1}{2} - \widehat{c} \right] + N \int_0^{\widehat{c}} G(c) \, dc \, .$$

Therefore, (56) and Findings 2 – 4 imply:

$$\frac{\partial c_1}{\partial N} = \frac{\partial}{\partial N} \left(\widehat{c} - \frac{A_W}{N-T} \right) = \frac{\partial \widehat{c}}{\partial N} + \frac{A_W}{[\, N-T \,]^2} - \left[\frac{1}{N-T} \right] \frac{\partial A_W}{\partial N}$$

$$= - \frac{1}{T} \frac{[\, G(\widehat{c}) \,]^2}{g(\widehat{c})} + \frac{A_W}{[\, N-T \,]^2} - \frac{1}{N-T} \int_0^{\widehat{c}} G(c) \, dc$$

$$= - \frac{1}{T} \frac{[\, G(\widehat{c}) \,]^2}{g(\widehat{c})} - \frac{1}{N-T} \int_0^{\widehat{c}} G(c) \, dc + \frac{1}{[\, N-T \,]^2} \left[T \left(\frac{1}{2} - \widehat{c} \right) + N \int_0^{\widehat{c}} G(c) \, dc \right]$$

$$= - \frac{1}{T} \frac{[\, G(\widehat{c}) \,]^2}{g(\widehat{c})} + \frac{T}{[\, N-T \,]^2} \left[\frac{1}{2} - \widehat{c} \right] + \int_0^{\widehat{c}} G(c) \, dc \left[\frac{N}{(N-T)^2} - \frac{1}{N-T} \right]$$

$$= - \frac{1}{T} \frac{[\, G(\widehat{c}) \,]^2}{g(\widehat{c})} + \frac{T}{[\, N-T \,]^2} \left[\frac{1}{2} - \widehat{c} \right] + \frac{T}{[\, N-T \,]^2} \int_0^{\widehat{c}} G(c) \, dc$$

$$= \frac{1}{T \, [\, N-T \,]^2 \, g(\widehat{c})} \left[- \; (N-T)^2 \, (G(\widehat{c}))^2 + T^2 \, g(\widehat{c}) \left(\frac{1}{2} - \widehat{c} \right) + T^2 \, g(\widehat{c}) \int_0^{\widehat{c}} G(c) \, dc \right]$$

$$= \frac{N^2}{T \, [\, N-T \,]^2 \, g(\widehat{c})} \left[- \; \left(1 - \frac{T}{N} \right)^2 (G(\widehat{c}))^2 + \left(\frac{T}{N} \right)^2 g(\widehat{c}) \left(\frac{1}{2} - \widehat{c} \right) + \left(\frac{T}{N} \right)^2 g(\widehat{c}) \int_0^{\widehat{c}} G(c) \, dc \right]$$

$$= \frac{N^2}{T \, [\, N-T \,]^2 \, g(\widehat{c})} \left[- \; (1 - G(\widehat{c}))^2 \, (G(\widehat{c}))^2 + (G(\widehat{c}))^2 \, g(\widehat{c}) \left(\frac{1}{2} - \widehat{c} \right) + (G(\widehat{c}))^2 \; g(\widehat{c}) \int_0^{\widehat{c}} G(c) \, dc \right]$$

$$= \frac{N^2 \, [\, G(\widehat{c}) \,]^2}{T \, [\, N-T \,]^2 \, g(\widehat{c})} \left[- \; (1 - G(\widehat{c}))^2 + g(\widehat{c}) \left(\frac{1}{2} - \widehat{c} \right) + g(\widehat{c}) \int_0^{\widehat{c}} G(c) \, dc \right]$$

$$= \frac{N^2 \, [\, G(\widehat{c}) \,]^2}{T \, [\, N-T \,]^2 \, g \, (\widehat{c})} \left[- \; 1 + 2 \, G(\widehat{c}) + g(\widehat{c}) \left(\frac{1}{2} - \widehat{c} \right) - (G(\widehat{c}))^2 + g(\widehat{c}) \int_0^{\widehat{c}} G(c) \, dc \right]$$

$$= \frac{N^2 \, [\, G(\widehat{c}) \,]^2}{T \, [\, N-T \,]^2 \, g(\widehat{c})} \left[- \; 1 + 2 \, G(\widehat{c}) + g(\widehat{c}) \left(\frac{1}{2} - \widehat{c} \right) - \int_0^{\widehat{c}} G(\widehat{c}) \, g \, (c) \, dc + \int_0^{\widehat{c}} g(\widehat{c}) G(c) \, dc \right]$$

$$= \frac{[\,G(\widehat{c})\,]^2}{T\left[\,1-\frac{T}{N}\,\right]^2 g(\widehat{c})} \left[\, -\,1 + 2\,G(\widehat{c}) + g(\widehat{c})\left(\frac{1}{2} - \widehat{c}\right) - \int_0^{\widehat{c}} [\,G(\widehat{c})\,g\,(c) - g(\widehat{c})\,G\,(c)\,]\,dc \right]$$

$$= \frac{[\,G(\widehat{c})\,]^2}{T\,[\,1-G(\widehat{c})\,]^2\,g(\widehat{c})} \left[\, -\,1 + 2\,G(\widehat{c}) + g(\widehat{c})\left(\frac{1}{2} - \widehat{c}\right) - \int_0^{\widehat{c}} [\,G(\widehat{c})\,g\,(c) - g(\widehat{c})\,G\,(c)\,]\,dc \right]. \tag{62}$$

Because $\frac{\partial}{\partial c}\left(\frac{G(c)}{g(c)}\right) > 0$, Finding 5 implies that $G(\widehat{c})\,g(c) - g(\widehat{c})\,G\,(c) > 0$ for $c < \widehat{c}$. Therefore, the expression in (62) is negative if $-\,1 + 2\,G(\widehat{c}) + g(\widehat{c})\,\left[\,\frac{1}{2} - \widehat{c}\,\right] < 0$.

Define

$$f(x) \;=\; -\,1 + 2\,G\,(x) + g\,(x)\left[\frac{1}{2} - x\right] \quad \text{for } x \in \left[0, \frac{1}{2}\right].$$

Observe that $f(0) = -\,1 + \frac{g(0)}{2} < 0$. This inequality holds because g is unimodal and symmetric around $\frac{1}{2}$. Also:

$$f\left(\frac{1}{2}\right) \;=\; -\,1 + 2\,G\left(\frac{1}{2}\right) + g\left(\frac{1}{2}\right)\left[\frac{1}{2} - \frac{1}{2}\right] \;=\; 0\,, \text{ and}$$

$$f'(x) \;=\; 2\,g(x) + g'(x)\left[\frac{1}{2} - x\right] - g(x) \;=\; g(x) + g'(x)\left[\frac{1}{2} - x\right]. \tag{63}$$

(63) implies that $f'(x) > 0$ for all $x \in \left(0, \frac{1}{2}\right]$ because $g(x)$ is increasing on $\left[\,0, \frac{1}{2}\,\right]$. Therefore, $f(0) < 0$, $f\left(\frac{1}{2}\right) = 0$, and $f'(x) > 0$ for all $x \in \left(0, \frac{1}{2}\right]$, which implies that $f(x) < 0$ for all $x \in \left[\,0, \frac{1}{2}\,\right]$. Hence, $\frac{\partial c_1}{\partial N} < 0$, from (62). ☐

Finding 8. $\left[\,\frac{1}{2} - G\,(c)\,\right] \int_0^c G\,(y)\,dy \;\geq\; \frac{1}{2}\,[\,G\,(c)\,]^2 \left[\,\frac{1}{2} - c\,\right]$ *for all* $c \in [\,0, \frac{1}{2}\,]$.

Proof. The intermediate value theorem ensures there exists $\eta \in (\,c, \frac{1}{2}\,)$ such that

$$\frac{1}{2} - G(c) \;=\; G(\frac{1}{2}) - G\,(c) \;=\; \left[\frac{1}{2} - c\right] G'(\eta) \;=\; \left[\frac{1}{2} - c\right] g(\eta)\,. \tag{64}$$

$g(\eta) \geq g(c)$ because $g(c)$ is increasing in c and $\eta \in (\,c, \frac{1}{2}\,)$. Therefore, (64) implies:

$$\frac{1}{2} - G(c) \;\geq\; \left[\frac{1}{2} - c\right] g(c)$$

$$\Rightarrow \quad \left[\frac{1}{2} - G(c)\right] \int_0^c G(y)\,dy \;\geq\; \left[\frac{1}{2} - c\right] g(c) \int_0^c G(y)\,dy\,. \tag{65}$$

(65) implies the Finding holds if:

$$\left[\frac{1}{2} - c\right] g(c) \int_0^c G(y)\,dy \;\geq\; \frac{1}{2}\,[\,G(c)\,]^2 \left[\frac{1}{2} - c\right]$$

$$\Leftrightarrow \quad R(c) \;\equiv\; g(c) \int_0^c G(y)\, dy - \frac{1}{2} \left[G(c) \right]^2 \;\geq\; 0\,. \tag{66}$$

Differentiating $R(c)$ provides:

$$R'(c) \;=\; g(c)\, G(c) + g'(c) \int_0^c G(y)\, dy - G(c)\, g(c) \;=\; g'(c) \int_0^c G(y)\, dy \;\geq\; 0\,. \tag{67}$$

The inequality in (67) holds because $g(c)$ is increasing for $c < \frac{1}{2}$. Because $R(0) = 0$ from (66), (67) implies $R(c) \geq 0$ for all $c \in \left[0, \frac{1}{2}\right]$. $\square$

Finding 9. $\left| \frac{\partial c_1}{\partial N} \right| \geq \left| \frac{\partial c_2}{\partial N} \right|$

Proof. From (56):

$$c_2 - c_1 \;=\; \widehat{c} + \frac{A_W}{T} - \left(\widehat{c} - \frac{A_W}{N-T} \right) \;=\; \frac{A_W}{T} \left[1 + \frac{T}{N-T} \right] \;=\; \frac{A_W}{T} \left[\frac{N}{N-T} \right]$$

$$\Rightarrow \quad \frac{\partial\,(c_2 - c_1)}{\partial N} \;=\; \left[\frac{N}{N-T} \right] \frac{\partial}{\partial N} \left(\frac{A_W}{T} \right) - \frac{A_W}{T} \left[\frac{T}{(N-T)^2} \right]$$

$$=\; \frac{1}{N-T} \left[\frac{N}{T} \frac{\partial}{\partial N} (A_W) - \frac{A_W}{N-T} \right]$$

$$\Rightarrow \quad \frac{\partial\,(c_2 - c_1)}{\partial N} \;\geq\; 0 \quad \Leftrightarrow \quad \frac{N}{T} \frac{\partial}{\partial N} (A_W) \;\geq\; \frac{A_W}{N-T}\,. \tag{68}$$

(68) and Findings 3 and 4 imply:

$$\frac{\partial\,(c_2 - c_1)}{\partial N} \;\geq\; 0 \quad \Leftrightarrow \quad \frac{N}{T} \int_0^{\widehat{c}} G(c)\, dc \;\geq\; \frac{T}{N-T} \left[\frac{1}{2} - \widehat{c} + \frac{N}{T} \int_0^{\widehat{c}} G(c)\, dc \right]$$

$$\Leftrightarrow \quad \frac{N}{T} \left[1 - \frac{T}{N-T} \right] \int_0^{\widehat{c}} G(c)\, dc \;\geq\; \frac{T}{N-T} \left[\frac{1}{2} - \widehat{c} \right]$$

$$\Leftrightarrow \quad \frac{N}{T} \left[N - 2\,T \right] \int_0^{\widehat{c}} G(c)\, dc \;\geq\; T \left[\frac{1}{2} - \widehat{c} \right]$$

$$\Leftrightarrow \quad \left[\frac{N - 2\,T}{N} \right] \int_0^{\widehat{c}} G(c)\, dc \;\geq\; \frac{T^2}{N^2} \left[\frac{1}{2} - \widehat{c} \right]$$

$$\Leftrightarrow \quad \left[1 - 2\,\frac{T}{N} \right] \int_0^{\widehat{c}} G(c)\, dc \;\geq\; \left[\frac{T}{N} \right]^2 \left[\frac{1}{2} - \widehat{c} \right]$$

$$\Leftrightarrow \quad \left[1 - 2\,G(\widehat{c}\,) \right] \int_0^{\widehat{c}} G(c)\, dc \;\geq\; \left[G(\widehat{c}\,) \right]^2 \left[\frac{1}{2} - \widehat{c} \right]$$

$$\Leftrightarrow \quad \left[\frac{1}{2} - G(\widehat{c}\,) \right] \int_0^{\widehat{c}} G(c)\, dc \;\geq\; \frac{1}{2} \left[G(\widehat{c}\,) \right]^2 \left[\frac{1}{2} - \widehat{c} \right].$$

Finding 8 implies that this inequality holds. □

Finding 10. *$A_W > A_M$ if $N = 2T$.*

<u>Proof</u>. $\widehat{c} = G(\frac{T}{N}) = G(\frac{1}{2}) = \frac{1}{2}$ when $N = 2T$. Furthermore, from (56), when $N = 2T$:

$$c_1 = \widehat{c} - \frac{A_W}{N - T} = \frac{1}{2} - \frac{A_W}{T}.$$

Therefore, (56) and (57) imply:

$$A_W > A_M \Leftrightarrow G\left(\frac{1}{2} + \frac{A_W}{T}\right) - G\left(\frac{1}{2} - \frac{A_W}{T}\right) > \frac{1}{2}. \tag{69}$$

From (58):

$$\frac{A_W}{T} = \frac{1}{2} - \frac{\int_0^{\widehat{c}} c\, g\,(c)\; dc}{G(\widehat{c})} \quad \Rightarrow \quad \frac{1}{2} + \frac{A_W}{T} = 1 - \frac{\int_0^{\frac{1}{2}} c\, g(c)\, dc}{G\left(\frac{1}{2}\right)} = 1 - 2 \int_0^{\frac{1}{2}} c\, g(c)\, dc$$

$$\Rightarrow \frac{1}{2} - \frac{A_W}{T} = \frac{1}{2} - \frac{1}{2} + \frac{\int_0^{\widehat{c}} c\, g(c)\, dc}{G(\widehat{c})} = \frac{\int_0^{\frac{1}{2}} c\, g(c)\, dc}{G\left(\frac{1}{2}\right)} = 2 \int_0^{\frac{1}{2}} c\, g(c)\, dc. \tag{70}$$

$G(c) = 1 - G(1 - c)$ because $g\,(c)$ is symmetric. Therefore:

$$G\left(1 - 2\int_0^{\frac{1}{2}} c\, g(c)\, dc\right) = 1 - G\left(2\int_0^{\frac{1}{2}} c\, g(c)\, dc\right). \tag{71}$$

(69) – (71) imply:

$$A_W > A_M \Leftrightarrow G\left(\frac{1}{2} + \frac{A_W}{T}\right) - G\left(\frac{1}{2} - \frac{A_W}{T}\right) > \frac{1}{2}$$

$$\Leftrightarrow \quad 1 - G\left(2\int_0^{\frac{1}{2}} c\, g(c)\, dc\right) - G\left(2\int_0^{\frac{1}{2}} c\, g(c)\, dc\right) > \frac{1}{2}$$

$$\Leftrightarrow \quad 1 - 2\,G\left(2\int_0^{\frac{1}{2}} c\, g(c)\, dc\right) > \frac{1}{2} \quad \Leftrightarrow \quad 2\,G\left(2\int_0^{\frac{1}{2}} c\, g(c)\, dc\right) < \frac{1}{2}. \tag{72}$$

Define $H(c) = 2\,G(c)$ for $0 \leq c \leq \frac{1}{2}$. Then:

$$h(c) \equiv H'(c) = 2\,G'(c) = 2\,g(c) \text{ for } 0 \leq c \leq \frac{1}{2}. \tag{73}$$

(73) implies that $H(c)$ is a distribution function on $\left[\,0, \frac{1}{2}\,\right]$ with corresponding density func-

tion $h(c)$.

Let Y be a random variable on $\left[0, \frac{1}{2}\right]$ with density function $h(c)$. Then:

$$E(Y) \;=\; \int_0^{\frac{1}{2}} c\, h(c)\, dc \;=\; 2\int_0^{\frac{1}{2}} c\, g(c)\, dc\,. \tag{74}$$

(73) and (74) imply that (72) holds if and only if

$$H(E(Y)) \;<\; \frac{1}{2}\,. \tag{75}$$

Let $M(Y)$ denote the median of Y, so $H(M(Y)) = \frac{1}{2}$. Then:

$$H(E(Y)) \;<\; \frac{1}{2} \;\Leftrightarrow\; H(E(Y)) \;<\; H(M(Y)) \;\Leftrightarrow\; E(Y) \;<\; M(Y)\,. \tag{76}$$

$E(Y) < M(Y)$ because $h(c)$ is strictly increasing in c for $c \in \left[0, \frac{1}{2}\right]$ (e.g., van Zwet 1979; Dharmadhikari and Joag-Dev, 1983). Therefore, $A_W > A_M$. □

We have shown that c_1 and c_2 both decline as N increases. We now determine how $G(c_2) - G(c_1)$ changes when c_1 and c_2 decline by the same amount (x).

Finding 11.

$$\frac{\partial}{\partial x}\left[\, G(c_2 - x) - G(c_1 - x)\,\right] \;<\; 0 \;\; \textit{for} \;\; x \in \left(0,\, c_2 - \frac{1}{2}\right). \tag{77}$$

<u>Proof</u>.

$$\frac{\partial}{\partial x}\left[G(c_2 - x) - G(c_1 - x)\right] \;=\; -\, g(c_2 - x) + g(c_1 - x) \;<\; 0\,.$$

The inequality holds here because (56) implies that when $N = 2T$:

$$c_2 \;=\; \frac{1}{2} + \frac{A_W}{T} > \frac{1}{2} \;\;\text{and}\;\; c_1 \;=\; \frac{1}{2} - \frac{A_W}{T} < \frac{1}{2}\,.$$

Therefore, $g(c_2) = g(c_1)$ when $N = 2T$ because g is symmetric around $\frac{1}{2}$. Furthermore, $g(c)$ is strictly increasing for $c \leq \frac{1}{2}$ and strictly decreasing for $c \geq \frac{1}{2}$. Hence, if both c_2 and c_1 decrease by x, $g(c_2 - x) > g(c_2)$, and $g(c_1) > g(c_1 - x)$. Consequently, $g(c_2 - x) > g(c_1 - x)$. □

Finding 12. *There exists a* $\widetilde{N} > 2T$ *such that* $G(c_2) - G(c_1) > \frac{1}{2}$ *for all* $N \in [\,2T, \widetilde{N}\,)$.

<u>Proof</u>. Express c_1 and c_2 as functions of N, as in (8). (57) and Finding 10 imply:

$$G(c_2(2T)) - G(c_1(2T)) \;>\; \frac{1}{2}\,. \tag{78}$$

Furthermore, Findings 6 and 7 imply $c_2(N) < c_2(2T)$ and $c_1(N) < c_1(2T)$ for $N > 2T$.

Define:

$$\widetilde{c}_1(N) \;\equiv\; c_1(2T) - [\, c_2(2T) - c_2(N)\,] \;\text{ for }\; N \,\geq\, 2T\,. \tag{79}$$

(79) implies $\widetilde{c}_1(2T) = c_1(2T)$. For $N > 2T$, $c_2(N)$ is less than $c_2(2T)$ by the amount $c_2(2T) - c_2(N)$. (79) implies that $\widetilde{c}_1(N)$ is less than $c_1(2T)$ by the identical amount because $c_1(2T) - \widetilde{c}_1(N) = c_2(2T) - c_2(N)$. Because $\widetilde{c}_1(N)$ and $c_2(N)$ decline by the same amount as N increases above $2T$, $G(c_2(N)) - G(\widetilde{c}_1(N))$ is a decreasing function of N, from Finding 11.

Finding 9 implies that $c_1(N)$ declines more rapidly than $c_2(N)$ declines as N increases above $2T$. Therefore, $\widetilde{c}_1(N) > c_1(N)$ for $N > 2T$. Consequently, because $G(c)$ is strictly increasing in c:

$$G(c_2(N)) - G(c_1(N)) \;>\; \frac{1}{2} \;\text{ if }\; G(c_2(N)) - G(\widetilde{c}_1(N)) \;\geq\; \frac{1}{2} \;\text{ for } N \,>\, 2T. \tag{80}$$

We next prove:

$$\text{There exists a finite } \widetilde{N} \text{ such that } \; G(c_2(N)) - G(\widetilde{c}_1(N)) \;=\; \frac{1}{2}\,. \tag{81}$$

To prove (81), observe initially that because $\widetilde{c}_1(2T) = c_1(2T)$, (78) implies:

$$G(c_2(2T)) - G(\widetilde{c}_1(2T)) \;=\; G(c_2(2T)) - G(c_1(2T)) \;>\; \frac{1}{2}\,. \tag{82}$$

Because $G(c_2(N)) - G(\widetilde{c}_1(N))$ is a decreasing function of N, (82) implies that (81) holds if:

$$\lim_{N \to \infty} \;\{\, G(c_2(N)) - G(\widetilde{c}_1(N))\,\} \;\leq\; \frac{1}{2}\,. \tag{83}$$

To show that (83) holds, first observe that when $N = 2T$, $G(\widehat{c}\,) = \frac{T}{N} = \frac{1}{2} \Rightarrow \widehat{c} = \frac{1}{2}$. Therefore, Finding 2 implies:

$$c_2(N) \geq \frac{1}{2} \;\text{ for all }\; N \,\geq\, 2T \;\;\text{ and }\;\; \lim_{N \to \infty} c_2(N) \;=\; \frac{1}{2}\,. \tag{84}$$

Next observe that Finding 3 implies:

$$\frac{A_W(2T)}{T} \;=\; 2\int_0^{\frac{1}{2}} G(c)\, dc \;\leq\; \frac{1}{4}\,. \tag{85}$$

The inequality in (85) holds because:

$$G(c) \;\leq\; c \;\text{ for all }\; c \,\in\, [\,0, \frac{1}{2}\,] \tag{86}$$

$$\Rightarrow \;\; \int_0^{\frac{1}{2}} G(c)\, dc \;\leq\; \int_0^{\frac{1}{2}} c\, dc \;=\; \frac{1}{2}\, c^2 \Big|_0^{\frac{1}{2}} \;=\; \frac{1}{8}\,.$$

To prove that (86) holds, define $\xi(c) \equiv G(c) - c$. Observe that:

$$\xi(0) \;=\; \xi(\frac{1}{2}) \;=\; 0\,. \tag{87}$$

In addition:

$$\xi'(c) \ = \ g(c) - 1 \quad \Rightarrow \quad \xi''(c) \ = \ g'(c) \ \geq \ 0\,, \ \text{ and}$$
$$\xi'(0) \ = \ g(0) - 1 \ \leq \ 0\,. \tag{88}$$

(87) and (88) imply that $\xi(c) \leq 0$ for all $c \in \left[0, \frac{1}{2}\right]$, so (86) holds.

(56) and (85) imply:

$$c_2(2T) \ = \ \widehat{c}(2T) + \frac{A_W(2T)}{T} \ \leq \ \frac{1}{2} + \frac{1}{4} \ = \ \frac{3}{4}\,, \ \text{ and}$$
$$c_1(2T) \ = \ \widehat{c}(2T) - \frac{A_W(2T)}{T} \ \geq \ \frac{1}{2} - \frac{1}{4} \ = \ \frac{1}{4}$$
$$\Rightarrow \quad c_1(2T) \geq c_2(2T) - \frac{1}{2}\,. \tag{89}$$

Because $\lim_{N \to \infty} c_2(N) \ = \ \frac{1}{2}$, (79) and (89) imply:

$$\lim_{N \to \infty} \widetilde{c}_1(N) \ = \ c_1(2T) - \left[c_2(2T) - \lim_{N \to \infty} c_2(N) \right] \ = \ c_1(2T) - [\, c_2(2T) - \frac{1}{2} \,] \ \geq \ 0\,. \tag{90}$$

(84) and (90) imply that (83) holds. □ ■

II.A. Additional Numerical Solutions.

Table A1 supplements Table 3 in the text by considering additional Beta densities with $c^e > c^d$. Table A1 provides further evidence that majority rule often favors MJS whenever N/T is even slightly larger than 1. The first two columns in Table A1 specify the relevant values of the parameters of the Beta density, α and β. The third and fourth columns present the mean (c^e) and median (c^d) of the density. The last column reports the smallest value of N/T ($\frac{\widetilde{N}}{T}$) for which $\frac{A_m}{A_w} > 1$ for $N/T \in (1, \frac{\widetilde{N}}{T})$ and $\frac{A_m}{A_w} < 1$ for $N/T > \frac{\widetilde{N}}{T}$ (so majority rule favors MJS when $N/T > \frac{\widetilde{N}}{T}$).

α	β	c^e	c^d	$\frac{\widetilde{N}}{T}$
1	2	0.33333	0.29289	1.03030
1	3	0.25000	0.20630	1.00660
1	4	0.20000	0.15910	1.00180
2	3	0.40000	0.38573	1.00070
2	4	0.33333	0.31381	1.00040
2	5	0.28571	0.26445	1.00015
3	4	0.42857	0.42141	1.00002
3	5	0.37500	0.36412	1.00002
3	6	0.33333	0.32052	1.00001

Table A1. Favoritism of MJS when $N/T > \frac{\widetilde{N}}{T}$.

Table A2 considers the same variables presented in Table A1, this time for settings where the Beta density has $c^e < c^d$. Proposition 3 in the text suggests that majority rule may favor VJS in these settings when N/T is large. Table A2 indicates that when c^d/c^e is close to 1 (e.g., when $\alpha = 4$ and $\beta = 3$ or when $\alpha = 5$ and $\beta = 4$), favoritism of VJS emerges only when N/T is pronounced (e.g., greater than $40,000$). In contrast, when c^d/c^e is considerably larger than 1 (e.g., when $\alpha = 2$ and $\beta = 1$), favoritism of VJS can arise for substantially smaller values of N/T (e.g., 34).

α	β	c^e	c^d	$\widetilde{\frac{N}{T}}$
2	1	0.66667	0.70711	34
3	2	0.60000	0.61427	1,323
3	1	0.75000	0.79370	151
4	3	0.57134	0.57859	44,320
4	2	0.66667	0.68619	2,424
4	1	0.80000	0.84090	531
5	4	0.55556	0.55985	1,766,300
5	3	0.62500	0.63588	48,320
5	2	0.71429	0.73555	5,883
5	1	0.83333	0.87055	1,754

Table A2. Favoritism of VJS when $N/T > \widetilde{\frac{N}{T}}$.

II.B. Additional Analytic Results.

Conclusion. $G(\widehat{c}) = \frac{T}{N}$ under an optimal VJS policy.

Proof. From (3):

$$w - \widehat{c} = -\frac{F\,N\,G(\widehat{c})}{T} \quad \Leftrightarrow \quad T\,[\,w - \widehat{c}\,] + F\,N\,G(\widehat{c}) = 0$$

$$\Rightarrow \quad \frac{d\widehat{c}}{dF} = \frac{N\,G(\widehat{c})}{T - F\,N\,g(\widehat{c})} \quad \text{and} \quad \frac{d\widehat{c}}{dw} = \frac{T}{T - F\,N\,g(\widehat{c})}\,. \tag{91}$$

Per-capita expected welfare is:

$$J \equiv \int_{\underline{c}}^{\widehat{c}} \frac{T}{N\,G(\widehat{c})}\,[\,w - c\,]\,dG(c) - [\,1 - G(\widehat{c})\,]\,F - \frac{A}{N}$$

$$= \frac{T}{N}\,w - \frac{T}{N\,G(\widehat{c})} \int_{\underline{c}}^{\widehat{c}} c\,dG(c) - [\,1 - G(\widehat{c})\,]\,F - \frac{A}{N} \tag{92}$$

$$\Rightarrow \quad \frac{\partial J}{\partial \widehat{c}} \;=\; -\,\frac{T}{N}\left[\frac{G(\widehat{c})\,\widehat{c}\,g(\widehat{c}) - g(\widehat{c})\int_{\underline{c}}^{\widehat{c}} c\,dG(c)}{[\,G(\widehat{c})\,]^2}\right] + g(\widehat{c})\,F$$

$$= \; g(\widehat{c})\left[F - \frac{T}{N\,[\,G(\widehat{c})\,]^2}\int_{\underline{c}}^{\widehat{c}} [\,\widehat{c} - c\,]\,dG(c)\right]. \tag{93}$$

An optimal VJS policy is the solution to the following problem, [P]:

$$\underset{w,\,F}{\text{Maximize}} \;\; J$$

subject to the adequate jury pool constraint and the financing constraint.

Let λ_1 and λ_2 denote the Lagrange multipliers associated with the adequate jury pool constraint and the financing constraint, respectively. Then the necessary conditions for a solution to [P] include:

$$w: \quad \frac{T}{N}\,[\,1 - \lambda_2\,N\,] + \left[\frac{\partial J}{\partial \widehat{c}} + \lambda_1\,N\,g(\widehat{c}) - \lambda_2\,F\,N\,g(\widehat{c})\right]\frac{d\widehat{c}}{dw} \;=\; 0\,; \;\text{ and} \tag{94}$$

$$F: \quad -\,[\,1 - G(\widehat{c})\,]\,[\,1 - \lambda_2\,N\,] + \left[\frac{\partial J}{\partial \widehat{c}} + \lambda_1\,N\,g(\widehat{c}) - \lambda_2\,F\,N\,g(\widehat{c})\right]\frac{d\widehat{c}}{dF} \;=\; 0\,. \tag{95}$$

(91) implies that $\frac{d\widehat{c}}{dF} \overset{s}{=} \frac{d\widehat{c}}{dw}$. Therefore, (94) and (95) imply:

$$\frac{T}{N}\,[\,1 - \lambda_2\,N\,] \;\overset{s}{=}\; -\,[\,1 - G(\widehat{c})\,]\,[\,1 - \lambda_2\,N\,] \;\;\Rightarrow\;\; \lambda_2 \;=\; \frac{1}{N} \;>\; 0\,.$$

Because $\lambda_2 = \frac{1}{N}$, (93) and (94) imply:

$$\frac{\partial J}{\partial \widehat{c}} + \lambda_1\,N\,g(\widehat{c}) - \lambda_2\,F\,N\,g(\widehat{c}) \;=\; 0$$

$$\Rightarrow \;\; \lambda_1\,N\,g(\widehat{c}) \;=\; F\,g(\widehat{c}) - g(\widehat{c})\,F + \frac{T\,g(\widehat{c})}{N\,[\,G(\widehat{c})\,]^2}\int_{\underline{c}}^{\widehat{c}} [\,\widehat{c} - c\,]\,dG(c)$$

$$= \; \frac{T\,g(\widehat{c})}{N\,[\,G(\widehat{c})\,]^2}\int_{\underline{c}}^{\widehat{c}} [\,\widehat{c} - c\,]\,dG(c) \;>\; 0 \;\;\Rightarrow\;\; \lambda_1 \;>\; 0\,.$$

Therefore, the adequate jury pool constraint binds, so $G(\widehat{c}) \;=\; \frac{T}{N}$. ■

Proposition B1 considers the setting where $g\,(c)$ is a piecewise linear density with an inverted-V shape. Formally, $[\,\underline{c}\,, \overline{c}\,]$ is normalized to be $[\,0, 2\,]$ without loss of generality,[1] and, $a \in [\,0, \frac{1}{2})$:

$$g\,(c) \;=\; \begin{cases} a + [\,1 - 2\,a\,]\,c & \text{if} \quad 0 \,\le\, c \,\le\, 1 \\ a + [\,1 - 2\,a\,]\,[\,2 - c\,] & \text{if} \quad 1 \,\le\, c \,\le\, 2\,. \end{cases} \tag{96}$$

This density increases at the constant rate $1 - 2\,a > 0$ on $[\,\underline{c}\,, c^e\,]$ and declines at the corresponding rate on $[\,c^e, \overline{c}\,]$.

Proposition B1. *For any finite $N/T > 1$ and $a \in [\,0, \frac{1}{2}\,)$, majority rule favors MJS (so $A_m < A_w$) when $g(c)$ is as specified in expression (96).*

Proof. We first prove that $[\,\underline{c}\,, \overline{c}\,]$ can be normalized to $[\,0, 2\,]$ without loss of generality. To do so, consider a random variable X that is distributed on $[\,\underline{c}\,, \overline{c}\,]$ with cumulative distribution function G_X. Define a random variable $Y \;=\; \frac{2\,[\,X - \underline{c}\,]}{\overline{c} - \underline{c}}$. (Observe that Y is distributed on $[\,0, 2\,]$.) Let G_Y be the cumulative distribution function for Y. Then, by definition:

$$\begin{aligned} G_Y\left(\frac{2\,[\,x - \underline{c}\,]}{\overline{c} - \underline{c}}\right) \;&=\; P\left[\,Y \,\le\, \frac{2\,[\,x - \underline{c}\,]}{\overline{c} - \underline{c}}\,\right] \;=\; P\left[\,\frac{2\,[\,X - \underline{c}\,]}{\overline{c} - \underline{c}} \;\le\; \frac{2\,[\,x - \underline{c}\,]}{\overline{c} - \underline{c}}\,\right] \\ &=\; P\,[\,X \,\le\, x\,] \;=\; G_X\,(x)\,. \end{aligned} \tag{97}$$

Define $\widehat{c}^{\,*}_X$ and $\widehat{c}^{\,*}_Y$ by:

$$G_X(\,\widehat{c}^{\,*}_X) \;=\; \frac{T}{N} \quad \text{and} \quad G_Y(\,\widehat{c}^{\,*}_Y) \;=\; \frac{T}{N} \quad \Rightarrow \quad G_X(\,\widehat{c}^{\,*}_X) \;=\; G_Y(\,\widehat{c}^{\,*}_Y)\,. \tag{98}$$

(97) and (98) imply:

$$\widehat{c}^{\,*}_Y \;=\; \frac{2\,[\,\widehat{c}^{\,*}_X - \underline{c}\,]}{\overline{c} - \underline{c}}\,. \tag{99}$$

Let g_X and g_Y be the density functions for the random variables X and Y, respectively. (97) implies:

$$g_Y\left(\frac{2\,[\,x - \underline{c}\,]}{\overline{c} - \underline{c}}\right)\left[\,\frac{2}{\overline{c} - \underline{c}}\,\right] \;=\; g_X(x) \;\;\Rightarrow\;\; g_Y\left(\frac{2\,[\,x - \underline{c}\,]}{\overline{c} - \underline{c}}\right) \;=\; \left[\,\frac{\overline{c} - \underline{c}}{2}\,\right] g_X(x)\,. \tag{100}$$

Define:

$$\frac{A_w(X)}{T} \;=\; E\,[X] - \frac{\int_{\underline{c}}^{\widehat{c}^{\,*}_X} t\,g_X(t)\,dt}{G_X(\,\widehat{c}^{\,*}_X)} \quad \text{and} \quad \frac{A_w\,(Y)}{T} \;=\; E\,[Y] - \frac{\int_0^{\widehat{c}^{\,*}_Y} t\,g_Y(t)\,dt}{G_Y(\,\widehat{c}^{\,*}_Y)}$$

$$\Rightarrow \quad \widehat{c}^{\,*}_Y + \frac{A_w(Y)}{T} \;=\; \widehat{c}^{\,*}_Y + E\,[Y] - \frac{\int_0^{\widehat{c}^{\,*}_Y} t\,g_Y(t)\,dt}{G_Y(\,\widehat{c}^{\,*}_Y)}$$

[1] The proof of Proposition B1 demonstrates that this normalization is without loss of generality.

$$= \frac{2\,[\,\widehat{c}_X^* - \underline{c}\,]}{\overline{c} - \underline{c}} + \frac{2\,[\,E\,[X] - \underline{c}\,]}{\overline{c} - \underline{c}} - \frac{\int_0^{\widehat{c}_Y^*} t\, g_Y(t)\, dt}{G_X\,(\,\widehat{c}_X^*)}\,. \tag{101}$$

$$\text{Define } \ t \ = \ \frac{2\,[\,x - \underline{c}\,]}{\overline{c} - \underline{c}} \quad \Rightarrow \quad dt \ = \ \left[\frac{2}{\overline{c} - \underline{c}} \right] dx$$

$$\Rightarrow \quad \int_0^{\widehat{c}_Y^*} t\, g_Y(t)\, dt \ = \ \int_{\underline{c}}^{\widehat{c}_X^*} \left(\frac{2\,[\,x - \underline{c}\,]}{\overline{c} - \underline{c}} \right) g_Y \left(\frac{2\,[\,x - \underline{c}\,]}{\overline{c} - \underline{c}} \right) \left[\frac{2}{\overline{c} - \underline{c}} \right] dx$$

$$= \int_{\underline{c}}^{\widehat{c}_X^*} \left(\frac{2\,[\,x - \underline{c}\,]}{\overline{c} - \underline{c}} \right) \left[\frac{\overline{c} - \underline{c}}{2} \right] g_X(x) \left[\frac{2}{\overline{c} - \underline{c}} \right] dx \ = \ \int_{\underline{c}}^{\widehat{c}_X^*} \frac{2\,[\,x - \underline{c}\,]}{\overline{c} - \underline{c}}\, g_X(x)\, dx$$

$$= \int_{\underline{c}}^{\widehat{c}_X^*} \left[\frac{2\,x}{\overline{c} - \underline{c}} \right] g_X(x) dx - \left[\frac{2\,\underline{c}}{\overline{c} - \underline{c}} \right] G_X\,(\,\widehat{c}_X^*)\,. \tag{102}$$

The second equality in (102) reflects (100). (101) and (102) imply:

$$\widehat{c}_Y^* + \frac{A_w\,(Y)}{T} \ = \ \frac{2\,[\,\widehat{c}_X^* - \underline{c}\,]}{\overline{c} - \underline{c}} + \frac{2\,[\,E\,[X] - \underline{c}\,]}{\overline{c} - \underline{c}}$$

$$- \ \frac{1}{G_X(\,\widehat{c}_X^*)} \left[\int_{\underline{c}}^{\widehat{c}_X^*} \left(\frac{2\,x}{\overline{c} - \underline{c}} \right) g_X(x)\, dx - \left[\frac{2\,\underline{c}}{\overline{c} - \underline{c}} \right] G_X(\,\widehat{c}_X^*) \right]$$

$$= \ \frac{2\,[\,\widehat{c}_X^* - \underline{c}\,]}{\overline{c} - \underline{c}} + \frac{2\,[\,E\,[X] - \underline{c}\,]}{\overline{c} - \underline{c}} - \left[\frac{2}{\overline{c} - \underline{c}} \right] \frac{\int_{\underline{c}}^{\widehat{c}_X^*} x\, g_X(x)\, dx}{G_X(\,\widehat{c}_X^*)} + \frac{2\,\underline{c}}{\overline{c} - \underline{c}}$$

$$= \ \frac{2}{\overline{c} - \underline{c}} \left[\widehat{c}_X^* - \underline{c} + E\,[X] - \frac{\int_{\underline{c}}^{\widehat{c}_X^*} x\, g_X(x)\, dx}{G_X(\,\widehat{c}_X^*)} \right]$$

$$\Rightarrow \ G_Y \left(\widehat{c}_Y^* + \frac{A_w\,(Y)}{T} \right) \ = \ P \left[Y \ \leq \ \widehat{c}_Y^* + \frac{A_w(Y)}{T} \right]$$

$$= \ P \left[\frac{2\,[\,X - \underline{c}\,]}{\overline{c} - \underline{c}} \ \leq \ \frac{2}{\overline{c} - \underline{c}} \left[\widehat{c}_X^* - \underline{c} + E\,[X] - \frac{\int_{\underline{c}}^{\widehat{c}_X^*} x\, g_X(x)\, dx}{G_X(\,\widehat{c}_X^*)} \right] \right]$$

$$= \ P \left[X \ \leq \ \widehat{c}_X^* + E\,[X] - \frac{\int_{\underline{c}}^{\widehat{c}_X^*} x\, g_X(x)\, dx}{G_X(\,\widehat{c}_X^*)} \right] \ = \ G_X \left(\widehat{c}_X^* + \frac{A_w(X)}{T} \right)\,. \tag{103}$$

Analogous arguments reveal:

$$G_Y\left(\widehat{c}_Y^* - \frac{A_w(Y)}{N-T}\right) \;=\; G_X\left(\widehat{c}_X^* - \frac{A_w(X)}{N-T}\right). \tag{104}$$

(103) and (104) imply:

$$\begin{aligned}
& G_Y\left(\widehat{c}_Y^* + \frac{A_w(Y)}{T}\right) - G_Y\left(\widehat{c}_Y^* - \frac{A_w(Y)}{N-T}\right) \\
& \quad = \; G_X\left(\widehat{c}_X^* + \frac{A_w(X)}{T}\right) - G_X\left(\widehat{c}_X^* - \frac{A_w(X)}{N-T}\right) \\
\Rightarrow \quad & G_Y\left(\widehat{c}_Y^* + \frac{A_w(Y)}{T}\right) - G_Y\left(\widehat{c}_Y^* - \frac{A_w(Y)}{N-T}\right) \;\gtreqqless\; \frac{1}{2} \\
\Leftrightarrow \quad & G_X\left(\widehat{c}_X^* + \frac{A_w(X)}{T}\right) - G_X\left(\widehat{c}_X^* - \frac{A_w(X)}{N-T}\right) \;\gtreqqless\; \frac{1}{2}.
\end{aligned} \tag{105}$$

(105) implies that the support of c can be taken to be $[\,0,2\,]$ without loss of generality when assessing whether majority rule favors MJS or VJS.

To specify the distribution function corresponding to $g(c)$, observe from (96) that when $c \in [\,0,1\,]$:

$$G\,(c) \;=\; \int_0^c (\,a + [\,1-2\,a\,]\,\widetilde{c}\,)\,d\,\widetilde{c} \;=\; \left[\,a\,\widetilde{c} + (\,1-2\,a\,)\left(\frac{\widetilde{c}^{\,2}}{2}\right)\right]_0^c \;=\; a\,c + [\,1-2\,a\,]\,\frac{c^2}{2}\,.$$

For $c \in [\,1,2\,]$:

$$\begin{aligned}
G\,(c) \;&=\; \int_0^1 (\,a + [\,1-2\,a\,]\,c\,)\,dc + \int_1^c (\,a + [\,1-2\,a\,]\,[\,2-\widetilde{c}\,]\,)\,d\,\widetilde{c} \\
&=\; \frac{1}{2} + a\,[\,c-1\,] + [\,1-2\,a\,]\left[\,2\,\widetilde{c} - \frac{\widetilde{c}^{\,2}}{2}\right]_1^c \\
&=\; \frac{1}{2} + a\,[\,c-1\,] + [\,1-2\,a\,]\left[\,2\,(c-1) - \frac{c^2-1}{2}\right] \\
&=\; \frac{1}{2} + a\,[\,c-1\,] + \left[\frac{1-2\,a}{2}\right][\,1 - \left(c^2 - 4\,c + 4\right)\,] \\
&=\; \frac{1}{2} + a\,[\,c-1\,] + \left[\frac{1-2\,a}{2}\right][\,1 - (2-c)^2\,]\,.
\end{aligned}$$

In summary, the distribution function for the density function in (96) is:

$$G(c) \;=\; \begin{cases} a\,c + [\,1 - 2\,a\,]\,\frac{c^2}{2} & \text{if} \quad 0 \,\le\, c \le\, 1 \\ \frac{1}{2} + a\,[\,c - 1\,] + \left[\,\frac{1-2\,a}{2}\,\right]\left[\,1 - (2 - c)^2\,\right] & \text{if} \quad 1 \,\le\, c \,\le\, 2\,. \end{cases} \tag{106}$$

Case 1. $N \,\ge\, 2\,T$.

Define $y \;\equiv\; \frac{T}{N}$. $\widehat{c} \le 1$ because: (i) $G\,(\,\widehat{c}\,) \;=\; y\,$; (ii) $y \;\le\; \frac{1}{2}$ by assumption; and (iii) $G(1) = \frac{1}{2}$ due to the symmetry in (96). Therefore, from (106):

$$a\,\widehat{c} + [\,1 - 2\,a\,]\,\frac{(\,\widehat{c}\,)^2}{2} \;=\; y \quad\Leftrightarrow\quad [\,1 - 2\,a\,]\,(\,\widehat{c}\,)^2 + 2\,a\,\widehat{c} - 2\,y \;=\; 0\,. \tag{107}$$

It is apparent from (107) that $\widehat{c} \;=\; 2\,y$ when $a = \frac{1}{2}$. If $a \neq \frac{1}{2}$, then (107) implies:

$$\widehat{c} \;=\; \frac{-\,2\,a + \sqrt{(2\,a)^2 + 8\,y\,[\,1 - 2\,a\,]}}{2\,[\,1 - 2\,a]} \;=\; \frac{-\,a + \sqrt{(a)^2 + 2\,y\,[\,1 - 2\,a\,]}}{1 - 2\,a}\,.$$

In summary:

$$\widehat{c} \;=\; \begin{cases} \frac{-\,a + \sqrt{a^2 + 2\,y[\,1 - 2\,a\,]}}{1 - 2\,a} & if \quad a \,\neq\, \frac{1}{2} \\ 2\,y & if \quad a \,=\, \frac{1}{2}\,. \end{cases} \tag{108}$$

(96) and (107) imply that when $a \neq \frac{1}{2}$:

$$\begin{aligned} \frac{A_w}{T} &= c^e - \frac{\int_0^{\widehat{c}} c\,dG(c)}{G\,(\,\widehat{c}\,)} \;=\; 1 - \frac{\int_0^{\widehat{c}} c\,(\,a + [\,1 - 2\,a\,]\,c\,)\,dc}{y} \\ &= 1 - \frac{1}{y}\left[\,a\left(\frac{c^2}{2}\right)_0^{\widehat{c}} + (1 - 2\,a)\left(\frac{c^3}{3}\right)_0^{\widehat{c}}\,\right] \;=\; 1 - \frac{1}{y}\left[\,\frac{a}{2}\,(\,\widehat{c}\,)^2 + \frac{[\,1 - 2\,a\,]\,(\,\widehat{c}\,)^3}{3}\,\right] \\ &= 1 - \frac{\widehat{c}}{y}\left[\,\frac{a}{2}\,(\,\widehat{c}\,) + \frac{[\,1 - 2\,a\,]\,(\,\widehat{c}\,)^2}{3}\,\right] \;=\; 1 - \frac{\widehat{c}}{y}\left[\,\frac{a}{2}\,(\,\widehat{c}\,) + \frac{2}{3}\,(\,y - a\,\widehat{c}\,)\,\right] \\ &= 1 - \frac{2\,\widehat{c}}{3} - \frac{a}{2\,y}\,(\,\widehat{c}\,)^2 + \frac{2\,a}{3\,y}\,(\,\widehat{c}\,)^2 \;=\; 1 - \frac{2\,\widehat{c}}{3} + \frac{a}{6\,y}\,(\,\widehat{c}\,)^2 \\ &= 1 - \frac{2\,\widehat{c}}{3} + \frac{a}{6\,y}\left[\,\frac{2\,(y - a\,\widehat{c}\,)}{1 - 2\,a}\,\right] \;=\; 1 - \frac{2\,\widehat{c}}{3} + \frac{a}{3\,y}\left[\,\frac{y - a\,\widehat{c}}{1 - 2\,a}\,\right] \\ &= 1 - \frac{2\,\widehat{c}}{3} + \frac{1}{3}\left[\,\frac{a}{1 - 2\,a}\,\right] - \frac{a^2\,\widehat{c}}{3\,y\,[\,1 - 2\,a\,]} \end{aligned}$$

$$= \; 1 + \frac{1}{3}\left[\frac{a}{1-2\,a}\right] - \frac{\widehat{c}}{3}\left[2 + \frac{a^2}{y\,(\,1-2\,a\,)}\right]$$

$$= \; 1 + \frac{a}{3\,[\,1-2\,a\,]} - \frac{\widehat{c}}{3\,y\,[\,1-2\,a\,]}\,[\,a^2 + 2\,y\,(1-2\,a)\,]\,. \tag{109}$$

Furthermore, (96) and (108) imply that when $a = \frac{1}{2}$:

$$\frac{A_w}{T} \;=\; c^e - \frac{\int_0^{\widehat{c}} c\,dG(c)}{G\,(\widehat{c})} \;=\; 1 - \frac{\int_0^{\widehat{c}} \frac{c}{2}\,dc}{y} \;=\; 1 - \frac{1}{y}\left[\frac{c^2}{4}\right]_0^{\widehat{c}} \;=\; 1 - \frac{1}{4\,y}\,(\,\widehat{c})^2$$

$$= \; 1 - \frac{1}{4\,y}\,[\,2\,y\,]^2 \;=\; 1 - y\,.$$

(109) implies that when $a \neq \frac{1}{2}$:

$$\widehat{c} + \frac{A_w}{T} \;=\; \widehat{c} + 1 + \frac{a}{3\,[\,1-2\,a\,]} - \frac{\widehat{c}}{3\,y\,[\,1-2\,a\,]}\,[\,a^2 + 2\,y\,(1-2\,a)\,]$$

$$= \; 1 + \frac{a}{3\,[\,1-2\,a\,]} + \widehat{c}\left[1 - \frac{a^2 + 2\,y\,(1-2a)}{3\,y\,(1-2a)}\right]$$

$$= \; 1 + \frac{a}{3\,[\,1-2\,a\,]} + \widehat{c}\left[\frac{3\,y\,(1-2\,a) - a^2 - 2\,y\,(1-2\,a)}{3\,y\,(1-2\,a)}\right]$$

$$= \; 1 + \frac{a}{3\,[\,1-2\,a\,]} + \widehat{c}\left[\frac{y\,(1-2\,a) - a^2}{3\,y\,(1-2\,a)}\right] \;=\; \beta_2 + \alpha_2\,\widehat{c},$$

$$\text{where} \quad \beta_2 \;\equiv\; 1 + \frac{a}{3\,[\,1-2\,a\,]} \quad \text{and} \quad \alpha_2 \;\equiv\; \frac{y\,[\,1-2\,a\,] - a^2}{3\,y\,[\,1-2\,a\,]}\,. \tag{110}$$

(109) also implies that when $a \neq \frac{1}{2}$:

$$\widehat{c} - \frac{A_w}{N-T} \;=\; \widehat{c} - \frac{A_w}{T}\left[\frac{T}{N-T}\right] \;=\; \widehat{c} - \frac{A_w}{T}\left[\frac{y}{1-y}\right]$$

$$= \; \widehat{c} - \frac{y}{1-y}\left[1 + \frac{a}{3\,(1-2\,a)} - \frac{\widehat{c}}{3\,y\,(1-2\,a)}\,[\,a^2 + 2y\,(1-2\,a)\,]\right]$$

$$= \; -\,\frac{y}{1-y}\left[1 + \frac{a}{3\,(1-2\,a)}\right] + \widehat{c}\left[1 + \frac{a^2 + 2\,y\,(1-2\,a)}{3\,(1-y)\,(1-2\,a)}\right]$$

$$= \; -\left[\frac{y}{1-y}\right]\beta_2 + \widehat{c}\left[\frac{3\,(1-y)\,(1-2\,a) + a^2 + 2\,y\,(1-2a)}{3\,(1-y)\,(1-2\,a)}\right]$$

$$= \; -\left[\frac{y}{1-y}\right]\beta_2 + \widehat{c}\left[\frac{(1-2\,a)\,(3-3\,y+2\,y) + a^2}{3\,(1-y)\,(1-2\,a)}\right]$$

$$= \; - \left[\frac{y}{1-y} \right] \beta_2 + \widehat{c} \left[\frac{(1-2\,a)\,(3-y) + a^2}{3\,(1-y)\,(1-2\,a)} \right] \; = \; - \left[\frac{y}{1-y} \right] \beta_2 + \alpha_1 \, \widehat{c} \, ,$$

$$\text{where} \;\; \alpha_1 \; \equiv \; \frac{[\,1-2\,a\,]\,(3-y) + a^2}{3\,[\,1-y\,]\,[\,1-2\,a\,]} \, . \tag{111}$$

$\widehat{c} \, \leq \, 1$ because $y \, \leq \, \frac{1}{2}$, by assumption. Therefore:

$$\widehat{c} - \frac{A_w}{N-T} \; \leq \; 1 \, . \tag{112}$$

(110) implies that for $a \neq \frac{1}{2}$:

$$\widehat{c} + \frac{A_w}{T} \; \geq \; 1 \quad \Leftrightarrow \quad 1 + \frac{1}{3} \left[\frac{a}{1-2\,a} \right] + \widehat{c} \left[\frac{y\,(1-2\,a) - a^2}{3\,y\,(1-2a)} \right] \; \geq \; 1$$

$$\Leftrightarrow \;\; \widehat{c} \left[\frac{y\,(1-2\,a) - a^2}{3\,y\,(1-2a)} \right] \; \geq \; - \frac{1}{3} \left[\frac{a}{1-2\,a} \right] \quad \Leftrightarrow \;\; \widehat{c} \left[\frac{y\,(1-2\,a) - a^2}{y} \right] \; \geq \; - \, a$$

$$\Leftrightarrow \;\; \widehat{c} \left[\, y\,(1-2\,a) - a^2 \, \right] \; \geq \; - \, a\, y \quad \Leftrightarrow \quad \widehat{c}\, y\,[\,1-2\,a\,] - \widehat{c}\, a^2 + a\, y \; \geq \; 0$$

$$\Leftrightarrow \;\; \widehat{c}\, y\,[\,1-2\,a\,] + a\,y - a^2\,\widehat{c} \; \geq \; 0 \, . \tag{113}$$

Because $a \leq \frac{1}{2}$, the inequality in (113) holds if:

$$a\, y - a^2 \, \widehat{c} \; \geq \; 0 \, .$$

(107) implies:

$$y - a\,\widehat{c} \; = \; \frac{[\,1-2\,a\,]\,(\widehat{c})^2}{2} \quad \Rightarrow \quad a\,y - a^2\,\widehat{c} \; = \; \frac{a\,[\,1-2\,a\,]\,(\,\widehat{c}\,)^2}{2} \; \geq \; 0 \, .$$

Therefore, the inequality in (113) holds, so:

$$\widehat{c} + \frac{A_w}{T} \; \geq \; 1 \, . \tag{114}$$

Because $\widehat{c} + \frac{A_w}{T} \, \geq \, 1$ from (114), (106) and (110) imply:

$$G\left(\widehat{c} + \frac{A_w}{T} \right) \; = \; G\,(\beta_2 + \alpha_2 \, \widehat{c})$$

$$= \; \frac{1}{2} + a\,[\,\beta_2 + \alpha_2 \; \widehat{c} - 1\,] + \left[\frac{1-2\,a}{2} \right] \left[\, 1 - (\,2 - \beta_2 - \alpha_2 \; \widehat{c})^2 \,\right] . \tag{115}$$

Because $\widehat{c} - \frac{A_w}{N-T} \, \leq \, 1$ from (112), (106) and (111) imply:

$$G\left(\widehat{c} - \frac{A_w}{N-T} \right) \; = \; G\left(- \left[\frac{y}{1-y} \right] \beta_2 + \alpha_1 \; \widehat{c} \right)$$

$$= a\left[-\frac{y\,\beta_2}{1-y}+\alpha_1\,\widehat{c}\right]+\left[\frac{1-2\,a}{2}\right]\left[-\frac{y\,\beta_2}{1-y}+\alpha_1\,\widehat{c}\right]^2. \tag{116}$$

(115) and (116) imply:

$$G\left(\widehat{c}+\frac{A_w}{T}\right)-G\left(\widehat{c}-\frac{A_w}{N-T}\right)$$

$$= \frac{1}{2}+a\left[\beta_2+\alpha_2\,\widehat{c}-1\right]+\frac{1}{2}-a-\left[\frac{1-2\,a}{2}\right]\left[2-\beta_2-\alpha_2\,\widehat{c}\right]^2$$
$$-\;a\left[-\frac{y\,\beta_2}{1-y}+\alpha_1\,\widehat{c}\right]-\left[\frac{1-2\,a}{2}\right]\left[-\frac{y\,\beta_2}{1-y}+\alpha_1\,\widehat{c}\right]^2$$

$$= 1+a\left[\beta_2+\alpha_2\,\widehat{c}-2-\alpha_1\,\widehat{c}+\frac{y\,\beta_2}{1-y}\right]$$
$$-\left[\frac{1-2\,a}{2}\right]\left[(2-\beta_2-\alpha_2\,\widehat{c})^2+\left(-\frac{y\,\beta_2}{1-y}+\alpha_1\,\widehat{c}\right)^2\right]$$

$$= 1-2\,a+a\left[\alpha_2-\alpha_1\right]\widehat{c}+a\,\beta_2\left[1+\frac{y}{1-y}\right]$$
$$-\left[\frac{1-2\,a}{2}\right]\left[(2-\beta_2-\alpha_2\,\widehat{c})^2+\left(-\frac{y\,\beta_2}{1-y}+\alpha_1\,\widehat{c}\right)^2\right]$$

$$= 1-2\,a+a\left[\alpha_2-\alpha_1\right]\widehat{c}+\frac{a\,\beta_2}{1-y}$$
$$-\left[\frac{1-2\,a}{2}\right]\left[(2-\beta_2-\alpha_2\,\widehat{c})^2+\left(-\frac{y\,\beta_2}{1-y}+\alpha_1\,\widehat{c}\right)^2\right]\;\equiv\;Z_1. \tag{117}$$

$A_w > A_m$ if $Z_1 > \frac{1}{2}$. *Mathematica* reveals that this is the case for all $y \in \left(0, \frac{1}{2}\right]$ and $a \in \left[0, \frac{1}{2}\right)$. Therefore, $A_w > A_m$ for any finite $N \geq 2T$ and $a \in \left[0, \frac{1}{2}\right)$.

<u>Case 2. $N < 2T$.</u>

$\widehat{c} \geq 1$ because: (i) $G(\widehat{c}) = y$; (ii) $y \equiv \frac{T}{N} > \frac{1}{2}$ by assumption; and (iii) $G(1) = \frac{1}{2}$ due to the symmetry in (96). Therefore:

$$\frac{1}{2}+a\left[\widehat{c}-1\right]+\frac{1-2\,a}{2}\left[1-(2-\widehat{c})^2\right]-y \;=\; 0$$

$$\Rightarrow\quad \left[1-2\,a\right]\left[1-(2-\widehat{c})^2\right]+2\,a\left[\widehat{c}-1\right]+1-2\,y \;=\; 0$$

$$\Rightarrow\quad -\left[1-2\,a\right]\left[2-\widehat{c}\right]^2+2\,a\left[\widehat{c}-1\right]+2-2\,a-2\,y \;=\; 0$$

$$\Rightarrow\quad -\left[1-2\,a\right](\widehat{c})^2+2\,a\left[\widehat{c}-1\right]+4\,\widehat{c}\left[1-2\,a\right]-4\left[1-2\,a\right]+2-2\,a-2\,y \;=\; 0$$

$$\Rightarrow \quad -[1-2a](\widehat{c})^2 + \widehat{c}[4-6a] + 4a - 2 - 2y \;=\; 0$$

$$\Rightarrow \quad \frac{1}{2}[2a-1](\widehat{c})^2 + [2-3a]\,\widehat{c} + 2a - 1 - y \;=\; 0\,. \tag{118}$$

Observe that:

$$[2-3a]^2 - 4\left[\frac{1}{2}\right][2a-1][2a-1-y] \;=\; 4 - 12a + 9a^2 + [2-4a][2a-1-y]$$

$$=\; 4 - 12a + 9a^2 + 4a - 2 - 2y - 8a^2 + 4a + 4ay \;=\; a^2 + 4ay - 4a - 2y + 2\,.$$

Therefore, because $a \neq \frac{1}{2}$, (118) implies:

$$\widehat{c} \;=\; \frac{-2 + 3a + \sqrt{a^2 + 4ay - 4a - 2y + 2}}{2a - 1}\,.$$

(96) implies:

$$\frac{A_w}{T} \;=\; c^e - \frac{\int_0^{\widehat{c}} c\, dG(c)}{G(\widehat{c})} \;=\; 1 - \frac{\int_0^1 c\,[\,a + (1-2a)\,c\,]\,dc + \int_1^{\widehat{c}} c\,[\,a + (1-2a)(2-c)\,]\,dc}{y}$$

$$=\; 1 - \frac{1}{y}\left[\frac{1}{2}\,a\,c^2\Big|_0^1 + \frac{1}{3}[1-2a]\,c^3\Big|_0^1 + \frac{1}{2}(a + 2[1-2a])\,c^2\Big|_1^{\widehat{c}} - \frac{1}{3}[1-2a]\,c^3\Big|_1^{\widehat{c}}\right]$$

$$=\; 1 - \frac{1}{y}\left[\frac{1}{2}\,a + \frac{1}{3}(1-2a) + \frac{1}{2}(a + 2[1-2a])(\widehat{c})^2 - \frac{1}{2}(a + 2[1-2a]) \right.$$
$$\left. - \frac{1}{3}(1-2a)(\widehat{c})^3 + \frac{1}{3}(1-2a)\right]$$

$$=\; 1 - \frac{1}{y}\left[-\frac{1}{3}(1-2a) + (\widehat{c})^2(1 - \frac{3a}{2}) - \frac{1}{3}(1-2a)(\widehat{c})^3\right]. \tag{119}$$

(119) implies:

$$c_2 \;=\; \widehat{c} + \frac{A_w}{T} \;=\; \widehat{c} + 1 + \frac{1}{y}\left[\frac{1}{3}(1-2a) - (\widehat{c})^2\left(1 - \frac{3a}{2}\right) + \frac{1}{3}(1-2a)(\widehat{c})^3\right]$$

$$=\; \widehat{c} + 1 + \frac{1}{6y}\left[2(1-2a) - 6(\widehat{c})^2\left(1 - \frac{3a}{2}\right) + 2(1-2a)(\widehat{c})^3\right]$$

$$=\; \frac{1}{6y}\left[2 - 4a + 6y(1+\widehat{c}) - 6(\widehat{c})^2 + 9a(\widehat{c})^2 + 2(1-2a)(\widehat{c})^3\right]$$

$$=\; \frac{1}{6y}\left[2(1-2a)(\widehat{c})^3 + 3(3a-2)(\widehat{c})^2 + 6y\,\widehat{c} + 2(1-2a+3y)\right] \tag{120}$$

and

$$c_1 \;=\; \widehat{c} - \frac{A_w}{N-T} \;=\; \widehat{c} - \frac{A_w}{T}\left[\frac{T}{N-T}\right] \;=\; \widehat{c} - \left[\frac{y}{1-y}\right]\frac{A_w}{T}$$

$$= \widehat{c} - \frac{y}{1-y} + \frac{1}{1-y}\left[-\frac{1}{3}(1-2a) + (\widehat{c})^2(1-\frac{3a}{2}) - \frac{1}{3}(1-2a)(\widehat{c})^3\right]$$

$$= \frac{1}{1-y}\left[-\frac{1}{3}(1-2a) + (\widehat{c})^2(1-\frac{3a}{2}) - \frac{1}{3}(1-2a)(\widehat{c})^3 + (1-y)\widehat{c} - y\right]$$

$$= \frac{1}{6[y-1]}\left[2(1-2a) - 6(\widehat{c})^2(1-\frac{3a}{2}) + 2(1-2a)(\widehat{c})^3 - 6(1-y)\widehat{c} + 6y\right]$$

$$= \frac{1}{6[y-1]}\left[2 - 4a - 6(\widehat{c})^2 + 9a(\widehat{c})^2 + 2(1-2a)(\widehat{c})^3 - 6(1-y)\widehat{c} + 6y\right]$$

$$= \frac{1}{6[y-1]}\left[2(1-2a)(\widehat{c})^3 - 3(2-3a)(\widehat{c})^2 - 6(1-y)\widehat{c} + 2(1-2a+3y)\right]. \qquad (121)$$

From (106):

$$G\left(\widehat{c} + \frac{A_w}{T}\right) - G\left(\widehat{c} - \frac{A_w}{N-T}\right) - \frac{1}{2} \;=\; Z_2\,,$$

where

$$Z_2 \;\equiv\; a[c_2 - 1] + \frac{1-2a}{2}\left[1 - (2-c_2)^2\right] - a\,c_1 - \left[\frac{1-2a}{2}\right](c_1)^2. \qquad (122)$$

and where c_1 and c_2 are defined in (120) and (121).

Mathematica reveals that $Z_2 > \frac{1}{2}$ (so $A_m < A_w$) for all $y \in (\frac{1}{2}, 1)$. ■

Proposition B2 considers the setting where the piecewise linear density in (96) approaches the uniform density. Proposition B2 develops a parallel with Proposition 4 in the text.

Proposition B2. $A_m \to A_w$ as $\alpha \to \frac{1}{2}$ when $g(c)$ is as specified in (96) with $a \in [0, \frac{1}{2})$.

Proof. (107) implies that for $a \neq \frac{1}{2}$:

$$2y - 2a\,\widehat{c} \;=\; [1-2a](\widehat{c})^2 \quad \Rightarrow \quad \widehat{c} \;\to\; 2y \;\text{ as }\; a \;\to\; \frac{1}{2}. \qquad (123)$$

(109) implies:

$$\frac{A_w}{T} \;=\; 1 + \frac{a}{3[1-2a]} - \frac{a^2\,\widehat{c}}{3[1-2a]\,y} - \frac{2\,\widehat{c}}{3}$$

$$= 1 - \frac{2\,\widehat{c}}{3} + \frac{a}{3[1-2a]}\left[1 - \frac{a(\widehat{c})^2}{y}\right]$$

$$= 1 - \frac{2\,\widehat{c}}{3} + \frac{a}{6y}\left[\frac{2y - 2a\,\widehat{c}}{1-2a}\right] \;=\; 1 - \frac{2\,\widehat{c}}{3} + \frac{a}{6y}(\widehat{c})^2\,. \qquad (124)$$

The equality in (124) reflects (123). (123) and (124) imply that as $a \to \frac{1}{2}$:

$$\frac{A_w}{T} \;\to\; 1 - \frac{2}{3}\,[\,2\,y\,] + \frac{1}{12\,y}\,[\,2\,y\,]^2 \;=\; 1 - \frac{4\,y}{3} + \frac{y}{3} \;=\; 1 - y\,. \tag{125}$$

(123) and (125) imply that as $a \to \frac{1}{2}$:

$$\widehat{c} + \frac{A_w}{T} \;\to\; 2\,y + 1 - y \;=\; 1 + y\,, \text{ and}$$

$$\widehat{c} - \left[\frac{y}{1-y}\right]\frac{A_w}{T} \;\to\; 2\,y - \left[\frac{y}{1-y}\right][\,1-y\,] \;=\; y\,. \tag{126}$$

(106) and (126) imply that as $a \to \frac{1}{2}$:

$$G\left(\widehat{c} + \frac{A_w}{T}\right) \;\to\; G\,(1+y) \;=\; \frac{1}{2} + \frac{1}{2}\,[\,1 + y - 1\,] \;=\; \frac{1+y}{2}\,, \text{ and}$$

$$G\left(\widehat{c} - \frac{A_w}{N-T}\right) \;\to\; G\,(y) \;=\; \frac{y}{2}$$

$$\Rightarrow\; G\left(\widehat{c} + \frac{A_w}{T}\right) - G\left(\widehat{c} - \frac{A_w}{N-T}\right) \;\to\; \frac{1+y}{2} - \frac{y}{2} \;=\; \frac{1}{2}\,. \quad \blacksquare$$

Proposition B3 considers a piecewise linear, symmetric, V-shaped density (with $c^e = c^d$) on the normalized support $[\,0, 2\,]$.[2] Formally, for $a \in [\,0, \frac{1}{2})$:

$$g\,(c) \;=\; \begin{cases} 1 - a + [\,2\,a - 1\,]\,c & \text{if } \;0 \,\le\, c \,\le\, 1 \\ 3\,a - 1 + [\,1 - 2\,a\,]\,c & \text{if } \;1 \,\le\, c \,\le\, 2\,. \end{cases} \tag{127}$$

This density declines at the constant rate $1 - 2\,a$ on $[\,0, 1\,]$ and increases at the corresponding rate on $[\,1, 2\,]$. The two segments of the symmetric density become more steeply sloped as a declines to 0.

Proposition B3 indicates that majority rule favors VJS in this setting when $g(c)$ has a moderate slope (i.e., for $a \in [\,0.042, 0.5)$). However, majority rule may favor MJS when $g(c)$ has a more pronounced slope if $N/T > 1$ is relatively small or if N/T is finite and sufficiently large.

Proposition B3. *If* $a \in [\,0.042, 0.5)$ *for the density specified in (127), then* $A_m > A_w$ *for all finite* $N/T > 1$*. If* $a = 0$ *for this density, then: (i)* $A_m > A_w$ *if* $N/T \in (1.7, 2.414)$*; whereas (ii)* $A_m < A_w$ *if* $N/T \in (1, 1.7)$ *or if finite* $N/T \ge 2.44$.[3]

[2] This focus on the $[\,0, 2\,]$ support is without loss of generality.

[3] If $a \in (\,0, 0.042)$, $A_m - A_w$ can be either positive or negative, depending on the value of N/T. To illustrate, when $a = 0.02$, $A_m < A_w$ when $\frac{N}{T} \in (1.098, 1.6) \cup (2.665, 11.161)$, and $A_m > A_w$ otherwise.

Proof. For the density in (127):

$$G(c) \;=\; \int_0^c (1 - a + [2a-1]\,\eta)\, d\eta \;=\; [1-a]\,c + \frac{c^2}{2}\,[2a-1] \;\text{ for } c \in [0,1]. \tag{128}$$

$$G(c) \;=\; \int_0^1 (1 - a + [2a-1]\,c)\, dc + \int_1^c (3a - 1 + [1-2a]\,\eta)\, d\eta$$

$$= \frac{1}{2} + [3a-1][c-1] + \frac{1}{2}[1-2a]\left[c^2 - 1\right] \;\text{ for } c \in [1,2]. \tag{129}$$

Case 1. $N \geq 2T$.

Define $y \equiv \frac{T}{N}$. $\widehat{c} \leq 1$ because: (i) $G(\widehat{c}) = y$; (ii) $y \leq \frac{1}{2}$ by assumption; and (iii) $G(1) = \frac{1}{2}$ due to the symmetry in (127). Therefore, from (128):

$$G(\widehat{c}) \;=\; y \quad \Rightarrow \quad [1-a]\,\widehat{c} + \frac{1}{2}[2a-1](\widehat{c})^2 \;=\; y$$

$$\Rightarrow \quad \frac{2[1-a]}{2a-1}\,\widehat{c} + (\widehat{c})^2 \;=\; \frac{2y}{2a-1} \quad \Rightarrow \quad (\widehat{c})^2 \;=\; \frac{2y}{2a-1} - \frac{2[1-a]}{2a-1}\,\widehat{c}. \tag{130}$$

Observe that:

$$\frac{A_w}{T} \;=\; c^e - \frac{\int_0^{\widehat{c}} c\, g(c)\, dc}{G(\widehat{c})}$$

$$\Rightarrow \quad \widehat{c} + \frac{A_w}{T} \;=\; \widehat{c} + c^e - \frac{\int_0^{\widehat{c}} c\, g(c)\, dc}{G(\widehat{c})} \;=\; c^e + \widehat{c} - \frac{\int_0^{\widehat{c}} c\, g(c)\, dc}{G(\widehat{c})} \tag{131}$$

$$= c^e + \frac{1}{G(\widehat{c})}\left[\widehat{c}\,G(\widehat{c}) - \int_0^{\widehat{c}} c\, g(c)\, dc\right] \;\geq\; c^e \;=\; 1. \tag{132}$$

Because $\widehat{c} \leq 1$, (127) implies:

$$\int_0^{\widehat{c}} c\, g(c)\, dc \;=\; \int_0^{\widehat{c}} c\,[1 - a + (2a-1)\,c]\, dc$$

$$= \frac{1}{2}[1-a](\widehat{c})^2 + \frac{1}{3}[2a-1](\widehat{c})^3. \tag{133}$$

(130), (131), and (133) imply:

$$c_2 \;\equiv\; \widehat{c} + \frac{A_w}{T} \;=\; 1 + \widehat{c} - \frac{1}{y}\left[\frac{1}{2}(1-a)(\widehat{c})^2 + \frac{1}{3}(2a-1)(\widehat{c})^3\right]$$

$$= \widehat{c} + 1 - \frac{(\widehat{c})^2}{y}\left[\frac{1}{2}(1-a) + \frac{1}{3}(2a-1)\,\widehat{c}\right]$$

$$= \widehat{c} + 1 - \frac{1}{y}\left[\frac{1-a}{2} + \left(\frac{2a-1}{3}\right)\widehat{c}\right]\left[\frac{2y}{2a-1} - \frac{2(1-a)}{2a-1}\,\widehat{c}\right]$$

$$= \widehat{c} + 1 - \frac{1}{y}\left[\frac{1-a}{2}\right]\left[\frac{2\,y}{2\,a-1}\right] - \frac{1}{y}\left[\frac{2\,a-1}{3}\right]\widehat{c}\left[\frac{2\,y}{2\,a-1}\right]$$

$$+ \frac{1}{y}\left[\frac{1-a}{2}\right]\left[\frac{2\,(1-a)}{2\,a-1}\right]\widehat{c} + \frac{1}{y}\left[\frac{2\,a-1}{3}\right]\left[\frac{2\,(1-a)}{2\,a-1}\right](\widehat{c})^2$$

$$= \widehat{c} + 1 - \frac{1-a}{2\,a-1} - \frac{2}{3}\,\widehat{c} + \frac{[\,1-a\,]^2}{y\,[\,2\,a-1\,]}\,\widehat{c} + \frac{2\,[\,1-a\,]}{3\,y}\,(\widehat{c})^2$$

$$= 1 - \frac{1-a}{2\,a-1} + \widehat{c}\left[\frac{1}{3} + \frac{(\,1-a\,)^2}{y\,(\,2\,a-1\,)}\right] + \frac{2\,[\,1-a\,]}{3\,y}\left[\frac{2\,y}{2\,a-1} - \frac{2\,(\,1-a\,)}{2\,a-1}\,\widehat{c}\right]$$

$$= 1 - \frac{1-a}{2\,a-1} + \frac{4}{3}\left[\frac{1-a}{2\,a-1}\right] + \widehat{c}\left[\frac{1}{3} + \frac{(\,1-a\,)^2}{y\,(\,2\,a-1\,)} - \frac{4\,(\,1-a\,)^2}{3\,y\,(\,2\,a-1\,)}\right]$$

$$= 1 + \frac{1}{3}\left[\frac{1-a}{2\,a-1}\right] + \widehat{c}\left[\frac{1}{3} - \frac{(\,1-a\,)^2}{3\,y\,(\,2\,a-1\,)}\right]$$

$$= 1 + \frac{1}{3}\left[\frac{1-a}{2\,a-1}\right] + \frac{\widehat{c}}{3}\left[1 - \frac{(\,1-a\,)^2}{y\,(\,2\,a-1\,)}\right] = A_2 + B_2\,\widehat{c} \tag{134}$$

where $A_2 \equiv 1 + \frac{1}{3}\left[\frac{1-a}{2\,a-1}\right]$ and $B_2 \equiv \frac{1}{3}\left[1 - \frac{(\,1-a\,)^2}{y\,(\,2\,a-1\,)}\right]$.

(134) implies:

$$c_1 \equiv \widehat{c} - \frac{A_w}{N-T} = \widehat{c} - \frac{A_w}{T}\left[\frac{T}{N-T}\right] = \widehat{c} - \frac{A_w}{T}\left[\frac{\frac{T}{N}}{1-\frac{T}{N}}\right]$$

$$= \widehat{c} - \frac{A_w}{T}\left[\frac{y}{1-y}\right] = \widehat{c} - \left[\frac{y}{1-y}\right][\,A_2 + B_2\,\widehat{c} - \widehat{c}\,] \tag{135}$$

$$= \left[1 + (1-B_2)\left(\frac{y}{1-y}\right)\right]\widehat{c} - \left[\frac{y}{1-y}\right]A_2 = A_1 + B_1\,\widehat{c} \tag{136}$$

where $A_1 \equiv -\left[\frac{y}{1-y}\right]A_2$ and $B_1 \equiv 1 + [\,1-B_2\,]\left[\frac{y}{1-y}\right]$.

(129) and (134) imply:

$$G\,(c_2) = \frac{1}{2} + [\,3\,a-1\,]\,[\,A_2 + B_2\,\widehat{c} - 1\,] + \frac{1}{2}\,[\,1-2\,a\,]\left[\,(A_2 + B_2\,\widehat{c})^2 - 1\,\right]. \tag{137}$$

(128) and (136) imply:

$$G\,(c_1) = [\,1-a\,]\,[\,A_1 + B_1\,\widehat{c}\,] + \frac{1}{2}\,[\,2\,a-1\,]\,[\,A_1 + B_1\,\widehat{c}\,]^2. \tag{138}$$

(137) and (138) imply:

$$G(c_2) - G(c_1) = \frac{1}{2} + [3a - 1][A_2 + B_2 \widehat{c} - 1] + \left[\frac{1-2a}{2}\right]\left[(A_2 + B_2 \widehat{c})^2 - 1\right]$$

$$- [1-a][A_1 + B_1 \widehat{c}] - \left[\frac{2a-1}{2}\right][A_1 + B_1 \widehat{c}]^2$$

$$= \frac{1}{2} - (3a-1) - \left(\frac{1-2a}{2}\right) + [3a-1][A_2 + B_2 \widehat{c}] + \left[\frac{1-2a}{2}\right][A_2 + B_2 \widehat{c}]^2$$

$$- [1-a][A_1 + B_1 \widehat{c}] + \left[\frac{1-2a}{2}\right][A_1 + B_1 \widehat{c}]^2 \equiv \Psi_1(a),$$

where $\widehat{c}$ is the solution to (130), so:

$$\widehat{c} = \frac{1}{2}\left[-\frac{2[1-a]}{2a-1} + \sqrt{\frac{4[1-a]^2}{[2a-1]^2} + \frac{8y}{2a-1}}\right]$$

$$= \frac{1}{2}\left[-\frac{2[1-a]}{2a-1} + \frac{1}{2a-1}\sqrt{4[1-a]^2 + 8y[2a-1]}\right]$$

$$= \frac{-(1-a) + \sqrt{[1-a]^2 + 2y[2a-1]}}{2a-1}. \tag{139}$$

Mathematica reveals that for all $a \in [0.042, 0.5)$, $\Psi_1(a) < \frac{1}{2}$ (so $A_m > A_w$) for all $y \in (0, \frac{1}{2})$. *Mathematica* also reveals that if $a = 0$, then: (i) $\Psi_1(a) > \frac{1}{2}$ for all $y \in (0, 0.41)$, i.e., for all finite $\frac{N}{T} \geq 2.44$; and (ii) $\Psi_1(a) < \frac{1}{2}$ for all $y \in [0.41, 0.5)$, i.e., for $\frac{N}{T} \in (1, 2.44)$.

Case 2. $N < 2T$.

$\widehat{c} \geq 1$ because: (i) $G(\widehat{c}) = y$; (ii) $y \equiv \frac{T}{N} > \frac{1}{2}$ by assumption; and (iii) $G(1) = \frac{1}{2}$ due to the symmetry in (127). Therefore, (127) implies:

$$\int_0^{\widehat{c}} c\, g(c)\, dc = \int_0^1 c[1 - a + (2a-1)c]\, dc + \int_1^{\widehat{c}} c[3a - 1 + (1-2a)c]\, dc$$

$$= [1-a]\left[\frac{c^2}{2}\right]_0^1 + [2a-1]\left[\frac{c^3}{3}\right]_0^1 + [3a-1]\left[\frac{c^2}{2}\right]_1^{\widehat{c}} + [1-2a]\left[\frac{c^3}{3}\right]_1^{\widehat{c}}$$

$$= \frac{1-a}{2} + \frac{2a-1}{3} - \frac{3a-1}{2} - \frac{1-2a}{3} + \frac{3a-1}{2}(\widehat{c})^2 + \frac{1-2a}{3}(\widehat{c})^3$$

$$= \frac{3 - 3a + 4a - 2 - 9a + 3 - 2 + 4a}{6} + \frac{3a-1}{2}(\widehat{c})^2 + \frac{1-2a}{3}(\widehat{c})^3$$

$$= \frac{1-2a}{3} + \frac{3a-1}{2}(\widehat{c})^2 + \frac{1-2a}{3}(\widehat{c})^3. \tag{140}$$

(131) and (140) imply:

$$c_2 \;\equiv\; \widehat{c} + \frac{A_w}{T} \;=\; \widehat{c} + 1 - \frac{1}{y}\left[\frac{1-2\,a}{3} + \frac{3\,a-1}{2}\,(\widehat{c})^2 + \frac{1-2\,a}{3}\,(\widehat{c})^3\right], \tag{141}$$

where $\widehat{c}$ is determined by:

$$G(\widehat{c}) \;=\; \frac{1}{2} + [\,3\,a-1\,]\,[\,\widehat{c}-1\,] + \left[\frac{1-2\,a}{2}\right]\left[\,(\widehat{c})^2 - 1\,\right] \;=\; y$$

$$\Rightarrow\;\; \frac{1}{2} + [\,3\,a-1\,]\,\widehat{c} - (3\,a-1) - \frac{1-2\,a}{2} + \left[\frac{1-2\,a}{2}\right](\widehat{c})^2 \;=\; y$$

$$\Rightarrow\;\; \left[\frac{1-2\,a}{2}\right](\widehat{c})^2 + [\,3\,a-1\,]\,\widehat{c} + 1 - 2\,a - y \;=\; 0$$

$$\Rightarrow\;\; (\widehat{c})^2 + \frac{2\,[\,3\,a-1\,]}{1-2\,a}\,\widehat{c} + \frac{2\,[\,1-2\,a-y\,]}{1-2\,a} \;=\; 0$$

$$\Rightarrow\;\; \widehat{c} \;=\; \frac{1}{2}\left[-\frac{2\,[\,3\,a-1\,]}{1-2\,a} + \sqrt{\frac{4\,[\,3\,a-1\,]^2}{[\,1-2\,a\,]^2} - \frac{8\,[\,1-2\,a-y\,]}{1-2\,a}}\,\right]$$

$$=\; \frac{1}{2}\left[-\frac{2\,[\,3\,a-1\,]}{1-2\,a} + \frac{2}{1-2\,a}\sqrt{[\,3\,a-1\,]^2 - 2\,[\,1-2\,a-y\,]\,[\,1-2\,a\,]}\,\right]$$

$$=\; \frac{-\,(3\,a-1) + \sqrt{a^2 + 2\,a + 2\,y - 4\,a\,y - 1}}{1-2\,a}\,. \tag{142}$$

(135) and (141) imply:

$$c_1 \;\equiv\; \widehat{c} - \frac{A_w}{T}\left[\frac{y}{1-y}\right] \;=\; \widehat{c} - \frac{y}{1-y}\,[\,c_2 - \widehat{c}\,]$$

$$=\; \widehat{c}\left[1 + \frac{y}{1-y}\right] - \frac{y\,c_2}{1-y} \;=\; \frac{\widehat{c}}{1-y} - \frac{y\,c_2}{1-y}\,. \tag{143}$$

(129) and (141) imply:

$$G\,(c_2) \;=\; \frac{1}{2} + [\,3\,a-1\,]\,[\,c_2-1\,] + \left[\frac{1-2\,a}{2}\right]\left[\,(c_2)^2 - 1\,\right]. \tag{144}$$

(128) and (143) imply:

$$G\,(c_1) \;=\; [\,1-a\,]\,c_1 + \left[\frac{2\,a-1}{2}\right](c_1)^2\;. \tag{145}$$

(144) and (145) imply:

$$G\,(c_2) - G\,(c_1) \;=\; \frac{1}{2} + [\,3\,a-1\,]\,[\,c_2-1\,] + \left[\frac{1-2\,a}{2}\right]\left[\,(c_2)^2-1\,\right] - [\,1-a\,]\,c_1 - \left[\frac{2\,a-1}{2}\right](c_1)^2$$

$$= \frac{1}{2} - (3a-1) - \left(\frac{1-2a}{2}\right) + [3a-1]c_2 + \left[\frac{1-2a}{2}\right](c_2)^2 - [1-a]c_1 - \left[\frac{2a-1}{2}\right](c_1)^2$$

$$= 1-2a + [3a-1]c_2 - [1-a]c_1 + \left[\frac{1-2a}{2}\right]\left[(c_2)^2 + (c_1)^2\right] \equiv \Psi_2(a).$$

Mathematica reveals that for all $a \in [0.042, 0.5)$, $\Psi_2(a) < \frac{1}{2}$ (so $A_w < A_m$) for all $y \in \left[\frac{1}{2}, 1\right)$. *Mathematica* also reveals that if $a = 0$, then: (i) $\Psi_2(a) > \frac{1}{2}$ for all $y \geq 0.586$, i.e., for $\frac{N}{T} \in (1, 1.7]$; and (ii) $\Psi_2(a) < \frac{1}{2}$ for all $y \in (0.5, 0.586)$, i.e., for $\frac{N}{T} \in (1, 1.7)$. ■

II.C. The Modified VJS Policy.

The ensuing analysis pertains to the modified VJS policy in which an individual's request for exemption from jury service is approved with probability $p \in (0, \overline{p}]$, where $\overline{p} < 1$. The proofs of the primary findings that follow, Propositions C1 – C3, rely upon Lemmas 5 – 10 and Conclusions 1 – 5.

Under the modified VJS policy, a "type c" individual (i.e., one who incurs cost c if he performs jury service) will "opt out" (i.e., request an exemption from jury service) if, when $\widetilde{N}$ individuals remain eligible for jury service:

$$p[-F] + [1-p]\frac{T}{\widetilde{N}}[w-c] > \frac{T}{\widetilde{N}}[w-c] \Leftrightarrow -pF > p\frac{T}{\widetilde{N}}[w-c]. \tag{146}$$

In contrast, a type c individual will "opt in" (i.e., not request an exemption) if, when $\widetilde{N}$ individuals are eligible for jury service:

$$\frac{T}{\widetilde{N}}[w-c] \geq p[-F] + [1-p]\frac{T}{\widetilde{N}}[w-c] \Leftrightarrow p\frac{T}{\widetilde{N}}[w-c] \geq -pF. \tag{147}$$

Lemma 5. *Suppose $p > 0$, w, and F are such that some type $\widehat{c} \in [\underline{c}, \overline{c}]$ is indifferent between opting in and opting out under VJS. Then types $c \in [\underline{c}, \widehat{c}]$ will opt in and types $c \in (\widehat{c}, \overline{c}]$ will opt out.*

<u>Proof</u>. (146) and (147) imply that if $p > 0$ and if $\widetilde{N} \in (0, N)$ individuals are eligible for jury service, then the type that is indifferent between opting in and opting out ($\widehat{c}$) is given by:

$$F = \frac{T}{\widetilde{N}}[\widehat{c} - w]. \tag{148}$$

Observe that when $p > 0$ and $\widetilde{N} \in (0, N)$, (146) will be satisfied for all $c \in (\widehat{c}, \overline{c}]$, whereas (147) will be satisfied for all $c \in [\underline{c}, \widehat{c}]$. ■

Lemma 5 implies that the (expected) number of individuals that are eligible for jury service is:

$$\widehat{N} \equiv N[G(\widehat{c}) + (1-p)(1-G(\widehat{c}))] = N[1 - p(1 - G(\widehat{c}))]. \tag{149}$$

Therefore, from (148), the type $\widehat{c} \in [\,\underline{c}, \overline{c}\,]$ that is indifferent between opting in and opting out is given by:

$$T\,[\,\widehat{c} - w\,] \;=\; F\,\widehat{N}\,. \tag{150}$$

Observe from (149) that $\frac{d\widehat{N}}{d\widehat{c}} = N\,p\,g(\widehat{c})$. Therefore, differentiating (150) provides:

$$[\,T - F\,N\,p\,g(\widehat{c})\,]\,d\widehat{c} - T\,dw \;=\; 0 \;\;\Rightarrow\;\; \left.\frac{d\widehat{c}}{dw}\right|_{dF\,=\,dp\,=\,0} \;=\; \frac{T}{T - F\,N\,p\,g(\widehat{c})}\,; \tag{151}$$

$$[\,T - F\,N\,p\,g(\widehat{c})\,]\,d\widehat{c} - \widehat{N}\,dF \;=\; 0 \;\;\Rightarrow\;\; \left.\frac{d\widehat{c}}{dF}\right|_{dw\,=\,dp\,=\,0} \;=\; \frac{\widehat{N}}{T - F\,N\,p\,g(\widehat{c})}\,; \tag{152}$$

$$\begin{aligned} &[\,T - F\,N\,p\,g(\widehat{c})\,]\,d\widehat{c} + F\,N\,[\,1 - G(\widehat{c})\,]\;dp \;=\; 0 \\ &\qquad\qquad\Rightarrow\;\; \left.\frac{d\widehat{c}}{dp}\right|_{dw\,=\,dF\,=\,0} \;=\; -\,\frac{F\,N\,[\,1 - G(\widehat{c})\,]}{T - F\,N\,p\,g(\widehat{c})}\,. \end{aligned} \tag{153}$$

Expected social welfare (per capita) under VJS given $\widehat{c}$ is:

$$\begin{aligned} W \;&=\; \int_{\underline{c}}^{\widehat{c}} \frac{T}{\widehat{N}}\,[\,w - c\,]\,dG(c) + \int_{\widehat{c}}^{\overline{c}} \left(p\,[\,-F\,] + [\,1 - p\,]\,\frac{T}{\widehat{N}}\,[\,w - c\,] \right) dG(c) - \frac{A}{N} \\ &=\; \frac{T}{\widehat{N}} \int_{\underline{c}}^{\overline{c}} [\,w - c\,]\,dG(c) - \frac{T}{\widehat{N}} \int_{\widehat{c}}^{\overline{c}} [\,w - c\,]\,dG(c) \\ &\qquad - \,p\,F \int_{\widehat{c}}^{\overline{c}} dG(c) \,+ [\,1 - p\,]\,\frac{T}{\widehat{N}} \int_{\widehat{c}}^{\overline{c}} [\,w - c\,]\,dG(c) - \frac{A}{N} \\ &=\; \frac{T}{\widehat{N}} \int_{\underline{c}}^{\overline{c}} [\,w - c\,]\,dG(c) - p \int_{\widehat{c}}^{\overline{c}} \left(F + \frac{T}{\widehat{N}}\,[\,w - c\,] \right) dG(c) - \frac{A}{N}\,. \end{aligned} \tag{154}$$

A modified VJS policy in which only types $c \in [\,\widehat{c}, \overline{c}\,]$ attempt to opt out will be self-financing in the sense that the expected payments to jurors and the administrative cost (A) do not exceed the expected revenue from opt-out fees if:

$$p\,[\,1 - G(\widehat{c})\,]\,N\,F \;\geq\; T\,w + A\,. \tag{155}$$

The expression in (154) can be written as:

$$\frac{T}{\widehat{N}}\,Z - p\,F \int_{\widehat{c}}^{\overline{c}} dG(c) - \frac{A}{N} \tag{156}$$

where:

$$Z \;\equiv\; \int_{\underline{c}}^{\widehat{c}} [\, w - c \,] \, dG(c) + [\, 1 - p \,] \int_{\widehat{c}}^{\overline{c}} [\, w - c \,] \, dG(c) \, . \tag{157}$$

(155) and (157) imply that [P], the social problem in this setting, is:

$$\underset{w,\, F,\, p \in [\, 0, \overline{p} \,]}{Maximize} \quad \frac{T}{\widehat{N}} \, Z \,-\, p \, F \int_{\widehat{c}}^{\overline{c}} dG(c) - \frac{A}{N}$$

subject to:

$$p \, [\, 1 - G(\, \widehat{c} \,) \,] \, N \, F \;\geq\; T \, w + A \, , \quad \text{and} \tag{158}$$

$$\widehat{N} \;=\; N \, [\, 1 - p \, (\, 1 - G(\, \widehat{c} \,) \,) \,] \;\geq\; T \, , \tag{159}$$

where $\widehat{c}$ is defined by (150).

Let λ_f denote the Lagrange multiplier associated with the "self-financing" constraint (158), and let λ_t denote the Lagrange multiplier associated with the "adequate jury pool" constraint, (159). Then the necessary conditions for a solution to [P] include:

$$F: \quad - \, p \int_{\widehat{c}}^{\overline{c}} dG(c) - \frac{T \, Z}{\left(\widehat{N} \right)^2} \, N \, p \, g(\, \widehat{c} \,) \, \frac{d \, \widehat{c}}{dF} + \; \frac{T}{\widehat{N}} \, p \, [\, w - \widehat{c} \,] \, g(\, \widehat{c} \,) \, \frac{d \widehat{c}}{dF} + p \, F \, g(\, \widehat{c} \,) \, \frac{d \widehat{c}}{dF}$$

$$+ \; \lambda_f \, p \, [\, 1 - G(\, \widehat{c} \,) \,] \, N - \lambda_f \, p \, N \, F \, g(\, \widehat{c} \,) \, \frac{d \, \widehat{c}}{dF} + \lambda_t \, p \, N \, g(\, \widehat{c} \,) \, \frac{d \, \widehat{c}}{dF} \;=\; 0$$

$$\Rightarrow \quad - \, p \int_{\widehat{c}}^{\overline{c}} dG(c) + \lambda_f \, p \, [\, 1 - G(\, \widehat{c} \,) \,] \, N - \; p \, N \, g(\, \widehat{c} \,) \, \frac{d \, \widehat{c}}{dF} \left[\frac{T \, Z}{\left(\widehat{N} \right)^2} + \lambda_f \, F - \lambda_t \right]$$

$$+ \;\; g(\, \widehat{c} \,) \, \frac{d \widehat{c}}{dF} \, p \left[\frac{T}{\widehat{N}} \, (\, w - \, \widehat{c} \,) + F \right] \;=\; 0 \, ; \tag{160}$$

$$w: \quad \frac{T}{\widehat{N}} \left[\int_{\underline{c}}^{\overline{c}} dG(c) - p \int_{\widehat{c}}^{\overline{c}} dG(c) \right] - \frac{T \, Z}{\left(\widehat{N} \right)^2} \, N \, p \, g(\, \widehat{c} \,) \, \frac{d \, \widehat{c}}{dw} + \; \frac{T}{\widehat{N}} \, p \, [\, w - \, \widehat{c} \,] \, g(\, \widehat{c} \,) \, \frac{d \, \widehat{c}}{dw}$$

$$+ \;\; p \, F \, g(\, \widehat{c} \,) \, \frac{d \, \widehat{c}}{dw} - \lambda_f \, T - \lambda_f \, p \, N \, F \, g(\, \widehat{c} \,) \, \frac{d \, \widehat{c}}{dw} + \lambda_t \, p \, N \, g(\, \widehat{c} \,) \, \frac{d \, \widehat{c}}{dw} \;=\; 0$$

$$\Rightarrow \quad \int_{\underline{c}}^{\overline{c}} dG(c) - p \int_{\widehat{c}}^{\overline{c}} dG(c) - \lambda_f \, \widehat{N} - p \, N \, g(\, \widehat{c} \,) \, \frac{d \, \widehat{c}}{dw} \, \frac{\widehat{N}}{T} \left[\frac{T \, Z}{\left(\widehat{N} \right)^2} + \lambda_f \, F - \lambda_t \right]$$

$$+ \; g(\widehat{c}\,)\,\frac{d\widehat{c}}{dw}\,\frac{\widehat{N}}{T}\,p\left[\frac{T}{\widehat{N}}\,(\,w-\,\widehat{c}\,)+F\right] \;=\; 0\,. \tag{161}$$

(151) and (152) imply that (161) can be written as:

$$\int_{\underline{c}}^{\overline{c}} dG(c) - p\int_{\widehat{c}}^{\overline{c}} dG(c) - \lambda_f\,\widehat{N} - p\,N\,g(\widehat{c}\,)\,\frac{d\widehat{c}}{dF}\left[\frac{T\,Z}{\left(\widehat{N}\right)^2} + \lambda_f\,F - \lambda_t\right]$$

$$+ \; g(\widehat{c}\,)\,\frac{d\widehat{c}}{dF}\,p\left[\frac{T}{\widehat{N}}\,(\,w-\,\widehat{c}\,)+F\right] \;=\; 0\,. \tag{162}$$

Subtracting (160) from (162) and using (150) provides:

$$\int_{\underline{c}}^{\overline{c}} dG(c) - \lambda_f\left[\widehat{N} + p\,(\,1-G(\widehat{c}\,)\,)\,N\right] \;=\; 0 \;\;\Rightarrow\;\; \lambda_f \;=\; \frac{1}{N}\int_{\underline{c}}^{\overline{c}} dG(c) \;>\; 0\,. \tag{163}$$

Because the self-financing constraint binds ($\lambda_f > 0$), (158) implies:

$$p\,[\,1-G(\widehat{c}\,)\,]\,N\,F \;=\; T\,w + A \;\;\Rightarrow\;\; w \;=\; p\,[\,1-G(\widehat{c}\,)\,]\,\frac{N}{T}\,F - \frac{A}{T}\,. \tag{164}$$

Also, from (150):

$$F \;=\; \frac{T}{\widehat{N}}\,[\,\widehat{c} - w\,]\,. \tag{165}$$

Combining (164) and (165) and using (149) provides:

$$w \;=\; p\,[\,1-G(\widehat{c}\,)\,]\,\frac{N}{\widehat{N}}\,[\,\widehat{c} - w\,] - \frac{A}{T}$$

$$\Rightarrow\;\; w\left[1 + p\,(\,1-G(\widehat{c}\,)\,)\,\frac{N}{\widehat{N}}\right] \;=\; p\,[\,1-G(\widehat{c}\,)\,]\,\frac{N}{\widehat{N}}\,\widehat{c} - \frac{A}{T}$$

$$\Rightarrow\;\; w\left[\widehat{N} + p\,(\,1-G(\widehat{c}\,)\,)\,N\right] \;=\; p\,[\,1-G(\widehat{c}\,)\,]\,N\,\widehat{c} - \frac{A}{T}\,\widehat{N}$$

$$\Rightarrow\;\; w \;=\; p\,[\,1-G(\widehat{c}\,)\,]\,\widehat{c} - \frac{A}{N}\,\frac{\widehat{N}}{T}\,. \tag{166}$$

(150) and (166) imply:

$$\widehat{c} - w \;=\; [\,1 - p\,(\,1-G(\widehat{c}\,)\,)\,]\,\widehat{c} + \frac{A}{N}\,\frac{\widehat{N}}{T} \;=\; \frac{\widehat{N}}{N}\left[\widehat{c} + \frac{A}{T}\right]. \tag{167}$$

(165) and (167) provide:

$$F \;=\; \frac{T}{N}\left[\widehat{c} + \frac{A}{T}\right]. \tag{168}$$

Letting λ_p denote the Lagrange multiplier associated with the constraint $p \leq \overline{p}$, the necessary condition for an optimum with respect to p is:

$$p: \; -\frac{T}{\widehat{N}} \int_{\widehat{c}}^{\overline{c}} [\, w - c\,]\, dG(c) - F \int_{\widehat{c}}^{\overline{c}} dG(c) - \frac{T\,Z}{\left(\widehat{N}\right)^2} \left[\frac{\partial \widehat{N}}{\partial p} + \frac{\partial \widehat{N}}{\partial \widehat{c}} \frac{d\widehat{c}}{dp} \right] + \frac{T}{\widehat{N}}\, p\, [\, w - \widehat{c}\,]\, g(\widehat{c}\,) \frac{d\widehat{c}}{dp}$$

$$+ \, p\, F\, g(\widehat{c}\,) \frac{d\widehat{c}}{dp} + \lambda_f\, N\, F \left[1 - G(\widehat{c}\,) - p\, g(\widehat{c}\,) \frac{d\widehat{c}}{dp} \right] + \lambda_t \left[\frac{\partial \widehat{N}}{\partial p} + \frac{\partial \widehat{N}}{\partial \widehat{c}} \frac{d\widehat{c}}{dp} \right] - \lambda_p \;=\; 0\,. \quad (169)$$

Using (165), (169) can be written as:

$$p: \quad -\frac{T}{\widehat{N}} \int_{\widehat{c}}^{\overline{c}} [\, w - c + \widehat{c} - w\,]\, dG(c) - \frac{T\,Z}{\left(\widehat{N}\right)^2} \left[-N\,[\,1 - G(\widehat{c}\,)\,] + p\,N\,g(\widehat{c}\,) \frac{d\widehat{c}}{dp} \right]$$

$$+ \; g(\widehat{c}\,) \frac{d\widehat{c}}{dp}\, p \left[-\frac{T}{\widehat{N}} (\,\widehat{c} - w\,) + F \right] + \lambda_t\, p\, N\, g(\widehat{c}\,) \frac{d\widehat{c}}{dp} - \lambda_p$$

$$+ \; \lambda_f\, N\, F\, [\,1 - G(\widehat{c}\,)\,] - \lambda_t\, N\, [\,1 - G(\widehat{c}\,)\,] - \lambda_f\, N\, F\, p\, g(\widehat{c}\,) \frac{d\widehat{c}}{dp} \;=\; 0$$

$$\Rightarrow \quad \frac{T}{\widehat{N}} \int_{\widehat{c}}^{\overline{c}} [\, c - \widehat{c}\,]\, dG(c) + N\,[\,1 - G(\widehat{c}\,)\,] \left[\frac{T\,Z}{\left(\widehat{N}\right)^2} + \lambda_f\, F - \lambda_t \right]$$

$$- \; p\, N\, g(\widehat{c}\,) \frac{d\widehat{c}}{dp} \left[\frac{T\,Z}{\left(\widehat{N}\right)^2} + \lambda_f\, F - \lambda_t \right] - \lambda_p \;=\; 0$$

$$\Rightarrow \quad \frac{T}{\widehat{N}} \int_{\widehat{c}}^{\overline{c}} [\, c - \widehat{c}\,]\, dG(c) - \frac{d\widehat{N}}{dp} \left[\frac{T\,Z}{\left(\widehat{N}\right)^2} + \lambda_f\, F - \lambda_t \right] - \lambda_p \;=\; 0 \qquad (170)$$

where

$$\frac{d\widehat{N}}{dp} \;=\; -\,N\,[\,1 - G(\widehat{c}\,)\,] + p\,N\,g(\widehat{c}\,) \frac{d\widehat{c}}{dp}\,.$$

(160), (163), and (165) imply:

$$-\,p \int_{\widehat{c}}^{\overline{c}} dG(c) + \lambda_f\, p\, [\,1 - G(\widehat{c}\,)\,]\, N \;=\; p\, N\, g(\widehat{c}\,) \frac{d\widehat{c}}{dF} \left[\frac{T\,Z}{\left(\widehat{N}\right)^2} + \lambda_f\, F - \lambda_t \right]$$

$$\Rightarrow \quad p\, N\, g(\widehat{c}\,) \frac{d\widehat{c}}{dF} \left[\frac{T\,Z}{\left(\widehat{N}\right)^2} + \lambda_f\, F - \lambda_t \right] \;=\; p\,[\,1 - G(\widehat{c}\,)\,] \int_{\underline{c}}^{\overline{c}} dG(c) - p \int_{\widehat{c}}^{\overline{c}} dG(c)$$

$$= \; p \left[\int_{\underline{c}}^{\widehat{c}} dG(c) - G(\widehat{c}) \int_{\underline{c}}^{\overline{c}} dG(c) \right]. \tag{171}$$

(171) implies that if $p > 0$ and $\frac{d\,\widehat{c}}{dF}$ is well-defined, then:

$$\frac{T\,Z}{\left(\widehat{N}\right)^2} + \lambda_f\,F - \lambda_t \;=\; \frac{\int_{\underline{c}}^{\widehat{c}} dG(c) - G(\widehat{c}) \int_{\underline{c}}^{\overline{c}} dG(c)}{N\,g(\widehat{c})\,\frac{d\,\widehat{c}}{dF}} \;=\; 0\,. \tag{172}$$

Conclusion 1. *If $G(\widehat{c}) < 1$, then $p = \overline{p}$ at the solution to [P].*

Proof. (170) and (172) imply that $\lambda_p > 0$ under the specified condition. Therefore, $p = \overline{p}$, by complementary slackness. ■

From (166) and (168), the expected net payoff of a type $c \in (\,\widehat{c}\,, \overline{c}\,]$ individual is:

$$\begin{aligned}
u(c) \;&=\; p\,[-F\,] + [\,1-p\,]\,\frac{T}{\widehat{N}}\,[\,w-c\,] \\
&=\; -\,p\,\frac{T}{N}\left[\,\widehat{c} + \frac{A}{T}\,\right] + [\,1-p\,]\,\frac{T}{\widehat{N}}\left[\,p\,[\,1-G(\widehat{c}\,)\,]\,\widehat{c} - \frac{A}{N}\,\frac{\widehat{N}}{T} - c\,\right] \\
&=\; \widehat{c}\,p\,T\left[\left(\frac{1-p}{\widehat{N}}\right)[\,1-G(\widehat{c}\,)\,] - \frac{1}{N}\right] - [\,1-p\,]\,\frac{T}{\widehat{N}}\,c - \,p\,\frac{A}{N} - [\,1-p\,]\,\frac{A}{N} \\
&=\; p\,\frac{T}{N\,\widehat{N}}\left[\,(\,1-p\,)\,[\,1-G(\widehat{c}\,)\,]\,N - \widehat{N}\,\right]\widehat{c} - [\,1-p\,]\,\frac{T}{\widehat{N}}\,c - \frac{A}{N} \\
&=\; p\,\frac{T}{N\,\widehat{N}}\,[\,-G(\widehat{c}\,)\,N\,]\,\widehat{c} - [\,1-p\,]\,\frac{T}{\widehat{N}}\,c - \frac{A}{N} \\
&=\; -\,\frac{T}{\widehat{N}}\,[\,p\,G(\widehat{c}\,)\,\widehat{c} + (\,1-p\,)\,c\,] - \frac{A}{N}\,.
\end{aligned} \tag{173}$$

The fifth equality in (173) holds because, from (149):

$$\begin{aligned}
&[\,1-p\,]\,[\,1-G(\widehat{c}\,)\,]\,N - \widehat{N} \;=\; [\,1-p\,]\,[\,1-G(\widehat{c}\,)\,]\,N - N\,[\,1-p\,(\,1-G(\widehat{c}\,)\,)\,] \\
&=\; N\,\{\,[\,1-p\,]\,[\,1-G(\widehat{c}\,)\,] - 1 + p\,[\,1-G(\widehat{c}\,)\,]\,\} \;=\; N\,[\,1-G(\widehat{c}\,)-1\,] \;=\; -\,N\,G(\widehat{c}\,)\,.
\end{aligned}$$

From (166) and using (149), the expected net payoff of a type $c \in [\,\underline{c}\,, \widehat{c}\,]$ individual is:

$$u(c) \;=\; \frac{T}{\widehat{N}}\left[\,p\,[\,1-G(\widehat{c}\,)\,]\,\widehat{c} - \frac{A}{N}\,\frac{\widehat{N}}{T} - c\,\right] \;=\; \frac{T}{\widehat{N}}\,[\,p\,(\,1-G(\widehat{c}\,)\,)\,\widehat{c} - c\,] - \frac{A}{N}\,. \tag{174}$$

Conclusion 2. *The individuals whose expected net payoff increases when an optimal modified VJS policy is implemented are those for whom* $c > \widehat{c} + \frac{A}{T} \frac{\widehat{N}}{\overline{p}\, G(\widehat{c})\, N}$ *and* $c < \widehat{c} - \frac{A}{T} \frac{\widehat{N}}{\overline{p}\,[1 - G(\widehat{c})] N}$.

Proof. From (173) and using (149), the expected net payoff of a type $c \in (\widehat{c}, \overline{c}\,]$ individual increases when an optimal VJS policy is implemented if:

$$- \left[\frac{T}{\widehat{N}} [\, \overline{p}\, G(\widehat{c})\, \widehat{c} + (1 - \overline{p}\,)\, c \,] + \frac{A}{N} \right] \;>\; - \frac{T}{N}\, c$$

$$\Leftrightarrow \;\; \frac{T}{N}\, c \;>\; \frac{T}{\widehat{N}} [\, \overline{p}\, G(\widehat{c})\, \widehat{c} + (1 - \overline{p}\,)\, c \,] + \frac{A}{N}$$

$$\Leftrightarrow \;\; \frac{T}{\widehat{N}\, N}\, c \left[\widehat{N} - N\, (1 - \overline{p}\,) \right] \;>\; \frac{T}{\widehat{N}\, N}\, \overline{p}\, N\, G(\widehat{c})\, \widehat{c} + \frac{A\, \widehat{N}}{\widehat{N}\, N}$$

$$\Leftrightarrow \;\; c\, [\, N\, G(\widehat{c}) + N\, (1 - \overline{p}\,)\, (1 - G(\widehat{c})\,) - N\, (1 - \overline{p}\,) \,] \;>\; \overline{p}\, N\, G(\widehat{c})\, \widehat{c} + \frac{A\, \widehat{N}}{T}$$

$$\Leftrightarrow \;\; \overline{p}\, N\, G(\widehat{c})\, c \;>\; \overline{p}\, N\, G(\widehat{c})\, \widehat{c} + \frac{A\, \widehat{N}}{T} \quad \Leftrightarrow \quad c \;>\; \widehat{c} + \frac{A}{T}\, \frac{\widehat{N}}{\overline{p}\, G(\widehat{c})\, N}\,.$$

From (174) and using (149), the expected net payoff of a type $c \in [\, \underline{c}, \widehat{c}\,]$ individual increases when an optimal VJS policy is implemented if:

$$\frac{T}{\widehat{N}} [\, \overline{p}\, (1 - G(\widehat{c})\,)\, \widehat{c} - c \,] - \frac{A}{N} \;>\; - \frac{T}{N}\, c$$

$$\Leftrightarrow \;\; \frac{T}{N}\, c \;>\; \frac{T}{\widehat{N}} [\, c - \overline{p}\, (1 - G(\widehat{c})\,)\, \widehat{c} \,] + \frac{A}{N}$$

$$\Leftrightarrow \;\; \frac{T}{\widehat{N}\, N} \left[\widehat{N} - N \right] c \;>\; - \frac{T\, N}{\widehat{N}\, N}\, \overline{p}\, [1 - G(\widehat{c})\,]\, \widehat{c} \; + \frac{A\, \widehat{N}}{\widehat{N}\, N}$$

$$\Leftrightarrow \;\; [\, N - G(\widehat{c})\, N - (1 - \overline{p}\,)\, (1 - G(\widehat{c})\,)\, N \,]\, c \;<\; \overline{p}\, N\, [1 - G(\widehat{c})\,]\, \widehat{c} \; - \frac{A\, \widehat{N}}{T}$$

$$\Leftrightarrow \;\; \overline{p}\, N\, [1 - G(\widehat{c})\,]\, c \;<\; \overline{p}\, N\, [1 - G(\widehat{c})\,]\, \widehat{c} \; - \frac{A\, \widehat{N}}{T}$$

$$\Leftrightarrow \;\; c \;<\; \widehat{c} - \frac{A}{T}\, \frac{\widehat{N}}{\overline{p}\, [1 - G(\widehat{c})\,]\, N}\,. \quad \blacksquare$$

(173) and (174) imply that expected social welfare per capita, given p, is:

$$W(p) \;=\; \int_{\underline{c}}^{\widehat{c}} \left[\frac{T}{\widehat{N}} [\, p\, (1 - G(\widehat{c})\,)\, \widehat{c} - c \,] - \frac{A}{N} \right] dG(c)$$

$$- \int_{\widehat{c}}^{\overline{c}} \left[\frac{T}{\widehat{N}} \left[p\, G(\widehat{c})\ \widehat{c} + (1-p)\ c \right] + \frac{A}{N} \right] dG(c)$$

$$= \; - \, p\, \widehat{c}\, G(\widehat{c})\, \frac{T}{\widehat{N}} \int_{\underline{c}}^{\overline{c}} dG(c) \; + p\, \widehat{c}\, \frac{T}{\widehat{N}} \int_{\underline{c}}^{\widehat{c}} dG(c)$$

$$- \; \frac{T}{\widehat{N}} \int_{\underline{c}}^{\overline{c}} c\, dG(c) + p\, \frac{T}{\widehat{N}} \int_{\widehat{c}}^{\overline{c}} c\, dG(c) - \frac{A}{N} \int_{\underline{c}}^{\overline{c}} dG(c)$$

$$= \; p\, \widehat{c}\, \frac{T}{\widehat{N}} \left[\int_{\underline{c}}^{\widehat{c}} dG(c) - G(\widehat{c}) \int_{\underline{c}}^{\overline{c}} dG(c) \right]$$

$$+ \; p\, \frac{T}{\widehat{N}} \int_{\widehat{c}}^{\overline{c}} c\, dG(c) - \frac{T}{\widehat{N}} \int_{\underline{c}}^{\overline{c}} c\, dG(c) - \frac{A}{N} \int_{\underline{c}}^{\overline{c}} dG(c)\, . \tag{175}$$

(175) implies:

$$W(p) \; = \; p\, \widehat{c}\, \frac{T}{\widehat{N}} \left[\int_{\underline{c}}^{\widehat{c}} dG(c) - G(\widehat{c}) \int_{\underline{c}}^{\overline{c}} dG(c) \right]$$

$$+ \; \frac{T}{\widehat{N}} \left[p \int_{\widehat{c}}^{\overline{c}} c\, dG(c) - \int_{\underline{c}}^{\overline{c}} c\, dG(c) \right] - \frac{A}{N} \int_{\underline{c}}^{\overline{c}} dG(c)$$

$$= \; \frac{T}{\widehat{N}} \left[p \int_{\widehat{c}}^{\overline{c}} c\, dG(c) - \int_{\underline{c}}^{\overline{c}} c\, dG(c) \right] - \frac{A}{N}$$

$$= \; - \, \frac{T}{\widehat{N}} \left[\int_{\underline{c}}^{\widehat{c}} c\, dG(c) + [\, 1 - p\,] \int_{\widehat{c}}^{\overline{c}} c\, dG(c) \right] - \frac{A}{N}\, . \tag{176}$$

Expected jury service cost is the product of the probability of being called for jury service ($\frac{T}{\widehat{N}}$) and $E\{\, c \,|\, I\, ;\, \widehat{c} \}$, the expected personal cost of an individual who is in the jury pool, i.e.,

$$E\{\, c \,|\, I\, ;\, \widehat{c} \} \; \equiv \; \int_{\underline{c}}^{\widehat{c}} c\, dG(c) + [\, 1 - \overline{p}\,] \int_{\widehat{c}}^{\overline{c}} c\, dG(c)\, . \tag{177}$$

(176) and Conclusion 1 imply that maximizing average aggregate surplus is equivalent to minimizing

$$-W \;=\; \frac{T}{\widehat{N}} \left[\int_{\underline{c}}^{\widehat{c}} c \, dG(c) + [\, 1 - \overline{p} \,] \int_{\widehat{c}}^{\overline{c}} c \, dG(c) \right] + \frac{A}{N} \,. \tag{178}$$

The ensuing analysis assumes that $\overline{p} < 1$ and $N > \frac{T}{1-\overline{p}}$, so the adequate jury pool constraint (159) does not bind.

Lemma 6. *Average surplus is maximized when $-W$ is minimized with respect to $\widehat{c}$.*

Proof. If $\overline{p} < 1$, then $1 - p\,[\,1 - G(\widehat{c}\,)\,] > 0$ for all $p \leq \overline{p}$ and $\widehat{c}$. Therefore, $\frac{T}{1-p\,[\,1-G(\widehat{c})\,]}$ is a finite number. Hence, if N exceeds $\frac{T}{1-\overline{p}}$ (which weakly exceeds $\frac{T}{1-p\,[\,1-G(\widehat{c})\,]}$ for all $p \leq \overline{p}$), then (159) holds as a strict inequality. When (159) does not bind, average surplus is maximized when $\widehat{c}$ is chosen to minimize $-W$. If the optimal $\widehat{c}$ lies in $(\underline{c}\,, \overline{c})$, then the corresponding F and w are uniquely determined by (148), (149), and $p\,[\,1 - G(\widehat{c}\,)\,]\,N\,F \;=\; T\,w + A$ (which is (158) with equality), with $p = \overline{p}$. ■

Conclusion 3. $-\widetilde{W}$ *is minimized at* $\widehat{c}^{*}$, *where*

$$\widehat{c}^{*} \;=\; \frac{\delta\,(\widehat{c}^{*})}{\beta\,(\widehat{c}^{*})} \;=\; \frac{\int_{\underline{c}}^{\widehat{c}^{*}} c \, dG(c) + [\, 1 - \overline{p} \,] \int_{\widehat{c}^{*}}^{\overline{c}} c \, dG(c)}{G(\widehat{c}^{*}) + [\, 1 - \overline{p} \,]\,[\, 1 - G(\widehat{c}^{*}\,) \,]} \,. \tag{179}$$

Proof. To minimize $-W$ with respect to $\widehat{c}$, observe from (159) and (178) that:

$$-W \;=\; \frac{T}{N} \left[\frac{\int_{\underline{c}}^{\widehat{c}} c \, dG(c) + [\, 1 - \overline{p} \,] \int_{\widehat{c}}^{\overline{c}} c \, dG(c)}{G(\widehat{c}\,) + [\, 1 - \overline{p} \,]\,[\, 1 - G(\widehat{c}\,) \,]} \right] + \frac{A}{N} \,. \tag{180}$$

Therefore, to maximize average surplus, it suffices to minimize:

$$-\widetilde{W} \;=\; \frac{\delta\,(\widehat{c}\,)}{\beta\,(\widehat{c}\,)} \tag{181}$$

where:

$$\delta\,(\widehat{c}\,) \;\equiv\; \int_{\underline{c}}^{\widehat{c}} c \, dG(c) + [\, 1 - \overline{p} \,] \int_{\widehat{c}}^{\overline{c}} c \, dG(c) \;\;\text{and}\;\; \beta\,(\widehat{c}\,) \;\equiv\; G(\widehat{c}\,) + [\, 1 - \overline{p} \,]\,[\, 1 - G(\widehat{c}\,) \,] \,. \tag{182}$$

From (181):

$$\log\left(-\widetilde{W}\right) \;=\; \log\,(\,\delta\,(\widehat{c}\,)\,) - \log\,(\,\beta\,(\widehat{c}\,)\,)$$

$$\Rightarrow \;\; \frac{\partial}{\partial \widehat{c}} \left\{ \log\left(-\widetilde{W}\right) \right\} \;=\; \frac{\delta'\,(\widehat{c}\,)}{\delta\,(\widehat{c}\,)} - \frac{\beta'\,(\widehat{c}\,)}{\beta\,(\widehat{c}\,)} \,. \tag{183}$$

From (182):

$$\delta'(\widehat{c}) \;=\; \widehat{c}\, g(\widehat{c}) - [\,1-\overline{p}\,]\, \widehat{c}\, g(\widehat{c}) \;=\; \overline{p}\; \widehat{c}\; g(\widehat{c})\;, \text{ and}$$

$$\beta'(\widehat{c}) \;=\; g(\widehat{c}) - [\,1-\overline{p}\,]\, g(\widehat{c}) \;=\; \overline{p}\; g(\widehat{c})\,. \tag{184}$$

(183) and (184) imply:

$$\frac{\partial}{\partial \widehat{c}} \left\{ \log\left(-\widetilde{W}\right) \right\} \;=\; \overline{p}\; g(\widehat{c}) \left[\frac{\widehat{c}}{\delta(\widehat{c})} - \frac{1}{\beta(\widehat{c})} \right] \;=\; \frac{\overline{p}\; g(\widehat{c})\,[\,\widehat{c}\,\beta(\widehat{c}) - \delta(\widehat{c})\,]}{\delta(\widehat{c})\; \beta(\widehat{c})}\,. \tag{185}$$

Define $\gamma(\widehat{c}) \equiv \widehat{c}\,\beta(\widehat{c}) - \delta(\widehat{c})$. Differentiating $\gamma(\widehat{c})$, using (184), provides:

$$\frac{\partial \gamma(\widehat{c})}{\partial \widehat{c}} \;=\; \beta(\widehat{c}) + \widehat{c}\; \overline{p}\; g(\widehat{c}) - \overline{p}\; \widehat{c}\, g(\widehat{c}) \;=\; \beta(\widehat{c}) \;>\; 0\,. \tag{186}$$

Also, from (182):

$$\gamma(\underline{c}) \;=\; \underline{c}\; \beta(\underline{c}) - \delta(\underline{c}) \;=\; \underline{c}\,[\,1-\overline{p}\,] - E\{c\}\,[\,1-\overline{p}\,] \;=\; [\,1-\overline{p}\,]\,[\,\underline{c} - E\{c\}\,] \;<\; 0\,;$$

$$\gamma(\overline{c}) \;=\; \overline{c}\; \beta(\overline{c}) - \delta(\overline{c}) \;=\; \overline{c} - E\{c\} \;>\; 0\,. \tag{187}$$

(186) and (187) imply that there exists a unique $\widehat{c}^{*} \in (\underline{c}, \overline{c})$ such that: (i) $\gamma(\widehat{c}) < 0$ for $\widehat{c} < \widehat{c}^{*}$; (ii) $\gamma(\widehat{c}^{*}) = 0$; and (iii) $\gamma(\widehat{c}) > 0$ for $\widehat{c} > \widehat{c}^{*}$. Therefore, (185) implies that $-\widetilde{W}$ is minimized at $\widehat{c}^{*}$. ■

Conclusion 4. *$\widehat{c}^{*}$ is independent of A. Furthermore, $\widehat{c}^{*} < E\{c\} = c^{e}$, $\frac{\partial}{\partial \overline{p}}\{\widehat{c}^{*}\} < 0$, and $\widehat{c}^{*} \to \underline{c}$ as $\overline{p} \to 1$.*

<u>Proof</u>. (180) and Conclusion 3 imply that under an optimal modified VJS policy, average surplus is:

$$W^{*} \;=\; -\frac{T}{N} \left[\frac{\int_{\underline{c}}^{\widehat{c}^{*}} c\; dG(c) + [\,1-\overline{p}\,] \int_{\widehat{c}^{*}}^{\overline{c}} c\; dG(c)}{G(\widehat{c}^{*}) + [\,1-\overline{p}\,]\,[\,1 - G(\widehat{c}^{*})\,]} \right] - \frac{A}{N} \;=\; -\frac{T}{N}\, \widehat{c}^{*} - \frac{A}{N}\,. \tag{188}$$

It is apparent from (179) that $\widehat{c}^{*}$ is independent of A. From (188), average surplus is $-\frac{T}{N}\,\widehat{c}^{*} - \frac{A}{N}$ under an optimal modified VJS policy. Average surplus is $-\frac{T}{N}\, c^{e}$ under mandatory jury service (MJS). Conclusion 2 implies that if $A = 0$, then all individuals are better off under the optimal modified VJS policy. Therefore:

$$-\frac{T}{N}\, \widehat{c}^{*} \;>\; -\frac{T}{N}\, c^{e} \quad\Leftrightarrow\quad \widehat{c}^{*} \;<\; c^{e}\,.$$

From (179):

$$\widehat{c}^{*}\,[\,G(\widehat{c}^{*}) + (\,1-\overline{p}\,)\,(\,1 - G(\widehat{c}^{*}\,)\,)\,] \;=\; \int_{\underline{c}}^{\widehat{c}^{*}} c\; dG(c) + [\,1-\overline{p}\,] \int_{\widehat{c}^{*}}^{\overline{c}} c\; dG(c)\,. \tag{189}$$

Differentiating (189) provides:

$$\frac{\partial \widehat{c}^*}{\partial \overline{p}} \left[G(\widehat{c}^*) + (1-\overline{p})(1-G(\widehat{c}^*)) \right] + \widehat{c}^* \left[g(\widehat{c}^*) \frac{\partial \widehat{c}^*}{\partial \overline{p}} - (1-\overline{p})\, g(\widehat{c}^*) \frac{\partial \widehat{c}^*}{\partial \overline{p}} - (1-G(\widehat{c}^*)) \right]$$

$$= \widehat{c}^* g(\widehat{c}^*) \frac{\partial \widehat{c}^*}{\partial \overline{p}} - [1-\overline{p}]\, \widehat{c}^* g(\widehat{c}^*) \frac{\partial \widehat{c}^*}{\partial \overline{p}} - \int_{\widehat{c}^*}^{\overline{c}} c \, dG(c)$$

$$\Rightarrow \frac{\partial \widehat{c}^*}{\partial \overline{p}} \left[G(\widehat{c}^*) + (1-\overline{p})(1-G(\widehat{c}^*)) \right] = \widehat{c}^* [1-G(\widehat{c}^*)] - \int_{\widehat{c}^*}^{\overline{c}} c \, dG(c)$$

$$\Rightarrow \frac{\partial \widehat{c}^*}{\partial \overline{p}} = \frac{\widehat{c}^* [1-G(\widehat{c}^*)] - \int_{\widehat{c}^*}^{\overline{c}} c \, dG(c)}{G(\widehat{c}^*) + [1-\overline{p}][1-G(\widehat{c}^*)]} < 0 . \tag{190}$$

The inequality in (190) holds because: (i) $\widehat{c}^* [1-G(\widehat{c}^*)] - \int_{\widehat{c}^*}^{\overline{c}} c \, dG(c) = \int_{\widehat{c}^*}^{\overline{c}} [\widehat{c}^* - c] \, dG(c) < 0$; and (ii) $G(\widehat{c}^*) + [1-\overline{p}][1-G(\widehat{c}^*)] > 0$.

Finally, observe from (189) that if $\overline{p} \rightarrow 1$, then:

$$\widehat{c}^* G(\widehat{c}^*) \rightarrow \int_{\underline{c}}^{\widehat{c}^*} c \, dG(c) \Leftrightarrow \int_{\underline{c}}^{\widehat{c}^*} [\widehat{c}^* - c] \, dG(c) \rightarrow 0 \Leftrightarrow \widehat{c}^* \rightarrow \underline{c} . \quad \blacksquare$$

From (149) and Conclusion 2, an individual prefers VJS to MJS if:

$$c > \widehat{c}^* + \frac{A}{T} \frac{\widehat{N}}{\overline{p}\, G(\widehat{c}^*)\, N} \quad \text{or} \quad c < \widehat{c}^* - \frac{A}{T} \frac{\widehat{N}}{\overline{p}\, [1-G(\widehat{c}^*)]\, N}$$

$$\Leftrightarrow \quad c > \widehat{c}^* + \frac{A}{T} a_2 \quad \text{or} \quad c < \widehat{c}^* - \frac{A}{T} a_1$$

$$\text{where} \quad a_1 \equiv \frac{1-\overline{p}\,[1-G(\widehat{c}^*)]}{\overline{p}\,[1-G(\widehat{c}^*)]} \quad \text{and} \quad a_2 \equiv \frac{1-\overline{p}\,[1-G(\widehat{c}^*)]}{\overline{p}\, G(\widehat{c}^*)} . \tag{191}$$

(191) implies that the fraction of the population that prefers VJS to MJS is:

$$J^O(A) \equiv G\left(\widehat{c}^* - \frac{A}{T} a_1\right) + 1 - G\left(\widehat{c}^* + \frac{A}{T} a_2\right)$$

whereas the fraction of the population that prefers MJS to VJS is:

$$J^M(A) \equiv 1 - \left[G\left(\widehat{c}^* + \frac{A}{T} a_2\right) + 1 - G\left(\widehat{c}^* + \frac{A}{T} a_2\right) \right]$$

$$= G\left(\widehat{c}^* + \frac{A}{T}a_2\right) - G\left(\widehat{c}^* + \frac{A}{T}a_2\right).$$

Therefore, the difference between the fraction of individuals that prefer the optimal VJS policy and the fraction that prefer MJS is:

$$J^O(A) - J^M(A) = G\left(\widehat{c}^* - \frac{A}{T}a_1\right) + 1 - G\left(\widehat{c}^* + \frac{A}{T}a_2\right) + G\left(\widehat{c}^* - \frac{A}{T}a_1\right) - G\left(\widehat{c}^* + \frac{A}{T}a_2\right)$$

$$= 1 - 2\left[G\left(\widehat{c}^* + \frac{A}{T}a_2\right) - G\left(\widehat{c}^* - \frac{A}{T}a_1\right)\right] \equiv J(A). \quad (192)$$

If $J(A) > 0$, then a majority of the population prefers the optimal VJS policy. If $J(A) < 0$, then a majority of the population prefers MJS.

Lemma 7. *There exists a unique $A_m > 0$ such that: (i) $J(A) > 0$ for all $A < A_m$; (ii) $J(A) < 0$ for all $A > A_m$; and (iii) $J(A_m) = 0$.*

Proof. The conclusion holds because it is apparent from (192) that $J(A)$ is a decreasing function of A, $J(0) = 1$, and $J(A) \to -1$ as $A \to \infty$. ■

From (188), average surplus is $-\frac{T}{N}\widehat{c}^* - \frac{A}{N}$ under the optimal modified VJS policy. Average surplus is $-\frac{T}{N}c^e$ under MJS. Therefore, aggregate surplus is greater under the optimal modified VJS policy than under MJS if and only if:

$$-\frac{T}{N}\widehat{c}^* - \frac{A}{N} > -\frac{T}{N}c^e \Leftrightarrow \frac{T}{N}\widehat{c}^* + \frac{A}{N} < \frac{T}{N}c^e \Leftrightarrow \widehat{c}^* + \frac{A}{T} < c^e. \quad (193)$$

Define $$H(A) \equiv \widehat{c}^* + \frac{A}{T} - c^e. \quad (194)$$

Lemma 8. *There exists a unique $A_w > 0$, such that: (i) $H(A) < 0$ for all $A < A_w$; (ii) $H(A) > 0$ for all $A > A_w$; and (iii) $H(A_w) = 0$.*

Proof. It is apparent from (194) that $H'(A) > 0$ and $H(\infty) > 0$. Conclusion 4 implies $H(0) = \widehat{c}^* - c^e < 0$. ■

Lemmas 7 and 8 provide the following conclusions.

Lemma 9. *If $A_m < A_w$, then:*

1. If $A < A_m$, then a majority of individuals prefer the optimal modified VJS policy, which provides a higher level of aggregate surplus than MJS.

2. If $A \in (A_m, A_w)$, then only a minority of individuals prefer the optimal modified VJS policy even though it secures a higher level of aggregate surplus than MJS.

3. If $A > A_w$, then a majority of individuals prefer MJS, which secures a higher level of aggregate surplus than the modified VJS policy.

Lemma 10. *If* $A_m > A_w$*. Then:*

1. If $A < A_w$, then a majority of individuals prefer the optimal modified VJS policy, which provides a higher level of aggregate surplus than MJS.
2. If $A \in (A_w, A_m)$, then a majority of individuals prefer the optimal modified VJS policy, even though it secures a lower level of aggregate surplus than MJS.
3. If $A > A_m$, then a majority of individuals prefer MJS, which secures a higher level of aggregate surplus than the optimal modified VJS policy.

Conclusion 5. *A majority of the population prefers the surplus-maximizing policy if* $A < Min\{A_m, A_w\}$ *or* $A > Max\{A_m, A_w\}$*. Only a minority of the population prefers the surplus-maximizing policy if* $A \in (\,Min\{A_m, A_w\},\, Max\{A_m, A_w\}\,)$*.*

Proposition C1. *Majority rule favors neither MJS nor the modified VJS policy (so* $A_m = A_w$*) for all* $\overline{p} \in (0,1)$ *if* $g(c)$ *is the uniform density.*

Proof. From (179), when $g(c) = \frac{1}{\overline{c} - \underline{c}}$ for all $c \in [\,\underline{c}, \overline{c}\,]$:

$$\widehat{c}^*\,[\,G(\widehat{c}^*) + (\,1 - \overline{p}\,)\,(\,1 - G(\widehat{c}^*\,)\,)\,] \;=\; \int_{\underline{c}}^{\widehat{c}^*} c\; dG(c) + [\,1 - \overline{p}\,] \int_{\widehat{c}^*}^{\overline{c}} c\; dG(c)$$

$$\Leftrightarrow\;\; \widehat{c}^* \left[\frac{\widehat{c}^* - \underline{c}}{\overline{c} - \underline{c}} + (\,1 - \overline{p}\,)\left(1 - \frac{\widehat{c}^* - \underline{c}}{\overline{c} - \underline{c}}\right)\right] \;=\; \frac{c^2}{2\,[\,\overline{c} - \underline{c}\,]}\Bigg|_{\underline{c}}^{\widehat{c}^*} + [\,1 - \overline{p}\,]\; \frac{c^2}{2\,[\,\overline{c} - \underline{c}\,]}\Bigg|_{\widehat{c}^*}^{\overline{c}}$$

$$= \frac{[\,\widehat{c}^* - \underline{c}\,]\,[\,\widehat{c}^* + \underline{c}\,]}{2\,[\,\overline{c} - \underline{c}\,]} + [\,1 - \overline{p}\,]\,\frac{[\,\overline{c} - \widehat{c}^*]\,[\,\overline{c} + \widehat{c}^*\,]}{2\,[\,\overline{c} - \underline{c}\,]}$$

$$\Leftrightarrow\;\; \widehat{c}^* \left[\frac{\widehat{c}^* - \underline{c}}{\overline{c} - \underline{c}} + (\,1 - \overline{p}\,)\;\frac{\overline{c} - \widehat{c}^*}{\overline{c} - \underline{c}}\right] \;=\; \frac{[\,\widehat{c}^* - \underline{c}\,]\,[\,\widehat{c}^* + \underline{c}\,]}{2\,[\,\overline{c} - \underline{c}\,]} + [\,1 - \overline{p}\,]\,\frac{[\,\overline{c} - \widehat{c}^*]\,[\,\overline{c} + \widehat{c}^*\,]}{2\,[\,\overline{c} - \underline{c}\,]}$$

$$\Leftrightarrow\;\; \widehat{c}^* \left[\frac{\widehat{c}^* - \underline{c}}{\overline{c} - \underline{c}}\right] - \frac{[\,\widehat{c}^* - \underline{c}\,]\,[\,\widehat{c}^* + \underline{c}\,]}{2\,[\,\overline{c} - \underline{c}\,]} \;=\; [\,1 - \overline{p}\,]\,\frac{[\,\overline{c} - \widehat{c}^*\,]\,[\,\overline{c} + \widehat{c}^*\,]}{2\,[\,\overline{c} - \underline{c}\,]} - [\,1 - \overline{p}\,]\;\widehat{c}^* \left[\frac{\overline{c} - \widehat{c}^*}{\overline{c} - \underline{c}}\right]$$

$$\Leftrightarrow\;\; [\,\widehat{c}^* - \underline{c}\,] \left[\frac{\widehat{c}^*}{\overline{c} - \underline{c}} - \frac{\widehat{c}^* + \underline{c}}{2\,[\,\overline{c} - \underline{c}\,]}\right] \;=\; [\,1 - \overline{p}\,]\,[\,\overline{c} - \widehat{c}^*\,] \left[\frac{\overline{c} + \widehat{c}^*}{2\,[\,\overline{c} - \underline{c}\,]} - \frac{\widehat{c}^*}{\overline{c} - \underline{c}}\right]$$

$$\Leftrightarrow \quad [\,\widehat{c}^* - \underline{c}\,]\,\frac{2\,\widehat{c}^* - [\,\widehat{c}^* + \underline{c}\,]}{2\,[\,\overline{c} - \underline{c}\,]} \;=\; [\,1 - \overline{p}\,]\,[\,\overline{c} - \widehat{c}^*\,]\,\frac{\overline{c} + \widehat{c}^* - 2\,\widehat{c}^*}{2\,[\,\overline{c} - \underline{c}\,]}$$

$$\Leftrightarrow \quad [\,\widehat{c}^* - \underline{c}\,]\,\frac{\widehat{c}^* - \underline{c}}{2\,[\,\overline{c} - \underline{c}\,]} \;=\; [\,1 - \overline{p}\,]\,[\,\overline{c} - \widehat{c}^*\,]\,\frac{\overline{c} - \widehat{c}^*}{2\,[\,\overline{c} - \underline{c}\,]}$$

$$\Leftrightarrow \quad [\,\widehat{c}^* - \underline{c}\,]^2 \;=\; [\,1 - \overline{p}\,]\,[\,\overline{c} - \widehat{c}^*\,]^2 \quad \Leftrightarrow \quad \widehat{c}^* - \underline{c} \;=\; \sqrt{1 - \overline{p}}\;[\,\overline{c} - \widehat{c}^*\,]$$

$$\Leftrightarrow \quad \widehat{c}^* + \widehat{c}^* \sqrt{1 - \overline{p}} \;=\; \overline{c}\,\sqrt{1 - \overline{p}} + \underline{c} \quad \Leftrightarrow \quad \widehat{c}^* \left[1 + \sqrt{1 - \overline{p}} \,\right] \;=\; \underline{c} + \overline{c}\,\sqrt{1 - \overline{p}}$$

$$\Leftrightarrow \quad \widehat{c}^* \;=\; \left[\frac{1}{1 + \sqrt{1 - \overline{p}}} \right] \underline{c} + \left[\frac{\sqrt{1 - \overline{p}}}{1 + \sqrt{1 - \overline{p}}} \right] \overline{c}\,. \tag{195}$$

From (192), when $g(c) \;=\; \frac{1}{\overline{c} - \underline{c}}$ for all $c \in [\,\underline{c}, \overline{c}\,]$:

$$G\left(\widehat{c}^* + \frac{A_m}{T}\,a_2 \right) - G\left(\widehat{c}^* - \frac{A_m}{T}\,a_1 \right) \;=\; \frac{1}{2}$$

$$\Leftrightarrow \quad \frac{\widehat{c}^* + \frac{A_m}{T}\,a_2 - \underline{c}}{\overline{c} - \underline{c}} - \frac{\widehat{c}^* - \frac{A_m}{T}\,a_1 - \underline{c}}{\overline{c} - \underline{c}} \;=\; \frac{1}{2} \quad \Leftrightarrow \quad \left[\frac{1}{\overline{c} - \underline{c}} \right] \frac{A_m}{T}\,[\,a_1 + a_2\,] \;=\; \frac{1}{2}\,. \tag{196}$$

From (191):

$$a_1 + a_2 \;=\; \frac{1 - \overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]}{\overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]} + \frac{1 - \overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]}{\overline{p}\;G\,(\widehat{c}^*)}$$

$$= \; \frac{1 - \overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]}{\overline{p}} \left[\frac{1}{1 - G\,(\widehat{c}^*)} + \frac{1}{G\,(\widehat{c}^*)} \right] = \; \frac{1 - \overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]}{\overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]\,G\,(\widehat{c}^*)}\,. \tag{197}$$

(196) and (197) imply:

$$\left[\frac{1}{\overline{c} - \underline{c}} \right] \frac{A_m}{T}\,\frac{1 - \overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]}{\overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]\,G\,(\widehat{c}^*)} \;=\; \frac{1}{2}$$

$$\Leftrightarrow \quad \frac{A_m}{T} \;=\; \left[\frac{\overline{c} - \underline{c}}{2} \right] \frac{\overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]\,G\,(\widehat{c}^*)}{1 - \overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]} \;=\; \frac{\overline{c} - \underline{c}}{2} \left[\frac{\overline{p}\left[1 - \frac{\widehat{c}^* - \underline{c}}{\overline{c} - \underline{c}} \right] \frac{\widehat{c}^* - \underline{c}}{\overline{c} - \underline{c}}}{1 - \overline{p}\left[1 - \frac{\widehat{c}^* - \underline{c}}{\overline{c} - \underline{c}} \right]} \right]$$

$$= \; \frac{\overline{c} - \underline{c}}{2} \left[\frac{\overline{p}\left[\frac{\overline{c} - \widehat{c}^*}{\overline{c} - \underline{c}} \right] \frac{\widehat{c}^* - \underline{c}}{\overline{c} - \underline{c}}}{1 - \overline{p}\left[\frac{\overline{c} - \widehat{c}^*}{\overline{c} - \underline{c}} \right]} \right] = \; \frac{1}{2} \left[\frac{\overline{p}\,[\,\overline{c} - \widehat{c}^*\,]\,[\,\widehat{c}^* - \underline{c}\,]}{\overline{c} - \underline{c} - \overline{p}\,[\,\overline{c} - \widehat{c}^*\,]} \right]. \tag{198}$$

From (195):

$$\overline{c} - \widehat{c}^* \;=\; \overline{c} - \left[\frac{1}{1 + \sqrt{1 - \overline{p}}} \right] \underline{c} - \left[\frac{\sqrt{1 - \overline{p}}}{1 + \sqrt{1 - \overline{p}}} \right] \overline{c}$$

$$= \left[\frac{1}{1+\sqrt{1-\overline{p}}} \right] \overline{c} - \left[\frac{1}{1+\sqrt{1-\overline{p}}} \right] \underline{c} \;=\; \frac{\overline{c}-\underline{c}}{1+\sqrt{1-\overline{p}}} \,. \tag{199}$$

Also:

$$\widehat{c}^{*} - \underline{c} \;=\; \left[\frac{1}{1+\sqrt{1-\overline{p}}} \right] \underline{c} + \left[\frac{\sqrt{1-\overline{p}}}{1+\sqrt{1-\overline{p}}} \right] \overline{c} - \underline{c}$$

$$= \left[\frac{\sqrt{1-\overline{p}}}{1+\sqrt{1-\overline{p}}} \right] \overline{c} - \left[\frac{\sqrt{1-\overline{p}}}{1+\sqrt{1-\overline{p}}} \right] \underline{c} \;=\; \frac{[\,\overline{c}-\underline{c}\,]\sqrt{1-\overline{p}}}{1+\sqrt{1-\overline{p}}} \,. \tag{200}$$

(198), (199), and (200) imply:

$$\frac{A_m}{T} \;=\; \frac{1}{2}\left(\frac{\overline{p}\,[\,\overline{c}-\widehat{c}^{*}\,]\,[\,\widehat{c}^{*}-\underline{c}\,]}{\overline{c}-\underline{c}-\overline{p}\,[\,\overline{c}-\widehat{c}^{*}\,]} \right) \;=\; \frac{1}{2}\left(\frac{\overline{p}\left[\frac{1}{1+\sqrt{1-\overline{p}}}\right]\frac{\sqrt{1-\overline{p}}}{1+\sqrt{1-\overline{p}}}\,[\,\overline{c}-\underline{c}\,]^{2}}{\overline{c}-\underline{c}-\overline{p}\left[\frac{1}{1+\sqrt{1-\overline{p}}}\right][\,\overline{c}-\underline{c}\,]} \right)$$

$$= \frac{1}{2}\left(\frac{\overline{p}\left[\frac{1}{1+\sqrt{1-\overline{p}}}\right]\left[\frac{\sqrt{1-\overline{p}}}{1+\sqrt{1-\overline{p}}}\right][\,\overline{c}-\underline{c}\,]}{1-\overline{p}\left[\frac{1}{1+\sqrt{1-\overline{p}}}\right]} \right) \;=\; \frac{1}{2}\left(\frac{\overline{p}\left[\frac{\sqrt{1-\overline{p}}}{1+\sqrt{1-\overline{p}}}\right][\,\overline{c}-\underline{c}\,]}{1-\overline{p}+\sqrt{1-\overline{p}}} \right)$$

$$= \frac{1}{2}\left(\frac{\overline{p}\left[\frac{\sqrt{1-\overline{p}}}{1+\sqrt{1-\overline{p}}}\right][\,\overline{c}-\underline{c}\,]}{\sqrt{1-\overline{p}}\,\left[\,1+\sqrt{1-\overline{p}}\,\right]} \right) \;=\; \frac{\overline{p}\,[\,\overline{c}-\underline{c}\,]}{2\left[\,1+\sqrt{1-\overline{p}}\,\right]^{2}} \,. \tag{201}$$

(194) and (195) imply:

$$\frac{A_w}{T} \;=\; c^{e} - \widehat{c}^{*} \;=\; \frac{\underline{c}+\overline{c}}{2} - \left[\frac{1}{1+\sqrt{1-\overline{p}}} \right] \underline{c} - \left[\frac{\sqrt{1-\overline{p}}}{1+\sqrt{1-\overline{p}}} \right] \overline{c}$$

$$= \frac{[\,\underline{c}+\overline{c}\,]\left[\,1+\sqrt{1-\overline{p}}\,\right] - 2\,\underline{c} - 2\,\overline{c}\sqrt{1-\overline{p}}}{2\left[\,1+\sqrt{1-\overline{p}}\,\right]}$$

$$= \frac{\overline{c}\left[\,1+\sqrt{1-\overline{p}}-2\sqrt{1-\overline{p}}\,\right] + \underline{c}\left[\,1+\sqrt{1-\overline{p}}-2\,\right]}{2\left[\,1+\sqrt{1-\overline{p}}\,\right]}$$

$$= \frac{\overline{c}\left[\,1-\sqrt{1-\overline{p}}\,\right] - \underline{c}\left[\,1-\sqrt{1-\overline{p}}\,\right]}{2\left[\,1+\sqrt{1-\overline{p}}\,\right]} \;=\; \frac{\left[\,1-\sqrt{1-\overline{p}}\,\right][\,\overline{c}-\underline{c}\,]}{2\left[\,1+\sqrt{1-\overline{p}}\,\right]}$$

$$= \frac{\left[\,1-\sqrt{1-\overline{p}}\,\right][\,\overline{c}-\underline{c}\,]\left[\,1+\sqrt{1-\overline{p}}\,\right]}{2\left[\,1+\sqrt{1-\overline{p}}\,\right]^{2}} \;=\; \frac{\overline{p}\,[\,\overline{c}-\underline{c}\,]}{2\left[\,1+\sqrt{1-\overline{p}}\,\right]^{2}} \,. \tag{202}$$

(201) and (202) imply:

$$\frac{A_m}{T} \;=\; \frac{\overline{p}\,[\,\overline{c}-\underline{c}\,]}{2\left[\,1+\sqrt{1-\overline{p}}\,\right]^{2}} \;=\; \frac{A_w}{T} \;\Rightarrow\; A_m \;=\; A_w. \quad \blacksquare$$

Proposition C2 refers to the piecewise linear density with an inverted-V shape specified in (96).

Proposition C2. *Majority rule favors MJS over the modified VJS policy (so $A_m < A_w$) for all $\overline{p} \in (0,1)$ if $g(c)$ is as specified in (96) with $a \in (0, \frac{1}{2})$.*[4]

Proof. The analytic proof proceeds for the case where $a = 0$, so:

$$g(c) \;=\; \begin{cases} c & \text{if} \quad 0 \le c \le 1 \\ 2 - c & \text{if} \quad 1 \le c \le 2\,. \end{cases} \tag{203}$$

Mathematica demonstrates that the conclusion holds for $a \in (0, \frac{1}{2})$.[5]

When (203) holds, the numerator in the expression for $\widehat{c}^* < 1$ in (179) is:

$$\int_0^{\widehat{c}^*} c\, dG(c) + [\,1 - \overline{p}\,] \left[\int_{\widehat{c}^*}^{1} c\, dG(c) + \int_1^2 c\, dG(c) \right]$$

$$= \int_0^{\widehat{c}^*} c^2\, dc + [\,1 - \overline{p}\,] \left[\int_{\widehat{c}^*}^{1} c^2\, dc + \int_1^2 c\,(\,2 - c\,)\, dc \right]$$

$$= \left[\frac{c^3}{3} \right]_0^{\widehat{c}^*} + [\,1 - \overline{p}\,] \left[\left(\frac{c^3}{3} \right)_{\widehat{c}^*}^{1} + \left(c^2\right)_1^2 - \left(\frac{c^3}{3} \right)_1^2 \right]$$

$$= \frac{(\widehat{c}^*)^3}{3} + [\,1 - \overline{p}\,] \left[\frac{1}{3} - \frac{(\widehat{c}^*)^3}{3} + 3 - \frac{7}{3} \right] \;=\; \frac{(\widehat{c}^*)^3}{3} + [\,1 - \overline{p}\,] \left[1 - \frac{(\widehat{c}^*)^3}{3} \right]. \tag{204}$$

When (203) holds, the denominator in the expression for $\widehat{c}^*$ in (179) is:

$$G(\widehat{c}^*) + [\,1 - \overline{p}\,]\,[\,1 - G(\widehat{c}^*\,)\,] \;=\; \int_0^{\widehat{c}^*} c\, dc + [\,1 - \overline{p}\,] \left[1 - \int_0^{\widehat{c}^*} c\, dc \right]$$

$$= \frac{(\widehat{c}^*)^2}{2} + [\,1 - \overline{p}\,] \left[1 - \frac{(\widehat{c}^*)^2}{2} \right]. \tag{205}$$

(179), (204), and (205) imply that when (203) holds:

$$\widehat{c}^* \left[\frac{(\widehat{c}^*)^2}{2} + [\,1 - \overline{p}\,] \left(1 - \frac{(\widehat{c}^*)^2}{2} \right) \right] \;=\; \frac{(\widehat{c}^*)^3}{3} + [\,1 - \overline{p}\,] \left[1 - \frac{(\widehat{c}^*)^3}{3} \right]$$

[4] Proposition C2 implies that $A_m = A_w$ *for all* $\overline{p} \in (0,1)$ if $a = \frac{1}{2}$ when $g(c)$ is as specified in (96).

[5] The *Mathematica* analysis has been conducted for all a between 0.001 and 0.499 (in increments of 0.001) and for all $\overline{p}$ between 0.001 and 0.999 (in increments of 0.001).

$$\Rightarrow \quad \frac{(\widehat{c}^*)^3}{2} - \frac{(\widehat{c}^*)^3}{3} \; = \; [\, 1 - \overline{p}\,] \left[1 - \frac{(\widehat{c}^*)^3}{3} - \widehat{c}^* + \frac{(\widehat{c}^*)^3}{2} \right]$$

$$\Rightarrow \quad \frac{(\widehat{c}^*)^3}{6} - [\, 1 - \overline{p}\,] \left[1 + \frac{(\widehat{c}^*)^3}{6} - \widehat{c}^* \right] \; = \; 0$$

$$\Rightarrow \quad \frac{\overline{p}\,(\widehat{c}^*)^3}{6} - [\, 1 - \overline{p}\,][\, 1 - \widehat{c}^*\,] \; = \; 0 \quad \Rightarrow \quad \overline{p}\,(\widehat{c}^*)^3 \; = \; 6\,[\, 1 - \overline{p}\,][\, 1 - \widehat{c}^*\,]\,. \tag{206}$$

From (192), A_m is defined by:

$$1 - 2 \left[G\left(\widehat{c}^* + \frac{A_m}{T}\, a_2 \right) - G\left(\widehat{c}^* - \frac{A_m}{T}\, a_1 \right) \right] \; = \; 0\,. \tag{207}$$

Observe that:

$$A_m \; < \; A_w \quad \Leftrightarrow \quad 1 - 2 \left[G\left(\widehat{c}^* + \frac{A_w}{T}\, a_2 \right) - G\left(\widehat{c}^* - \frac{A_w}{T}\, a_1 \right) \right] \; < \; 0$$

$$\Leftrightarrow \quad G\left(\widehat{c}^* + \frac{A_w}{T}\, a_2 \right) - G\left(\widehat{c}^* - \frac{A_w}{T}\, a_1 \right) \; > \; \frac{1}{2}\,. \tag{208}$$

The first equivalence in (208) holds because the last inequality states that more than half of the population prefers MJS to VJS when $A = A_w$. By definition, the same number of individuals prefer MJS and VJS if $A = A_m$. Therefore, A_w must exceed A_m and so for $A \in (A_m, A_w)$, the majority will favor MJS even though welfare would be higher under VJS.

Because $\frac{A_w}{T} = c^e - \widehat{c}^*$ from (194) and $a_2 \equiv \frac{1 - \overline{p}\,[\,1 - G(\widehat{c}^*)\,]}{\overline{p}\,G(\widehat{c}^*)}$ from (191):

$$\widehat{c}^* + \frac{A_w}{T}\, a_2 \; = \; \widehat{c}^* + [\, c^e - \widehat{c}^*\,] \left[\frac{1 - \overline{p}\,[\, 1 - G\,(\widehat{c}^*)\,]}{\overline{p}\; G\,(\widehat{c}^*)} \right] \; = \; \widehat{c}^* + [\, c^e - \widehat{c}^*\,] \left[1 + \frac{1 - \overline{p}}{\overline{p}\; G\,(\widehat{c}^*)} \right]$$

$$= \; \widehat{c}^* + c^e - \widehat{c}^* + [\, c^e - \widehat{c}^*\,] \left[\frac{1 - \overline{p}}{\overline{p}\; G\,(\widehat{c}^*)} \right] \; = \; c^e + [\, c^e - \widehat{c}^*\,] \left[\frac{1 - \overline{p}}{\overline{p}\; G\,(\widehat{c}^*)} \right]. \tag{209}$$

Because $\frac{A_w}{T} = c^e - \widehat{c}^*$ from (194) and $a_1 \equiv \frac{1 - \overline{p}\,[\,1 - G(\widehat{c}^*)\,]}{\overline{p}\,[\,1 - G(\widehat{c}^*)\,]}$ from (191):

$$\widehat{c}^* - \frac{A_w}{T}\, a_1 \; = \; \widehat{c}^* - [\, c^e - \widehat{c}^*\,] \; \frac{1 - \overline{p}\,[\, 1 - G\,(\widehat{c}^*)\,]}{\overline{p}\,[\, 1 - G\,(\widehat{c}^*)\,]} \; = \; \widehat{c}^* - [\, c^e - \widehat{c}^*\,] \left[\frac{1}{\overline{p}\,[\, 1 - G\,(\widehat{c}^*)\,]} - 1 \right]$$

$$= \; \widehat{c}^* + c^e - \widehat{c}^* - [\, E\,\{c\} - \widehat{c}^*\,] \; \frac{1}{\overline{p}\,[\, 1 - G\,(\widehat{c}^*)\,]} \; = \; c^e - [\, c^e - \widehat{c}^*\,] \; \frac{1}{\overline{p}\,[\, 1 - G\,(\widehat{c}^*)\,]}\,. \tag{210}$$

(208), (209), and (210) imply:

$$A_m \; < \; A_w \quad \text{if} \quad G\,(c^e + [\, c^e - \widehat{c}^*\,]\, \alpha_2) - \; G\,(c^e - [\, c^e - \widehat{c}^*\,]\, \alpha_1) \; > \; \frac{1}{2} \tag{211}$$

$$\text{where} \quad \alpha_1 \equiv \frac{1}{\overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]} \quad \text{and} \quad \alpha_2 \equiv \frac{1-\overline{p}}{\overline{p}\;G\,(\widehat{c}^*)}\,. \tag{212}$$

The left hand side of the second inequality in (211) is the area under $g(c)$ for c between $c^e - [\,c^e - \widehat{c}^*\,]\,\alpha_1$ and $c^e + [\,c^e - \widehat{c}^*\,]\,\alpha_2$. This area is the sum of the areas under $g(c)$ for c between: (i) $c^e - [\,c^e - \widehat{c}^*\,]\,\alpha_1$ and 1; and (ii) 1 and $c^e + [\,c^e - \widehat{c}^*\,]\,\alpha_2$.

From (203), the area under $g(c)$ for c between $c^e - [\,c^e - \widehat{c}^*\,]\,\alpha_1$ and 1 is:

$$\int_{1-[1-\widehat{c}^*]\alpha_1}^{1} c\,dc \;=\; \frac{c^2}{2}\bigg|_{1-[1-\widehat{c}^*]\alpha_1}^{1} \;=\; \frac{1}{2} - \frac{1}{2}\,[\,1 - (1-\widehat{c}^*)\,\alpha_1\,]^2\,. \tag{213}$$

From (203), the area under $g(c)$ for c between 1 and $c^e + [\,c^e - \widehat{c}^*\,]\,\alpha_2$ is:

$$\int_{1}^{1+[1-\widehat{c}^*]\alpha_2} [\,2-c\,]\,dc \;=\; 2\,c\big|_1^{1+[1-\widehat{c}^*]\alpha_2} - \frac{c^2}{2}\bigg|_1^{1+[1-\widehat{c}^*]\alpha_2}$$

$$= 2\,[\,1-\widehat{c}^*\,]\,\alpha_2 - \frac{1}{2}\,[\,1 + (\,1-\widehat{c}^*\,)\,\alpha_2\,]^2 + \frac{1}{2}$$

$$= 2\,[\,1-\widehat{c}^*\,]\,\alpha_2 - \frac{1}{2}\left[\,1 + 2\,(\,1-\widehat{c}^*\,)\,\alpha_2 + (\,1-\widehat{c}^*)^2\,(\alpha_2)^2\,\right] + \frac{1}{2}$$

$$= \frac{1}{2} - \frac{1}{2}\left[\,1 - 2\,(\,1-\widehat{c}^*\,)\,\alpha_2 + (\,1-\widehat{c}^*\,)^2\,(\alpha_2)^2\,\right] \;=\; \frac{1}{2} - \frac{1}{2}\,[\,1 - (1-\widehat{c}^*)\,\alpha_2\,]^2\,. \tag{214}$$

(211), (213), and (214) imply:

$$A_m < A_w \;\Leftrightarrow\; 1 - \frac{1}{2}\,[\,c^e - (c^e - \widehat{c}^*)\,\alpha_1\,]^2 - \frac{1}{2}\,[\,c^e - (c^e - \widehat{c}^*)\,\alpha_2\,]^2 \;>\; \frac{1}{2}$$

$$\Leftrightarrow\; [\,c^e - (c^e - \widehat{c}^*)\,\alpha_1\,]^2 + [\,c^e - (c^e - \widehat{c}^*)\,\alpha_2\,]^2 \;<\; 1$$

$$\Leftrightarrow\; [\,1 - (1-\widehat{c}^*)\,\alpha_1\,]^2 + [\,1 - (1-\widehat{c}^*)\,\alpha_2\,]^2 \;<\; 1$$

$$\Leftrightarrow\; \left[\,1 - 2\,(1-\widehat{c}^*)\,\alpha_1 + (1-\widehat{c}^*)^2\,\alpha_1^2\,\right] + \left[\,1 - 2\,(1-\widehat{c}^*)\,\alpha_2 + (1-\widehat{c}^*)^2\,\alpha_2^2\,\right] \;<\; 1$$

$$\Leftrightarrow\; 2 - 2\,[\,1-\widehat{c}^*\,]\,[\,\alpha_1+\alpha_2\,] + [\,1-\widehat{c}^*\,]^2\left[\,\alpha_1^2+\alpha_2^2\,\right] \;<\; 1$$

$$\Leftrightarrow\; 1 - 2\,[\,1-\widehat{c}^*\,]\,[\,\alpha_1+\alpha_2\,] + [\,1-\widehat{c}^*\,]^2\left[\,\alpha_1^2+\alpha_2^2\,\right] \;<\; 0\,. \tag{215}$$

Observe that:

$$1 - 2\,[\,1-\widehat{c}^*\,]\,[\,\alpha_1+\alpha_2\,] + [\,1-\widehat{c}^*\,]^2\left[\,\alpha_1^2+\alpha_2^2\,\right]$$

$$= 1 - 2\,[\,1-\widehat{c}^*\,]\,[\,\alpha_1+\alpha_2\,] + [\,1-\widehat{c}^*\,]^2\left[\,\alpha_1^2+\alpha_2^2 + 2\,\alpha_1\,\alpha_2 - 2\,\alpha_1\,\alpha_2\,\right]$$

$$= 1 - 2\,[\,1-\widehat{c}^*\,]\,[\,\alpha_1+\alpha_2\,] + [\,1-\widehat{c}^*\,]^2\left[\,(\alpha_1+\alpha_2)^2 - 2\,\alpha_1\,\alpha_2\,\right]$$

$$= \; 1 - 2\,[\,1 - \widehat{c}^*\,]\,[\,\alpha_1 + \alpha_2\,] + [\,1 - \widehat{c}^*\,]^2\,[\,\alpha_1 + \alpha_2\,]^2 - 2\,[\,1 - \widehat{c}^*\,]^2\,\alpha_1\,\alpha_2$$

$$= \; [\,1 - (1 - \widehat{c}^*)\,(\alpha_1 + \alpha_2)\,]^2 - 2\,[\,1 - \widehat{c}^*\,]^2\,\alpha_1\,\alpha_2\,. \tag{216}$$

(215) and (216) imply:

$$A_m \; < \; A_w \quad \Leftrightarrow \quad [\,1 - (1 - \widehat{c}^*)\,(\alpha_1 + \alpha_2)\,]^2 \; < \; 2\,[\,1 - \widehat{c}^*\,]^2\,\alpha_1\,\alpha_2$$

$$\Leftrightarrow \quad 1 - [\,1 - \widehat{c}^*\,]\,[\,\alpha_1 + \alpha_2\,] < \; \sqrt{2}\;[\,1 - \widehat{c}^*\,]\,\sqrt{\alpha_1\,\alpha_2}\,. \tag{217}$$

(212) implies:

$$2\,[\,1 - \widehat{c}^*\,]^2\,\alpha_1\,\alpha_2 \; = \; 2\,[\,1 - \widehat{c}^*\,]^2 \; \frac{1}{\overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]}\;\frac{1 - \overline{p}}{\overline{p}\;G\,(\widehat{c}^*)}$$

$$= \; 2\,[\,1 - \widehat{c}^*\,]^2 \; \frac{1 - \overline{p}}{(\overline{p})^2}\;\frac{1}{[\,1 - G\,(\widehat{c}^*)\,]\,G\,(\widehat{c}^*)} \; = \; 2\,[\,1 - \widehat{c}^*\,]^2 \; \frac{1 - \overline{p}}{(\overline{p})^2}\;\frac{1}{\left[\,1 - \frac{(\widehat{c}^*)^2}{2}\,\right]\,\frac{(\widehat{c}^*)^2}{2}}$$

$$= \; 2\,[\,1 - \widehat{c}^*\,]^2 \; \frac{1 - \overline{p}}{(\overline{p})^2}\;\frac{4}{\left[\,2 - (\widehat{c}^*)^2\,\right]\,(\widehat{c}^*)^2}$$

$$\Rightarrow \quad \sqrt{2}\;[\,1 - \widehat{c}^*\,]\,\sqrt{\alpha_1\,\alpha_2} \; = \; 2\,\sqrt{2}\;\frac{[\,1 - \widehat{c}^*\,]}{\widehat{c}^*}\;\frac{\sqrt{1 - \overline{p}}}{\overline{p}}\;\frac{1}{\sqrt{2 - (\widehat{c}^*)^2}}\,. \tag{218}$$

(212) also implies:

$$\alpha_1 + \alpha_2 \; = \; \frac{1}{\overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]} + \frac{1 - \overline{p}}{\overline{p}\;G\,(\widehat{c}^*)} \; = \; \frac{G\,(\widehat{c}^*) + [\,1 - \overline{p}\,]\,[\,1 - G\,(\widehat{c}^*)\,]}{\overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]\,G\,(\widehat{c}^*)}$$

$$= \; \frac{G\,(\widehat{c}^*) + 1 - \overline{p} - G\,(\widehat{c}^*) + \overline{p}\;G\,(\widehat{c}^*)}{\overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]\,G\,(\widehat{c}^*)} \; = \; \frac{1 - \overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]}{\overline{p}\,[\,1 - G\,(\widehat{c}^*)\,]\,G\,(\widehat{c}^*)}$$

$$= \; \frac{1 - \overline{p}\left[\,1 - \frac{(\widehat{c}^*)^2}{2}\,\right]}{\overline{p}\left[\,1 - \frac{(\widehat{c}^*)^2}{2}\,\right]\,\frac{(\widehat{c}^*)^2}{2}} \; = \; \frac{2\left[\,2 - \overline{p}\left(2 - (\widehat{c}^*)^2\right)\,\right]}{\overline{p}\left[\,2 - (\widehat{c}^*)^2\,\right]\,(\widehat{c}^*)^2}$$

$$\Rightarrow \quad 1 - [\,1 - \widehat{c}^*\,]\,[\,\alpha_1 + \alpha_2\,] \; = \; 1 - [\,1 - \widehat{c}^*\,]\;\frac{2\left[\,2 - \overline{p}\left(2 - (\widehat{c}^*)^2\right)\,\right]}{\overline{p}\left[\,2 - (\widehat{c}^*)^2\,\right]\,(\widehat{c}^*)^2}\,. \tag{219}$$

(217), (218), and (219) imply:

$$A_m \; < \; A_w \quad \Leftrightarrow \quad 1 - [\,1 - \widehat{c}^*\,]\,\frac{2\left[\,2 - \overline{p}\left(2 - (\widehat{c}^*)^2\right)\,\right]}{\overline{p}\left[\,2 - (\widehat{c}^*)^2\,\right]\,(\widehat{c}^*)^2} \; < \; \frac{2\,\sqrt{2}\;[\,1 - \widehat{c}^*\,]\,\sqrt{1 - \overline{p}}}{\widehat{c}^*\,\overline{p}\,\sqrt{2 - (\widehat{c}^*)^2}}$$

$$\Leftrightarrow \quad \overline{p}\left[\,2 - (\widehat{c}^*)^2\,\right]\,(\widehat{c}^*)^2 - 2\,[\,1 - \widehat{c}^*\,]\left[\,2 - \overline{p}\left(2 - (\widehat{c}^*)^2\right)\,\right]$$

$$< \frac{\overline{p}\,(\widehat{c}^*)^2\left[2-(\widehat{c}^*)^2\right]}{\overline{p}\,\widehat{c}^*\sqrt{2-(\widehat{c}^*)^2}}\;2\sqrt{2}\,[\,1-\widehat{c}^*\,]\sqrt{1-\overline{p}}$$

$$= 2\sqrt{2}\sqrt{1-\overline{p}}\;[\,\widehat{c}^*\,]\,[\,1-\widehat{c}^*\,]\sqrt{2-(\widehat{c}^*)^2}\,. \tag{220}$$

Observe that:

$$\overline{p}\left[2-(\widehat{c}^*)^2\right](\widehat{c}^*)^2 - 2\,[\,1-\widehat{c}^*\,]\left[2-\overline{p}\left(2-(\widehat{c}^*)^2\right)\right]$$

$$= \overline{p}\left[2-(\widehat{c}^*)^2\right](\widehat{c}^*)^2 - 4\,[\,1-\widehat{c}^*\,] + 2\,[\,1-\widehat{c}^*\,]\,\overline{p}\left[2-(\widehat{c}^*)^2\right]$$

$$= \overline{p}\left[2-(\widehat{c}^*)^2\right](\widehat{c}^*)^2 - 4\,[\,1-\widehat{c}^*\,] + 4\,\overline{p}\,[\,1-\widehat{c}^*\,] - 2\,\overline{p}\,[\,1-\widehat{c}^*\,]\,(\widehat{c}^*)^2$$

$$= \overline{p}\,(\widehat{c}^*)^2\left[2-(\widehat{c}^*)^2-2\,(1-\widehat{c}^*)\right] - 4\,[\,1-\widehat{c}^*\,]\,[\,1-\overline{p}\,]$$

$$= \overline{p}\,(\widehat{c}^*)^3\,[\,2-\widehat{c}^*\,] - 4\,[\,1-\widehat{c}^*\,]\,[\,1-\overline{p}\,]$$

$$= 6\,[\,1-\overline{p}\,]\,[\,1-\widehat{c}^*\,]\,[\,2-\widehat{c}^*\,] - 4\,[\,1-\widehat{c}^*\,]\,[\,1-\overline{p}\,] \tag{221}$$

$$= 2\,[\,1-\widehat{c}^*\,]\,[\,1-\overline{p}\,]\,[\,3\,(2-\widehat{c}^*)-2\,] = 2\,[\,1-\widehat{c}^*\,]\,[\,1-\overline{p}\,]\,[\,4-3\,\widehat{c}^*\,]\,. \tag{222}$$

The equality in (221) follows from (206). (220) and (222) imply:

$$A_m < A_w \;\Leftrightarrow\; 2\,[\,1-\widehat{c}^*\,]\,[\,1-\overline{p}\,]\,[\,4-3\,\widehat{c}^*\,]$$

$$< 2\sqrt{2}\sqrt{1-\overline{p}}\;[\,\widehat{c}^*\,]\,[\,1-\widehat{c}^*\,]\sqrt{2-(\widehat{c}^*)^2}\,. \tag{223}$$

From (206):

$$\overline{p}\;(\widehat{c}^*)^3 - 6\,[\,1-\overline{p}\,]\,[\,1-\widehat{c}^*\,] = 0 \;\Rightarrow\; \overline{p}\;(\widehat{c}^*)^3 + 6\,\overline{p}\,[\,1-\widehat{c}^*\,] - 6\,[\,1-\widehat{c}^*\,] = 0$$

$$\Rightarrow\; \overline{p}\left[(\widehat{c}^*)^3 + 6\,(1-\widehat{c}^*)\right] = 6\,[\,1-\widehat{c}^*\,]$$

$$\Rightarrow\; \overline{p} = \frac{6\,[\,1-\widehat{c}^*\,]}{(\widehat{c}^*)^3+6\,[\,1-\widehat{c}^*\,]} \;\Rightarrow\; 1-\overline{p} = \frac{(\widehat{c}^*)^3}{(\widehat{c}^*)^3+6\,[\,1-\widehat{c}^*\,]}\,. \tag{224}$$

(224) implies:

$$\frac{d\overline{p}}{d\widehat{c}^*} \stackrel{s}{=} -\left[(\widehat{c}^*)^3+6\,(1-\widehat{c}^*)\right] - [\,1-\widehat{c}^*\,]\left[3\,(\widehat{c}^*)^2-6\right]$$

$$= -\,(\widehat{c}^*)^3 - 3\,[\,1-\widehat{c}^*\,]\,(\widehat{c}^*)^2 < 0 \quad \text{for } \widehat{c}^* \in (\,0,1\,]\,;$$

$$\widehat{c}^* \to 1 \text{ as } \overline{p} \to 0\,; \text{ and } \widehat{c}^* \to 0 \text{ as } \overline{p} \to 1\,. \tag{225}$$

(224) and (225) imply that $\widehat{c}^* \in (0,1)$ if $\overline{p} \in (0,1)$. Consequently, (223) implies that for $\overline{p} \in (0,1)$:

$$A_m < A_w \iff [1-\overline{p}]\,[4-3\,\widehat{c}^*] < \sqrt{2}\sqrt{1-\overline{p}}\;[\widehat{c}^*]\sqrt{2-(\widehat{c}^*)^2}$$

$$\iff \sqrt{1-\overline{p}}\;[4-3\,\widehat{c}^*] < \sqrt{2}\,[\widehat{c}^*]\sqrt{2-(\widehat{c}^*)^2}$$

$$\Rightarrow \quad A_m < A_w \text{ if } f(\widehat{c}^*) \equiv 2(\widehat{c}^*)^2\left[2-(\widehat{c}^*)^2\right] - [1-\overline{p}]\,[4-3\,\widehat{c}^*]^2 > 0\,. \tag{226}$$

(224) and (226) imply that for $\overline{p} \in (0,1)$:

$$f(\widehat{c}^*) = 2(\widehat{c}^*)^2\left[2-(\widehat{c}^*)^2\right] - \frac{(\widehat{c}^*)^3\,[4-3\,\widehat{c}^*]^2}{(\widehat{c}^*)^3 + 6\,[1-\widehat{c}^*]} > 0$$

$$\iff \eta(\widehat{c}^*) \equiv 2\left[(\widehat{c}^*)^3 + 6\,(1-\widehat{c}^*)\right]\left[2-(\widehat{c}^*)^2\right] - \widehat{c}^*\,[4-3\,\widehat{c}^*]^2 > 0\,. \tag{227}$$

Observe that:

$$\begin{aligned}
\eta(\widehat{c}^*) &= 2\left[2(\widehat{c}^*)^3 + 12\,(1-\widehat{c}^*) - (\widehat{c}^*)^5 - 6\,(1-\widehat{c}^*)\,(\widehat{c}^*)^2\right] \\
&\quad - \widehat{c}^*\left[16 - 24\,\widehat{c}^* + 9\,(\widehat{c}^*)^2\right] \\
&= 4\,(\widehat{c}^*)^3 + 24 - 24\,\widehat{c}^* - 2\,(\widehat{c}^*)^5 - 12\,(\widehat{c}^*)^2 + 12\,(\widehat{c}^*)^3 \\
&\qquad - 16\,\widehat{c}^* + 24\,(\widehat{c}^*)^2 - 9\,(\widehat{c}^*)^3 \\
&= 24 - 40\,\widehat{c}^* + 12\,(\widehat{c}^*)^2 + 7\,(\widehat{c}^*)^3 - 2\,(\widehat{c}^*)^5\,.
\end{aligned} \tag{228}$$

Differentiating (228) provides:

$$\begin{aligned}
\eta'(\widehat{c}^*) &= -40 + 24\,\widehat{c}^* + 21\,(\widehat{c}^*)^2 - 10\,(\widehat{c}^*)^4 \\
\Rightarrow \quad \eta''(\widehat{c}^*) &= 24 + 42\,\widehat{c}^* - 40\,(\widehat{c}^*)^3 = 24 + 2\,\widehat{c}^* + 40\,\widehat{c}^*\left[1-(\widehat{c}^*)^2\right] \\
&\qquad > 0 \text{ for all } \widehat{c}^* \in [0,1].
\end{aligned} \tag{229}$$

(229) implies that $\eta(\widehat{c}^*)$ is a strictly convex function of $\widehat{c}^*$ for all $\widehat{c}^* \in (0,1)$. Also, from (228) and (229):

$$\eta(0) = 24 > 0\,;\ \eta(1) = 24 - 40 + 12 + 7 - 2 = 1 > 0\,;$$

$$\eta'(0) = -40 < 0\,;\text{ and } \eta'(1) = -40 + 24 + 21 - 10 = -5 < 0\,. \tag{230}$$

(229) and (230) imply that $\eta(\widehat{c}^*) > 0$ for all $\widehat{c}^* \in (0,1)$, so (226) holds. Therefore, (226) and (227) imply that $A_m < A_w$ if $\overline{p} \in (0,1)$. ■

Proposition C3 considers the setting where $g(c)$ is a piecewise linear, V-shaped density.

Proposition C3. *$A_m \lesseqqgtr A_w$ as $\overline{p} \gtreqqless 0.75$ if:*

$$g(c) \;=\; \begin{cases} 1-c & \text{if} \quad 0 \leq c \leq 1 \\ c-1 & \text{if} \quad 1 \leq c \leq 2\,. \end{cases} \tag{231}$$

Proof. When (231) holds, the numerator in the expression for $\widehat{c}^* < 1$ in (179) is:

$$\int_0^{\widehat{c}^*} c\, dG(c) + [1-\overline{p}] \left[\int_{\widehat{c}^*}^1 c\, dG(c) + \int_1^2 c\, dG(c) \right]$$

$$= \int_0^{\widehat{c}^*} c\,[1-c]\, dc + [1-\overline{p}] \left[\int_{\widehat{c}^*}^1 c\,(1-c)\, dc + \int_1^2 c\,(c-1)\, dc \right]$$

$$= \int_0^{\widehat{c}^*} c\; dc - \int_0^{\widehat{c}^*} c^2\; dc + [1-\overline{p}] \left[\int_{\widehat{c}^*}^1 c\, dc - \int_{\widehat{c}^*}^1 c^2\, dc + \int_1^2 \left(c^2 - c\right) dc \right]$$

$$= \left[\frac{c^2}{2}\right]_0^{\widehat{c}^*} - \left[\frac{c^3}{3}\right]_0^{\widehat{c}^*} + [1-\overline{p}] \left[\left(\frac{c^2}{2}\right)_{\widehat{c}^*}^1 - \left(\frac{c^3}{3}\right)_{\widehat{c}^*}^1 + \left(\frac{c^3}{3}\right)_1^2 - \left(\frac{c^2}{2}\right)_1^2 \right]$$

$$= \frac{(\widehat{c}^*)^2}{2} - \frac{(\widehat{c}^*)^3}{3} + [1-\overline{p}] \left[\frac{1}{2} - \frac{(\widehat{c}^*)^2}{2} - \frac{1}{3} + \frac{(\widehat{c}^*)^3}{3} + \frac{8}{3} - \frac{1}{3} - 2 + \frac{1}{2} \right]$$

$$= \frac{(\widehat{c}^*)^2}{2} - \frac{(\widehat{c}^*)^3}{3} + [1-\overline{p}] \left[1 - \frac{(\widehat{c}^*)^2}{2} + \frac{(\widehat{c}^*)^3}{3} \right]. \tag{232}$$

When (231) holds, the denominator in the expression for $\widehat{c}^*$ in (179) is:

$$G(\widehat{c}^*) + [1-\overline{p}]\,[1 - G(\widehat{c}^*)] \;=\; \int_0^{\widehat{c}^*} [1-c]\, dc + [1-\overline{p}] \left[1 - \int_0^{\widehat{c}^*} (1-c)\, dc \right]$$

$$= \widehat{c}^* - \frac{(\widehat{c}^*)^2}{2} + [1-\overline{p}] \left[1 - \widehat{c}^* + \frac{(\widehat{c}^*)^2}{2} \right]. \tag{233}$$

(179), (232), and (233) imply that when (231) holds, the welfare-maximizing value of $\widehat{c}^*$ is determined by:

$$\widehat{c}^* \left[\widehat{c}^* - \frac{(\widehat{c}^*)^2}{2} + [1-\overline{p}] \left(1 - \widehat{c}^* + \frac{(\widehat{c}^*)^2}{2} \right) \right]$$

$$= \frac{(\widehat{c}^*)^2}{2} - \frac{(\widehat{c}^*)^3}{3} + [\,1 - \overline{p}\,]\left[1 - \frac{(\widehat{c}^*)^2}{2} + \frac{(\widehat{c}^*)^3}{3}\right]$$

$$\Rightarrow \quad (\widehat{c}^*)^2 - \frac{(\widehat{c}^*)^2}{2} - \frac{(\widehat{c}^*)^3}{2} + \frac{(\widehat{c}^*)^3}{3}$$

$$= [\,1 - \overline{p}\,]\left[1 - \frac{(\widehat{c}^*)^2}{2} + \frac{(\widehat{c}^*)^3}{3} + (\widehat{c}^*)^2 - \widehat{c}^* - \frac{(\widehat{c}^*)^3}{2}\right]$$

$$\Rightarrow \quad \frac{(\widehat{c}^*)^2}{2} - \frac{(\widehat{c}^*)^3}{6} - [\,1 - \overline{p}\,]\left[1 + \frac{(\widehat{c}^*)^2}{2} - \frac{(\widehat{c}^*)^3}{6} - \widehat{c}^*\right] = 0$$

$$\Rightarrow \quad \frac{(\widehat{c}^*)^2}{2} - \frac{(\widehat{c}^*)^3}{6} - 1 - \frac{(\widehat{c}^*)^2}{2} + \frac{(\widehat{c}^*)^3}{6} + \widehat{c}^* + \overline{p}\left[1 + \frac{(\widehat{c}^*)^2}{2} - \frac{(\widehat{c}^*)^3}{6} - \widehat{c}^*\right] = 0$$

$$\Rightarrow \quad -\,1 + \widehat{c}^* + \overline{p}\left[1 + \frac{(\widehat{c}^*)^2}{2} - \frac{(\widehat{c}^*)^3}{6} - \widehat{c}^*\right] = 0$$

$$\Rightarrow \quad \overline{p}\left[1 + \frac{(\widehat{c}^*)^2}{2} - \frac{(\widehat{c}^*)^3}{6} - \widehat{c}^*\right] = 1 - \widehat{c}^*$$

$$\Rightarrow \quad \overline{p}\,\left[\,6\,(1 - \widehat{c}^*) + (\widehat{c}^*)^2\,(3 - \widehat{c}^*)\,\right] = 6\,[\,1 - \widehat{c}^*\,]$$

$$\Rightarrow \quad \overline{p} = \frac{6\,[\,1 - \widehat{c}^*\,]}{6\,[\,1 - \widehat{c}^*\,] + (\widehat{c}^*)^2\,[\,3 - \widehat{c}^*\,]}\,. \tag{234}$$

From (192), A_m is defined by:

$$1 - 2\left[G\left(\widehat{c}^* + \frac{A_m}{T}\,a_2\right) - G\left(\widehat{c}^* - \frac{A_m}{T}\,a_1\right)\right] = 0\,. \tag{235}$$

Observe that:

$$A_m \gtreqless A_w \quad \Leftrightarrow \quad 1 - 2\left[G\left(\widehat{c}^* + \frac{A_w}{T}\,a_2\right) - G\left(\widehat{c}^* - \frac{A_w}{T}\,a_1\right)\right] \gtreqless 0$$

$$\Leftrightarrow \quad G\left(\widehat{c}^* + \frac{A_w}{T}\,a_2\right) - G\left(\widehat{c}^* - \frac{A_w}{T}\,a_1\right) \lesseqgtr \frac{1}{2}\,. \tag{236}$$

The first equivalence in (236) holds because the last inequality states that more than half of the population prefers MJS to VJS when $A = A_w$. By definition, the same number of individuals prefer MJS and VJS if $A = A_m$. Therefore, A_m must exceed A_w and so for $A \in (A_w, A_m)$, the majority will favor VJS even though welfare would be higher under MJS.

Because $\frac{A_w}{T} = c^e - \widehat{c}^*$ from (194) and $a_2 \equiv \frac{1-\overline{p}\,[\,1-G(\widehat{c}^*)\,]}{\overline{p}\,G(\widehat{c}^*)}$ from (191):

$$\widehat{c}^* + \frac{A_w}{T}\,a_2 \;=\; \widehat{c}^* + [\,c^e - \widehat{c}^*\,]\left[\frac{1-\overline{p}\,[\,1-G(\widehat{c}^*)\,]}{\overline{p}\,G(\widehat{c}^*)}\right] \;=\; \widehat{c}^* + [\,c^e - \widehat{c}^*\,]\left[1 + \frac{1-\overline{p}}{\overline{p}\,G(\widehat{c}^*)}\right]$$

$$= \; \widehat{c}^* + c^e - \widehat{c}^* + [\,c^e - \widehat{c}^*\,]\left[\frac{1-\overline{p}}{\overline{p}\,G(\widehat{c}^*)}\right] \;=\; c^e + [\,c^e - \widehat{c}^*\,]\left[\frac{1-\overline{p}}{\overline{p}\,G(\widehat{c}^*)}\right]. \tag{237}$$

Because $\frac{A_w}{T} = c^e - \widehat{c}^*$ from (194) and $a_1 \equiv \frac{1-\overline{p}\,[\,1-G(\widehat{c}^*)\,]}{\overline{p}\,[\,1-G(\widehat{c}^*)\,]}$ from (191):

$$\widehat{c}^* - \frac{A_w}{T}\,a_1 \;=\; \widehat{c}^* - [\,c^e - \widehat{c}^*\,]\;\frac{1-\overline{p}\,[\,1-G(\widehat{c}^*)\,]}{\overline{p}\,[\,1-G(\widehat{c}^*)\,]} \;=\; \widehat{c}^* - [\,c^e - \widehat{c}^*\,]\left[\frac{1}{\overline{p}\,[\,1-G(\widehat{c}^*)\,]} - 1\right]$$

$$= \; \widehat{c}^* + c^e - \widehat{c}^* - [\,c^e - \widehat{c}^*\,]\;\frac{1}{\overline{p}\,[\,1-G(\widehat{c}^*)\,]} \;=\; c^e - [\,c^e - \widehat{c}^*\,]\;\frac{1}{\overline{p}\,[\,1-G(\widehat{c}^*)\,]}\,. \tag{238}$$

(236), (237), and (238) imply:

$$A_m \;\gtreqless\; A_w \;\; \text{as} \;\; G\,(c^e + [\,c^e - \widehat{c}^*\,]\,\alpha_2) - \,G\,(c^e - [\,c^e - \widehat{c}^*\,]\,\alpha_1) \;\;\lesseqgtr\; \frac{1}{2} \tag{239}$$

$$\text{where} \quad \alpha_1 \;\equiv\; \frac{1}{\overline{p}\,[\,1-G(\widehat{c}^*)\,]} \quad \text{and} \quad \alpha_2 \;\equiv\; \frac{1-\overline{p}}{\overline{p}\,G(\widehat{c}^*)}\,. \tag{240}$$

The left hand side of the second inequality in (239) is the area under $g(c)$ for c between $c^e - [\,c^e - \widehat{c}^*\,]\,\alpha_1$ and $c^e + [\,c^e - \widehat{c}^*\,]\,\alpha_2$. This area is the sum of the areas under $g(c)$ for c between: (i) $c^e - [\,c^e - \widehat{c}^*\,]\,\alpha_1$ and 1; and (ii) 1 and $c^e + [\,c^e - \widehat{c}^*\,]\,\alpha_2$.

From (231), the area under $g(c)$ for c between $c^e - [\,c^e - \widehat{c}^*\,]\,\alpha_1$ and 1 is:

$$\int_{1-[1-\widehat{c}^*]\,\alpha_1}^{1} [\,1-c\,]\,dc \;=\; c\,\Big|_{1-[1-\widehat{c}^*]\alpha_1}^{1} - \frac{c^2}{2}\Big|_{1-[1-\widehat{c}^*]\alpha_1}^{1}$$

$$= \; 1 - (1 - [\,1-\widehat{c}^*\,]\,\alpha_1) - \frac{1}{2} + \frac{(\,1 - [\,1-\widehat{c}^*\,]\,\alpha_1)^2}{2}$$

$$= \; [\,1-\widehat{c}^*\,]\,\alpha_1 - \frac{1}{2} + \frac{1}{2} - \frac{2\,[\,1-\widehat{c}^*\,]\,\alpha_1}{2} + \frac{(\,[\,1-\widehat{c}^*\,]\,\alpha_1)^2}{2}$$

$$= \; [\,1-\widehat{c}^*\,]\,\alpha_1 - [\,1-\widehat{c}^*\,]\,\alpha_1 + \frac{(\,[\,1-\widehat{c}^*\,]\,\alpha_1)^2}{2} \;=\; \frac{(\,[\,1-\widehat{c}^*\,]\,\alpha_1)^2}{2}\,. \tag{241}$$

From (231), the area under $g(c)$ for c between 1 and $c^e + [\,c^e - \widehat{c}^*\,]\,\alpha_2$ is:

$$\int_{1}^{1+[1-\widehat{c}^*]\,\alpha_2} [\,c-1\,]\,dc \;=\; \frac{c^2}{2}\Big|_{1}^{1+[1-\widehat{c}^*]\alpha_2} \;-\; c\,\Big|_{1}^{1+[1-\widehat{c}^*]\alpha_2}$$

$$= \frac{(1 + [1 - \widehat{c}^*]\,\alpha_2)^2}{2} - \frac{1}{2} - (1 + [1 - \widehat{c}^*]\,\alpha_2) + 1$$

$$= \frac{1}{2} + \frac{2\,[1 - \widehat{c}^*]\,\alpha_2}{2} + \frac{([1 - \widehat{c}^*]\,\alpha_2)^2}{2} - \frac{1}{2} - 1 - [1 - \widehat{c}^*]\,\alpha_2 + 1$$

$$= [1 - \widehat{c}^*]\,\alpha_2 + \frac{([1 - \widehat{c}^*]\,\alpha_2)^2}{2} - [1 - \widehat{c}^*]\,\alpha_2 = \frac{([1 - \widehat{c}^*]\,\alpha_2)^2}{2}\,. \tag{242}$$

(239), (241), and (242) imply:

$$A_m \gtreqless A_w \Leftrightarrow \frac{([1 - \widehat{c}^*]\,\alpha_1)^2}{2} + \frac{([1 - \widehat{c}^*]\,\alpha_2)^2}{2} \lesseqgtr \frac{1}{2}$$

$$\Leftrightarrow [1 - \widehat{c}^*]^2\,(\alpha_1)^2 + [1 - \widehat{c}^*]^2\,(\alpha_2)^2 \lesseqgtr 1$$

$$\Leftrightarrow [1 - \widehat{c}^*]^2\left[(\alpha_1)^2 + (\alpha_2)^2\right] \lesseqgtr 1\,. \tag{243}$$

Recall from (233) that $G(\widehat{c}^*) = \widehat{c}^* - \frac{(\widehat{c}^*)^2}{2}$ when (231) holds. Therefore, From (240):

$$(\alpha_1)^2 + (\alpha_2)^2 = \left[\frac{1}{\overline{p}\,[1 - G(\widehat{c}^*)]}\right]^2 + \left[\frac{1 - \overline{p}}{\overline{p}\;G(\widehat{c}^*)}\right]^2$$

$$= \frac{1}{(\overline{p})^2}\left\{\left[\frac{1}{[1 - G(\widehat{c}^*)]^2}\right] + \frac{(1 - \overline{p})^2}{[G(\widehat{c}^*)]^2}\right\}$$

$$= \frac{1}{(\overline{p})^2}\left\{\left[\frac{1}{\left[1 - \widehat{c}^* + \frac{(\widehat{c}^*)^2}{2}\right]^2}\right] + \left[\frac{(1 - \overline{p})^2}{\left[\widehat{c}^* - \frac{(\widehat{c}^*)^2}{2}\right]^2}\right]\right\}$$

$$= \frac{1}{(\overline{p})^2}\left\{\left[\frac{1}{\left[\frac{2 - 2\,\widehat{c}^* + (\widehat{c}^*)^2}{2}\right]^2}\right] + \left[\frac{(1 - \overline{p})^2}{\left[\frac{2\,\widehat{c}^* - (\widehat{c}^*)^2}{2}\right]^2}\right]\right\}$$

$$= \frac{4}{(\overline{p})^2\left[2 - 2\,\widehat{c}^* + (\widehat{c}^*)^2\right]^2} + \frac{4\,[1 - \overline{p}]^2}{(\overline{p})^2\left[2\,\widehat{c}^* - (\widehat{c}^*)^2\right]^2}\,. \tag{244}$$

From (234):

$$\frac{1 - \overline{p}}{\overline{p}} = \frac{1 - \frac{6[1 - \widehat{c}^*]}{6\,[1 - \widehat{c}^*] + (\widehat{c}^*)^2[3 - \widehat{c}^*]}}{\frac{6\,[1 - \widehat{c}^*]}{6\,[1 - \widehat{c}^*] + (\widehat{c}^*)^2[3 - \widehat{c}^*]}} = \frac{(\widehat{c}^*)^2\,[3 - \widehat{c}^*]}{6\,[1 - \widehat{c}^*]}\,. \tag{245}$$

(234), (244), and (245) imply:

$$(\alpha_1)^2 + (\alpha_2)^2 = \frac{4}{(\overline{p})^2 \left[1 - 2\widehat{c}^* + (\widehat{c}^*)^2 \right]^2} + \left[\frac{(\widehat{c}^*)^2 (3 - \widehat{c}^*)}{6(1 - \widehat{c}^*)} \right]^2 \frac{4}{\left[2\widehat{c}^* - (\widehat{c}^*)^2 \right]^2}$$

$$= \frac{4}{(\overline{p})^2 \left[2 - 2\widehat{c}^* + (\widehat{c}^*)^2 \right]^2} + \frac{(\widehat{c}^*)^2 [3 - \widehat{c}^*]^2}{9[1 - \widehat{c}^*]^2 [2 - \widehat{c}^*]^2}$$

$$= \frac{\left[6(1 - \widehat{c}^*) + (\widehat{c}^*)^2 (3 - \widehat{c}^*) \right]^2}{36[1 - \widehat{c}^*]^2} \frac{4}{\left[2 - 2\widehat{c}^* + (\widehat{c}^*)^2 \right]^2} + \frac{(\widehat{c}^*)^2 [3 - \widehat{c}^*]^2}{9[1 - \widehat{c}^*]^2 [2 - \widehat{c}^*]^2}$$

$$= \frac{\left[6(1 - \widehat{c}^*) + (\widehat{c}^*)^2 (3 - \widehat{c}^*) \right]^2}{9[1 - \widehat{c}^*]^2 \left[2 - 2\widehat{c}^* + (\widehat{c}^*)^2 \right]^2} + \frac{(\widehat{c}^*)^2 [3 - \widehat{c}^*]^2}{9[1 - \widehat{c}^*]^2 [2 - \widehat{c}^*]^2}$$

$$\Rightarrow \quad [1 - \widehat{c}^*]^2 \left[(\alpha_1)^2 + (\alpha_2)^2 \right]$$

$$= \frac{\left[6(1 - \widehat{c}^*) + (\widehat{c}^*)^2 (3 - \widehat{c}^*) \right]^2}{9\left[2 - 2\widehat{c}^* + (\widehat{c}^*)^2 \right]^2} + \frac{(\widehat{c}^*)^2 [3 - \widehat{c}^*]^2}{9[2 - \widehat{c}^*]^2} . \tag{246}$$

(243) and (246) imply:

$$[1 - \widehat{c}^*]^2 \left[(\alpha_1)^2 + (\alpha_2)^2 \right] \lesseqqgtr 1 \Leftrightarrow \varphi(\widehat{c}^*) \gtreqqless 0, \text{ where, for } \widehat{c}^* \in [0, 1], \tag{247}$$

$$\varphi(\widehat{c}^*) \equiv 1 - \frac{\left[6(1 - \widehat{c}^*) + (\widehat{c}^*)^2 (3 - \widehat{c}^*) \right]^2}{9\left[2 - 2\widehat{c}^* + (\widehat{c}^*)^2 \right]^2} - \frac{(\widehat{c}^*)^2 [3 - \widehat{c}^*]^2}{9[2 - \widehat{c}^*]^2} . \tag{248}$$

(243) implies that $A_m \lesseqqgtr A_w$ as $\varphi(\widehat{c}^*) \lesseqqgtr 0$.

(248) implies:

$$\varphi(0) = 1 - \frac{[6]^2}{9[2]^2} = 0 \quad \text{and} \quad \varphi(1) = 1 - \frac{[2]^2}{9} - \frac{[2]^2}{9} = \frac{1}{9} > 0 . \tag{249}$$

Furthermore, it can be verified that for $\widehat{c}^* \in (0, 1]$, $\varphi(\widehat{c}^*) \lesseqqgtr 0$ as $\widehat{c}^* \lesseqqgtr \widetilde{c}_1 \approx 0.585786$. In addition, (234) implies that $\overline{p} = 0.75$ when $\widehat{c}^* = \widetilde{c}_1$. Also, from (234):

$$\frac{\partial \overline{p}}{\partial \widehat{c}^*} \overset{s}{=} -6[1 - \widehat{c}^*] - (\widehat{c}^*)^2 [3 - \widehat{c}^*] - [1 - \widehat{c}^*] \left[-6 + 6\widehat{c}^* - 3(\widehat{c}^*)^2 \right]$$

$$= -(\widehat{c}^*)^2 [3 - \widehat{c}^*] - [1 - \widehat{c}^*] \left[6\widehat{c}^* - 3(\widehat{c}^*)^2 \right]$$

$$= -3(\widehat{c}^*)^2 + (\widehat{c}^*)^3 - 6\widehat{c}^* + 3(\widehat{c}^*)^2 + 6(\widehat{c}^*)^2 - 3(\widehat{c}^*)^3$$

$$= -6\widehat{c}^* [1 - \widehat{c}^*] - 2(\widehat{c}^*)^3 < 0 .$$

Because $\overline{p}$ and $\widehat{c}^*$ vary inversely, it follows that $\varphi(\widehat{c}^*) \lesseqqgtr 0$ as $\overline{p} \gtreqqless 0.75$. ■

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